J. reine angew. Math. **662** (2012), 1—56
DOI 10.1515/CRELLE.2011.133

Journal für die reine und
angewandte Mathematik

Berezin–Toeplitz quantization on Kähler manifolds

Dedicated to Professor Jochen Brüning on the occasion of his 65th birthday

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Abstract. We study Berezin–Toeplitz quantization on Kähler manifolds. We explain first how to compute various associated asymptotic expansions, then we compute explicitly the first terms of the expansion of the kernel of the Berezin–Toeplitz operators, and of the composition of two Berezin–Toeplitz operators. As an application, we estimate the norm of Donaldson's Q-operator.

Contents

0. Introduction

Berezin–Toeplitz operators are important in geometric quantization and the properties of their kernels turn out to be deeply related to various problems in Kähler geometry (see e.g. [16], [17]). In this paper, we will study the precise asymptotic expansion of these kernels. We refer the reader to the book [24] for a comprehensive study of the Bergman kernel, Berezin–Toeplitz quantization and its applications. See also the survey [23].

The setting of Berezin–Toeplitz quantization on Kähler manifolds is the following. Let (X,ω,J) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ with Kähler form ω and complex structure J. Let (L,h^L) be a holomorphic Hermitian line bundle on X, and let (E,h^E) be a holomorphic Hermitian vector bundle on X. Let ∇^L, ∇^E be the holomorphic Hermitian connections on (L,h^L), (E,h^E) with curvatures $R^L = (\nabla^L)^2$, $R^E = (\nabla^E)^2$, respectively. We assume that (L,h^L,∇^L) is a prequantum line bundle, i.e.,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.1}$$

Let $g^{TX}(\cdot,\cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω and J. The Riemannian volume form dv_X of (X,g^{TX}) has the form $dv_X = \omega^n/n!$. The L^2-Hermitian product on the space $\mathscr{C}^\infty(X, L^p\otimes E)$ of smooth sections of $L^p\otimes E$ on X, with $L^p := L^{\otimes p}$, is given by

$$\langle s_1, s_2\rangle = \int_X \langle s_1, s_2\rangle(x)\, dv_X(x). \tag{0.2}$$

We denote the corresponding norm by $\|\cdot\|_{L^2}$ and by $L^2(X, L^p\otimes E)$ the completion of $\mathscr{C}^\infty(X, L^p\otimes E)$ with respect to this norm.

Given a continuous smoothing linear operator $K : L^2(X, L^p\otimes E) \to L^2(X, L^p\otimes E)$, the Schwartz kernel theorem [24], Theorem B.2.7, guarantees the existence of an integral kernel with respect to dv_X, denoted by $K(x,x') \in (L^p\otimes E)_x \otimes (L^p\otimes E)^*_{x'}$, for $x, x'\in X$, i.e.,

$$(KS)(x) = \int_X K(x,x')S(x')\, dv_X(x'), \quad S\in L^2(X, L^p\otimes E). \tag{0.3}$$

Consider now the space $H^0(X, L^p\otimes E)$ of holomorphic sections of $L^p\otimes E$ on X and let $P_p : L^2(X, L^p\otimes E) \to H^0(X, L^p\otimes E)$ be the orthogonal (Bergman) projection. Its kernel $P_p(x,x')$ with respect to $dv_X(x')$ is smooth; it is called the *Bergman kernel*. The *Berezin–Toeplitz quantization* of a section $f\in \mathscr{C}^\infty\big(X, \mathrm{End}(E)\big)$ is the *Berezin–Toeplitz operator* $\{T_{f,p}\}_{p\in\mathbb{N}}$ which is a sequence of linear operators $T_{f,p}$ defined by

$$T_{f,p} : L^2(X, L^p\otimes E) \to L^2(X, L^p\otimes E), \quad T_{f,p} = P_p f P_p. \tag{0.4}$$

The kernel $T_{f,p}(x,x')$ of $T_{f,p}$ with respect to $dv_X(x')$ is also smooth. Since $\mathrm{End}(L) = \mathbb{C}$, we have $T_{f,p}(x,x) \in \mathrm{End}(E)_x$ for $x\in X$.

We introduce now the relevant geometric objects used in Theorems 0.1, 0.2 and 0.3. Let $T^{(1,0)}X$ be the holomorphic tangent bundle on X, and $T^{*(1,0)}X$ its dual bundle. Let ∇^{TX} be the Levi–Civita connection on (X, g^{TX}). We denote by $R^{TX} = (\nabla^{TX})^2$ the curvature, by Ric the Ricci curvature and by $\boldsymbol{r}$ the scalar curvature of ∇^{TX} (cf. (3.3)).

We still denote by ∇^E the connection on $\mathrm{End}(E)$ induced by ∇^E. Consider the (positive) Laplacian Δ acting on the functions on (X, g^{TX}) and the Bochner Laplacian Δ^E on $\mathscr{C}^\infty(X, E)$ and on $\mathscr{C}^\infty\big(X, \mathrm{End}(E)\big)$. Let $\{e_k\}$ be a (local) orthonormal frame of (TX, g^{TX}). Then

$$\Delta^E = -\sum_k (\nabla^E_{e_k}\nabla^E_{e_k} - \nabla^E_{\nabla^{TX}_{e_k} e_k}). \tag{0.5}$$

Let $\Omega^{q,r}\big(X, \mathrm{End}(E)\big)$ be the space of (q,r)-forms on X with values in $\mathrm{End}(E)$, and let

$$\nabla^{1,0} : \Omega^{q,*}\big(X, \mathrm{End}(E)\big) \to \Omega^{q+1,*}\big(X, \mathrm{End}(E)\big) \tag{0.6}$$

be the $(1,0)$-component of the connection ∇^E. Let $(\nabla^E)^*$, $\nabla^{1,0*}, \bar\partial^{E*}$ be the adjoints of ∇^E, $\nabla^{1,0}$, $\bar\partial^E$, respectively. Let $D^{1,0}$, $D^{0,1}$ be the $(1,0)$ and $(0,1)$ components of the connection $\nabla^{T^*X} : \mathscr{C}^\infty(X, T^*X) \to \mathscr{C}^\infty(X, T^*X \otimes T^*X)$ induced by ∇^{TX}.

In the following, we denote by

$$\langle \cdot, \cdot \rangle_\omega : \Omega^{*,*}\big(X, \mathrm{End}(E)\big) \times \Omega^{*,*}\big(X, \mathrm{End}(E)\big) \to \mathscr{C}^\infty\big(X, \mathrm{End}(E)\big)$$

the $\mathbb{C}$-bilinear pairing $\langle \alpha \otimes f, \beta \otimes g\rangle_\omega = \langle \alpha, \beta\rangle f \cdot g$, for forms $\alpha, \beta \in \Omega^{*,*}(X)$ and sections $f, g \in \mathscr{C}^\infty\big(X, \mathrm{End}(E)\big)$ (cf. (0.14), (0.16), (0.17)). Put

$$R^E_\Lambda = \langle R^E, \omega\rangle_\omega. \tag{0.7}$$

Let $\mathrm{Ric}_\omega = \mathrm{Ric}(J\cdot, \cdot)$ be the $(1,1)$-form associated to Ric. Set

$$|\mathrm{Ric}_\omega|^2 = \sum_{i<j} \mathrm{Ric}_\omega(e_i, e_j)^2, \quad |R^{TX}|^2 = \sum_{i<j}\sum_{k<l} \langle R^{TX}(e_i, e_j)e_k, e_l\rangle^2,$$

and let

$$\begin{aligned}
\boldsymbol{b}_{2\mathbb{C}} &= -\frac{\Delta \boldsymbol{r}}{48} + \frac{1}{96}|R^{TX}|^2 - \frac{1}{24}|\mathrm{Ric}_\omega|^2 + \frac{1}{128}\boldsymbol{r}^2,\\
\boldsymbol{b}_{2E} &= \frac{\sqrt{-1}}{32}(2\boldsymbol{r}R^E_\Lambda - 4\langle \mathrm{Ric}_\omega, R^E\rangle_\omega + \Delta^E R^E_\Lambda) - \frac{1}{8}(R^E_\Lambda)^2\\
&\quad + \frac{1}{8}\langle R^E, R^E\rangle_\omega + \frac{3}{16}\bar\partial^{E*}\nabla^{1,0*}R^E,\\
\boldsymbol{b}_1 &= \frac{\boldsymbol{r}}{8\pi} + \frac{\sqrt{-1}}{2\pi}R^E_\Lambda, \quad \boldsymbol{b}_2 = \frac{1}{\pi^2}(\boldsymbol{b}_{2\mathbb{C}} + \boldsymbol{b}_{2E}).
\end{aligned} \tag{0.8}$$

We use now the notation from (3.6). By our convention (cf. (3.5)), we have at $x_0 \in X$,

$$\langle \alpha_{\ell\bar m}\, dz_\ell \wedge d\bar z_m, \beta_{k\bar q}\, dz_k \wedge d\bar z_q\rangle = -4\alpha_{\ell\bar m}\beta_{m\bar\ell}, \quad \langle \alpha_{\bar m\bar q}\, d\bar z_m \otimes d\bar z_q, \beta_{k\ell}\, dz_k \otimes dz_\ell\rangle = 4\alpha_{\bar m\bar q}\beta_{mq}$$

(note that $|dz_q|^2 = 2$). Then by Lemma 3.1, (5.3) and (5.4), we have at $x_0 \in X$,

$$\mathrm{Ric}_{\ell\bar{k}} = 2R_{\ell\bar{k}q\bar{q}} = 2R_{\ell\bar{q}q\bar{k}}, \quad \boldsymbol{r} = 8R_{\ell\bar{\ell}q\bar{q}}, \quad \mathrm{Ric}_\omega = \sqrt{-1}\,\mathrm{Ric}_{\ell\bar{k}}\,dz_\ell \wedge d\bar{z}_k,$$

$$\sqrt{-1}R^E_\Lambda = 2R^E_{k\bar{k}}, \quad \boldsymbol{b}_1 = \frac{1}{\pi}(R_{k\bar{k}m\bar{m}} + R^E_{m\bar{m}}),$$

$$(0.9) \qquad \boldsymbol{b}_{2\mathbb{C}} = -\frac{\Delta \boldsymbol{r}}{48} + \frac{1}{6}R_{k\bar{\ell}m\bar{q}}R_{\ell\bar{k}q\bar{m}} - \frac{2}{3}R_{\ell\bar{\ell}m\bar{q}}R_{k\bar{k}q\bar{m}} + \frac{1}{2}R_{\ell\bar{\ell}q\bar{q}}R_{k\bar{k}m\bar{m}},$$

$$\boldsymbol{b}_{2E} = R^E_{q\bar{q}}R_{k\bar{k}m\bar{m}} - R^E_{m\bar{q}}R_{k\bar{k}q\bar{m}} + \frac{1}{2}(R^E_{q\bar{q}}R^E_{m\bar{m}} - R^E_{m\bar{q}}R^E_{q\bar{m}}) + \frac{1}{4}(-R^E_{k\bar{k};m\bar{m}} + 3R^E_{m\bar{k};k\bar{m}}).$$

We say that a sequence $\Theta_p \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has an asymptotic expansion of the form

$$(0.10) \qquad \Theta_p(x) = \sum_{r=0}^{\infty} \boldsymbol{A}_r(x)p^{n-r} + \mathscr{O}(p^{-\infty}), \quad \boldsymbol{A}_r \in \mathscr{C}^\infty(X, \mathrm{End}(E)),$$

if for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}^*$, we have

$$(0.11) \qquad \left|\Theta_p(x) - \sum_{r=0}^{k} \boldsymbol{A}_r(x)p^{n-r}\right|_{\mathscr{C}^l(X)} \leqq C_{k,l}p^{n-k-1},$$

where $|\cdot|_{\mathscr{C}^l(X)}$ is the $\mathscr{C}^l$-norm on X.

Theorem 0.1. *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we have*

$$(0.12) \qquad T_{f,p}(x,x) = \sum_{r=0}^{\infty} \boldsymbol{b}_{r,f}(x)p^{n-r} + \mathscr{O}(p^{-\infty}), \quad \boldsymbol{b}_{r,f} \in \mathscr{C}^\infty(X, \mathrm{End}(E)).$$

Moreover,

$$(0.13) \qquad \boldsymbol{b}_{0,f} = f, \quad \boldsymbol{b}_{1,f} = \frac{\boldsymbol{r}}{8\pi}f + \frac{\sqrt{-1}}{4\pi}(R^E_\Lambda f + fR^E_\Lambda) - \frac{1}{4\pi}\Delta^E f.$$

If $f \in \mathscr{C}^\infty(X)$, then

$$(0.14) \qquad \pi^2\boldsymbol{b}_{2,f} = \pi^2\boldsymbol{b}_2 f + \frac{1}{32}\Delta^2 f - \frac{1}{32}\boldsymbol{r}\Delta f - \frac{\sqrt{-1}}{8}\langle \mathrm{Ric}_\omega, \partial\bar{\partial}f\rangle + \frac{\sqrt{-1}}{24}\langle df, \nabla^E R^E_\Lambda\rangle_\omega + \frac{1}{24}\langle \partial f, \nabla^{1,0*}R^E\rangle_\omega - \frac{1}{24}\langle \bar{\partial}f, \bar{\partial}^{E*}R^E\rangle_\omega - \frac{\sqrt{-1}}{8}(\Delta f)R^E_\Lambda + \frac{1}{4}\langle \partial\bar{\partial}f, R^E\rangle_\omega.$$

Theorem 0.2. *For any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the kernel of the composition $T_{f,p} \circ T_{g,p}$ has an asymptotic expansion on the diagonal*

$$(0.15) \qquad (T_{f,p} \circ T_{g,p})(x,x) = \sum_{r=0}^{\infty} \boldsymbol{b}_{r,f,g}(x)p^{n-r} + \mathscr{O}(p^{-\infty}), \quad \boldsymbol{b}_{r,f,g} \in \mathscr{C}^\infty(X, \mathrm{End}(E)),$$

in the sense of (0.11). *Moreover*, $\boldsymbol{b}_{0,f,g} = fg$ *and*

$$\boldsymbol{b}_{1,f,g} = \frac{1}{8\pi}\boldsymbol{r}fg + \frac{\sqrt{-1}}{4\pi}(R^E_\Lambda fg + fgR^E_\Lambda) - \frac{1}{4\pi}\big(f\Delta^E g + (\Delta^E f)g\big) + \frac{1}{2\pi}\langle\bar{\partial}^E f, \nabla^{1,0} g\rangle_\omega. \tag{0.16}$$

If $f, g \in \mathscr{C}^\infty(X)$, *then*

$$\begin{aligned}\boldsymbol{b}_{2,f,g} &= f\boldsymbol{b}_{2,g} + g\boldsymbol{b}_{2,f} - fg\boldsymbol{b}_2 \\ &\quad + \frac{1}{\pi^2}\Big\{-\frac{1}{8}\langle\bar{\partial}f, \partial\Delta g\rangle - \frac{1}{8}\langle\bar{\partial}\Delta f, \partial g\rangle + \frac{1}{2}\langle\bar{\partial}f, \partial g\rangle\pi\boldsymbol{b}_1 \\ &\quad - \frac{1}{4}\langle\bar{\partial}f \wedge \partial g, R^E\rangle_\omega + \frac{1}{16}\Delta f \cdot \Delta g + \frac{1}{8}\langle D^{0,1}\bar{\partial}f, D^{1,0}\partial g\rangle\Big\}.\end{aligned} \tag{0.17}$$

The existence of the expansions (0.12) and (0.15) and the formulas for the leading terms hold in fact in the symplectic setting and are consequences of [27], Lemma 4.6 and (4.79), or [24], Lemma 7.2.4 and (7.4.6), (cf. Lemma 2.2). The novel point of Theorems 0.1, 0.2 is the calculation of the coefficients $\boldsymbol{b}_{1,f}$, $\boldsymbol{b}_{2,f}$, $\boldsymbol{b}_{1,f,g}$ and $\boldsymbol{b}_{2,f,g}$. Note that the precise formula $\boldsymbol{b}_{1,f}$ for a function $f \in \mathscr{C}^\infty(X)$ was already given in [24], Problem 7.2. In Theorem 5.1, we find a general formula of $\boldsymbol{b}_{2,f}$ for any $f \in \mathscr{C}^\infty\big(X, \mathrm{End}(E)\big)$.

If $f = 1$, then $T_{f,p} = P_p$, and the existence of the expansion (0.12) and the form of the leading term was proved by [30], [6], [33]. The terms $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ were computed by Lu [22] (for $E = \mathbb{C}$, the trivial line bundle with trivial metric), X. Wang [32], L. Wang [31], in various degree of generality. The method of these authors is to construct appropriate peak sections as in [30], using Hörmander's L^2 $\bar{\partial}$-method. In [8], §5.1, Dai–Liu–Ma computed $\boldsymbol{b}_1$ by using the heat kernel, and in [25], §2, [26], §2 (cf. also [24], §4.1.8, §8.3.4), we computed $\boldsymbol{b}_1$ in the symplectic case.

The expansion of the Bergman kernel $P_p(x,x)$ on the diagonal, for $E = \mathbb{C}$, was rederived by Douglas and Klevtsov [11] by using path integral and perturbation theory. They give physics interpretations in terms of supersymmetric quantum mechanics, the quantum Hall effect and black holes (cf. also [12]).

An interesting consequence of Theorem 0.2 is the following precise computation of the expansion of the composition of two Berezin–Toeplitz operators.

Theorem 0.3. *Let* $f, g \in \mathscr{C}^\infty\big(X, \mathrm{End}(E)\big)$. *The product of the Toeplitz operators* $T_{f,p}$ *and* $T_{g,p}$ *is a Toeplitz operator*, *more precisely*, *it admits the asymptotic expansion*

$$T_{f,p} \circ T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{0.18}$$

where C_r *are bidifferential operators*, *in the sense that for any* $k \geqq 0$, *there exists* $c_k > 0$ *with*

$$\Big\| T_{f,p} \circ T_{g,p} - \sum_{l=0}^{k} p^{-l} T_{C_r(f,g),p} \Big\| \leqq c_k p^{-k-1}, \tag{0.19}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators. We have

$$
\begin{aligned}
C_0(f,g) &= fg,\\
C_1(f,g) &= -\frac{1}{2\pi}\langle \nabla^{1,0}f, \overline{\partial}^E g\rangle_\omega \in \mathscr{C}^\infty\big(X, \mathrm{End}(E)\big),\\
C_2(f,g) &= \boldsymbol{b}_{2,f,g} - \boldsymbol{b}_{2,fg} - \boldsymbol{b}_{1,\,C_1(f,g)}.
\end{aligned}
\tag{0.20}
$$

If $f, g \in \mathscr{C}^\infty(X)$, then

$$
\begin{aligned}
C_2(f,g) = \frac{1}{8\pi^2}\langle D^{1,0}\partial f, D^{0,1}\overline{\partial} g\rangle &+ \frac{\sqrt{-1}}{4\pi^2}\langle \mathrm{Ric}_\omega, \partial f\wedge\overline{\partial} g\rangle\\
&- \frac{1}{4\pi^2}\langle \partial f\wedge\overline{\partial} g, R^E\rangle_\omega.
\end{aligned}
\tag{0.21}
$$

The existence of the expansion (0.18) is a special case of [27], Theorem 1.1, (cf. also [24], Theorems 7.4.1 and 8.1.10), where we found a symplectic version in which the Toeplitz operators (0.4) are constructed by using the projection to the kernel of the Dirac operator. Note that the precise values of $\boldsymbol{b}_2$ are not used to derive (0.21) (cf. Section 5.3).

The existence of the expansion (0.18) for $E = \mathbb{C}$ was first established by Bordemann, Meinrenken and Schlichenmaier [3], Schlichenmaier [29] (cf. also [19]) using the theory of Toeplitz structures by Boutet de Monvel and Guillemin [4]. Charles [7] calculated $C_1(f,g)$ for $E = \mathbb{C}$. The asymptotic expansion (0.18) with a twisting bundle E was derived by Hawkins [18], Lemma 4.1, up to order one (i.e., (0.19) for $k = 0$).

Also, there is related work of Engliš [13], [14] dealing with expressing asymptotic expansions of Bergman kernel and coefficients of the Berezin–Toeplitz expansion (0.18) in terms of the metric. Engliš [14], Corollary 15, computed $C_1(f,g)$ and $C_2(f,g)$ for a smoothly bounded pseudoconvex domain $X = \{z \in \mathbb{C}^n : \varphi(z) > 0\}$, where φ is a defining function such that $-\log\varphi$ is strictly plurisubharmonic, and for the trivial line bundle $L = \mathbb{C}$ over X, equipped with the nontrivial metric $h^L = \varphi$ of positive curvature.

Note that we work throughout the paper with a non-trivial twisting bundle E. Moreover, we have shown in [27], §5–6, (cf. also [24], §7.5) that Berezin–Toeplitz quantization holds for complete Kähler manifolds and orbifolds endowed with a prequantum line bundle. The calculations of the coefficients in the present paper being local in nature, they hold also for the above cases.

For some applications of the results of this paper to Kähler geometry see the paper [17] by Fine.

We close the introduction with some remarks about the Berezin–Toeplitz star-product. Following the ground-breaking work of Berezin [1], one can define a star-product by using Toeplitz operators. Note that formal star-products are known to exist on symplectic manifolds by [9], [15]. The Berezin–Toeplitz star-product gives a very concrete and geometric realization of such product. For general symplectic manifolds this was realized in [24], [27] by using Toeplitz operators obtained by projecting on the kernel of the Dirac operator.

Consider now a compact Kähler manifold (X,ω) and a prequantum line bundle L. For every $f,g \in \mathscr{C}^\infty(X)$ one defines the Berezin–Toeplitz star-product (cf. [19], [29] and [24], [27] for the symplectic case) by

$$f * g := \sum_{k=0}^{\infty} C_k(f,g)\hbar^k \in \mathscr{C}^\infty(X)[\![\hbar]\!]. \tag{0.22}$$

This star-product is associative. Moreover, for $f,g \in \mathscr{C}^\infty(X)$ we have (cf. [24], (7.4.3), [27], (4.89))

$$C_0(f,g) = fg = C_0(g,f), \quad C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}, \tag{0.23}$$

where $\{f,g\}$ is the Poisson bracket associated to $2\pi\omega$. Therefore

$$[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \quad p \to \infty. \tag{0.24}$$

Consider a twisting holomorphic Hermitian vector bundle E and $f,g \in \mathscr{C}^\infty\big(X, \operatorname{End}(E)\big)$ as in Theorem 0.3. This corresponds to matrix-valued Berezin–Toeplitz quantization, which models a quantum system with $r = \operatorname{rank} E$ degrees of freedom. By (0.18), this Berezin–Toeplitz quantization has the expected semi-classical behaviour. Moreover, by [24], Theorem 7.4.2, [27], Theorem 4.19, we have

$$\lim_{p\to\infty} \|T_{f,p}\| = \|f\|_\infty := \sup_{0 \neq u \in E_x, x \in X} |f(x)(u)|_{h^E}/|u|_{h^E}. \tag{0.25}$$

Corollary 0.4. *Let $f,g \in \mathscr{C}^\infty\big(X, \operatorname{End}(E)\big)$. Set*

$$f * g := \sum_{k=0}^{\infty} C_k(f,g)\hbar^k \in \mathscr{C}^\infty\big(X, \operatorname{End}(E)\big)[\![\hbar]\!], \tag{0.26}$$

where $C_r(f,g)$ are determined by (0.18). *Then* (0.26) *defines an associative star-product on $\mathscr{C}^\infty\big(X, \operatorname{End}(E)\big)$. Set moreover*

$$\{\{f,g\}\} := \frac{1}{2\pi\sqrt{-1}}\big(\langle \nabla^{1,0} g, \bar\partial^E f\rangle_\omega - \langle \nabla^{1,0} f, \bar\partial^E g\rangle_\omega\big). \tag{0.27}$$

If $fg = gf$ on X we have

$$[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{\{f,g\}\},p} + \mathcal{O}(p^{-2}), \quad p \to \infty. \tag{0.28}$$

The associativity of the star-product (0.26) follows immediately from the associativity rule for the composition of Toeplitz operators, $(T_{f,p} \circ T_{g,p}) \circ T_{k,p} = T_{f,p} \circ (T_{g,p} \circ T_{k,p})$ for any $f,g,k \in \mathscr{C}^\infty\big(X, \operatorname{End}(E)\big)$, and from the asymptotic expansion (0.18) applied to both sides of the latter equality.

Due to the fact that $\{\{f,g\}\} = \{f,g\}$ if E is trivial and comparing (0.24) to (0.28), one can regard $\{\{f,g\}\}$ defined in (0.27) as a non-commutative Poisson bracket.

Remark 0.5. Throughout the paper we suppose that $g^{TX}(u,v) = \omega(u,Jv)$. The results presented so far still hold for a general non-Kähler J-invariant Riemannian metric g^{TX}. To explain this point we follow [24, §4.1.9].

Let us denote the metric associated to ω by $g^{TX}_{\omega} := \omega(\cdot, J\cdot)$. We identify the 2-form R^L with the Hermitian matrix $\dot{R}^L \in \operatorname{End}(T^{(1,0)}X)$ via g^{TX}. Then the Riemannian volume form of g^{TX}_{ω} is given by $dv_{X,\omega} = (2\pi)^{-n}\det(\dot{R}^L)\, dv_X$ (where dv_X is the Riemannian volume form of g^{TX}). Moreover, $h^E_{\omega} := \det\left(\frac{\dot{R}^L}{2\pi}\right)^{-1} h^E$ defines a metric on E. We add a subscript ω to indicate the objects associated to g^{TX}_{ω}, h^L and h^E_{ω}. Hence $\langle\cdot,\cdot\rangle_{\omega}$ denotes the L^2 Hermitian product on $\mathscr{C}^{\infty}(X, L^p \otimes E)$ induced by g^{TX}_{ω}, h^L, h^E_{ω}. This product is equivalent to the product $\langle\cdot,\cdot\rangle$ induced by g^{TX}, h^L, h^E.

Moreover, $H^0(X, L^p \otimes E)$ does not depend on the Riemannian metric on X or on the Hermitian metrics on L, E. Therefore, the orthogonal projections from $\big(\mathscr{C}^{\infty}(X, L^p \otimes E), \langle\cdot,\cdot\rangle_{\omega}\big)$ and $\big(\mathscr{C}^{\infty}(X, L^p \otimes E), \langle\cdot,\cdot\rangle\big)$ onto $H^0(X, L^p \otimes E)$ are the same. Hence $P_p = P_{p,\omega}$ and therefore $T_{f,p} = T_{f,p,\omega}$ as operators. However, their kernels are different. If $T_{f,p,\omega}(x,x')$, $(x,x' \in X)$, denotes the smooth kernels of $T_{f,p,\omega}$ with respect to $dv_{X,\omega}(x')$, we have

$$T_{f,p}(x,x') = (2\pi)^{-n}\det(\dot{R}^L)(x')\, T_{f,p,\omega}(x,x'). \tag{0.29}$$

For the kernel $T_{f,p,\omega}(x,x')$, we can apply Theorem 0.1 since $g^{TX}_{\omega}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX. We obtain in this way the expansion of $T_{f,p}(x,x)$ for a non-Kähler metric g^{TX} on X. By (0.29), the coefficients of these expansions (0.12), (0.18) satisfy

$$\begin{aligned} \boldsymbol{b}_{r,f} &= (2\pi)^{-n}\det(\dot{R}^L)\boldsymbol{b}_{r,f,\omega}, \\ C_r(f,g) &= C_{r,\omega}(f,g). \end{aligned} \tag{0.30}$$

This paper is organized as follows. In Section 1, we recall the formal calculus on $\mathbb{C}^n$ for the model operator $\mathscr{L}$, which is the main ingredient of our approach. In Section 2, we review the asymptotic expansion of the kernel of Berezin–Toeplitz operators and explain the strategy of our computation. In Section 3, we obtain explicitly the first terms of the Taylor expansion of our rescaled operator $\mathscr{L}_t$. In Section 4, we study in detail the contribution of $\mathcal{O}_2$, $\mathcal{O}_4$ to the term $\mathscr{F}_4$ from (2.20). In Section 5, by applying the formal calculi on $\mathbb{C}^n$ and the results from Section 4, we establish Theorems 0.1, 0.2 and 0.3. We also verify that our calculations are compatible with the Riemann–Roch–Hirzebruch Theorem. In Section 6, we estimate the $\mathscr{C}^m$-norm of Donaldson's Q-operator, thus continuing [20], [21].

We shall use the following notations. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^m$, $Z \in \mathbb{C}^m$, we set $|\alpha| := \sum_{j=1}^{m} \alpha_j$ and $Z^{\alpha} := Z_1^{\alpha_1} \cdots Z_m^{\alpha_m}$. Moreover, when an index variable appears twice in a single term, it means that we are summing over all its possible values.

Acknowledgments. We would like to thank Joel Fine for motivating and helpful discussions which led to the writing of this paper. We are grateful to the referee for pointing out some interesting references. X. M. thanks the Institut Universitaire de France for support. G. M. was partially supported by SFB/TR 12 and Fondation Sciences Mathématiques de Paris. We were also supported by the DAAD Procope Program.

1. Kernel calculus on $\mathbb{C}^n$

In this section we recall the formal calculus on $\mathbb{C}^n$ for the model operator $\mathscr{L}$ introduced in [27], §2, [24], §7.1, (with $a_j = 2\pi$ therein), and we derive the properties of the calculus of the kernels $(F\mathscr{P})(Z, Z')$, where $F \in \mathbb{C}[Z, Z']$ and $\mathscr{P}(Z, Z')$ is the kernel of the projection on the null space of the model operator $\mathscr{L}$. This calculus is the main ingredient of our approach.

Let us consider the canonical coordinates $(Z_1, \dots, Z_{2n})$ on the real vector space $\mathbb{R}^{2n}$. On the complex vector space $\mathbb{C}^n$ we consider the complex coordinates $(z_1, \dots, z_n)$. The two sets of coordinates are linked by the relation $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$, $j = 1, \dots, n$.

We consider the L^2-norm $\|\cdot\|_{L^2} = \left(\int_{\mathbb{R}^{2n}} |\cdot|^2\, dZ\right)^{1/2}$ on $\mathbb{R}^{2n}$, where $dZ = dZ_1 \cdots dZ_{2n}$ is the standard Euclidean volume form. We define the differential operators:

$$b_i = -2\frac{\partial}{\partial z_i} + \pi \bar{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i, \quad b = (b_1, \dots, b_n), \quad \mathscr{L} = \sum_i b_i b_i^+, \tag{1.1}$$

which extend to closed densely defined operators on $\left(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2}\right)$. As such, b_i^+ is the adjoint of b_i and $\mathscr{L}$ defines as a densely defined self-adjoint operator on $\left(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2}\right)$. The following result was established in [25], Theorem 1.15, (cf. also [24], Theorem 4.1.20).

Theorem 1.1. *The spectrum of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathscr{L}) = \{4\pi|\alpha| : \alpha \in \mathbb{N}^n\}. \tag{1.2}$$

Each $\lambda \in \operatorname{Spec}(\mathscr{L})$ is an eigenvalue of infinite multiplicity and an orthogonal basis of the corresponding eigenspace is given by

$$B_\lambda = \{b^\alpha(z^\beta e^{-\pi\sum_i |z_i|^2/2}) : \alpha \in \mathbb{N}^n \text{ with } 4\pi|\alpha| = \lambda, \beta \in \mathbb{N}^n\} \tag{1.3}$$

and $\bigcup\{B_\lambda : \lambda \in \operatorname{Spec}(\mathscr{L})\}$ forms a complete orthogonal basis of $L^2(\mathbb{R}^{2n})$. In particular, an orthonormal basis of $\operatorname{Ker}(\mathscr{L})$ *is*

$$\left\{\varphi_\beta(z) = \left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} z^\beta e^{-\pi\sum_i |z_i|^2/2} : \beta \in \mathbb{N}^n\right\}. \tag{1.4}$$

Let $\mathscr{P}(Z, Z')$ denote the kernel of the orthogonal projection $\mathscr{P} : L^2(\mathbb{R}^{2n}) \to \operatorname{Ker}(\mathscr{L})$ with respect to dZ'. Let $\mathscr{P}^\perp = \operatorname{Id} - \mathscr{P}$. We call $\mathscr{P}(\cdot,\cdot)$ the Bergman kernel of $\mathscr{L}$.

Obviously $\mathscr{P}(Z, Z') = \sum_\beta \varphi_\beta(z)\overline{\varphi_\beta(z')}$ so we infer from (1.4) that

$$\mathscr{P}(Z, Z') = \exp\left(-\frac{\pi}{2}\sum_{i=1}^n (|z_i|^2 + |z_i'|^2 - 2z_i\bar{z}_i')\right). \tag{1.5}$$

In the calculations involving the kernel $\mathscr{P}(\cdot,\cdot)$, we prefer however to use the orthogonal decomposition of $L^2(\mathbb{R}^{2n})$ given in Theorem 1.1 and the fact that $\mathscr{P}$ is an orthogonal projection, rather than integrating against the expression (1.5) of $\mathscr{P}(\cdot,\cdot)$. This point of view helps simplify a lot the computations and understand better the operations. As an example, if $\varphi(Z) = b^{\alpha}(z^{\beta}e^{-\pi\sum_i |z_i|^2/2})$ with $\alpha, \beta \in \mathbb{N}^n$, then Theorem 1.1 implies immediately that

$$(\mathscr{P}\varphi)(Z) = \begin{cases} z^{\beta}e^{-\pi\sum_i |z_i|^2/2}, & \text{if } |\alpha| = 0, \\ 0, & \text{if } |\alpha| > 0. \end{cases} \tag{1.6}$$

The following commutation relations are very useful in the computations. Namely, for any polynomial $g(z,\bar{z})$ in z and $\bar{z}$, we have

$$\begin{aligned} [b_i, b_j^+] &= b_i b_j^+ - b_j^+ b_i = -4\pi\delta_{ij}, \\ [b_i, b_j] &= [b_i^+, b_j^+] = 0, \\ [g(z,\bar{z}), b_j] &= 2\frac{\partial}{\partial z_j} g(z,\bar{z}), \\ [g(z,\bar{z}), b_j^+] &= -2\frac{\partial}{\partial \bar{z}_j} g(z,\bar{z}). \end{aligned} \tag{1.7}$$

For a polynomial F in Z, Z', we denote by $F\mathscr{P}$ the operator on $L^2(\mathbb{R}^{2n})$ defined by the kernel $F(Z,Z')\mathscr{P}(Z,Z')$ and the volume form dZ according to (0.3).

The following very useful lemma ([24], Lemma 7.1.1), describes the calculus of the kernels $(F\mathscr{P})(Z,Z') := F(Z,Z')\mathscr{P}(Z,Z')$.

Lemma 1.2. *For any $F, G \in \mathbb{C}[Z,Z']$ there exists a polynomial $\mathscr{K}[F,G] \in \mathbb{C}[Z,Z']$ with degree $\deg \mathscr{K}[F,G]$ of the same parity as $\deg F + \deg G$, such that*

$$\big((F\mathscr{P}) \circ (G\mathscr{P})\big)(Z,Z') = \mathscr{K}[F,G](Z,Z')\mathscr{P}(Z,Z'). \tag{1.8}$$

2. Expansion of the kernel of Berezin–Toeplitz operators

In this section, we review some results from [25], [27] (cf. also [24], §7.2). We explain then how to compute the coefficients of various expansions considered in this paper. We keep the notations and assumptions from the Introduction.

Kodaira–Laplace operator. Let $\bar{\partial}^{L^p\otimes E,*}$ be the adjoint of the Dolbeault operator $\bar{\partial}^{L^p\otimes E}$. Let $\square_p = \bar{\partial}^{L^p\otimes E,*}\bar{\partial}^{L^p\otimes E}$ be the restriction of the Kodaira Laplacian to $\mathscr{C}^{\infty}(X, L^p \otimes E)$. Let $\Delta^{L^p\otimes E}$ be the Bochner Laplacian on $\mathscr{C}^{\infty}(X, L^p \otimes E)$ associated to ∇^L, ∇^E, g^{TX}, defined as in (0.5). Then we have (cf. [24], Remark 1.4.8)

$$2\square_p = \Delta^{L^p\otimes E} - \frac{\sqrt{-1}}{2}R^E(e_i, Je_i) - 2np. \tag{2.1}$$

Moreover, by Hodge theory (cf. [24], Theorem 1.4.1) we have

$$\operatorname{Ker}(\square_p) = H^0(X, L^p \otimes E). \tag{2.2}$$

This identification is important since the computations are performed by rescaling $\square_p$ and expanding the rescaled operator.

Normal coordinates. Let a^X be the injectivity radius of (X, g^{TX}). We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. Then the exponential map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ onto $B^X(x,\varepsilon)$ for $\varepsilon \leqq a^X$. From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ via the exponential map for $\varepsilon \leqq a^X$. Throughout what follows, ε runs in the fixed interval $]0, a^X/4[$.

Basic trivialization. We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0,\varepsilon)$ we identify (L_Z, h_Z^L), (E_Z, h_Z^E) and $(L^p \otimes E)_Z$ to $(L_{x_0}, h_{x_0}^L)$, $(E_{x_0}, h_{x_0}^E)$ and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z : [0,1] \ni u \to \exp_{x_0}^X(uZ)$. This is the basic trivialization we use in this paper.

Using this trivialization we identify $f \in \mathscr{C}^\infty\big(X, \operatorname{End}(E)\big)$ to a family $\{f_{x_0}\}_{x_0\in X}$ where f_{x_0} is the function f in normal coordinates near x_0, i.e., $f_{x_0} : B^{T_{x_0}X}(0,\varepsilon) \to \operatorname{End}(E_{x_0})$, $f_{x_0}(Z) = f \circ \exp_{x_0}^X(Z)$. In general, for functions in the normal coordinates, we will add a subscript x_0 to indicate the base point $x_0 \in X$. Similarly, $P_p(x,x')$ induces in terms of the basic trivialization a smooth section $(Z,Z') \mapsto P_{p,x_0}(Z,Z')$ of $\pi^*\operatorname{End}(E)$ over $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0. Here we identify a section $S \in \mathscr{C}^\infty\big(TX \times_X TX, \pi^*\operatorname{End}(E)\big)$ with the family $(S_x)_{x\in X}$, where $S_x = S|_{\pi^{-1}(x)}$.

Coordinates on $T_{x_0}X$. Let us choose an orthonormal basis $\{w_i\}_{i=1}^n$ of $T_{x_0}^{(1,0)}X$. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \dots, n$, form an orthonormal basis of $T_{x_0}X$. We use coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ given by the identification

$$\mathbb{R}^{2n} \ni (Z_1, \dots, Z_{2n}) \mapsto \sum_i Z_i e_i \in T_{x_0}X. \tag{2.3}$$

In what follows we also use complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Volume form on $T_{x_0}X$. If dv_{TX} is the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, there exists a smooth positive function $\kappa_{x_0} : T_{x_0}X \to \mathbb{R}$, $Z \mapsto \kappa_{x_0}(Z)$, defined by

$$dv_X(Z) = \kappa_{x_0}(Z)\, dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1, \tag{2.4}$$

where the subscript x_0 of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$.

Sequences of operators. Let $\Theta_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)$ be a sequence of continuous linear operators with smooth kernel $\Theta_p(\cdot,\cdot)$ with respect to dv_X (e.g. $\Theta_p = T_{f,p}$). Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. In terms of our basic trivialization, $\Theta_p(x,y)$ induces a family of smooth sections $Z, Z' \mapsto \Theta_{p,x_0}(Z,Z')$ of $\pi^*\operatorname{End}(E)$ over $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0.

We denote by $|\Theta_{p,x_0}(Z,Z')|_{\mathscr{C}^l(X)}$ the $\mathscr{C}^l$ norm with respect to the parameter $x_0 \in X$. We say that $\Theta_{p,x_0}(Z,Z') = \mathcal{O}(p^{-\infty})$ if for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that $|\Theta_{p,x_0}(Z,Z')|_{\mathscr{C}^m(X)} \leqq C_{l,m} p^{-l}$.

Notation 2.1. Recall that $\mathscr{P}_{x_0} = \mathscr{P}$ was defined in (1.5). Fix $k \in \mathbb{N}$ and $\varepsilon' \in]0, a^X[$. Let $\{Q_{r,x_0}\}_{0 \leqq r \leqq k, x_0 \in X}$ be a family of polynomials $Q_{r,x_0} \in \operatorname{End}(E)_{x_0}[Z, Z']$ in Z, Z', which is smooth with respect to the parameter $x_0 \in X$. We say that

$$p^{-n}\Theta_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{2.5}$$

on $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon'\}$ if there exist $C_0 > 0$ and a decomposition

$$\begin{aligned} p^{-n}\Theta_{p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} \\ = \Psi_{p,k,x_0}(Z,Z') + \mathcal{O}(p^{-\infty}), \end{aligned} \tag{2.6}$$

where Ψ_{p,k,x_0} satisfies the following estimate on $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon'\}$: for every $l \in \mathbb{N}$ there exist $C_{k,l} > 0$, $M > 0$ such that for all $p \in \mathbb{N}^*$

$$|\Psi_{p,k,x_0}(Z,Z')|_{\mathscr{C}^l(X)} \leqq C_{k,l}p^{-(k+1)/2}(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M e^{-C_0\sqrt{p}|Z-Z'|}. \tag{2.7}$$

The sequence P_p. By [8], Proposition 4.1, we know that the Bergman kernel decays very fast outside the diagonal of $X \times X$. Namely, for any $l, m \in \mathbb{N}$, $\varepsilon > 0$, there exists $C_{l,m,\varepsilon} > 0$ such that for all $p \geqq 1$ we have

$$|P_p(x,x')|_{\mathscr{C}^m} \leqq C_{l,m,\varepsilon}p^{-l} \quad \text{on } \{(x,x') \in X \times X : d(x,x') \geqq \varepsilon\}. \tag{2.8}$$

Here the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E, ∇^{TX} and h^L, h^E, g^{TX}.

By [8], Theorem 4.18′, there exist polynomials $J_{r,x_0}(Z,Z') \in \operatorname{End}(E)_{x_0}$ in Z, Z' with the same parity as r, such that for any $k \in \mathbb{N}$, $\varepsilon \in]0, a^X/4[$, we have

$$p^{-n}P_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{2.9}$$

on the set $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < 2\varepsilon\}$, in the sense of Notation 2.1.

The sequence $T_{f,p}$. From (2.9), we get the following result (cf. [27], Lemma 4.6, [24], Lemma 7.2.4).

Lemma 2.2. *Let $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$. There exists a family $\{Q_{r,x_0}(f)\}_{r \in \mathbb{N}, x_0 \in X}$, depending smoothly on the parameter $x_0 \in X$, where $Q_{r,x_0}(f) \in \operatorname{End}(E)_{x_0}[Z, Z']$ are polynomials with the same parity as r and such that for every $k \in \mathbb{N}$, $\varepsilon \in]0, a^X/4[$,*

$$p^{-n}T_{f,p,x_0}(Z,Z') \cong \sum_{r=0}^{k} \big(Q_{r,x_0}(f)\mathscr{P}_{x_0}\big)(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{2.10}$$

on the set $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < 2\varepsilon\}$, in the sense of Notation 2.1. Moreover, $Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\left[J_{r_1,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\right]. \tag{2.11}$$

Especially,

$$Q_{0,x_0}(f) = f(x_0). \tag{2.12}$$

Our goal is of course to compute the coefficients $Q_{r,x_0}(f)$. For this we need J_{r,x_0}, which are obtained by computing the operators $\mathscr{F}_{r,x_0}$ defined by the smooth kernels

$$\mathscr{F}_{r,x_0}(Z,Z') = J_{r,x_0}(Z,Z')\mathscr{P}(Z,Z') \tag{2.13}$$

with respect to dZ'. Our strategy (already used in [24], [25]) is to rescale the Kodaira–Laplace operator, take the Taylor expansion of the rescaled operator and apply resolvent analysis. In the remaining of this section we outline the main steps and continue the calculation in Section 3.

Rescaling $\square_p$ and Taylor expansion. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$, $Z \in \mathbb{R}^{2n}$, $|Z| \leqq 2\varepsilon$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}
&(S_t s)(Z) := s(Z/t),\\
&\nabla'_t := S_t^{-1} t \nabla^{L^p\otimes E} S_t,\\
&\nabla_t := S_t^{-1} t \kappa^{1/2} \nabla^{L^p\otimes E} \kappa^{-1/2} S_t = \kappa^{1/2}(tZ)\nabla'_t \kappa^{-1/2}(tZ),\\
&\mathscr{L}_t := S_t^{-1}\kappa^{1/2} t^2 (2\square_p)\kappa^{-1/2} S_t.
\end{aligned} \tag{2.14}$$

Then by [24], Theorem 4.1.7, there exist second order differential operators $\mathscr{O}_r$ such that we have an asymptotic expansion in t when $t \to 0$,

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathscr{O}_r + \mathscr{O}(t^{m+1}). \tag{2.15}$$

From [24], Theorems 4.1.21 and 4.1.25 (cf. also Theorem 3.2), we obtain

$$\mathscr{L}_0 = \sum_j b_j b_j^+ = \mathscr{L}, \quad \mathscr{O}_1 = 0. \tag{2.16}$$

Resolvent analysis. We define by recurrence $f_r(\lambda) \in \operatorname{End}\big(L^2(\mathbb{R}^{2n}, E_{x_0})\big)$ by

$$f_0(\lambda) = (\lambda - \mathscr{L}_0)^{-1}, \quad f_r(\lambda) = (\lambda - \mathscr{L}_0)^{-1}\sum_{j=1}^{r} \mathscr{O}_j f_{r-j}(\lambda). \tag{2.17}$$

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\pi/2$. Then by [25], (1.110), (cf. also [24], (4.1.91))

$$\mathscr{F}_{r,x_0} = \frac{1}{2\pi\sqrt{-1}} \int_\delta f_r(\lambda)\, d\lambda. \tag{2.18}$$

Since the spectrum of $\mathscr{L}$ is well understood we can calculate the coefficients $\mathscr{F}_{r,x_0}$. Recall than $\mathscr{P}^{\perp} = \mathrm{Id} - \mathscr{P}$. From Theorem 1.1, (2.16) and (2.18), we get

$$\begin{aligned} &\mathscr{F}_{0,x_0} = \mathscr{P}, \quad \mathscr{F}_{1,x_0} = 0,\\ &\mathscr{F}_{2,x_0} = -\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp},\\ &\mathscr{F}_{3,x_0} = -\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_3\mathscr{P} - \mathscr{P}\mathcal{O}_3\mathscr{L}^{-1}\mathscr{P}^{\perp}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \mathscr{F}_{4,x_0} = \frac{1}{2\pi\sqrt{-1}}\int_{\delta}\Big[&(\lambda-\mathscr{L})^{-1}\mathscr{P}^{\perp}(\mathcal{O}_2 f_2 + \mathcal{O}_4 f_0)(\lambda)\\ &+\frac{1}{\lambda}\mathscr{P}(\mathcal{O}_2 f_2 + \mathcal{O}_4 f_0)(\lambda)\Big]\,d\lambda\\ = \mathscr{L}^{-1}&\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P} - \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_4\mathscr{P}\\ &+ \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp} - \mathscr{P}\mathcal{O}_4\mathscr{L}^{-1}\mathscr{P}^{\perp}\\ &+ \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-2}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P}\\ &- \mathscr{P}\mathcal{O}_2\mathscr{P}\mathcal{O}_2\mathscr{L}^{-2}\mathscr{P}^{\perp} - \mathscr{P}^{\perp}\mathscr{L}^{-2}\mathcal{O}_2\mathscr{P}\mathcal{O}_2\mathscr{P}. \end{aligned} \tag{2.20}$$

In particular, the first two identities of (2.19) imply

$$J_{0,x_0} = 1, \quad J_{1,x_0} = 0. \tag{2.21}$$

Remark 2.3. $\mathscr{L}_t$ is a formally self-adjoint elliptic operator on $\mathscr{C}^{\infty}(\mathbb{R}^{2n}, E_{x_0})$ with respect to the norm $\|\cdot\|_{L^2}$ induced by $h^{E_{x_0}}$, dZ. Thus $\mathscr{L}_0$ and $\mathcal{O}_r$ are also formally self-adjoint with respect to $\|\cdot\|_{L^2}$. Therefore the third and fourth terms in (2.20) are the adjoints of the first and second terms, respectively. In Lemma 4.1, we will show that $\mathscr{P}\mathcal{O}_2\mathscr{P} = 0$, hence the last two terms in (2.20) vanish. Set

$$\mathscr{F}_{41} = \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P} - \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_4\mathscr{P}. \tag{2.22}$$

3. Taylor expansion of the rescaled operator $\mathscr{L}_t$

In this section we compute the operators $\mathscr{L}_0$ and $\mathcal{O}_i$ (for $1 \leqq i \leqq 4$) from (2.15) (see Theorem 3.2), which will be used in Sections 4, 5 for the evaluation of the coefficients of the expansion of the kernels of the Berezin–Toeplitz operators.

We denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induced by g^{TX}. Let R^{TX} be the curvature of the Levi–Civita connection ∇^{TX}. Let Ric and $\boldsymbol{r}$ be the Ricci and scalar curvature of ∇^{TX}. Then we have the following well know facts: for U, V, W, Y vector fields on X,

$$\begin{aligned} &R^{TX}(U,V)W + R^{TX}(V,W)U + R^{TX}(W,U)V = 0, \\ &\langle R^{TX}(U,V)W, Y\rangle = \langle R^{TX}(W,Y)U, V\rangle = -\langle R^{TX}(V,U)W, Y\rangle. \end{aligned} \tag{3.1}$$

Now we work on $T_{x_0}X \simeq \mathbb{R}^{2n}$ as in (2.3). Recall that we have trivialized L, E. Let ∇_U denote the ordinary differentiation operator on $T_{x_0}X$ in the direction U.

We adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$ and most of time, we will omit the subscript x_0.

For $W \in T_{x_0}X$, $Z \in \mathbb{R}^{2n}$, let $\tilde{W}(Z)$ be the parallel transport of W with respect to ∇^{TX} along the curve $[0,1] \ni u \to uZ$. Because the complex structure J is parallel with respect to ∇^{TX}, we know that

$$J_Z \tilde{W}(Z) = \widetilde{J_{x_0} W}(Z). \tag{3.2}$$

Recall that $\{e_i\}$ is a fixed orthonormal basis of $(T_{x_0}X, g^{TX})$. Then for $U, V \in T_{x_0}X$,

$$\mathrm{Ric}_{x_0}(U,V) = -\langle R^{TX}(U,e_j)V, e_j\rangle_{x_0}, \quad \boldsymbol{r}_{x_0} = -\langle R^{TX}(e_i,e_j)e_i, e_j\rangle_{x_0}. \tag{3.3}$$

We define

$$\begin{aligned} R^{TX}_{;\bullet} &\in \big(T^*X \otimes \Lambda^2(T^*X) \otimes \mathrm{End}(TX)\big)_{x_0}, \\ R^{TX}_{;(\bullet,\bullet)} &\in \big((T^*X)^{\otimes 2} \otimes \Lambda^2(T^*X) \otimes \mathrm{End}(TX)\big)_{x_0}, \\ \mathrm{Ric}_{;\bullet} &\in \big(T^*X \otimes (T^*X)^{\otimes 2}\big)_{x_0}, \\ R^{E}_{;\bullet} &\in \big(T^*X \otimes \Lambda^2(T^*X) \otimes \mathrm{End}(E)\big)_{x_0}, \\ R^{E}_{;(\bullet,\bullet)} &\in \big((T^*X)^{\otimes 2} \otimes \Lambda^2(T^*X) \otimes \mathrm{End}(E)\big)_{x_0}, \end{aligned}$$

by

$$\begin{aligned} \langle R^{TX}_{;e_k}(e_m,e_j)e_q, e_i\rangle &= \big(\nabla_{e_k}\langle R^{TX}(\tilde{e}_m,\tilde{e}_j)\tilde{e}_q, \tilde{e}_i\rangle\big)_{x_0}, \\ \langle R^{TX}_{;(e_k,e_\ell)}(e_m,e_j)e_q, e_i\rangle &= \big(\nabla_{e_\ell}\nabla_{e_k}\langle R^{TX}(\tilde{e}_m,\tilde{e}_j)\tilde{e}_q, \tilde{e}_i\rangle\big)_{x_0}, \\ \mathrm{Ric}_{;e_k}(e_i,e_j) &= \big(\nabla_{e_k}\mathrm{Ric}(\tilde{e}_i,\tilde{e}_j)\big)_{x_0}, \\ R^{E}_{;e_k}(e_i,e_j) &= \big(\nabla_{e_k}R^{E}(\tilde{e}_i,\tilde{e}_j)\big)_{x_0}, \\ R^{E}_{;(e_k,e_\ell)}(e_i,e_j) &= \big(\nabla_{e_\ell}\nabla_{e_k}R^{E}(\tilde{e}_i,\tilde{e}_j)\big)_{x_0}. \end{aligned} \tag{3.4}$$

We will also use the complex coordinates $z = (z_1, \dots, z_n)$. Note that

$$e_{2j-1} = \frac{\partial}{\partial Z_{2j-1}} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}, \quad e_{2j} = \frac{\partial}{\partial Z_{2j}} = \sqrt{-1}\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j}\right), \quad \left|\frac{\partial}{\partial z_j}\right|^2 = \frac{1}{2}. \tag{3.5}$$

Set

$$(3.6)\qquad \begin{aligned} &R_{k\bar m\ell\bar q} = \left\langle R^{TX}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar z_m}\right)\frac{\partial}{\partial z_\ell},\frac{\partial}{\partial \bar z_q}\right\rangle_{x_0}, \quad R^E_{k\bar\ell} = R^E_{x_0}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar z_\ell}\right),\\ &\mathrm{Ric}_{k\bar\ell} = \mathrm{Ric}_{x_0}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar z_\ell}\right),\\ &R_{k\bar m\ell\bar q;s} = \left\langle R^{TX}_{;\frac{\partial}{\partial z_s}}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar z_m}\right)\frac{\partial}{\partial z_\ell},\frac{\partial}{\partial \bar z_q}\right\rangle, \quad R^E_{k\bar q;s} = R^E_{;\frac{\partial}{\partial z_s}}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \bar z_q}\right),\end{aligned}$$

and in the same way, we define $R_{k\bar m\ell\bar q;\bar s}$, $R_{k\bar m\ell\bar q;t\bar s}$, $\mathrm{Ric}_{k\bar q;\bar s}$, $R^E_{k\bar q;\bar s}$, $R^E_{k\bar q;t\bar s}$.

Since R^{TX} is a $(1,1)$-form and ∇^E is the Chern connection on (E,h^E), we deduce from (3.1)–(3.3) the following.

Lemma 3.1. (1) $R_{k\bar m\ell\bar q} = R_{\ell\bar m k\bar q} = R_{k\bar q\ell\bar m} = R_{\ell\bar q k\bar m}$, $\boldsymbol{r} = 8R_{m\bar q q\bar m}$, $(R^E_{k\bar q})^* = R^E_{q\bar k}$.

(2) $R_{k\bar m\ell\bar q;\bar s} = R_{\ell\bar m k\bar q;\bar s} = R_{k\bar q\ell\bar m;\bar s} = R_{\ell\bar q k\bar m;\bar s}$.

(3) Ric *is a symmetric* $(1,1)$-*tensor and* $\mathrm{Ric}_{m\bar q} = 2R_{m\bar k k\bar q}$, $\mathrm{Ric}_{m\bar q;\bar s} = 2R_{m\bar k k\bar q;\bar s}$.

(4) $R^{TX}_{;e_k}$, $R^{TX}_{;(e_k,e_\ell)}$ *are* $(1,1)$-*forms with values in* $\mathrm{End}(T_{x_0}X)$ *which commute with* J_{x_0}.

(5) $R^E_{;e_k}$, $R^E_{;(e_k,e_\ell)} \in \mathrm{End}(E_{x_0})$.

Let div(Ric) be the divergence of Ric. By [28], §2.3.4, Proposition 6,

$$(3.7)\qquad d\boldsymbol{r} = 2\,\mathrm{div}(\mathrm{Ric}) = 2(\nabla^{T^*X}_{\tilde e_m}\mathrm{Ric})(\tilde e_m,\cdot).$$

Lemma 3.1 and (3.7) entail

$$(3.8)\qquad \begin{aligned} &R_{\ell\bar\ell m\bar m;\bar k} = R_{\ell\bar\ell m\bar k;\bar m}, \qquad R_{\ell\bar\ell m\bar m;k} = R_{\ell\bar\ell k\bar m;m},\\ &-(\Delta\boldsymbol{r})_{x_0} = 2e_qe_m\big(\mathrm{Ric}(\tilde e_q,\tilde e_m)\big)_{x_0} = 32R_{k\bar m q\bar q;m\bar k} = 32R_{m\bar m q\bar q;k\bar k}.\end{aligned}$$

Set

$$(3.9)\qquad \mathcal{R} := \sum_i Z_ie_i = Z, \quad \nabla_{0,\bullet} := \nabla_\bullet + \frac{1}{2}R^L_{x_0}(\mathcal{R},\bullet).$$

Thus $\mathcal{R}$ is the radial vector field on $\mathbb{R}^{2n}$. We also introduce the vector fields $z = \sum_i z_i\frac{\partial}{\partial z_i}$ and $\bar z = \sum_i \bar z_i\frac{\partial}{\partial \bar z_i}$. By [24], Proposition 1.2.2, (3.2), we have

$$(3.10)\qquad \mathcal{R} = \sum_i Z_i\tilde e_i, \quad z = \sum_i z_i\widetilde{\frac{\partial}{\partial z_i}}, \quad \bar z = \sum_i \bar z_i\widetilde{\frac{\partial}{\partial \bar z_i}}.$$

By (0.1) and (1.1), we get

$$(3.11)\qquad b_i = -2\nabla_{0,\frac{\partial}{\partial z_i}}, \quad b_i^+ = 2\nabla_{0,\frac{\partial}{\partial \bar z_i}}, \quad R^L_{x_0} = -2\pi\sqrt{-1}\langle J\bullet,\bullet\rangle_{x_0}.$$

Let $A_{1Z}, A_{2Z} \in (T^*X)^{\otimes 2}$ be polynomials in Z with values symmetric tensors, defined by

$$
\begin{aligned}
A_{1Z}(e_i, e_j) &= \langle R^{TX}_{;(Z,Z)}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0},\\
A_{2Z}(e_i, e_j) &= \langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}\rangle_{x_0}.
\end{aligned}
\tag{3.12}
$$

Recall that the operator $\mathscr{L}$ was defined in (1.1). Set

$$
\begin{aligned}
\mathcal{O}'_2 = {}& \frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle \nabla_{0,e_i}\nabla_{0,e_j} - 2R^E_{k\bar{k}}\\
&+ \left(\left\langle \frac{\pi}{3} R^{TX}_{x_0}(z,\bar{z})\mathcal{R}, e_j\right\rangle + \frac{2}{3}\operatorname{Ric}_{x_0}(\mathcal{R}, e_j) - R^E_{x_0}(\mathcal{R}, e_j)\right)\nabla_{0,e_j},
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
\mathcal{O}_{41} &= \frac{1}{20}\left(A_{1Z} - \frac{4}{3}A_{2Z}\right)(e_i, e_j)\nabla_{0,e_i}\nabla_{0,e_j},\\
\mathcal{O}_{42} &= \left[\mathscr{L}, -\left(\frac{1}{80}A_{1Z} - \frac{1}{360}A_{2Z}\right)(e_j, e_j) - \frac{1}{288}\operatorname{Ric}(\mathcal{R},\mathcal{R})^2\right]\\
&\quad + \frac{\mathscr{L}}{144}\operatorname{Ric}(\mathcal{R},\mathcal{R})^2,\\
\mathcal{O}_{43} &= -\frac{1}{144}\operatorname{Ric}(\mathcal{R},\mathcal{R})\mathscr{L}\operatorname{Ric}(\mathcal{R},\mathcal{R}),\\
\mathcal{O}_{44} &= \left\{\frac{\pi}{30}A_{1Z}(\bar{z}, e_i) - \frac{\pi}{10}A_{2Z}(\bar{z}, e_i) + \frac{\partial}{\partial Z_j}\left(\frac{1}{20}A_{1Z} + \frac{2}{45}A_{2Z}\right)(e_i, e_j)\right.\\
&\quad \left. - \frac{\partial}{\partial Z_i}\left(\frac{1}{40}A_{1Z} + \frac{1}{45}A_{2Z}\right)(e_j, e_j)\right\}\nabla_{0,e_i},
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
\mathcal{O}_{45} &= \left\{\frac{2}{9}\langle R^{TX}_{x_0}(\mathcal{R}, e_k)\mathcal{R}, R^{TX}_{x_0}(\mathcal{R}, e_k)e_\ell\rangle_{x_0}\right.\\
&\quad - \frac{1}{9}\langle R^{TX}_{x_0}(\mathcal{R}, e_\ell)\mathcal{R}, e_k\rangle_{x_0}\operatorname{Ric}(\mathcal{R}, e_k)\\
&\quad \left. + \frac{1}{4}\langle R^{TX}_{x_0}(\mathcal{R}, e_\ell)\mathcal{R}, e_m\rangle_{x_0} R^E_{x_0}(\mathcal{R}, e_m) - \frac{1}{4}R^E_{;(Z,Z)}(\mathcal{R}, e_\ell)\right\}\nabla_{0,e_\ell},\\
\mathcal{O}_{46} &= -\frac{\pi^2}{36}A_{2Z}(\bar{z},\bar{z}) + \frac{\pi}{30}\langle R^{TX}_{;(Z,e_\ell)}(z,\bar{z})\mathcal{R}, e_\ell\rangle_{x_0}\\
&\quad - \frac{\pi}{20}\langle R^{TX}_{x_0}(z,\bar{z})\mathcal{R}, e_m\rangle_{x_0}\operatorname{Ric}(\mathcal{R}, e_m) + \frac{4}{9}\operatorname{Ric}_{k\bar{m}}\operatorname{Ric}_{m\bar{\ell}} z_k\bar{z}_\ell\\
&\quad - \frac{4}{9}R_{k\bar{\ell}m\bar{q}}\operatorname{Ric}_{\ell\bar{m}} z_k\bar{z}_q + \frac{1}{6}\langle \pi R^{TX}_{x_0}(z,\bar{z})\mathcal{R}, e_m\rangle R^E_{x_0}(\mathcal{R}, e_m)
\end{aligned}
\tag{3.15}
$$

$$+\frac{1}{8}\operatorname{Ric}(\mathcal{R},e_m)R^E_{x_0}(\mathcal{R},e_m)+\frac{1}{2}(R^E_{k\bar{m}}R^E_{m\bar{\ell}}+R^E_{m\bar{\ell}}R^E_{k\bar{m}})z_k\bar{z}_\ell$$
$$-\frac{1}{4}R^E_{;(Z,e_\ell)}(\mathcal{R},e_\ell)-R^E_{;(Z,Z)}\left(\frac{\partial}{\partial z_\ell},\frac{\partial}{\partial \bar{z}_\ell}\right).$$

The following result extends [24], Theorem 4.1.25, where $\mathscr{L}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$ were computed.

Theorem 3.2. *The following identities hold for the operators* $\mathcal{O}_r$ *introduced in* (2.15):

(3.16a) $$\mathscr{L}_0=\sum_j b_j b_j^+=\mathscr{L}=-\sum_i \nabla_{0,e_i}\nabla_{0,e_i}-2\pi n,\quad \mathcal{O}_1=0,$$

(3.16b) $$\mathcal{O}_2=\mathcal{O}_2'-\frac{1}{3}\operatorname{Ric}_{x_0}(\mathcal{R},e_j)\nabla_{0,e_j}-\frac{\boldsymbol{r}_{x_0}}{6},$$

and

(3.17a) $$\mathcal{O}_3=\frac{1}{6}\langle R^{TX}_{;Z}(\mathcal{R},e_i)\mathcal{R},e_j\rangle_{x_0}\nabla_{0,e_i}\nabla_{0,e_j}$$
$$+\bigg[\frac{2\pi}{15}\langle R^{TX}_{;Z}(z,\bar{z})\mathcal{R},e_i\rangle_{x_0}+\frac{1}{6}\operatorname{Ric}_{;Z}(\mathcal{R},e_i)$$
$$+\frac{1}{6}\langle R^{TX}_{;e_j}(\mathcal{R},e_j)\mathcal{R},e_i\rangle_{x_0}-\frac{2}{3}R^E_{;Z}(\mathcal{R},e_i)\bigg]\nabla_{0,e_i}$$
$$+\frac{\pi}{15}\langle R^{TX}_{;e_j}(z,\bar{z})\mathcal{R},e_j\rangle_{x_0}-\frac{1}{6}\operatorname{Ric}_{;e_i}(\mathcal{R},e_i)$$
$$-\frac{1}{12}\operatorname{Ric}_{;Z}(e_i,e_i)-\frac{1}{3}R^E_{;e_i}(\mathcal{R},e_i)-\frac{\sqrt{-1}}{2}R^E_{;Z}(e_i,Je_i),$$

(3.17b) $$\mathcal{O}_4=\mathcal{O}_{41}+\mathcal{O}_{42}+\mathcal{O}_{43}+\mathcal{O}_{44}+\mathcal{O}_{45}+\mathcal{O}_{46}.$$

Proof. Recall that $\tilde{e}_i(Z)$ is the parallel transport of e_i with respect to ∇^{TX} along the curve $[0,1]\ni u\to uZ$. Let $\tilde{\theta}(Z)=\big(\theta^i_j(Z)\big)^{2n}_{i,j=1}$ be the $2n\times 2n$-matrix such that

(3.18) $$e_i=\sum_j\theta^j_i(Z)\tilde{e}_j(Z),\quad \tilde{e}_j(Z)=\big(\tilde{\theta}(Z)^{-1}\big)^k_j e_k.$$

Taking into account the Taylor expansion of θ^i_j at 0 we have (cf. [24], (1.2.27))

(3.19) $$\sum_{|\alpha|\geqq 1}(|\alpha|^2+|\alpha|)(\partial^\alpha\theta^i_j)(0)\frac{Z^\alpha}{\alpha!}=\langle R^{TX}(\mathcal{R},e_j)\mathcal{R},\tilde{e}_i\rangle_Z.$$

From this equation, we obtain first that

(3.20) $$e_j(Z)=\tilde{e}_j(Z)+\frac{1}{6}\langle R^{TX}_{x_0}(\mathcal{R},e_j)\mathcal{R},e_k\rangle_{x_0}\tilde{e}_k(Z)+\mathcal{O}(|Z|^3).$$

From (3.4), (3.10), (3.12), (3.19) and (3.20), we get further

$$\theta_j^i = \delta_{ij} + \frac{1}{6}\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0} + \frac{1}{12}\langle R_{;Z}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0} + \frac{1}{20}\left(\frac{1}{2}A_{1Z}(e_i, e_j) + \frac{1}{6}A_{2Z}(e_i, e_j)\right) + \mathcal{O}(|Z|^5). \tag{3.21}$$

Set $g_{ij}(Z) = g^{TX}(e_i, e_j)(Z) = \langle e_i, e_j\rangle_Z$ and let $\big(g^{ij}(Z)\big)$ be the inverse of the matrix $\big(g_{ij}(Z)\big)$. Then by (3.12), (3.18) and (3.21), we have

$$\begin{aligned} g_{ij}(Z) &= \theta_i^k(Z)\theta_j^k(Z) \\ &= \delta_{ij} + \frac{1}{3}\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0} + \frac{1}{6}\langle R_{;Z}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0} \\ &\quad + \frac{1}{20}A_{1Z}(e_i, e_j) + \frac{2}{45}A_{2Z}(e_i, e_j) + \mathcal{O}(|Z|^5). \end{aligned} \tag{3.22}$$

In view of the expansion $(1+a)^{-1} = 1 - a + a^2 + \cdots$, we obtain

$$\begin{aligned} g^{ij}(Z) &= \delta_{ij} - \frac{1}{3}\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0} - \frac{1}{6}\langle R_{;Z}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0} \\ &\quad - \frac{1}{20}A_{1Z}(e_i, e_j) + \frac{1}{15}A_{2Z}(e_i, e_j) + \mathcal{O}(|Z|^5). \end{aligned} \tag{3.23}$$

If Γ_{ij}^ℓ are the Christoffel symbols of ∇^{TX} with respect to the frame $\{e_i\}$, then $(\nabla_{e_i}^{TX} e_j)(Z) = \Gamma_{ij}^\ell(Z)e_\ell$. By the explicit formula for ∇^{TX}, we get (cf. [24], (4.1.102)) with $\partial_j := \dfrac{\partial}{\partial Z_j}$

$$\begin{aligned} \Gamma_{ij}^\ell(Z) &= \frac{1}{2}g^{\ell k}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})(Z) \\ &= \frac{1}{3}[\langle R_{x_0}^{TX}(\mathcal{R}, e_j)e_i, e_\ell\rangle_{x_0} + \langle R_{x_0}^{TX}(\mathcal{R}, e_i)e_j, e_\ell\rangle_{x_0}] + \mathcal{O}(|Z|^2). \end{aligned} \tag{3.24}$$

For j fixed, $\Gamma_{jj}^\ell(Z) = \dfrac{1}{2}g^{\ell k}(2\partial_j g_{jk} - \partial_k g_{jj})(Z)$, thus by (3.22) and (3.23),

$$\begin{aligned} \Gamma_{jj}^\ell(Z) &= \frac{2}{3}\langle R_{x_0}^{TX}(\mathcal{R}, e_j)e_j, e_\ell\rangle_{x_0} + \frac{1}{12}[4\langle R_{;Z}^{TX}(\mathcal{R}, e_j)e_j, e_\ell\rangle_{x_0} \\ &\quad + 2\langle R_{;e_j}^{TX}(\mathcal{R}, e_j)\mathcal{R}, e_\ell\rangle_{x_0} + \langle R_{;e_\ell}^{TX}(\mathcal{R}, e_j)e_j, \mathcal{R}\rangle_{x_0}] \\ &\quad - \frac{2}{9}\langle R_{x_0}^{TX}(\mathcal{R}, e_\ell)\mathcal{R}, R_{x_0}^{TX}(\mathcal{R}, e_j)e_j\rangle_{x_0} \\ &\quad + \frac{\partial}{\partial Z_j}\left(\frac{1}{20}A_{1Z} + \frac{2}{45}A_{2Z}\right)(e_j, e_\ell) \\ &\quad - \frac{1}{2}\frac{\partial}{\partial Z_l}\left(\frac{1}{20}A_{1Z} + \frac{2}{45}A_{2Z}\right)(e_j, e_j) + \mathcal{O}(|Z|^4). \end{aligned} \tag{3.25}$$

Note that

$$\det(\delta_{ij} + a_{ij}) = 1 + \sum_i a_{ii} + \sum_{i<j} (a_{ii}a_{jj} - a_{ij}a_{ji}) + \cdots$$

and

$$(1+a)^{1/4} = 1 + \frac{1}{4}a - \frac{3}{32}a^2 + \cdots.$$

By (3.3) and (3.22), we get

$$\begin{aligned}
\kappa(Z)^{1/2} &= \left|\det\bigl(g_{ij}(Z)\bigr)\right|^{1/4} \\
&= 1 + \frac{1}{12}\langle R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}, e_j\rangle_{x_0} \\
&\quad - \frac{1}{24}\operatorname{Ric}_{;Z}(\mathcal{R},\mathcal{R}) + \frac{1}{80}A_{1Z}(e_j,e_j) + \frac{1}{90}A_{2Z}(e_j,e_j) \\
&\quad + \frac{1}{36}\sum_{i<j}\bigl(\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\rangle_{x_0}\langle R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}, e_j\rangle_{x_0} - \langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle^2_{x_0}\bigr) \\
&\quad - \frac{1}{96}\left(\sum_j \langle R^{TX}_{x_0}(\mathcal{R}, e_j)\mathcal{R}, e_j\rangle_{x_0}\right)^2 + \mathcal{O}(|Z|^5) \\
&= 1 - \frac{1}{12}\operatorname{Ric}(\mathcal{R},\mathcal{R}) - \frac{1}{24}\operatorname{Ric}_{;Z}(\mathcal{R},\mathcal{R}) \\
&\quad + \left(\frac{1}{80}A_{1Z} - \frac{1}{360}A_{2Z}\right)(e_j,e_j) \\
&\quad + \frac{1}{288}\operatorname{Ric}(\mathcal{R},\mathcal{R})^2 + \mathcal{O}(|Z|^5).
\end{aligned} \tag{3.26}$$

Thus

$$\begin{aligned}
\kappa(Z)^{-1/2} = 1 &+ \frac{1}{12}\operatorname{Ric}(\mathcal{R},\mathcal{R}) + \frac{1}{24}\operatorname{Ric}_{;Z}(\mathcal{R},\mathcal{R}) \\
&- \left(\frac{1}{80}A_{1Z} - \frac{1}{360}A_{2Z}\right)(e_j,e_j) + \left(\frac{1}{144} - \frac{1}{288}\right)\operatorname{Ric}(\mathcal{R},\mathcal{R})^2 + \mathcal{O}(|Z|^5).
\end{aligned} \tag{3.27}$$

Observe that J is parallel with respect to ∇^{TX}, thus $\langle J\tilde{e}_i, \tilde{e}_j\rangle_Z = \langle Je_i, e_j\rangle_{x_0}$. From (3.9), (3.10), (3.18) and (3.21), we get

$$\begin{aligned}
\frac{\sqrt{-1}}{2\pi}R^L_Z(\mathcal{R}, e_\ell) &= \theta^j_\ell(Z)\langle J\tilde{e}_i, \tilde{e}_j\rangle_Z Z_i = \theta^j_\ell(Z)\langle J\mathcal{R}, e_j\rangle_{x_0} \\
&= \langle J\mathcal{R}, e_\ell\rangle_{x_0} + \frac{1}{6}\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_\ell\rangle_{x_0} \\
&\quad + \frac{1}{12}\langle R^{TX}_{;Z}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_\ell\rangle_{x_0}
\end{aligned} \tag{3.28}$$

$$+\frac{1}{40}\langle R^{TX}_{;(Z,Z)}(\mathcal{R},J\mathcal{R})\mathcal{R},e_\ell\rangle_{x_0}$$
$$+\frac{1}{120}\langle R^{TX}_{x_0}(\mathcal{R},J\mathcal{R})\mathcal{R},R^{TX}_{x_0}(\mathcal{R},e_\ell)\mathcal{R}\rangle_{x_0}+\mathcal{O}(|Z|^6).$$

Let $\Gamma^\bullet=\Gamma^E,\Gamma^L$ and $R^\bullet=R^E,R^L$, respectively. By [24], Lemma 1.2.4, the Taylor coefficients of $\Gamma^\bullet(e_\ell)(Z)$ at x_0 up to order r are only determined by those of $R^\bullet$ up to order $r-1$, and

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^\bullet)_{x_0}(e_\ell)\frac{Z^\alpha}{\alpha!}=\frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^\bullet)_{x_0}(\mathcal{R},e_\ell)\frac{Z^\alpha}{\alpha!}. \tag{3.29}$$

Thus by (3.28), (3.29) and since R^{TX} is a $(1,1)$-form, we obtain

$$R^{TX}(\mathcal{R},J\mathcal{R})=-2\sqrt{-1}R^{TX}(z,\bar z),\quad \langle R^{TX}_{;(Z,Z)}(\mathcal{R},J\mathcal{R})\mathcal{R},e_i\rangle_{x_0}=-2\sqrt{-1}A_{1Z}(\bar z,e_i)$$

and

$$\begin{aligned}t^{-1}\Gamma^L(e_i)(tZ)=&-\pi\sqrt{-1}\langle J\mathcal{R},e_i\rangle_{x_0}-t^2\frac{\pi}{6}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_i\rangle_{x_0}\\&-t^3\frac{\pi}{15}\langle R^{TX}_{;Z}(z,\bar z)\mathcal{R},e_i\rangle_{x_0}-t^4\frac{\pi}{60}A_{1Z}(\bar z,e_i)\\&-t^4\frac{\pi}{180}A_{2Z}(\bar z,e_i)+\mathcal{O}(t^5).\end{aligned} \tag{3.30}$$

By (2.14), (3.4), (3.20), (3.29) and (3.30), for $t=\frac{1}{\sqrt{p}}$, we get

$$\begin{aligned}\Gamma^E(e_i)(Z)=&\frac12 R^E_{x_0}(\mathcal{R},e_i)+\frac13 R^E_{;Z}(\mathcal{R},e_i)\\&+\frac18\Big(R^E_{;(Z,Z)}(\mathcal{R},e_i)+\frac13\langle R^{TX}(\mathcal{R},e_i)\mathcal{R},e_k\rangle_{x_0}R^E_{x_0}(\mathcal{R},e_k)\Big)\\&+\mathcal{O}(|Z|^4),\\ \nabla'_{t,e_i}=&\nabla_{e_i}+\frac1t\Gamma^L(e_i)(tZ)+t\Gamma^E(e_i)(tZ)\\=&\nabla_{0,e_i}-\frac{t^2}{6}\langle\pi R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_i\rangle_{x_0}+\frac{t^2}{2}R^E_{x_0}(\mathcal{R},e_i)+\mathcal{O}(t^3).\end{aligned} \tag{3.31}$$

By (2.1) and (2.14), we get

$$\begin{aligned}\mathscr{L}_t=&-\kappa(tZ)^{1/2}g^{ij}(tZ)[\nabla'_{t,e_i}\nabla'_{t,e_j}-t\Gamma^l_{ij}(t\bullet)\nabla'_{t,e_l}](Z)\kappa(tZ)^{-1/2}\\&-t^2\frac{\sqrt{-1}}{2}R^E(\tilde e_i,J\tilde e_i)(tZ)-2\pi n.\end{aligned} \tag{3.32}$$

We will derive now (3.16a) and (3.16b) (they were already obtained in [24], Theorem 4.1.25). By using the Taylor expansion of the expressions from (3.32) (see (3.23), (3.24), (3.26), (3.27), (3.31)) we obtain immediately the formulas for $\mathscr{L}_0$ and $\mathcal{O}_1$ given in (3.16a).

In order to compute $\mathcal{O}_2$, observe first that by (3.1) and the fact that R^{TX} is a $(1,1)$-form with values in $\operatorname{End}(TX)$, we get

$$\nabla_{e_j}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R}, e_j\rangle = 2\left(\frac{\partial}{\partial \bar z_j}\left\langle R^{TX}_{x_0}(z,\bar z)\bar z, \frac{\partial}{\partial z_j}\right\rangle + \frac{\partial}{\partial z_j}\left\langle R^{TX}_{x_0}(z,\bar z) z, \frac{\partial}{\partial \bar z_j}\right\rangle\right) = 0. \tag{3.33}$$

Thus from (3.13), (3.23), (3.24), (3.26), (3.31)–(3.33), we have

$$\mathcal{O}_2 = \mathcal{O}_2' + \left[\mathscr{L}_0, \frac{1}{12}\operatorname{Ric}(\mathcal{R},\mathcal{R})\right]. \tag{3.34}$$

By the formula of $\mathscr{L}_0$ (see (3.16a)) and since

$$[\mathscr{L}_0, \operatorname{Ric}(\mathcal{R},\mathcal{R})] = -4\operatorname{Ric}(\mathcal{R}, e_j)\nabla_{0,e_j} - 2\operatorname{Ric}(e_j, e_j),$$

we get from (3.34) the formula for $\mathcal{O}_2$ given in (3.16b).

From (3.32), we have also

$$\begin{aligned}\mathcal{O}_3 = {} & \frac{1}{6}\langle R^{TX}_{;Z}(\mathcal{R}, e_i)\mathcal{R}, e_j\rangle_{x_0}\nabla_{0,e_i}\nabla_{0,e_j} \\ & - \left[-\frac{2\pi}{15}\langle R^{TX}_{;Z}(z,\bar z)\mathcal{R}, e_i\rangle_{x_0} + \frac{2}{3}R^E_{;Z}(\mathcal{R}, e_i)\right]\nabla_{0,e_i} \\ & - \frac{\partial}{\partial Z_i}\left[-\frac{\pi}{15}\langle R^{TX}_{;Z}(z,\bar z)\mathcal{R}, e_i\rangle_{x_0} + \frac{1}{3}R^E_{;Z}(\mathcal{R}, e_i)\right] \\ & + \frac{1}{12}[4\operatorname{Ric}_{;Z}(\mathcal{R}, e_l) + 2\langle R^{TX}_{;e_j}(\mathcal{R}, e_j)\mathcal{R}, e_l\rangle_{x_0} + \operatorname{Ric}_{;e_l}(\mathcal{R},\mathcal{R})]\nabla_{0,e_l} \\ & + \left[\mathscr{L}_0, \frac{1}{24}\operatorname{Ric}_{;Z}(\mathcal{R},\mathcal{R})\right] - \frac{\sqrt{-1}}{2}R^E_{;Z}(e_i, Je_i).\end{aligned} \tag{3.35}$$

In (3.35), the first (resp. second and third, resp. fourth, resp. fifth) term is the contribution of the coefficient of t^3 in $g^{ij}(tZ)$ (resp. ∇'_{t,e_i}, resp. $t\Gamma^l_{ii}(t\cdot)$, resp. $\kappa^{1/2}(tZ)$). By the same argument in (3.33) and the formula of $\mathscr{L}_0$ given in (3.16a), we get

$$\begin{aligned}&\frac{\partial}{\partial Z_i}\langle R^{TX}_{;Z}(z,\bar z)\mathcal{R}, e_i\rangle_{x_0} = \langle R^{TX}_{;e_j}(z,\bar z)\mathcal{R}, e_j\rangle_{x_0}, \\ &[\mathscr{L}_0, \operatorname{Ric}_{;Z}(\mathcal{R},\mathcal{R})] = -2\big(\operatorname{Ric}_{;e_i}(\mathcal{R},\mathcal{R}) + 2\operatorname{Ric}_{;Z}(\mathcal{R}, e_i)\big)\nabla_{0,e_i} \\ &\qquad - 4\operatorname{Ric}_{;e_i}(\mathcal{R}, e_i) - 2\operatorname{Ric}_{;Z}(e_i, e_i).\end{aligned} \tag{3.36}$$

From (3.35) and (3.36) we get the formula for $\mathcal{O}_3$ asserted in (3.17a). Moreover,

$$
\begin{aligned}
(3.37)\qquad \mathcal{O}_4 = {} & \mathcal{O}_{42} + \left[\mathcal{O}_2', \frac{1}{12}\operatorname{Ric}(\mathcal{R},\mathcal{R})\right] + \mathcal{O}_{43} + \mathcal{O}_{41} + \frac{1}{3}\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_j\rangle_{x_0} \\
& \times \Big\{\Big(-\frac{1}{3}\langle \pi R^{TX}_{x_0}(z,\bar{z})\mathcal{R},e_j\rangle_{x_0} + R^E_{x_0}(\mathcal{R},e_j)\Big)\nabla_{0,e_i} \\
& \quad -\frac{1}{6}\frac{\partial}{\partial Z_i}\langle \pi R^{TX}_{x_0}(z,\bar{z})\mathcal{R},e_j\rangle_{x_0} - \frac{2}{3}\langle R^{TX}_{x_0}(\mathcal{R},e_i)e_j,e_l\rangle_{x_0}\nabla_{0,e_l}\Big\} \\
& - \Big(-\frac{1}{6}\langle \pi R^{TX}_{x_0}(z,\bar{z})\mathcal{R},e_i\rangle_{x_0} + \frac{1}{2}R^E_{x_0}(\mathcal{R},e_i) - \frac{2}{3}\langle R^{TX}_{x_0}(\mathcal{R},e_j)e_j,e_i\rangle_{x_0}\Big) \\
& \times \Big(-\frac{1}{6}\langle \pi R^{TX}_{x_0}(z,\bar{z})\mathcal{R},e_i\rangle_{x_0} + \frac{1}{2}R^E_{x_0}(\mathcal{R},e_i)\Big) \\
& - \Big\{-\mathcal{O}_{44} + \Big[-\frac{\pi}{9}A_{2Z}(\bar{z},e_i) + \frac{1}{12}\langle R^{TX}(\mathcal{R},e_i)\mathcal{R},e_m\rangle_{x_0}R^E_{x_0}(\mathcal{R},e_m) \\
& \quad + \frac{2}{9}\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_k\rangle_{x_0}\operatorname{Ric}(\mathcal{R},e_k) + \frac{1}{4}R^E_{;(Z,Z)}(\mathcal{R},e_i)\Big]\nabla_{0,e_i}\Big\} \\
& - \frac{\partial}{\partial Z_i}\Big(-\frac{\pi}{60}A_{1Z}(\bar{z},e_i) - \frac{\pi}{180}A_{2Z}(\bar{z},e_i) + \frac{1}{8}R^E_{;(Z,Z)}(\mathcal{R},e_i) \\
& \qquad + \frac{1}{24}\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_k\rangle_{x_0}R^E_{x_0}(\mathcal{R},e_k)\Big) \\
& - \frac{\sqrt{-1}}{4}R^E_{;(Z,Z)}(e_i,Je_i).
\end{aligned}
$$

Here

- $\mathcal{O}_{42}$ is the contribution of the coefficients of t^4 in $\kappa^{1/2}(tZ)$ and $\kappa^{-1/2}(tZ)$,

- the second term is the contribution of the coefficients of t^2 in $\kappa^{1/2}(tZ)$, $\kappa^{-1/2}(tZ)$ and in $-g^{ij}(tZ)\big(\nabla'_{t,e_i}\nabla'_{t,e_j} - t\Gamma^{\ell}_{ij}(tZ)\nabla'_{t,e_\ell}\big)$,

- $\mathcal{O}_{43}$ is the contribution of the coefficients of t^2 in $\kappa^{1/2}(tZ)$ and $\kappa^{-1/2}(tZ)$,

- $\mathcal{O}_{41}$ is the contribution of the coefficients of t^4 in $g^{ij}(tZ)$,

- the fifth term is the contribution of the coefficients of t^2 in $-g^{ij}(tZ)$ and in $\big(\nabla'_{t,e_i}\nabla'_{t,e_j} - t\Gamma^{\ell}_{ij}(tZ)\nabla'_{t,e_\ell}\big)$,

- the sixth, seventh and eight terms are the contributions of the coefficients of t^4 in $-\big(\nabla'_{t,e_i}\nabla'_{t,e_i} - t\Gamma^{\ell}_{ii}(tZ)\nabla'_{t,e_\ell}\big)$: the sixth term is the contribution of the coefficients of t^2 in ∇'_{t,e_i}, $-t\Gamma^{\ell}_{ii}(tZ)$ and t^2 in ∇'_{t,e_i}; the seventh and eighth terms are the contributions of the coefficients of t^4 in ∇'_{t,e_i} and $-t\Gamma^{\ell}_{ii}(tZ)$.

Now by (3.13),

$$\begin{aligned}(3.38)\qquad &\frac{1}{12}[\mathcal{O}_2', \mathrm{Ric}(\mathcal{R},\mathcal{R})]\\ &= \frac{1}{36}\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_j\rangle_{x_0}\big(4\,\mathrm{Ric}(\mathcal{R},e_j)\nabla_{0,e_i}+2\,\mathrm{Ric}(e_i,e_j)\big)\\ &\quad+\frac{1}{6}\bigg(\Big\langle\frac{\pi}{3}R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_j\Big\rangle+\frac{2}{3}\,\mathrm{Ric}(\mathcal{R},e_j)-R^E_{x_0}(\mathcal{R},e_j)\bigg)\mathrm{Ric}(\mathcal{R},e_j).\end{aligned}$$

And the same argument used to obtain (3.33) shows that

$$\begin{aligned}&\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_j\rangle_{x_0}\frac{\partial}{\partial Z_i}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_j\rangle=\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},R^{TX}_{x_0}(z,\bar z)e_i\rangle,\\ &\frac{\partial}{\partial Z_i}\big(A_{1Z}(\bar z,e_i)\big)=\frac{\partial}{\partial Z_i}\langle R^{TX}_{;(Z,Z)}(z,\bar z)\mathcal{R},e_i\rangle_{x_0}=2\langle R^{TX}_{;(Z,e_i)}(z,\bar z)\mathcal{R},e_i\rangle_{x_0},\\ (3.39)\qquad&\frac{\partial}{\partial Z_i}\big(A_{2Z}(\bar z,e_i)\big)=\frac{\partial}{\partial Z_i}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R},R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R}\rangle_{x_0}\\ &\qquad=\langle R^{TX}_{x_0}(z,\bar z)e_i,R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R}\rangle_{x_0}\\ &\qquad\quad+\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_j\rangle_{x_0}\,\mathrm{Ric}(\mathcal{R},e_j).\end{aligned}$$

Finally, by (3.1) and since R^{TX} is a $(1,1)$-form, we obtain

$$\begin{aligned}(3.40)\qquad&\langle R^{TX}_{x_0}(z,\bar z)e_\ell,R^{TX}_{x_0}(\mathcal{R},e_\ell)\mathcal{R}\rangle=\langle R^{TX}_{x_0}(z,\bar z)e_\ell,e_m\rangle\langle R^{TX}_{x_0}(\mathcal{R},e_\ell)\mathcal{R},e_m\rangle=0,\\ &\langle R^{TX}_{x_0}(\mathcal{R},e_\ell)\mathcal{R},e_m\rangle_{x_0}\,\mathrm{Ric}(e_\ell,e_m)=-8R_{k\bar\ell m\bar q}\,\mathrm{Ric}_{\ell\bar m}\,z_k\bar z_q.\end{aligned}$$

Thus

$$\begin{aligned}(3.41)\qquad\mathcal{O}_4=&\sum_{\alpha=1}^{5}\mathcal{O}_{4\alpha}-\frac{\pi^2}{36}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R},R^{TX}_{x_0}(z,\bar z)\mathcal{R}\rangle-\frac{1}{4}R^E_{x_0}(\mathcal{R},e_j)^2\\ &+\frac{1}{6}\langle\pi R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_j\rangle R^E_{x_0}(\mathcal{R},e_j)+\frac{\pi}{30}\langle R^{TX}_{;(Z,e_i)}(z,\bar z)\mathcal{R},e_i\rangle_{x_0}\\ &-\frac{\pi}{20}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R},e_j\rangle_{x_0}\,\mathrm{Ric}(\mathcal{R},e_j)+\frac{1}{9}\,\mathrm{Ric}(\mathcal{R},e_j)\,\mathrm{Ric}(\mathcal{R},e_j)\\ &+\frac{1}{18}\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_j\rangle_{x_0}\,\mathrm{Ric}(e_i,e_j)+\frac{1}{8}\,\mathrm{Ric}(\mathcal{R},e_j)R^E_{x_0}(\mathcal{R},e_j)\\ &-\frac{1}{4}R^E_{;(Z,e_i)}(\mathcal{R},e_i)-R^E_{;(Z,Z)}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial\bar z_i}\Big).\end{aligned}$$

Putting together Lemma 3.1, (3.12), (3.14), (3.40), (3.41) and the fact that R^{TX} is a $(1,1)$-form, we infer (3.17b). The proof of Theorem 3.2 is completed. □

4. Evaluation of $\mathscr{F}_4$ from (2.20)

We calculate in this section an explicit formula for the operator $\mathscr{F}_4$, defined by (2.13) and appearing in the Bergman kernel expansion (2.9). This is necessary in Section 5 in order to evaluate the expansion of the kernel of the Berezin–Toeplitz operators. We use formula (2.20) to achieve our aim. Recall that explicit formulas for the operators $\mathscr{L}$, $\mathcal{O}_2$, $\mathcal{O}_4$ appearing in (2.20) were given in Theorem 3.2.

This section is organized as follows. In Section 4.1, we determine the terms in (2.20) which involve $\mathcal{O}_2$. In Section 4.2, we calculate the terms in (2.20) which involve $\mathcal{O}_4$. In Section 4.3, we obtain the formula for $\mathscr{F}_4(0,0)$ (cf. Theorem 4.5).

We adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$ and most of time, we will omit the subscript x_0.

4.1. Contribution of $\mathcal{O}_2$ to $\mathscr{F}_4(0,0)$.

Lemma 4.1. *The following identities hold*:

$$\mathscr{P}\mathcal{O}_2\mathscr{P} = 0, \tag{4.1a}$$

$$(\mathscr{L}^{-1}\mathcal{O}_2\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1})(0,0) = \frac{1}{4\pi^2}\Big(\sum_{km} R_{m\bar{m}k\bar{k}} + \sum_k R^E_{k\bar{k}}\Big)^2, \tag{4.1b}$$

$$\begin{aligned}(\mathscr{P}\mathcal{O}_2\mathscr{L}^{-2}\mathcal{O}_2\mathscr{P})(0,0) &= \frac{1}{36\pi^2} R_{m\bar{k}q\bar{\ell}} R_{k\bar{m}\ell\bar{q}} \\ &\quad + \frac{1}{4\pi^2}\Big(\frac{4}{3}R_{q\bar{m}m\bar{\ell}} + R^E_{q\bar{\ell}}\Big)\Big(\frac{4}{3}R_{\ell\bar{k}k\bar{q}} + R^E_{\ell\bar{q}}\Big).\end{aligned} \tag{4.1c}$$

Proof. Note that by (1.1) and (1.5),

$$(b_i^+\mathscr{P})(Z,Z') = 0, \quad (b_i\mathscr{P})(Z,Z') = 2\pi(\bar{z}_i - \bar{z}'_i)\mathscr{P}(Z,Z'). \tag{4.2}$$

For $\phi \in T^*X$, by (3.5), (3.11) and (4.2), we have

$$\begin{aligned}&\phi(e_i)e_i = 2\phi\Big(\frac{\partial}{\partial z_j}\Big)\frac{\partial}{\partial \bar{z}_j} + 2\phi\Big(\frac{\partial}{\partial \bar{z}_j}\Big)\frac{\partial}{\partial z_j}, \quad \phi(e_i)\nabla_{0,e_i} = \phi\Big(\frac{\partial}{\partial z_j}\Big)b_j^+ - \phi\Big(\frac{\partial}{\partial \bar{z}_j}\Big)b_j, \\ &\phi(e_i)\nabla_{0,e_i}\mathscr{P}(Z,0) = -2\pi\phi(\bar{z})\mathscr{P}(Z,0).\end{aligned} \tag{4.3}$$

By Lemma 3.1, (3.1), (3.6), (3.16b), (4.3) and the fact that R^{TX} is a $(1,1)$-form, we get

$$\begin{aligned}\mathcal{O}_2 &= \frac{1}{3}R_{k\bar{m}\ell\bar{q}} z_k z_\ell b_m b_q + \frac{1}{3}R_{k\bar{q}\ell\bar{m}} z_k \bar{z}_m (b_q b_\ell^+ + b_\ell^+ b_q) + \frac{1}{3}R_{k\bar{m}\ell\bar{q}} \bar{z}_m \bar{z}_q b_k^+ b_\ell^+ \\ &\quad - \frac{4}{3}R_{m\bar{m}q\bar{q}} + \Big(\frac{2}{3}R_{k\bar{m}\ell\bar{k}}\bar{z}_m - \frac{\pi}{3}R_{k\bar{m}\ell\bar{q}} z_k \bar{z}_m \bar{z}_q\Big) b_l^+\end{aligned} \tag{4.4}$$

$$- \left(\frac{2}{3} R_{\ell \bar{k} k \bar{q}} z_\ell + \frac{\pi}{3} R_{k \bar{m} \ell \bar{q}} z_k \bar{z}_m z_\ell \right) b_q$$
$$- 2 R^E_{q \bar{q}} - R^E\left(\bar{z}, \frac{\partial}{\partial z_\ell} \right) b^+_\ell + R^E\left(z, \frac{\partial}{\partial \bar{z}_q} \right) b_q.$$

By Lemma 3.1, (1.7) and (4.4), we get $[R_{k\bar{m}\ell\bar{q}} z_k z_\ell, b_m b_q] = 8 b_q R_{k\bar{k}\ell\bar{q}} z_\ell + 8 R_{m\bar{m}q\bar{q}}$, and

$$\begin{aligned}
\mathcal{O}_2 = {} & \frac{1}{3} b_m b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell + b_q \left(-\frac{\pi}{3} R_{k\bar{m}\ell\bar{q}} z_k z_\ell \bar{z}_m + 2 R_{\ell\bar{k}k\bar{q}} z_\ell + R^E_{\ell\bar{q}} z_\ell \right) \\
& + \left(\frac{2b_q}{3} R_{k\bar{m}\ell\bar{q}} z_k \bar{z}_m - \frac{\pi}{3} R_{k\bar{m}\ell\bar{q}} z_k \bar{z}_m \bar{z}_q + 2 R_{k\bar{k}\ell\bar{m}} \bar{z}_m + R^E_{\ell\bar{m}} \bar{z}_m \right) b^+_\ell \\
& + \frac{1}{3} R_{k\bar{m}\ell\bar{q}} \bar{z}_m \bar{z}_q b^+_k b^+_\ell .
\end{aligned} \tag{4.5}$$

Thus Lemma 3.1, (1.7), (4.2) and (4.5) yield

$$\begin{aligned}
\mathcal{O}_2 \mathscr{P} = & \Bigg\{ \frac{1}{3} b_m b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell + b_q \Bigg(-\frac{\pi}{3} R_{k\bar{m}\ell\bar{q}} z_k z_\ell \left(\frac{b_m}{2\pi} + \bar{z}'_m \right) \\
& \qquad + 2 R_{\ell\bar{k}k\bar{q}} z_\ell + R^E_{\ell\bar{q}} z_\ell \Bigg) \Bigg\} \mathscr{P} \\
= & \left\{ \frac{1}{6} b_m b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell + \frac{4}{3} b_q R_{\ell\bar{k}k\bar{q}} z_\ell - \frac{\pi}{3} b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell \bar{z}'_m + b_q R^E_{\ell\bar{q}} z_\ell \right\} \mathscr{P}.
\end{aligned} \tag{4.6}$$

Now, (4.1a) follows from Theorem 1.1, (1.6) and (4.6). These imply also

$$\mathscr{L}^{-1} \mathcal{O}_2 \mathscr{P} = \left\{ \frac{b_m b_q}{48\pi} R_{k\bar{m}\ell\bar{q}} z_k z_l + \frac{b_q}{3\pi} R_{\ell\bar{k}k\bar{q}} z_\ell - \frac{b_q}{12} R_{k\bar{m}\ell\bar{q}} z_k z_\ell \bar{z}'_m + \frac{b_q}{4\pi} R^E_{\ell\bar{q}} z_\ell \right\} \mathscr{P}. \tag{4.7}$$

Due to (1.7) and (4.7) we have

$$\begin{aligned}
(\mathscr{L}^{-1} \mathcal{O}_2 \mathscr{P})(Z, 0) &= \left\{ \frac{b_m b_q}{48\pi} R_{k\bar{m}\ell\bar{q}} z_k z_\ell + \frac{b_q}{4\pi} \left(\frac{4}{3} R_{\ell\bar{k}k\bar{q}} + R^E_{\ell\bar{q}} \right) z_\ell \right\} \mathscr{P}(Z, 0), \\
(\mathscr{L}^{-1} \mathcal{O}_2 \mathscr{P})(0, Z) &= -\frac{1}{2\pi} (R_{m\bar{m}q\bar{q}} + R^E_{q\bar{q}}) \mathscr{P}(0, Z).
\end{aligned} \tag{4.8}$$

Since $\mathcal{O}_2$, $\mathscr{L}$ are symmetric (as explained in Remark 2.3) and $(R^E_{\ell\bar{q}})^* = R^E_{q\bar{\ell}}$, we get by (1.1), (3.6) and (4.8),

$$\begin{aligned}
(\mathscr{P} \mathcal{O}_2 \mathscr{L}^{-1})(0, Z) &= \left((\mathscr{L}^{-1} \mathcal{O}_2 \mathscr{P})(Z, 0) \right)^* \\
&= \left\{ \mathscr{P} \left[R_{u\bar{v}q\bar{s}} \bar{z}_v \bar{z}_s \frac{b^+_u b^+_q}{48\pi} + \left(\frac{4}{3} R_{q\bar{v}v\bar{s}} + R^E_{q\bar{s}} \right) \bar{z}_s \frac{b^+_q}{4\pi} \right] \right\} (0, Z), \\
(\mathscr{P} \mathcal{O}_2 \mathscr{L}^{-1})(Z, 0) &= \left((\mathscr{L}^{-1} \mathcal{O}_2 \mathscr{P})(0, Z) \right)^* = -\frac{1}{2\pi} (R_{k\bar{k}q\bar{q}} + R^E_{q\bar{q}}) \mathscr{P}(Z, 0).
\end{aligned} \tag{4.9}$$

Note that $\mathscr{P}(0,0) = 1$ by (1.5). From (1.6), (1.7), (4.2), (4.8) and (4.9), we get (4.1b), and

$$(\mathscr{P}\mathscr{O}_2\mathscr{L}^{-2}\mathscr{O}_2\mathscr{P})(0,0) = \mathscr{P}\bigg\{\frac{32\pi^2}{(48\pi)^2}R_{m\bar{s}q\bar{t}}\bar{z}_s\bar{z}_t R_{k\bar{m}\ell\bar{q}}z_k z_\ell + \frac{1}{4\pi}\left(\frac{4}{3}R_{q\bar{s}s\bar{t}} + R^E_{q\bar{t}}\right)\bar{z}_t\left(\frac{4}{3}R_{\ell\bar{k}k\bar{q}} + R^E_{\ell\bar{q}}\right)z_\ell\bigg\}\mathscr{P}(0,0). \tag{4.10}$$

Let $\phi \in \mathbb{C}[b,z]$ be a polynomial in b, z. By (1.7), (4.2), we have

$$\begin{aligned}(\bar{z}_k\phi(b,z)\mathscr{P})(Z,0) &= \phi(b,z)\frac{b_k}{2\pi}\mathscr{P}(Z,0) = \left(\frac{b_k}{2\pi}\phi + \frac{1}{\pi}\frac{\partial\phi}{\partial z_k}\right)\mathscr{P}(Z,0),\\ \big(\bar{z}_k\bar{z}_l\phi(b,z)\mathscr{P}\big)(Z,0) &= \big(\phi(b,z)\bar{z}_k\bar{z}_l\mathscr{P}\big)(Z,0)\\ &= \left(\frac{b_k b_l}{4\pi^2}\phi + \frac{b_k}{2\pi^2}\frac{\partial\phi}{\partial z_l} + \frac{b_l}{2\pi^2}\frac{\partial\phi}{\partial z_k} + \frac{1}{\pi^2}\frac{\partial^2\phi}{\partial z_k\partial z_l}\right)\mathscr{P}(Z,0).\end{aligned} \tag{4.11}$$

Let $F(Z)$ be a homogeneous degree 2 polynomial in Z. By (4.2),

$$\big(F(Z)\mathscr{P}\big)(Z,0) = \left(\frac{1}{2}\frac{\partial^2 F}{\partial z_i\partial z_j}z_i z_j + \frac{\partial^2 F}{\partial z_i\partial\bar{z}_j}z_i\frac{b_j}{2\pi} + \frac{1}{2}\frac{\partial^2 F}{\partial\bar{z}_i\partial\bar{z}_j}\frac{b_i b_j}{4\pi^2}\right)\mathscr{P}(Z,0). \tag{4.12}$$

Thus from (1.6), (4.11) and (4.12), we have

$$\begin{aligned}(\mathscr{P}F\mathscr{P})(Z,0) &= \left(\sum_{|\alpha|=2}\frac{\partial^2 F}{\partial z^\alpha}\frac{z^\alpha}{\alpha!} + \frac{1}{\pi}\frac{\partial^2 F}{\partial z_i\partial\bar{z}_i}\right)\mathscr{P}(Z,0),\\ (\mathscr{P}F\bar{z}_i\bar{z}_j\mathscr{P})(0,0) &= \frac{1}{\pi^2}\frac{\partial^2 F}{\partial z_i\partial z_j}.\end{aligned} \tag{4.13}$$

By (4.10) and (4.13), we get (4.1c). The proof of Lemma 4.1 is completed. □

Lemma 4.2. *The following identity holds*:

$$\begin{aligned}&\pi^2(\mathscr{L}^{-1}\mathscr{P}^\perp\mathscr{O}_2\mathscr{L}^{-1}\mathscr{O}_2\mathscr{P})(0,0)\\ &= -\frac{25}{2^3\cdot 3^3}R_{m\bar{k}q\bar{\ell}}R_{k\bar{m}\ell\bar{q}} - \frac{47}{54}R_{k\bar{k}q\bar{\ell}}R_{m\bar{m}\ell\bar{q}} + \frac{1}{8}R_{k\bar{k}\ell\bar{\ell}}R_{m\bar{m}q\bar{q}}\\ &\quad + \frac{1}{4}R^E_{\ell\bar{\ell}}R_{m\bar{m}q\bar{q}} - \frac{7}{6}R^E_{q\bar{\ell}}R_{m\bar{m}\ell\bar{q}} + \frac{1}{8}(R^E_{\ell\bar{\ell}}R^E_{q\bar{q}} - 3R^E_{q\bar{\ell}}R^E_{\ell\bar{q}}).\end{aligned} \tag{4.14}$$

Proof. Set

$$\begin{aligned}\boldsymbol{I}_1 = &\left\{\frac{1}{3}b_k b_\ell R_{\bar{s}k\bar{t}\bar{\ell}}z_s z_t + b_\ell\left[-\frac{\pi}{3}R_{\bar{s}k\bar{t}\bar{\ell}}z_s z_t\bar{z}_k + (2R_{t\bar{s}s\bar{\ell}} + R^E_{t\bar{\ell}})z_t\right]\right\}\\ &\times\left[\frac{b_m b_q}{48\pi}R_{i\bar{m}j\bar{q}}z_i z_j + \frac{b_q}{4\pi}\left(\frac{4}{3}R_{j\bar{i}i\bar{q}} + R^E_{j\bar{q}}\right)z_j\right],\end{aligned} \tag{4.15}$$

$$\boldsymbol{I}_2 = \left[\frac{2b_\ell}{3} R_{s\bar{k}q\bar{\ell}} z_s \bar{z}_k - \frac{\pi}{3} R_{s\bar{k}q\bar{\ell}} z_s \bar{z}_k \bar{z}_\ell + (2R_{s\bar{s}q\bar{k}} + R^E_{q\bar{k}})\bar{z}_k\right] \times \left(\frac{b_m}{6} R_{i\bar{m}j\bar{q}} z_i z_j + \frac{4}{3} R_{j\bar{i}i\bar{q}} z_j + R^E_{j\bar{q}} z_j\right). \tag{4.15}$$

Then by Lemma 3.1, (1.7), (4.2), (4.5) and (4.8), we get as in (4.10) that

$$(\mathcal{O}_2 \mathcal{L}^{-1} \mathcal{O}_2 \mathscr{P})(Z,0) = \left\{\boldsymbol{I}_1 + \boldsymbol{I}_2 + \frac{2\pi}{9} R_{m\bar{s}q\bar{t}} \bar{z}_s \bar{z}_t R_{k\bar{m}\ell\bar{q}} z_k z_\ell\right\} \mathscr{P}(Z,0). \tag{4.16}$$

Let $h(z) = \sum_i h_i z_i$, $h'(z) = \sum_i h'_i z_i$ with $h_i, h'_i \in \mathbb{C}$, and let $F(Z)$ be a homogeneous degree 2 polynomial in Z. By Theorem 1.1, (1.6), (1.7), (4.2), (4.11) and (4.12), we have

$$(\mathscr{P}^\perp F \mathscr{P})(0,0) = -\frac{1}{\pi} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i}, \tag{4.17a}$$

$$(\mathcal{L}^{-1} \mathscr{P}^\perp h b_i \mathscr{P})(0,0) = (\mathcal{L}^{-1} b_i h \mathscr{P})(0,0) = -\frac{1}{2\pi} h_i, \tag{4.17b}$$

$$(\mathcal{L}^{-1} \mathscr{P}^\perp F \mathscr{P})(0,0) = -\frac{1}{4\pi^2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i}, \tag{4.17c}$$

$$(\mathcal{L}^{-1} b_j F b_i \mathscr{P})(0,0) = -(\mathcal{L}^{-1} b_i b_j F \mathscr{P})(0,0) = -\frac{1}{2\pi} \frac{\partial^2 F}{\partial z_i \partial z_j}, \tag{4.17d}$$

$$(\mathcal{L}^{-1} \mathscr{P}^\perp F b_i b_j \mathscr{P})(0,0) = \mathcal{L}^{-1} \mathscr{P}^\perp \left(b_j F b_i + 2\frac{\partial F}{\partial z_j} b_i\right) \mathscr{P}(0,0) = -\frac{3}{2\pi} \frac{\partial^2 F}{\partial z_i \partial z_j}, \tag{4.17e}$$

$$(\mathcal{L}^{-1} \mathscr{P}^\perp F \bar{z}_i \bar{z}_j \mathscr{P})(0,0) = \mathcal{L}^{-1} \mathscr{P}^\perp F \frac{b_i b_j}{4\pi^2} \mathscr{P}(0,0) = -\frac{3}{8\pi^3} \frac{\partial^2 F}{\partial z_i \partial z_j}, \tag{4.17f}$$

$$(\mathcal{L}^{-1} b_k F \bar{z}_l \mathscr{P})(0,0) = \mathcal{L}^{-1} b_k F \frac{b_l}{2\pi} \mathscr{P}(0,0) = -\frac{1}{4\pi^2} \frac{\partial^2 F}{\partial z_k \partial z_l}. \tag{4.17g}$$

In (4.17c) and (4.17d), we have used $F b_i = b_i F + 2\frac{\partial F}{\partial z_i}$. Observe that (1.5), (1.7) imply that for every homogeneous degree k polynomial G in Z, and every $\alpha \in \mathbb{N}^n$, we have

$$(b^\alpha G \mathscr{P})(0,0) = \begin{cases} 0, & \text{if } |\alpha| \neq k, \\ (-2)^k \dfrac{\partial^\alpha G}{\partial z^\alpha}, & \text{if } |\alpha| = k. \end{cases} \tag{4.18}$$

By Theorem 1.1, (1.6) and (1.7), we also have

$$(\mathcal{L}^{-1} b_i h b_j h' \mathscr{P})(0,0) = \left(\frac{b_j b_i}{8\pi} h h' + \frac{b_i}{2\pi} h_j h'\right) \mathscr{P}(0,0) = \frac{1}{2\pi}(h_i h'_j - h_j h'_i), \tag{4.19a}$$

$$(\mathcal{L}^{-1} \mathscr{P}^\perp h b_i h' b_j \mathscr{P})(0,0) = -\frac{1}{2\pi} h_j h'_i - \frac{3}{2\pi} h_i h'_j, \tag{4.19b}$$

where we use in the last equation that $hb_ih'b_j = b_ihh'b_j + 2h_ih'b_j$ and further (4.17b), (4.17d).

Let $\phi(z) = \phi_{ij}z_iz_j$, $\psi = \psi_{ij}z_iz_j$ be degree 2 polynomials in z with symmetric matrices (ϕ_{ij}), (ψ_{ij}). Then by Theorem 1.1, (1.7), (4.11), (4.17d), (4.17e) and (4.19a), we obtain

$$\begin{aligned}(4.20a)\qquad (\mathscr{L}^{-1}b_\ell b_k\phi b_jh\mathscr{P})(0,0) &= \left(\frac{b_\ell b_k b_j}{12\pi}\phi h + \frac{b_\ell b_k}{4\pi}\frac{\partial\phi}{\partial z_j}h\right)\mathscr{P}(0,0)\\ &= \frac{-2}{3\pi}\frac{\partial^3(\phi h)}{\partial z_\ell\partial z_k\partial z_j} + \frac{2}{\pi}(\phi_{\ell j}h_k + \phi_{kj}h_\ell),\end{aligned}$$

$$\begin{aligned}(4.20b)\qquad (\mathscr{L}^{-1}b_k\phi b_\ell b_jh\mathscr{P})(0,0) &= \mathscr{L}^{-1}\left(b_\ell b_k\phi + 2b_k\frac{\partial\phi}{\partial z_\ell}\right)b_jh\mathscr{P}(0,0)\\ &= \frac{-2}{3\pi}\frac{\partial^3(\phi h)}{\partial z_\ell\partial z_k\partial z_j} + \frac{2}{\pi}(\phi_{kj}h_\ell + \phi_{\ell k}h_j),\end{aligned}$$

$$\begin{aligned}(4.20c)\qquad -\pi(\mathscr{L}^{-1}b_k\phi\bar z_\ell b_jh\mathscr{P})(0,0) &= -\frac{1}{2}\mathscr{L}^{-1}b_k\phi b_j(b_\ell h + 2h_\ell)\mathscr{P}(0,0)\\ &= \frac{1}{3\pi}\frac{\partial^3(\phi h)}{\partial z_\ell\partial z_k\partial z_j} - \frac{1}{\pi}\phi_{\ell k}h_j,\end{aligned}$$

and

$$\begin{aligned}(4.21a)\qquad (\mathscr{L}^{-1}b_khb_ib_j\psi\mathscr{P})(0,0) &= \left(\frac{b_kb_ib_j}{12\pi}h\psi + \frac{b_k}{4\pi}(b_ih_j + b_jh_i)\psi\right)\mathscr{P}(0,0)\\ &= \frac{-2}{3\pi}\frac{\partial^3(h\psi)}{\partial z_k\partial z_i\partial z_j} + \frac{2}{\pi}(\psi_{ki}h_j + \psi_{kj}h_i),\end{aligned}$$

$$\begin{aligned}(4.21b)\qquad (\mathscr{L}^{-1}b_kh\bar z_\ell b_j\psi\mathscr{P})(0,0) &= \mathscr{L}^{-1}b_khb_j\left(\frac{b_\ell}{2\pi}\psi + \frac{1}{\pi}\frac{\partial\psi}{\partial z_\ell}\right)\mathscr{P}(0,0)\\ &= \frac{-1}{3\pi^2}\frac{\partial^3(h\psi)}{\partial z_\ell\partial z_k\partial z_j} + \frac{1}{\pi^2}(\psi_{kj}h_\ell + \psi_{\ell j}h_k),\end{aligned}$$

$$(4.21c)\qquad -\pi(\mathscr{L}^{-1}h\bar z_k\bar z_\ell b_j\psi\mathscr{P})(0,0) = \frac{1}{6\pi^2}\frac{\partial^3(h\psi)}{\partial z_\ell\partial z_k\partial z_j} + \frac{3}{2\pi^2}\psi_{\ell k}h_j.$$

In fact, by (4.2), (4.11) implies

$$\pi(h\bar z_k\bar z_\ell b_j\psi\mathscr{P})(Z,0) = \frac{1}{2}\left(b_khb_j\psi\bar z_\ell + \left(2h_kb_j\psi + 2hb_j\frac{\partial\psi}{\partial z_k}\right)\frac{b_\ell}{2\pi}\right)\mathscr{P}(Z,0),$$

so from (4.17d), (4.19b) and (4.21b) we get (4.21c).

Set $R_{\bar m\bar q}(z) = R_{k\bar m l\bar q}z_kz_l$. By (1.7) and Lemma 3.1,

$$R_{\bar s\bar k}(z)b_mb_qR_{\bar m\bar q}(z) = \left(b_mb_qR_{\bar s\bar k}(z) + 8b_mR_{q\bar s\ell\bar k}z_\ell + 8R_{m\bar sq\bar k}\right)R_{\bar m\bar q}(z),$$

thus Theorem 1.1, (4.11), (4.20b) and (4.21a) show that

$$
\begin{aligned}
&(\mathscr{L}^{-1} b_s b_k R_{\bar{s}\bar{k}} b_m b_q R_{\bar{m}\bar{q}} \mathscr{P})(0,0) \\
&= b_s b_k \left(\frac{b_m b_q}{16\pi} R_{\bar{s}\bar{k}} + \frac{2 b_m}{3\pi} R_{q\bar{s}\ell\bar{k}} z_\ell + \frac{1}{\pi} R_{m\bar{s}q\bar{k}} \right) R_{\bar{m}\bar{q}} \mathscr{P}(0,0) \\
&= \frac{1}{\pi} \frac{\partial^4 (R_{\bar{s}\bar{k}} R_{\bar{m}\bar{q}})}{\partial z_s \partial z_k \partial z_m \partial z_q} - \frac{16}{3\pi} \frac{\partial^3 (R_{q\bar{s}\ell\bar{k}} z_\ell R_{\bar{m}\bar{q}})}{\partial z_s \partial z_k \partial z_m} + \frac{8}{\pi} R_{m\bar{s}q\bar{k}} R_{s\bar{m}k\bar{q}},
\end{aligned}
$$

$$
\begin{aligned}
(4.22)\qquad &-(\pi \mathscr{L}^{-1} b_k \bar{z}_s R_{\bar{s}\bar{k}} b_m b_q R_{\bar{m}\bar{q}} \mathscr{P})(0,0) \\
&= -\mathscr{L}^{-1} b_k \left(\frac{1}{2} b_s R_{\bar{s}\bar{k}} b_m b_q R_{\bar{m}\bar{q}} + \frac{\partial R_{\bar{s}\bar{k}}}{\partial z_s} b_m b_q R_{\bar{m}\bar{q}} + R_{\bar{s}\bar{k}} b_m b_q \frac{\partial R_{\bar{m}\bar{q}}}{\partial z_s} \right) \mathscr{P}(0,0) \\
&= \frac{1}{6\pi} \frac{\partial^4 (R_{\bar{s}\bar{k}} R_{\bar{m}\bar{q}})}{\partial z_s \partial z_k \partial z_m \partial z_q} + \frac{8}{3\pi} \frac{\partial^3 (R_{q\bar{s}\ell\bar{k}} z_\ell R_{\bar{m}\bar{q}})}{\partial z_s \partial z_k \partial z_m} \\
&\quad - \frac{4}{\pi} R_{m\bar{s}q\bar{k}} R_{s\bar{m}k\bar{q}} - \frac{16}{\pi} R_{s\bar{s}q\bar{k}} R_{m\bar{m}k\bar{q}}.
\end{aligned}
$$

Due to (4.15), (4.17a)–(4.22), we get

$$
\begin{aligned}
(4.23)\qquad &\pi^2 (\mathscr{L}^{-1} \boldsymbol{I}_1 \mathscr{P})(0,0) \\
&= \frac{1}{144} \Bigg[\frac{7}{6} \frac{\partial^4 (R_{\bar{s}\bar{t}} R_{\bar{m}\bar{q}})}{\partial z_s \partial z_t \partial z_m \partial z_q} - \frac{8}{3} \frac{\partial^3 (R_{q\bar{s}\ell\bar{t}} z_v R_{\bar{m}\bar{q}})}{\partial z_s \partial z_t \partial z_m} + 4 R_{m\bar{s}q\bar{t}} R_{s\bar{m}t\bar{q}} \\
&\qquad - 16 R_{s\bar{s}q\bar{t}} R_{m\bar{m}t\bar{q}} - 2 \frac{\partial^3 \left((2 R_{v\bar{u}u\bar{t}} + R^E_{v\bar{t}}) z_v R_{\bar{m}\bar{q}} \right)}{\partial z_t \partial z_m \partial z_q} \\
&\qquad + 12 (2 R_{q\bar{u}u\bar{t}} + R^E_{q\bar{t}}) R_{t\bar{m}m\bar{q}} \Bigg] \\
&\quad + \frac{1}{12} \Bigg[-\frac{1}{3} \frac{\partial^3 \left(R_{\bar{s}\bar{t}} \left(\frac{4}{3} R_{\ell\bar{k}k\bar{q}} + R^E_{\ell\bar{q}} \right) z_\ell \right)}{\partial z_s \partial z_t \partial z_q} + 4 R_{s\bar{s}q\bar{t}} \left(\frac{4}{3} R_{t\bar{k}k\bar{q}} + R^E_{t\bar{q}} \right) \\
&\qquad - R_{s\bar{s}t\bar{t}} \left(\frac{4}{3} R_{q\bar{k}k\bar{q}} + R^E_{q\bar{q}} \right) \Bigg] \\
&\quad + \frac{1}{8} \Bigg[(2 R_{t\bar{u}u\bar{t}} + R^E_{t\bar{t}}) \left(\frac{4}{3} R_{q\bar{k}k\bar{q}} + R^E_{q\bar{q}} \right) \\
&\qquad - (2 R_{q\bar{u}u\bar{t}} + R^E_{q\bar{t}}) \left(\frac{4}{3} R_{t\bar{k}k\bar{q}} + R^E_{t\bar{q}} \right) \Bigg].
\end{aligned}
$$

But from Lemma 3.1, we have

$$\frac{\partial^4(R_{\bar{s}\bar{t}}R_{\bar{m}\bar{q}})}{\partial z_s\partial z_t\partial z_m\partial z_q} = 4R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} + 4R_{s\bar{s}t\bar{t}}R_{m\bar{m}q\bar{q}} + 16R_{s\bar{s}m\bar{t}}R_{q\bar{m}t\bar{q}},$$

$$(4.24)\qquad \frac{\partial^3(R_{s\bar{s}v\bar{t}}z_vR_{\bar{m}\bar{q}})}{\partial z_t\partial z_m\partial z_q} = 2R_{s\bar{s}t\bar{t}}R_{m\bar{m}q\bar{q}} + 4R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}},$$

$$\frac{\partial^3(R_{m\bar{s}v\bar{t}}z_vR_{\bar{m}\bar{q}})}{\partial z_s\partial z_t\partial z_q} = 2R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} + 4R_{m\bar{s}s\bar{t}}R_{t\bar{m}q\bar{q}}.$$

Plugging (4.24) in (4.23) we see that the coefficient of $R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}}$ in the term $\frac{1}{144}[\cdots]$ of (4.23) is $\frac{1}{144}\left(\frac{56}{3}-\frac{32}{3}-16-16+24\right)=0$ and

$$(4.25)\qquad \pi^2(\mathscr{L}^{-1}\boldsymbol{I}_1\mathscr{P})(0,0)$$
$$= \frac{1}{144}\frac{10}{3}R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} - \frac{1}{27}R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}}$$
$$+ \left[\frac{1}{144}\cdot\frac{-10}{3}+\frac{1}{12}\cdot\frac{-20}{9}+\frac{1}{3}\right]R_{s\bar{s}t\bar{t}}R_{m\bar{m}q\bar{q}}$$
$$+ \left[\frac{-4}{144}+\frac{1}{12}\cdot\frac{-5}{3}+\frac{5}{12}\right]R^E_{t\bar{t}}R_{m\bar{m}q\bar{q}}$$
$$+ \left[\frac{4}{144}+\frac{1}{12}\cdot\frac{8}{3}-\frac{10}{3\cdot 8}\right]R^E_{q\bar{t}}R_{m\bar{m}t\bar{q}} + \frac{1}{8}(R^E_{t\bar{t}}R^E_{q\bar{q}} - R^E_{q\bar{t}}R^E_{t\bar{q}})$$
$$= \frac{5}{2^3\cdot 3^3}R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} - \frac{1}{27}R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}} + \frac{1}{8}R_{s\bar{s}t\bar{t}}R_{m\bar{m}q\bar{q}}$$
$$+ \frac{1}{4}R^E_{t\bar{t}}R_{m\bar{m}q\bar{q}} - \frac{1}{6}R^E_{q\bar{t}}R_{m\bar{m}t\bar{q}} + \frac{1}{8}(R^E_{t\bar{t}}R^E_{q\bar{q}} - R^E_{q\bar{t}}R^E_{t\bar{q}}).$$

From (4.15), (4.17c), (4.17f), (4.17g), (4.21b), (4.21c) and (4.24), we get

$$(4.26)\qquad \pi^2(\mathscr{L}^{-1}\boldsymbol{I}_2\mathscr{P})(0,0)$$
$$= \frac{1}{18}\left[-\frac{1}{2}\frac{\partial^3(R_{u\bar{s}q\bar{t}}z_uR_{\bar{m}\bar{q}})}{\partial z_s\partial z_t\partial z_m} + 4R_{s\bar{s}q\bar{t}}R_{t\bar{m}m\bar{q}}\right.$$
$$\left.+\frac{3}{2}R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} + 3(2R_{u\bar{u}q\bar{s}} + R^E_{q\bar{s}})\left(-\frac{2}{4}\right)R_{m\bar{m}s\bar{q}}\right]$$
$$+ \left[-\frac{1}{12}R_{t\bar{s}q\bar{t}} - \frac{1}{4}(2R_{u\bar{u}q\bar{s}} + R^E_{q\bar{s}})\right]\left(\frac{4}{3}R_{s\bar{k}k\bar{q}} + R^E_{s\bar{q}}\right)$$
$$= \frac{1}{36}R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} - \frac{5}{6}R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}} - R^E_{s\bar{q}}R_{t\bar{s}q\bar{t}} - \frac{1}{4}R^E_{q\bar{t}}R^E_{t\bar{q}}.$$

By (4.17f), we get

$$(4.27)\qquad \left(\mathscr{L}^{-1}\frac{2\pi}{9}R_{m\bar{s}q\bar{t}}\bar{z}_s\bar{z}_t R_{k\bar{m}\ell\bar{q}}z_k z_\ell\mathscr{P}\right)(0,0) = -\frac{1}{6\pi^2}R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}}.$$

Relations (4.16), (4.25), (4.26) and (4.27) imply the desired formula (4.14). □

4.2. Contribution of $\mathcal{O}_4$ to $\mathscr{F}_4(0,0)$. We will use the following remark repeatedly in our computation.

Remark 4.3. Let Φ be a polynomial in b^+, z, b, $\bar{z}$. Due to (1.7) and (4.2), the value of the kernels of $\mathscr{P}\Phi\mathscr{P}$, $\mathscr{P}^\perp\Phi\mathscr{P}$, $\mathscr{L}^{-1}\mathscr{P}^\perp\Phi\mathscr{P}$ at $(0,0)$ consists of the terms of Φ whose total degree in b and $\bar{z}$ is the same as the total degree in b^+ and z.

Lemma 4.4. *We have the following identity*:

$$(4.28)\qquad -\pi^2(\mathscr{L}^{-1}\mathcal{O}_4\mathscr{P})(0,0) = -\frac{\Delta \boldsymbol{r}}{96} + \frac{23}{108}R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} + \frac{41}{54}R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}}$$
$$+ R_{m\bar{m}q\bar{k}}R^E_{k\bar{q}} + \frac{1}{8}(-R^E_{m\bar{m};q\bar{q}} + 3R^E_{q\bar{m};m\bar{q}}) + \frac{1}{4}R^E_{k\bar{q}}R^E_{q\bar{k}}.$$

Proof. By (1.7), (3.14), (4.2) and (4.3), as in (4.4), we have

$$(4.29)\qquad -(\mathcal{O}_{41}\mathscr{P})(Z,0) = -\left\{\frac{1}{20}\left(A_{1Z} - \frac{4}{3}A_{2Z}\right)\left(\frac{\partial}{\partial\bar{z}_i}, \frac{\partial}{\partial\bar{z}_j}\right)b_i b_j\right.$$
$$\left. - \frac{1}{20}\left(A_{1Z} - \frac{4}{3}A_{2Z}\right)\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial\bar{z}_j}\right)b_i^+ b_j\right\}\mathscr{P}(Z,0)$$
$$= \left[-\frac{\pi^2}{5}\left(A_{1Z} - \frac{4}{3}A_{2Z}\right)(\bar{z},\bar{z})\right.$$
$$\left. + \frac{\pi}{5}\left(A_{1Z} - \frac{4}{3}A_{2Z}\right)\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial\bar{z}_j}\right)\right]\mathscr{P}(Z,0).$$

and

$$(4.30)\qquad -\mathscr{L}^{-1}\mathcal{O}_{42}\mathscr{P} = \mathscr{P}^\perp\left[4\left(\frac{1}{80}A_{1Z} - \frac{1}{360}A_{2Z}\right)\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial\bar{z}_j}\right) - \frac{1}{72}\operatorname{Ric}(z,\bar{z})^2\right]\mathscr{P}.$$

From (3.12), (3.14), (4.2) and (4.3), and since R^{TX} is $(1,1)$-form, we have

$$-(\mathcal{O}_{44}\mathscr{P})(Z,0) = 2\pi\bigg\{\frac{\pi}{30}A_{1Z}(\bar z,\bar z) + \bigg(\frac{\partial}{\partial Z_j}\Big(\frac{1}{20}A_{1Z} + \frac{2}{45}A_{2Z}\Big)\bigg)(\bar z, e_j) - \frac{\pi}{10}A_{2Z}(\bar z,\bar z) - \bar z_i\frac{\partial}{\partial \bar z_i}\Big(\frac{1}{10}A_{1Z} + \frac{4}{45}A_{2Z}\Big)\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_j}\Big)\bigg\}\mathscr{P}(Z,0),$$

(4.31)

$$-(\mathcal{O}_{45}\mathscr{P})(Z,0) = 2\pi\bigg\{\frac{2}{9}\langle R^{TX}(\mathcal{R},e_k)z, R^{TX}_{x_0}(\mathcal{R},e_k)\bar z\rangle - \frac{1}{9}\langle R^{TX}(z,\bar z)\mathcal{R},e_k\rangle\,\mathrm{Ric}(\mathcal{R},e_k) + \frac{1}{4}\langle R^{TX}(z,\bar z)\mathcal{R},e_j\rangle R^E(\mathcal{R},e_j) - \frac{1}{4}R^E_{;(Z,Z)}(z,\bar z)\bigg\}\mathscr{P}(Z,0).$$

Let ψ_{ijk} be degree 3 polynomials in z which are symmetric in i, j. By (1.7), (4.2), we get

$$(\psi_{ijk}\bar z_i\bar z_j\bar z_k\mathscr{P})(Z,0) = \frac{1}{8\pi^3}(\psi_{ijk}b_ib_jb_k\mathscr{P})(Z,0) = \frac{1}{8\pi^3}\bigg\{b_ib_jb_k\psi_{ijk} + 2b_ib_j\frac{\partial\psi_{ijk}}{\partial z_k} + 4b_ib_k\frac{\partial\psi_{ijk}}{\partial z_j} + 8b_i\frac{\partial^2\psi_{ijk}}{\partial z_j\partial z_k} + 4b_k\frac{\partial^2\psi_{ijk}}{\partial z_i\partial z_j} + 8\frac{\partial^3\psi_{ijk}}{\partial z_i\partial z_j\partial z_k}\bigg\}\mathscr{P}(Z,0). \tag{4.32}$$

Thus by Theorem 1.1, (1.1) and (4.32), we obtain

$$\begin{aligned}\pi^2(\mathscr{L}^{-1}\mathscr{P}^\perp\psi_{ijk}\bar z_i\bar z_j\bar z_k\mathscr{P})(0,0) &= \frac{1}{8\pi^2}\bigg\{\frac{b_ib_jb_k}{12}\psi_{ijk} + \frac{1}{4}b_ib_j\frac{\partial\psi_{ijk}}{\partial z_k} + \frac{1}{2}b_ib_k\frac{\partial\psi_{ijk}}{\partial z_j} + 2b_i\frac{\partial^2\psi_{ijk}}{\partial z_j\partial z_k} + b_k\frac{\partial^2\psi_{ijk}}{\partial z_i\partial z_j}\bigg\}\mathscr{P}(0,0)\\ &= -\frac{11}{24\pi^2}\frac{\partial^3\psi_{ijk}}{\partial z_i\partial z_j\partial z_k}.\end{aligned} \tag{4.33}$$

Since $\left|\frac{\partial}{\partial z_j}\right|^2 = \frac{1}{2}$, Lemma 3.1, (3.6), (3.12) and the fact that R^{TX} is a $(1,1)$-form entail

$$\begin{aligned} A_{1Z}(\bar z,\bar z) &= \langle R^{TX}_{;(Z,Z)}(z,\bar z)z,\bar z\rangle,\\ A_{2Z}(\bar z,\bar z) &= 2\langle R^{TX}(z,\bar z)z, R^{TX}(z,\bar z)\bar z\rangle = -4R_{u\bar s v\bar t}R_{k\bar m t\bar q}z_u\bar z_s z_v z_k\bar z_m\bar z_q,\\ A_{1Z}\Big(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar z_q}\Big) &= \Big\langle R^{TX}_{;(Z,Z)}\Big(\bar z,\frac{\partial}{\partial z_q}\Big)z,\frac{\partial}{\partial\bar z_q}\Big\rangle,\\ A_{2Z}\Big(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar z_q}\Big) &= \Big\langle R^{TX}\Big(\bar z,\frac{\partial}{\partial z_q}\Big)\mathcal{R}, R^{TX}\Big(z,\frac{\partial}{\partial\bar z_q}\Big)\mathcal{R}\Big\rangle\\ &= 2(R_{q\bar s k\bar t}R_{\ell\bar q t\bar u} + R_{q\bar s t\bar u}R_{\ell\bar q k\bar t})\bar z_s\bar z_u z_k z_\ell.\end{aligned} \tag{4.34}$$

By Remark 4.3, we can replace $A_{1Z}(\bar{z},\bar{z})$ by $2R_{k\bar{m}\ell\bar{q};s\bar{t}}z_k\bar{z}_m z_\ell \bar{z}_q z_s \bar{z}_t$ in our computation. We deduce from (4.33) and (4.34) that

$$
\begin{aligned}
&\pi^2\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}A_{1Z}(\bar{z},\bar{z})\mathscr{P}\big)(0,0) = -\frac{11}{6\pi^2}(R_{m\bar{m}q\bar{q};t\bar{t}} + 2R_{t\bar{m}q\bar{q};m\bar{t}}),\\
&\pi^2\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}A_{2Z}(\bar{z},\bar{z})\mathscr{P}\big)(0,0) = \frac{11}{3\pi^2}(R_{m\bar{s}q\bar{t}}R_{s\bar{m}t\bar{q}} + 2R_{s\bar{s}q\bar{t}}R_{m\bar{m}t\bar{q}}).
\end{aligned} \tag{4.35}
$$

Let F_{ij} be homogeneous degree 2 polynomials in Z. Then by (1.1), (4.11) and (4.17f), we obtain

$$(\mathscr{P}^{\perp}F_{ij}\bar{z}_i\bar{z}_j\mathscr{P})(0,0) = -\frac{1}{\pi^2}\frac{\partial^2 F_{ij}}{\partial z_i \partial z_j}, \tag{4.36a}$$

$$\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\bar{z}_k\frac{\partial(F_{ij}\bar{z}_i\bar{z}_j)}{\partial\bar{z}_k}\mathscr{P}\right)(0,0) = -\frac{3}{4\pi^3}\frac{\partial^2 F_{ij}}{\partial z_i\partial z_j}. \tag{4.36b}$$

(Note that by Remark 4.3 the contributions of $\bar{z}_k\dfrac{\partial(F_{ij}\bar{z}_i\bar{z}_j)}{\partial\bar{z}_k}$ and $2F_{ij}\bar{z}_i\bar{z}_j$ to (4.36b) are the same, so (4.36b) follows from (4.17f).)

By Remark 4.3 and (4.34), only the term $-2R_{q\bar{m}k\bar{q};\ell\bar{t}}\bar{z}_m z_k \bar{z}_t z_\ell$ from $A_{1Z}\left(\dfrac{\partial}{\partial z_q},\dfrac{\partial}{\partial\bar{z}_q}\right)$ has a nontrivial contribution in our computation at $(0,0)$, and from (4.17f), (4.36a) and (4.36b), we get

$$
\begin{aligned}
&\left(\mathscr{P}^{\perp}A_{1Z}\left(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar{z}_q}\right)\mathscr{P}\right)(0,0) = \frac{2}{\pi^2}(R_{q\bar{m}m\bar{q};t\bar{t}} + R_{q\bar{m}t\bar{q};m\bar{t}}),\\
&\frac{1}{2}\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\bar{z}_m\frac{\partial}{\partial\bar{z}_m}A_{1Z}\left(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar{z}_q}\right)\mathscr{P}\right)(0,0)\\
&\qquad = \left(\mathscr{L}^{-1}\mathscr{P}^{\perp}A_{1Z}\left(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar{z}_q}\right)\mathscr{P}\right)(0,0) = \frac{3}{4\pi^3}(R_{q\bar{m}m\bar{q};t\bar{t}} + R_{q\bar{m}t\bar{q};m\bar{t}}),\\
&\left(\mathscr{P}^{\perp}A_{2Z}\left(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar{z}_q}\right)\mathscr{P}\right)(0,0) = -\frac{2}{\pi^2}(R_{q\bar{s}s\bar{t}}R_{u\bar{q}t\bar{u}} + 3R_{q\bar{s}u\bar{t}}R_{s\bar{q}t\bar{u}}),\\
&\frac{1}{2}\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\bar{z}_m\frac{\partial}{\partial\bar{z}_m}A_{2Z}\left(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar{z}_q}\right)\mathscr{P}\right)(0,0)\\
&\qquad = \left(\mathscr{L}^{-1}\mathscr{P}^{\perp}A_{2Z}\left(\frac{\partial}{\partial z_q},\frac{\partial}{\partial\bar{z}_q}\right)\mathscr{P}\right)(0,0)\\
&\qquad = -\frac{3}{4\pi^3}(R_{q\bar{s}s\bar{t}}R_{u\bar{q}t\bar{u}} + 3R_{q\bar{s}u\bar{t}}R_{s\bar{q}t\bar{u}}).
\end{aligned} \tag{4.37}
$$

Remark 4.3, (4.3), (4.17c) and (4.17f) yield

$$
(4.38)\quad
\begin{aligned}
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\langle R^{TX}_{;(Z,e_m)}(z,\bar z)\mathcal{R}, e_m\rangle_{x_0}\mathscr{P}\big)(0,0)\\
&\qquad = 2\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}(R_{k\bar q\ell\bar m;m\bar s} - R_{k\bar q m\bar s;\ell\bar m}) z_k\bar z_q z_\ell \bar z_s\mathscr{P}\big)(0,0) = 0,\\
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\langle R^{TX}_{x_0}(z,\bar z)\mathcal{R}, e_k\rangle_{x_0}\operatorname{Ric}(\mathcal{R}, e_k)\mathscr{P}\big)(0,0)\\
&\qquad = 2\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}(-R_{\ell\bar s k\bar q}\operatorname{Ric}_{m\bar k} + R_{\ell\bar s m\bar k}\operatorname{Ric}_{k\bar q}) z_\ell\bar z_s\bar z_q z_m\mathscr{P}\big)(0,0) = 0,\\
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\operatorname{Ric}(\mathcal{R}, e_q) R^E_{x_0}(\mathcal{R}, e_q)\mathscr{P}\big)(0,0)\\
&\qquad = 2\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}(-\operatorname{Ric}_{k\bar q} R^E_{q\bar s} + \operatorname{Ric}_{q\bar s} R^E_{k\bar q}) z_k\bar z_s\mathscr{P}\big)(0,0) = 0,\\
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp} R^E_{;(Z,e_s)}(\mathcal{R}, e_s)\mathscr{P}\big)(0,0)\\
&\qquad = 2\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}(R^E_{k\bar s;s\bar q} - R^E_{s\bar q;k\bar s}) z_k\bar z_q\mathscr{P}\big)(0,0) = 0.
\end{aligned}
$$

By (3.39), (3.40), (4.3) and (4.38), we know that for $\alpha = 1, 2$,

$$
(4.39)\quad
\begin{aligned}
&\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\frac{\partial}{\partial Z_q}\big(A_{\alpha Z}(\bar z, e_q)\big)\mathscr{P}\right)(0,0) = 0,\\
&\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\left(\frac{\partial}{\partial Z_q}A_{\alpha Z}\right)(\bar z, e_q)\mathscr{P}\right)(0,0) = -2\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}A_{\alpha Z}\left(\frac{\partial}{\partial \bar z_q}, \frac{\partial}{\partial z_q}\right)\mathscr{P}\right)(0,0).
\end{aligned}
$$

By (3.5) and (4.17f), we have

$$
(4.40)\quad
\begin{aligned}
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\langle R^{TX}_{x_0}(\mathcal{R}, e_k)z, R^{TX}_{x_0}(\mathcal{R}, e_k)\bar z\rangle_{x_0}\mathscr{P}\big)(0,0)\\
&\qquad = 4\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}(R_{k\bar s\ell\bar u}R_{m\bar k u\bar q} + R_{m\bar k\ell\bar u}R_{k\bar s u\bar q})\bar z_s z_\ell z_m\bar z_q\mathscr{P}\big)(0,0)\\
&\qquad = -\frac{3}{2\pi^3}(R_{k\bar s s\bar u}R_{q\bar k u\bar q} + 3R_{s\bar k q\bar u}R_{k\bar s u\bar q}),\\
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\langle \pi R^{TX}_{x_0}(z,\bar z)\mathcal{R}, e_k\rangle R^E_{x_0}(\mathcal{R}, e_k)\mathscr{P}\big)(0,0)\\
&\qquad = 2\pi\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}(-R_{\ell\bar s k\bar q}R^E_{m\bar k} - R_{\ell\bar s m\bar k}R^E_{k\bar q}) z_\ell\bar z_s\bar z_q z_m\mathscr{P}\big)(0,0) = \frac{3}{\pi^2}R_{s\bar s q\bar k}R^E_{k\bar q},\\
&\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}R^E_{;(Z,Z)}(z,\bar z)\mathscr{P}\big)(0,0) = -\frac{3}{4\pi^3}(R^E_{s\bar s;q\bar q} + R^E_{q\bar s;s\bar q}).
\end{aligned}
$$

Note that by Lemma 3.1, $\operatorname{Ric}(\mathcal{R},\mathcal{R}) = 2\operatorname{Ric}_{k\bar q} z_k\bar z_q$, and by (1.2), (1.7) and (4.2),

$$
(4.41)\quad \mathscr{L}\operatorname{Ric}(\mathcal{R},\mathcal{R})\mathscr{P} = 2b_m b_m^+\operatorname{Ric}_{k\bar q} z_k\bar z_q\mathscr{P} = 4b_m\operatorname{Ric}_{k\bar m} z_k\mathscr{P}.
$$

Thus by (3.14), (4.19b) and (4.41), we obtain

$$
(4.42)\quad
\begin{aligned}
-(\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_{43}\mathscr{P})(0,0) &= \frac{1}{18}\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\operatorname{Ric}_{l\bar q} z_l b_m\operatorname{Ric}_{k\bar m} z_k\frac{b_q}{2\pi}\mathscr{P}\right)(0,0)\\
&= -\frac{1}{72\pi^2}(\operatorname{Ric}_{m\bar m}\operatorname{Ric}_{q\bar q} + 3\operatorname{Ric}_{m\bar q}\operatorname{Ric}_{q\bar m}).
\end{aligned}
$$

From (4.29)–(4.39) and (4.42), we get

$$(4.43)\qquad \begin{aligned}
&-\pi^2\big(\mathscr{L}^{-1}(\mathcal{O}_{41}+\mathcal{O}_{42}+\mathcal{O}_{43}+\mathcal{O}_{44})\mathscr{P}\big)(0,0)\\
&= -\frac{2}{15}\cdot\frac{-11}{6}(R_{\ell\bar{\ell}q\bar{q};u\bar{u}} + 2R_{u\bar{\ell}q\bar{q};\ell\bar{u}})\\
&\quad + \frac{1}{15}\cdot\frac{11}{3}(R_{\ell\bar{s}q\bar{u}}R_{s\bar{\ell}u\bar{q}} + 2R_{s\bar{s}q\bar{u}}R_{\ell\bar{\ell}u\bar{q}})\\
&\quad + \left(-\frac{2}{5}\cdot\frac{3}{4}+\frac{1}{10}\right)(R_{q\bar{m}m\bar{q};u\bar{u}} + R_{q\bar{m}u\bar{q};m\bar{u}})\\
&\quad - \left(\frac{-12}{15}\cdot\frac{3}{4}-\frac{2}{90}\right)(R_{q\bar{s}s\bar{u}}R_{v\bar{q}u\bar{v}} + 3R_{q\bar{s}v\bar{u}}R_{s\bar{q}u\bar{v}})\\
&\quad + \frac{1}{72}(\mathrm{Ric}_{\ell\bar{\ell}}\,\mathrm{Ric}_{q\bar{q}} + \mathrm{Ric}_{\ell\bar{q}}\,\mathrm{Ric}_{q\bar{\ell}}) - \frac{1}{72}(\mathrm{Ric}_{\ell\bar{\ell}}\,\mathrm{Ric}_{q\bar{q}} + 3\,\mathrm{Ric}_{\ell\bar{q}}\,\mathrm{Ric}_{q\bar{\ell}})\\
&= \frac{2}{45}R_{\ell\bar{\ell}q\bar{q};u\bar{u}} + \frac{13}{45}R_{u\bar{\ell}q\bar{q};\ell\bar{u}} + \frac{19}{9}R_{\ell\bar{s}q\bar{u}}R_{s\bar{\ell}u\bar{q}} + R_{s\bar{s}q\bar{u}}R_{\ell\bar{\ell}u\bar{q}}.
\end{aligned}$$

Moreover, (4.31), (4.38) and (4.40) yield

$$(4.44)\qquad \begin{aligned}
-\pi^2(\mathscr{L}^{-1}\mathcal{O}_{45}\mathscr{P})(0,0) &= -\frac{2}{3}(R_{k\bar{\ell}\ell\bar{u}}R_{q\bar{k}u\bar{q}} + 3R_{\ell\bar{k}q\bar{u}}R_{k\bar{\ell}u\bar{q}})\\
&\quad + \frac{3}{2}R_{\ell\bar{\ell}q\bar{k}}R^E_{k\bar{q}} + \frac{3}{8}(R^E_{\ell\bar{\ell};q\bar{q}} + R^E_{q\bar{\ell};\ell\bar{q}}).
\end{aligned}$$

Further, (3.15), (4.17c), (4.35), (4.38) and (4.40) imply

$$(4.45)\qquad \begin{aligned}
-\pi^2(\mathscr{L}^{-1}\mathcal{O}_{46}\mathscr{P})(0,0) &= \frac{11}{108}(R_{\ell\bar{s}q\bar{u}}R_{s\bar{\ell}u\bar{q}} + 2R_{s\bar{s}q\bar{u}}R_{\ell\bar{\ell}u\bar{q}}) + \frac{1}{9}\,\mathrm{Ric}_{k\bar{q}}\,\mathrm{Ric}_{q\bar{k}}\\
&\quad - \frac{1}{9}R_{k\bar{\ell}q\bar{k}}\,\mathrm{Ric}_{\ell\bar{q}} - \frac{1}{2}R_{\ell\bar{\ell}q\bar{k}}R^E_{k\bar{q}} + \frac{1}{4}R^E_{k\bar{q}}R^E_{q\bar{k}} - \frac{1}{2}R^E_{\ell\bar{\ell};q\bar{q}}\\
&= \frac{11}{108}R_{\ell\bar{s}q\bar{u}}R_{s\bar{\ell}u\bar{q}} + \frac{23}{54}R_{s\bar{s}q\bar{u}}R_{\ell\bar{\ell}u\bar{q}} - \frac{1}{2}R_{\ell\bar{\ell}q\bar{k}}R^E_{k\bar{q}}\\
&\quad + \frac{1}{4}R^E_{k\bar{q}}R^E_{q\bar{k}} - \frac{1}{2}R^E_{\ell\bar{\ell};q\bar{q}}.
\end{aligned}$$

By (3.8), (3.17b), (4.43)–(4.45), we get (4.28). The proof of Lemma 4.4 is completed. □

4.3. Evaluation of $\mathscr{F}_4(0,0)$.

Theorem 4.5. *The following identity holds*:

$$(4.46)\qquad \mathscr{F}_{4,x_0}(0,0) = \boldsymbol{b}_2.$$

Proof. By Lemmas 4.2, 4.4 and (2.22), we have

$$\begin{aligned}\pi^2\mathscr{F}_{41}(0,0) = &-\frac{\Delta \boldsymbol{r}}{96}+\frac{7}{72}R_{m\bar{s}q\bar{u}}R_{s\bar{m}u\bar{q}}-\frac{1}{9}R_{s\bar{s}q\bar{u}}R_{m\bar{m}u\bar{q}}+\frac{1}{8}R_{s\bar{s}u\bar{u}}R_{m\bar{m}q\bar{q}}\\ &+\frac{1}{4}R^E_{u\bar{u}}R_{m\bar{m}q\bar{q}}-\frac{1}{6}R^E_{q\bar{u}}R_{m\bar{m}u\bar{q}}\\ &+\frac{1}{8}(R^E_{u\bar{u}}R^E_{q\bar{q}}-R^E_{q\bar{u}}R^E_{u\bar{q}}-R^E_{m\bar{m};q\bar{q}}+3R^E_{q\bar{m};m\bar{q}}).\end{aligned} \tag{4.47}$$

Remark 2.3, Lemmas 3.1, 4.1, (0.9), (2.20), (4.47) and formula $\mathscr{P}(0,0)=1$ entail

$$\begin{aligned}J_{4,x_0}(0,0) &= \mathscr{F}_{41}(0,0)+\mathscr{F}_{41}(0,0)^*+\frac{1}{4\pi^2}\Big[\sum_{mq}R_{m\bar{m}q\bar{q}}+\sum_q R^E_{q\bar{q}}\Big]^2\\ &\quad-\frac{1}{36\pi^2}R_{m\bar{k}q\bar{\ell}}R_{k\bar{m}\ell\bar{q}}-\frac{1}{4\pi^2}\Big(\frac{4}{3}R_{q\bar{v}v\bar{\ell}}+R^E_{q\bar{\ell}}\Big)\Big(\frac{4}{3}R_{\ell\bar{k}k\bar{q}}+R^E_{\ell\bar{q}}\Big)\\ &= \boldsymbol{b}_2.\end{aligned} \tag{4.48}$$

The proof of Theorem 4.5 is completed. □

5. The first coefficients of the asymptotic expansion

The lay-out of this section is as follows. In Section 5.1, we explain the expansion of the kernel of Berezin–Toeplitz operators and verify its compatibility with the Riemann–Roch–Hirzebruch Theorem. In Section 5.2, we establish Theorem 0.1. The results from Sections 4.1, 4.2 play an important role here. In Section 5.3, we prove Theorems 0.2, 0.3, i.e., the expansion of the composition of two Berezin–Toeplitz operators.

We use the notations and assumptions from the Introduction and Section 3.

5.1. Expansion of the kernel of Berezin–Toeplitz operators. For $U \in TX$, we have (cf. (3.2))

$$\nabla^{T^*X}_U \widetilde{dz_j} = 2\Big\langle \nabla^{TX}_U \widetilde{\frac{\partial}{\partial \bar{z}_j}}, \widetilde{\frac{\partial}{\partial z_m}}\Big\rangle \widetilde{dz_m}, \quad \nabla^{T^*X}_U \widetilde{d\bar{z}_j} = 2\Big\langle \nabla^{TX}_U \widetilde{\frac{\partial}{\partial z_j}}, \widetilde{\frac{\partial}{\partial \bar{z}_m}}\Big\rangle \widetilde{d\bar{z}_m}. \tag{5.1}$$

For $\sigma = \sum_{kq}\sigma_{k\bar{q}}\,\widetilde{dz_k}\wedge\widetilde{d\bar{z}_q} \in \Omega^{1,1}(X,\mathrm{End}(E))$, by [24], Lemma 1.4.4, (0.6) and (5.1), we get

$$\begin{aligned}\nabla^{1,0*}\sigma &= -\Big(2\nabla^E_{\widetilde{\frac{\partial}{\partial \bar{z}_m}}}\sigma_{m\bar{q}}+4\sigma_{k\bar{q}}\Big\langle\nabla^{TX}_{\widetilde{\frac{\partial}{\partial \bar{z}_m}}}\widetilde{\frac{\partial}{\partial \bar{z}_k}},\widetilde{\frac{\partial}{\partial z_m}}\Big\rangle+4\sigma_{m\bar{l}}\Big\langle\nabla^{TX}_{\widetilde{\frac{\partial}{\partial \bar{z}_m}}}\widetilde{\frac{\partial}{\partial z_l}},\widetilde{\frac{\partial}{\partial \bar{z}_q}}\Big\rangle\Big)\widetilde{d\bar{z}_q},\\ \bar{\partial}^{E*}\sigma &= \Big(2\nabla^E_{\widetilde{\frac{\partial}{\partial z_m}}}\sigma_{k\bar{m}}+4\sigma_{k\bar{q}}\Big\langle\nabla^{TX}_{\widetilde{\frac{\partial}{\partial z_m}}}\widetilde{\frac{\partial}{\partial z_q}},\widetilde{\frac{\partial}{\partial \bar{z}_m}}\Big\rangle+4\sigma_{l\bar{m}}\Big\langle\nabla^{TX}_{\widetilde{\frac{\partial}{\partial \bar{z}_m}}}\widetilde{\frac{\partial}{\partial \bar{z}_l}},\widetilde{\frac{\partial}{\partial z_k}}\Big\rangle\Big)\widetilde{dz_k}.\end{aligned} \tag{5.2}$$

We evaluate now (5.2) at the point x_0 (identified to $0 \in \mathbb{R}^{2n}$). By using (3.29) applied for $r = 0, 1$ associated with the vector bundles E, $T^{(1,0)}X$, we get

$$(5.3)\qquad \begin{aligned}(\nabla^{1,0*}\sigma)_{x_0} &= -2\frac{\partial\sigma_{m\bar q}}{\partial\bar z_m}(0)\,d\bar z_q, \quad (\bar\partial^{E*}\sigma)_{x_0} = 2\frac{\partial\sigma_{k\bar m}}{\partial z_m}(0)\,dz_k,\\ (\bar\partial^{E*}\nabla^{1,0*}\sigma)_{x_0} &= 4\frac{\partial\sigma_{k\bar m}}{\partial z_m\partial\bar z_k}(0) + 2[R^E_{m\bar k}, \sigma_{k\bar m}(0)].\end{aligned}$$

Note that by (0.9), (3.6) and (5.3), we have at x_0,

$$(5.4)\qquad \begin{aligned}&\omega = \frac{\sqrt{-1}}{2}dz_q\wedge d\bar z_q, \quad \mathrm{Tr}[R^{T^{(1,0)}X}] = \mathrm{Ric}_{k\bar q}\,dz_k\wedge d\bar z_q = -\sqrt{-1}\,\mathrm{Ric}_\omega,\\ &R^E = R^E_{k\bar q}\,dz_k\wedge d\bar z_q, \quad \nabla^{1,0*}R^E = -2R^E_{m\bar q;\bar m}\,d\bar z_q, \quad \bar\partial^{E*}R^E = 2R^E_{k\bar q;q}\,dz_k.\end{aligned}$$

For $f\in\mathscr{C}^\infty\big(X,\mathrm{End}(E)\big)$, recall that $T_{f,p}(x,x')$ is the smooth kernel of the Berezin–Toeplitz operator $T_{f,p}$ defined according to (0.4). Then by (3.29), at x_0,

$$(5.5)\qquad \frac{\partial^2 f_{x_0}}{\partial z_q\partial\bar z_\ell}(0) = (\nabla^E_{\frac{\partial}{\partial z_q}}\nabla^E_{\frac{\partial}{\partial\bar z_\ell}}f)(x_0) - \frac12[R^E_{q\bar\ell}, f(x_0)], \quad \Delta^E f = -4\frac{\partial^2 f_{x_0}}{\partial z_q\partial\bar z_q}(0).$$

In view of Lemma 3.1, (0.7) and (5.5), we introduce the following coefficients:

$$(5.6)\qquad \begin{aligned}\boldsymbol{b}_{\mathbb{C}f} &:= R_{m\bar m q\bar q}\frac{\partial^2 f_{x_0}}{\partial z_k\partial\bar z_k}(0) - R_{\ell\bar k k\bar q}\frac{\partial^2 f_{x_0}}{\partial z_q\partial\bar z_\ell}(0)\\ &= -\frac{\boldsymbol{r}}{32}\Delta^E f - \frac{\sqrt{-1}}{8}\left\langle \mathrm{Ric}_\omega, \nabla^{1,0}\bar\partial^E f - \frac12[R^E,f]\right\rangle_\omega\end{aligned}$$

and

$$(5.7)\qquad \begin{aligned}\boldsymbol{b}_{Ef1} &:= \frac{\partial f_{x_0}}{\partial z_u}(0)\left(\frac16 R^E_{k\bar k;\bar u} - \frac{5}{12}R^E_{q\bar u;\bar q}\right) + \frac14 R^E_{m\bar u;\bar m}\frac{\partial f_{x_0}}{\partial z_u}(0)\\ &\quad + \left(\frac16 R^E_{k\bar k;u} - \frac{5}{12}R^E_{u\bar q;q}\right)\frac{\partial f_{x_0}}{\partial\bar z_u}(0) + \frac14\frac{\partial f_{x_0}}{\partial\bar z_u}(0)R^E_{u\bar m;m}\\ &= \frac{1}{48}\langle\nabla^{1,0}f, 2\sqrt{-1}\bar\partial^E R^E_\Lambda + 5\nabla^{1,0*}R^E\rangle_\omega - \frac{1}{16}\langle\nabla^{1,0*}R^E, \nabla^{1,0}f\rangle_\omega\\ &\quad + \frac{1}{16}\langle\bar\partial^E f, \bar\partial^{E*}R^E\rangle_\omega + \frac{1}{48}\langle 2\sqrt{-1}\nabla^{1,0}R^E_\Lambda - 5\bar\partial^{E*}R^E, \bar\partial^E f\rangle_\omega,\\ \boldsymbol{b}_{Ef2} &:= \frac12\frac{\partial^2 f_{x_0}}{\partial z_k\partial\bar z_k}(0)R^E_{q\bar q} - \frac12\frac{\partial^2 f_{x_0}}{\partial z_q\partial\bar z_\ell}(0)R^E_{\ell\bar q}\\ &\quad + \frac12 R^E_{q\bar q}\frac{\partial^2 f_{x_0}}{\partial z_k\partial\bar z_k}(0) - \frac12 R^E_{\ell\bar q}\frac{\partial^2 f_{x_0}}{\partial z_q\partial\bar z_\ell}(0)\\ &= -\frac{\sqrt{-1}}{16}[R^E_\Lambda\Delta^E f + (\Delta^E f)R^E_\Lambda] + \frac18\left\langle\nabla^{1,0}\bar\partial^E f - \frac12[R^E,f], R^E\right\rangle_\omega\\ &\quad + \frac18\left\langle R^E, \nabla^{1,0}\bar\partial^E f - \frac12[R^E,f]\right\rangle_\omega.\end{aligned}$$

The following result implies Theorem 0.1.

Theorem 5.1. *Let $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. There exist smooth sections $\boldsymbol{b}_{r,f}(x) \in \mathrm{End}(E)_x$ such that* (0.12) *and* (0.13) *hold and*

$$\pi^2 \boldsymbol{b}_{2,f} = \boldsymbol{b}_{2\mathbb{C}} f(x_0) + \frac{1}{2}\left(\boldsymbol{b}_{2E} + \frac{1}{16}(R^E_\Lambda)^2\right) f(x_0) + \frac{1}{2} f(x_0)\left(\boldsymbol{b}_{2E} + \frac{1}{16}(R^E_\Lambda)^2\right) - \frac{1}{16} R^E_\Lambda f(x_0) R^E_\Lambda + \frac{1}{32}(\Delta^E)^2 f + \boldsymbol{b}_{\mathbb{C}f} + \boldsymbol{b}_{Ef1} + \boldsymbol{b}_{Ef2}. \tag{5.8}$$

Before giving the proof, we verify that Theorem 0.1 is compatible with the Riemann–Roch–Hirzebruch Theorem. Note that by (0.1), the first Chern class $c_1(L)$ of L is represented by ω. By the Kodaira Vanishing Theorem and the Riemann–Roch–Hirzebruch Theorem, we have for p large enough:

$$\begin{aligned}\dim H^0(X, L^p \otimes E) &= \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E) e^{p\omega} \\ &= \mathrm{rk}(E) \int_X \frac{\omega^n}{n!} p^n + \int_X \left(c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(X)\right) \frac{\omega^{n-1} p^{n-1}}{(n-1)!} \\ &\quad + \int_X \left(\mathrm{rk}(E)\{\mathrm{Td}(T^{(1,0)}X)\}^{(4)} + \frac{1}{2} c_1(X) c_1(E) + \{\mathrm{ch}(E)\}^{(4)}\right) \frac{\omega^{n-2} p^{n-2}}{(n-2)!} \\ &\quad + \mathscr{O}(p^{n-3}).\end{aligned} \tag{5.9}$$

As usual, $\mathrm{ch}(\cdot)$, $c_1(\cdot)$, $\mathrm{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles, $\{\cdot\}^{(4)}$ is the degree 4-part of the corresponding differential forms. Note that

$$\frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} + \cdots,$$

thus $\{\mathrm{Td}(T^{(1,0)}X)\}^{(4)} = \frac{1}{12}\big(c_1(X)^2 + c_2(X)\big)$. Let $R^{T^{(1,0)}X}$ be the curvature of the Chern connection on $T^{(1,0)}X$ which is the restriction of the Levi–Civita connection in our case. Then by (5.4), we have the following identities at the cohomology level:

$$\begin{aligned}&\{\mathrm{ch}(E)\}^{(4)} = -\frac{1}{8\pi^2} \mathrm{Tr}[(R^E)^2], \quad c_1(X) = \frac{1}{2\pi} \mathrm{Ric}_\omega, \\ &\{\mathrm{Td}(T^{(1,0)}X)\}^{(4)} = \frac{1}{32\pi^2} (\mathrm{Ric}_\omega)^2 + \frac{1}{96\pi^2} \mathrm{Tr}[(R^{T^{(1,0)}X})^2].\end{aligned} \tag{5.10}$$

By Lemma 3.1 and (5.4), we have

$$\begin{aligned}\frac{1}{32}\langle (\mathrm{Ric}_\omega)^2, \omega^2/2\rangle &= \frac{1}{2}(R_{u\bar{u}v\bar{v}} R_{k\bar{k}q\bar{q}} - R_{u\bar{u}q\bar{v}} R_{k\bar{k}v\bar{q}}), \\ \frac{1}{96}\langle \mathrm{Tr}[(R^{T^{(1,0)}X})^2], \omega^2/2\rangle &= \frac{1}{6}(R_{k\bar{u}q\bar{v}} R_{u\bar{k}v\bar{q}} - R_{u\bar{u}q\bar{v}} R_{k\bar{k}v\bar{q}}),\end{aligned} \tag{5.11}$$

(5.11)
$$\langle\{\mathrm{ch}(E)\}^{(4)}, \omega^2/2\rangle = \frac{1}{2\pi^2}\,\mathrm{Tr}[R^E_{k\bar k}R^E_{q\bar q} - R^E_{k\bar q}R^E_{q\bar k}],$$
$$\left\langle \frac{1}{2}c_1(X)c_1(E), \omega^2/2\right\rangle = \frac{1}{2\pi^2}\,\mathrm{Tr}(R^E_{k\bar k}\,\mathrm{Ric}_{q\bar q} - \mathrm{Ric}_{q\bar k}\,R^E_{k\bar q}).$$

We set now $f = 1$ in Theorem 0.1, take the pointwise trace of the expansion (0.12) relative to E and then integrate the result over X with respect to the volume form $\omega^n/n!$. Taking into account (0.9) and (5.10), (5.11), we recover the expansion up to $\mathscr{O}(p^{n-3})$ given in (5.9) for the Hilbert polynomial. Thus the value of $\boldsymbol{b}_2$ obtained in Theorem 0.1 is compatible with the Riemann–Roch–Hirzebruch Theorem.

5.2. Proof of Theorem 5.1. The first part of Theorem 5.1 follows from Lemma 2.2. Moreover, by (2.10), we have for any $r \in \mathbb{N}$,

(5.12)
$$\boldsymbol{b}_{r,f}(x_0) = Q_{2r,x_0}(f)(0,0).$$

Thus by (2.12), the formula $\boldsymbol{b}_{0,f} = f$, and by (2.11) and (2.21), we get

(5.13)
$$Q_{2,x_0}(f) = \mathscr{K}[1, f(x_0)J_{2,x_0}] + \mathscr{K}[J_{2,x_0}, f(x_0)] + \sum_{|\alpha|=2}\mathscr{K}\left[1, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right].$$

Further, (1.8), (2.13) and (2.19), entail

(5.14)
$$\mathscr{K}[1, f(x_0)J_{2,x_0}]\mathscr{P} = -f(x_0)\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp,$$
$$\mathscr{K}[J_{2,x_0}, f(x_0)]\mathscr{P} = -(\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{P})f(x_0).$$

From (1.8) and (4.13), we deduce

(5.15)
$$\sum_{|\alpha|=2}\mathscr{K}\left[1, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right]\mathscr{P}(Z,0) = \left(\sum_{|\alpha|=2}\frac{\partial^\alpha f_{x_0}}{\partial z^\alpha}(0)\frac{z^\alpha}{\alpha!} + \frac{1}{\pi}\frac{\partial^2 f_{x_0}}{\partial z_i\partial\bar z_i}(0)\right)\mathscr{P}(Z,0).$$

Lemma 3.1, (4.8), (4.9), (5.5) and (5.12)–(5.15) yield the formula for $\boldsymbol{b}_{1,f}$ from (0.12).

It remains to compute $\boldsymbol{b}_{2,f}$ for a self-adjoint section $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ in order to complete the proof of Theorem 5.1. Set

(5.16)
$$\mathscr{K}_{2f} = \sum_{|\alpha|=2}\mathscr{K}\left[1, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{2,x_0}\right].$$

By (2.11) and (2.21), we get

(5.17)
$$\begin{aligned} Q_{4,x_0}(f) = {}& \mathscr{K}[1, f(x_0)J_{4,x_0}] + \mathscr{K}[J_{2,x_0}, f(x_0)J_{2,x_0}] + \mathscr{K}[J_{4,x_0}, f(x_0)] \\ & + \sum_{|\alpha|=2}\mathscr{K}\left[J_{2,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right] + \mathscr{K}_{2f} + \mathscr{K}\left[1, \frac{\partial f_{x_0}}{\partial Z_i}(0)Z_iJ_{3,x_0}\right] \\ & + \mathscr{K}\left[J_{3,x_0}, \frac{\partial f_{x_0}}{\partial Z_i}(0)Z_i\right] + \sum_{|\alpha|=4}\mathscr{K}\left[1, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right]. \end{aligned}$$

Since $\mathscr{L}_0$ and $\mathcal{O}_r$ are formally self-adjoint, (2.17) and (2.18) show that $(\mathscr{F}_{r,x_0})^* = \mathscr{F}_{r,x_0}$. Hence, in the right-hand side of (5.17), the first, fourth and sixth terms are adjoints of the third, fifth and seventh terms, respectively. When we take $f = 1$ in (5.17), we get

$$J_{4,x_0} = \mathscr{K}[1, J_{4,x_0}] + \mathscr{K}[J_{2,x_0}, J_{2,x_0}] + \mathscr{K}[J_{4,x_0}, 1], \tag{5.18}$$

which is also a direct consequence of (2.19), (2.20) and (4.1a), as by (1.8),

$$\begin{aligned} &\mathscr{K}[1, J_{4,x_0}]\mathscr{P} = \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp - \mathscr{P}\mathcal{O}_4\mathscr{L}^{-1}\mathscr{P}^\perp - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-2}\mathcal{O}_2\mathscr{P},\\ &\mathscr{K}[J_{4,x_0}, 1]\mathscr{P} = \mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{P} - \mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_4\mathscr{P} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-2}\mathcal{O}_2\mathscr{P},\\ &\mathscr{K}[J_{2,x_0}, J_{2,x_0}]\mathscr{P} = \mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp + \mathscr{P}\mathcal{O}_2\mathscr{L}^{-2}\mathcal{O}_2\mathscr{P}. \end{aligned} \tag{5.19}$$

Set

$$\begin{aligned} K_{41} :=& -\frac{\Delta \boldsymbol{r}}{96} + \frac{5}{72}R_{m\bar{u}q\bar{v}}R_{u\bar{m}v\bar{q}} - \frac{5}{9}R_{u\bar{u}q\bar{v}}R_{m\bar{m}v\bar{q}} + \frac{1}{8}R_{u\bar{u}v\bar{v}}R_{m\bar{m}q\bar{q}},\\ K_{42} :=& \frac{1}{4}R^E_{v\bar{v}}R_{m\bar{m}q\bar{q}} - \frac{5}{6}R^E_{q\bar{v}}R_{m\bar{m}v\bar{q}}\\ &+\frac{1}{8}(R^E_{v\bar{v}}R^E_{q\bar{q}} - 3R^E_{q\bar{v}}R^E_{v\bar{q}} - R^E_{m\bar{m};q\bar{q}} + 3R^E_{q\bar{m};m\bar{q}}),\\ K_{2f} :=& \frac{1}{4}(R_{m\bar{m}q\bar{q}} + R^E_{q\bar{q}})f(x_0)(R_{u\bar{u}v\bar{v}} + R^E_{v\bar{v}}) + \frac{1}{36}R_{m\bar{k}q\bar{\ell}}R_{k\bar{m}\ell\bar{q}}f(x_0)\\ &+\frac{1}{4}\left(\frac{4}{3}R_{q\bar{s}s\bar{\ell}} + R^E_{q\bar{\ell}}\right)f(x_0)\left(\frac{4}{3}R_{\ell\bar{k}k\bar{q}} + R^E_{\ell\bar{q}}\right). \end{aligned} \tag{5.20}$$

By (2.22), (4.1c), (4.47) and (5.19), we have

$$\mathscr{K}[J_{4,x_0}, 1](0,0) = \frac{1}{\pi^2}(K_{41} + K_{42}). \tag{5.21}$$

By (1.8) and (2.19), we see as in (5.19) that

$$\begin{aligned} \mathscr{K}[J_{2,x_0}, f(x_0)J_{2,x_0}]\mathscr{P} =& \mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2 f(x_0)\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp\\ &+ \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}f(x_0)\mathscr{L}^{-1}\mathcal{O}_2\mathscr{P}. \end{aligned} \tag{5.22}$$

Thus by (4.8), (4.9) and (5.22), as in Lemma 4.1, we get

$$\mathscr{K}[J_{2,x_0}, f(x_0)J_{2,x_0}](0,0) = \frac{1}{\pi^2}K_{2f}. \tag{5.23}$$

We next compute the fifth term in (5.17). From (2.19) and (5.16), we get

$$\mathscr{K}_{2f} = \mathscr{P}\sum_{|\alpha|=2}\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}(-\mathscr{L}^{-1}\mathcal{O}_2\mathscr{P} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp). \tag{5.24}$$

For a degree 2 polynomial $F(Z)$ we have by Remark 4.3, (1.6), (1.7), (4.2) and (4.8)

$$(5.25)\qquad -(\mathscr{P}F\mathscr{L}^{-1}\mathcal{O}_2\mathscr{P})(0,0)$$
$$= -\left(\mathscr{P}\frac{\partial^2 F}{\partial z_u \partial \bar{z}_v} z_u \bar{z}_v \left\{\frac{b_m b_q}{48\pi} R_{k\bar{m}l\bar{q}} z_k z_l + \frac{b_q}{4\pi}\left(\frac{4}{3}R_{l\bar{k}k\bar{q}} + R^E_{l\bar{q}}\right) z_l\right\}\mathscr{P}\right)(0,0)$$
$$= -\left(\mathscr{P}\frac{\partial^2 F}{\partial z_q \partial \bar{z}_v}\frac{\bar{z}_v}{2\pi}\left(\frac{4}{3}R_{l\bar{k}k\bar{q}} + R^E_{l\bar{q}}\right) z_l \mathscr{P}\right)(0,0)$$
$$= -\frac{1}{2\pi^2}\frac{\partial^2 F}{\partial z_q \partial \bar{z}_l}\left(\frac{4}{3}R_{l\bar{k}k\bar{q}} + R^E_{l\bar{q}}\right),$$

where we have used (1.6), (1.7) and (4.13) in the last two equalities.

By (4.9), (4.13), (5.24) and (5.25), we get

$$(5.26)\qquad \mathscr{K}_{2f}(0,0) = \frac{1}{2\pi^2}\frac{\partial^2 f_{x_0}}{\partial z_k \partial \bar{z}_k}(0)(R_{m\bar{m}q\bar{q}} + R^E_{q\bar{q}}) - \frac{1}{2\pi^2}\frac{\partial^2 f_{x_0}}{\partial z_q \partial \bar{z}_l}(0)\left(\frac{4}{3}R_{l\bar{k}k\bar{q}} + R^E_{l\bar{q}}\right).$$

By Lemma 3.1, (5.6), (5.7) and (5.26), we get

$$(5.27)\qquad \mathscr{K}_{2f}(0,0) + \mathscr{K}_{2f}(0,0)^* = \frac{1}{\pi^2}(\boldsymbol{b}_{\mathbb{C}f} + \boldsymbol{b}_{Ef2}) - \frac{1}{3\pi^2}R_{l\bar{k}k\bar{q}}\frac{\partial^2 f_{x_0}}{\partial z_q \partial \bar{z}_l}(0).$$

We compute now the last term in (5.17). By Remark 4.3, (1.8) and (4.13), we have

$$(5.28)\qquad \sum_{|\alpha|=4}\mathscr{K}\left[1, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right](0,0) = \frac{1}{2\pi^2}\frac{\partial^4 f_{x_0}}{\partial z_i \partial z_q \partial \bar{z}_i \partial \bar{z}_q}(0).$$

We next turn to the computation of the sixth term in (5.17). Set

$$(5.29)\qquad K_{3f} := \left(\frac{1}{6}R_{k\bar{k}m\bar{m};\bar{u}} - \frac{1}{3}R_{k\bar{k}m\bar{u};\bar{m}}\right)\frac{\partial f_{x_0}}{\partial z_u}(0) + \left(\frac{1}{6}R_{k\bar{k}m\bar{m};u} - \frac{1}{3}R_{k\bar{k}u\bar{m};m}\right)\frac{\partial f_{x_0}}{\partial \bar{z}_u}(0)$$
$$+ \frac{\partial f_{x_0}}{\partial z_u}(0)\left(\frac{1}{6}R^E_{k\bar{k};\bar{u}} - \frac{1}{2}R^E_{q\bar{u};\bar{q}}\right) + \frac{1}{3}R^E_{m\bar{u};\bar{m}}\frac{\partial f_{x_0}}{\partial z_u}(0)$$
$$+ \left(\frac{1}{6}R^E_{k\bar{k};u} - \frac{1}{2}R^E_{u\bar{q};q}\right)\frac{\partial f_{x_0}}{\partial \bar{z}_u}(0) + \frac{1}{3}\frac{\partial f_{x_0}}{\partial \bar{z}_u}(0)R^E_{u\bar{m};m}.$$

Lemma 5.2. *The following identity holds with K_{3f} defined in* (5.29):

$$(5.30)\qquad \mathscr{K}\left[J_{3,x_0}, \frac{\partial f_{x_0}}{\partial Z_u}(0)Z_u\right](0,0) + \mathscr{K}\left[1, \frac{\partial f_{x_0}}{\partial Z_u}(0)Z_u J_{3,x_0}\right](0,0) = \frac{1}{\pi^2}K_{3f}.$$

Proof of Lemma 5.2. Set

$$(5.31a)\qquad B_1(b,Z) := \Big\{\frac{1}{6}R_{k\bar{s}\ell\bar{q};\bar{m}}z_kz_\ell b_sb_q + \frac{2\pi}{3}R_{q\bar{s}k\bar{q};\bar{m}}\bar{z}_sz_k - \Big[\frac{2\pi}{15}R_{k\bar{q}\ell\bar{s};\bar{m}}z_\ell\bar{z}_q + \frac{1}{3}R_{\ell\bar{\ell}k\bar{s};\bar{m}} - \frac{1}{3}R_{k\bar{s}q\bar{m};\bar{q}} - \frac{2}{3}R^E_{k\bar{s};\bar{m}}\Big]z_kb_s - \frac{2\pi}{15}R_{k\bar{s}q\bar{m};\bar{q}}z_k\bar{z}_s - \frac{2}{3}R_{\ell\bar{\ell}q\bar{m};\bar{q}} - \frac{2}{3}R_{\ell\bar{\ell}q\bar{q};\bar{m}} + \frac{2}{3}R^E_{q\bar{m};\bar{q}} - 2R^E_{q\bar{q};\bar{m}}\Big\}\bar{z}_m,$$

$$(5.31b)\qquad \mathcal{B}_1(Z) := \frac{2\pi^2}{5}R_{k\bar{s}\ell\bar{q};\bar{m}}z_kz_\ell\bar{z}_s\bar{z}_q\bar{z}_m + \frac{8\pi}{15}R_{k\bar{s}q\bar{m};\bar{q}}z_k\bar{z}_s\bar{z}_m + \frac{4\pi}{3}R^E_{k\bar{s};\bar{m}}z_k\bar{z}_s\bar{z}_m - \frac{2}{3}[R_{\ell\bar{\ell}q\bar{m};\bar{q}} + R_{\ell\bar{\ell}q\bar{q};\bar{m}} - R^E_{q\bar{m};\bar{q}} + 3R^E_{q\bar{q};\bar{m}}]\bar{z}_m.$$

Then by Lemma 3.1, (4.2), we have

$$(5.32)\qquad \begin{aligned} B_1(b,Z)\mathscr{P}(Z,0) &= \Big\{\frac{4\pi^2}{6}R_{k\bar{s}\ell\bar{q};\bar{m}}z_kz_\ell\bar{z}_s\bar{z}_q + \frac{2\pi}{3}R_{q\bar{s}k\bar{q};\bar{m}}\bar{z}_sz_k \\ &\quad - \Big[\frac{2\pi}{15}R_{k\bar{q}\ell\bar{s};\bar{m}}z_\ell\bar{z}_q + \frac{1}{3}R_{\ell\bar{\ell}k\bar{s};\bar{m}} - \frac{1}{3}R_{k\bar{s}q\bar{m};\bar{q}} - \frac{2}{3}R^E_{k\bar{s};\bar{m}}\Big]z_k\cdot 2\pi\bar{z}_s \\ &\quad - \frac{2\pi}{15}R_{k\bar{s}q\bar{m};\bar{q}}z_k\bar{z}_s - \frac{2}{3}R_{\ell\bar{\ell}q\bar{m};\bar{q}} - \frac{2}{3}R_{\ell\bar{\ell}q\bar{q};\bar{m}} \\ &\quad + \frac{2}{3}R^E_{q\bar{m};\bar{q}} - 2R^E_{q\bar{q};\bar{m}}\Big\}\bar{z}_m\mathscr{P}(Z,0) \\ &= \mathcal{B}_1(Z)\mathscr{P}(Z,0). \end{aligned}$$

Observe that the commutation relations (1.7) imply that

$$R_{k\bar{s}\ell\bar{q};\bar{m}}z_kz_\ell b_sb_qb_m = b_sb_qb_mR_{k\bar{s}\ell\bar{q};\bar{m}}z_kz_\ell + b_sb_m(8R_{q\bar{s}k\bar{q};\bar{m}} + 4R_{k\bar{s}q\bar{m};\bar{q}})z_k + b_m(8R_{q\bar{s}s\bar{q};\bar{m}} + 16R_{s\bar{s}q\bar{m};\bar{q}}).$$

By (1.7), (4.2) and (5.31b), we have

$$(5.33)\qquad \begin{aligned} \mathcal{B}_1(Z)\mathscr{P}(Z,0) &= \frac{1}{\pi}\Big\{\Big(\frac{1}{20}R_{k\bar{s}l\bar{q};\bar{m}}z_kz_\ell b_q + \frac{2}{15}R_{k\bar{s}q\bar{m};\bar{q}}z_k + \frac{1}{3}R^E_{k\bar{s};\bar{m}}z_k\Big)b_sb_m \\ &\quad - \frac{1}{3}[R_{\ell\bar{\ell}q\bar{m};\bar{q}} + R_{\ell\bar{\ell}q\bar{q};\bar{m}} - R^E_{q\bar{m};\bar{q}} + 3R^E_{q\bar{q};\bar{m}}]b_m\Big\}\mathscr{P}(Z,0) \end{aligned}$$

$$= \frac{1}{\pi}\bigg\{\frac{b_s b_q b_m}{20} R_{k\bar{s}\ell\bar{q};\bar{m}} z_k z_\ell + b_s b_m \Big(\frac{2}{5} R_{q\bar{s}k\bar{q};\bar{m}} + \frac{1}{3} R_{k\bar{s}q\bar{m};\bar{q}} + \frac{1}{3} R^E_{k\bar{s};\bar{m}}\Big) z_k$$
$$+ b_m \Big(\frac{1}{15} R_{s\bar{s}q\bar{q};\bar{m}} + R_{s\bar{s}q\bar{m};\bar{q}} - \frac{1}{3} R^E_{s\bar{s};\bar{m}} + R^E_{q\bar{m};\bar{q}}\Big)\bigg\}\mathscr{P}(Z,0).$$

Note that by Theorem 1.1 and (4.2), we have $(\mathscr{P}^\perp z_u \mathscr{P})(Z,0) = (\mathscr{P}\bar{z}_u\mathscr{P})(Z,0) = 0$. Taking into account that $\mathscr{P}(0,0) = 1$ and relations (1.8), (2.13) and (2.19), we get

$$\mathscr{K}\Big[J_{3,x_0}, \frac{\partial f_{x_0}}{\partial Z_u}(0) Z_u\Big](0,0) = -\Big(\mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_3 \mathscr{P}\frac{\partial f_{x_0}}{\partial z_u}(0) z_u \mathscr{P}\Big)(0,0)$$
$$- \Big(\mathscr{P}\mathcal{O}_3 \mathscr{L}^{-1}\mathscr{P}^\perp \frac{\partial f_{x_0}}{\partial \bar{z}_u}(0)\bar{z}_u\mathscr{P}\Big)(0,0). \tag{5.34}$$

By Remark 2.3, $(\mathscr{P}\mathcal{O}_3\mathscr{L}^{-1}\bar{z}_u\mathscr{P})(0,0)$ is the adjoint of $(\mathscr{P}z_u\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_3\mathscr{P})(0,0)$, thus we will compute only the latter. By Lemma 3.1, (1.7), (3.17a) and (4.3), we get as in (4.4),

$$\mathcal{O}_3 = \frac{1}{6} R_{k\bar{s}\ell\bar{q};Z} z_k z_\ell b_s b_q + \frac{2\pi}{3} R_{q\bar{s}k\bar{q};Z}\bar{z}_s z_k + c_s(b,b^+,Z) b_s^+$$
$$- \Big(\frac{2\pi}{15} R_{\ell\bar{q}k\bar{s};Z} z_\ell \bar{z}_q + \frac{1}{3} R_{\ell\bar{\ell}k\bar{s};Z} + \frac{1}{3}(R_{\ell\bar{q}k\bar{s};q} z_\ell - R_{q\bar{m}k\bar{s};\bar{q}}\bar{z}_m) - \frac{2}{3} R^E_{k\bar{s};Z}\Big) z_k b_s$$
$$+ \frac{2\pi}{15}(R_{k\bar{s}\ell\bar{q};q} z_\ell - R_{k\bar{s}q\bar{m};\bar{q}}\bar{z}_m) z_k \bar{z}_s - \frac{2}{3}(R_{\ell\bar{\ell}k\bar{q};q} z_k + R_{\ell\bar{\ell}q\bar{m};\bar{q}}\bar{z}_m)$$
$$- \frac{2}{3} R_{\ell\bar{\ell}q\bar{q};Z} - \frac{2}{3}(R^E_{k\bar{q};q} z_k - R^E_{q\bar{m};\bar{q}}\bar{z}_m) - 2R^E_{q\bar{q};Z}, \tag{5.35}$$

and $c_s(b,b^+,Z)$ are polynomials in b, b^+, Z, whose precise formula will not be used, and $R_{k\bar{q}\ell\bar{s};Z}$, $R^E_{k\bar{s};Z}$ are defined by replacing $\dfrac{\partial}{\partial z_s}$ by Z in (3.6).

From Lemma 3.1, (1.7), (5.31a) and (5.35) we deduce that the only term in $\mathcal{O}_3$ not containing b^+ and having total degree in b, $\bar{z}$ bigger than its degree in z, is $B_1(b,Z)$. Now, Theorem 1.1, Remark 4.3, (1.6), (1.7), (5.32), (5.33) and (5.35) imply

$$-(\mathscr{P}z_u\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_3\mathscr{P})(0,0) = -\big(\mathscr{P}z_u\mathscr{L}^{-1}\mathscr{P}^\perp B_1(b,Z)\mathscr{P}\big)(0,0)$$
$$= -\bigg(\mathscr{P}z_u\frac{b_m}{4\pi^2}\Big(\frac{1}{15}R_{s\bar{s}q\bar{q};\bar{m}} + R_{s\bar{s}q\bar{m};\bar{q}} - \frac{1}{3}R^E_{s\bar{s};\bar{m}} + R^E_{q\bar{m};\bar{q}}\Big)\mathscr{P}\bigg)(0,0)$$
$$= -\frac{1}{2\pi^2}\Big(\frac{1}{15}R_{s\bar{s}q\bar{q};\bar{u}} + R_{s\bar{s}q\bar{u};\bar{q}} - \frac{1}{3}R^E_{s\bar{s};\bar{u}} + R^E_{q\bar{u};\bar{q}}\Big). \tag{5.36}$$

By Remark 4.3, (1.6), (1.7) and (5.35), we also have

$$-(\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_3\mathscr{P}z_u\mathscr{P})(0,0) = -\big(\mathscr{L}^{-1}\mathscr{P}^\perp B_1(b,Z) z_u\mathscr{P}\big)(0,0)$$
$$= -\Big\{\mathscr{L}^{-1}\mathscr{P}^\perp\Big[z_u B_1(b,Z) - 2\frac{\partial}{\partial b_u}B_1(b,Z)\Big]\mathscr{P}\Big\}(0,0). \tag{5.37}$$

Lemma 3.1, (4.17c), (4.17f), (4.33), (5.31b) and (5.32) yield

$$\begin{aligned}(5.38)\qquad & -\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_u B_1(b,Z)\mathscr{P}\big)(0,0)\\ &= -\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_u \mathcal{B}_1(Z)\mathscr{P}\big)(0,0)\\ &= \frac{11}{24\pi^2}\cdot\frac{2}{5}(4R_{s\bar{s}m\bar{u};\bar{m}}+2R_{s\bar{s}q\bar{q};\bar{u}})+\frac{3}{8\pi^2}\cdot\left(\frac{16}{15}R_{s\bar{s}q\bar{u};\bar{q}}+\frac{4}{3}R^E_{s\bar{s};\bar{u}}+\frac{4}{3}R^E_{m\bar{u};\bar{m}}\right)\\ &\quad -\frac{1}{6\pi^2}(R_{\ell\bar{\ell}q\bar{u};\bar{q}}+R_{\ell\bar{\ell}q\bar{q};\bar{u}}-R^E_{q\bar{u};\bar{q}}+3R^E_{q\bar{q};\bar{u}})\\ &= \frac{1}{\pi^2}\left(\frac{29}{30}R_{s\bar{s}m\bar{u};\bar{m}}+\frac{1}{5}R_{s\bar{s}q\bar{q};\bar{u}}+\frac{2}{3}R^E_{m\bar{u};\bar{m}}\right).\end{aligned}$$

Moreover, by (5.31a),

$$\begin{aligned}(5.39)\qquad \frac{\partial}{\partial b_u}B_1(b,Z) &= \frac{1}{3}R_{k\bar{s}\ell\bar{u};\bar{m}}z_k z_\ell b_s\bar{z}_m\\ &\quad -\left(\frac{2\pi}{15}R_{k\bar{q}\ell\bar{u};\bar{m}}z_\ell\bar{z}_q+\frac{1}{3}R_{\ell\bar{\ell}k\bar{u};\bar{m}}-\frac{1}{3}R_{k\bar{u}q\bar{m};\bar{q}}-\frac{2}{3}R^E_{k\bar{u};\bar{m}}\right)z_k\bar{z}_m.\end{aligned}$$

Lemma 3.1, (4.2), (4.17c), (4.17f) and (5.39) yield

$$\begin{aligned}(5.40)\qquad & \left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\left(\frac{\partial}{\partial b_u}B_1(b,Z)\right)\mathscr{P}\right)(0,0)\\ &= -\frac{3}{8\pi^2}\cdot\frac{8}{15}\cdot 2R_{s\bar{s}m\bar{u};\bar{m}}+\frac{1}{4\pi^2}\left(\frac{1}{3}R_{m\bar{q}q\bar{u};\bar{m}}-\frac{1}{3}R_{m\bar{u}q\bar{m};\bar{q}}-\frac{2}{3}R^E_{m\bar{u};\bar{m}}\right)\\ &= -\frac{2}{5\pi^2}R_{s\bar{s}m\bar{u};\bar{m}}-\frac{1}{6\pi^2}R^E_{m\bar{u};\bar{m}}.\end{aligned}$$

Formulas (5.37)–(5.40) entail altogether

$$(5.41)\qquad -\pi^2(\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_3\mathscr{P}z_u\mathscr{P})(0,0)=\frac{1}{6}R_{s\bar{s}m\bar{u};\bar{m}}+\frac{1}{5}R_{s\bar{s}q\bar{q};\bar{u}}+\frac{1}{3}R^E_{m\bar{u};\bar{m}}.$$

Combining (5.34), (5.36) with (5.41), we get

$$\begin{aligned}(5.42)\qquad & \pi^2\mathscr{K}\left[J_{3,x_0},\frac{\partial f_{x_0}}{\partial Z_u}(0)Z_u\right](0,0)\\ &= \left(\frac{1}{6}R_{s\bar{s}m\bar{u};\bar{m}}+\frac{1}{5}R_{s\bar{s}q\bar{q};\bar{u}}+\frac{1}{3}R^E_{m\bar{u};\bar{m}}\right)\frac{\partial f_{x_0}}{\partial z_u}(0)\\ &\quad +\left(-\frac{1}{30}R_{s\bar{s}q\bar{q};u}-\frac{1}{2}R_{s\bar{s}u\bar{q};q}+\frac{1}{6}R^E_{s\bar{s};u}-\frac{1}{2}R^E_{u\bar{q};q}\right)\frac{\partial f_{x_0}}{\partial\bar{z}_u}(0).\end{aligned}$$

Since $\mathscr{K}\left[1,\frac{\partial f_{x_0}}{\partial Z_u}(0)Z_uJ_{3,x_0}\right]$ is the adjoint of $\mathscr{K}\left[J_{3,x_0},\frac{\partial f_{x_0}}{\partial Z_u}(0)Z_u\right]$, Lemma 3.1 yields

$$\begin{aligned}(5.43)\qquad \pi^2\mathscr{K}\left[1,\frac{\partial f_{x_0}}{\partial Z_u}(0)Z_uJ_{3,x_0}\right](0,0)\\ =\frac{\partial f_{x_0}}{\partial \bar z_u}(0)\left(\frac16 R_{s\bar s u\bar m;m}+\frac15 R_{s\bar s q\bar q;u}+\frac13 R^E_{u\bar m;m}\right)\\ +\frac{\partial f_{x_0}}{\partial z_u}(0)\left(-\frac{1}{30}R_{s\bar s q\bar q;\bar u}-\frac12 R_{s\bar s q\bar u;\bar q}+\frac16 R^E_{s\bar s;\bar u}-\frac12 R^E_{q\bar u;\bar q}\right).\end{aligned}$$

Finally, (5.42) and (5.43) deliver (5.30). The proof of Lemma 5.2 is completed. □

We continue with the proof of Theorem 5.1. We'll write now the formulas in terms of connections. For $f\in\mathscr{C}^\infty(X,\mathrm{End}(E))$, we obtain by (2.1) (as in (3.32)) the following formula in normal coordinates:

$$(5.44)\qquad (\Delta^E f)(Z)=-g^{ij}(\nabla^E_{e_i}\nabla^E_{e_j}-\Gamma^l_{ij}\nabla^E_{e_l})f,\quad \nabla^E f=df+[\Gamma^E(\cdot),f].$$

By (3.31),

$$\begin{aligned}(5.45)\qquad \nabla^E_{e_i}\nabla^E_{e_i}=\frac{\partial^2}{\partial Z_i^2}+R^E(\mathcal{R},e_i)\frac{\partial}{\partial Z_i}+\frac14 R^E(\mathcal{R},e_i)R^E(\mathcal{R},e_i)\\ +\frac23 R^E_{;Z}(\mathcal{R},e_i)\frac{\partial}{\partial Z_i}+\frac13 R^E_{;e_i}(\mathcal{R},e_i)\\ +\frac18\left(2R^E_{;(Z,e_i)}(\mathcal{R},e_i)+\frac13\langle R^{TX}(\mathcal{R},e_i)e_i,e_k\rangle R^E(\mathcal{R},e_k)\right)+\mathcal{O}(|Z|^3).\end{aligned}$$

Note that by (3.4), $\frac{\partial^2}{\partial Z_m^2}R^E_{;(Z,e_i)}(\mathcal{R},e_i)=2R^E_{;(e_m,e_i)}(e_m,e_i)=0$. Thus from (3.1), (4.3), (5.45), and taking into account that $R^E(e_m,e_i)$ is anti-symmetric in m, i, and $\mathrm{Ric}(e_m,e_i)$ is symmetric in m, i, we infer

$$\begin{aligned}(5.46)\qquad \left(\frac{\partial^2}{\partial Z_m^2}\nabla^E_{e_i}\nabla^E_{e_i}f\right)(0)\\ =\frac{\partial^4 f_{x_0}}{\partial Z_m^2\partial Z_i^2}(0)+2\left[R^E(e_m,e_i),\frac{\partial^2 f_{x_0}}{\partial Z_i\partial Z_m}(0)\right]\\ +\frac12[R^E(e_m,e_i),[R^E(e_m,e_i),f(x_0)]]+\frac23\left[R^E_{;e_m}(e_m,e_i),\frac{\partial f_{x_0}}{\partial Z_i}(0)\right]\\ =16\frac{\partial^4 f_{x_0}}{\partial z_i\partial z_q\partial\bar z_i\partial\bar z_q}(0)-4[R^E_{k\bar\ell},[R^E_{\ell\bar k},f(x_0)]]\\ +\frac83\left[R^E_{m\bar q;\bar m},\frac{\partial f_{x_0}}{\partial z_q}(0)\right]-\frac83\left[R^E_{q\bar m;m},\frac{\partial f_{x_0}}{\partial\bar z_q}(0)\right].\end{aligned}$$

By (3.3), (3.25), (3.31) and (5.44),

$$(5.47)\qquad -\frac{\partial^2}{\partial Z_m^2}(\Gamma_{ii}^{\ell}\nabla_{e_\ell}^E f)(0)$$
$$= -\frac{4}{3}\operatorname{Ric}(e_m, e_\ell)\left(\frac{\partial^2 f_{x_0}}{\partial Z_\ell \partial Z_m}(0) + \frac{1}{2}[R^E(e_m, e_\ell), f(x_0)]\right)$$
$$-\frac{1}{6}\big(4\operatorname{Ric}_{;e_m}(e_m, e_\ell) - 2\operatorname{Ric}_{;e_q}(e_q, e_\ell) + \operatorname{Ric}_{;e_\ell}(e_m, e_m)\big)\frac{\partial f_{x_0}}{\partial Z_\ell}(0)$$
$$= -\frac{32}{3}\operatorname{Ric}_{m\bar\ell}\frac{\partial^2 f_{x_0}}{\partial z_\ell \partial \bar z_m}(0)$$
$$-\frac{4}{3}\left((\operatorname{Ric}_{m\bar\ell;\bar m} + \operatorname{Ric}_{m\bar m;\bar\ell})\frac{\partial f_{x_0}}{\partial z_\ell}(0) + (\operatorname{Ric}_{\ell\bar m;m} + \operatorname{Ric}_{m\bar m;\ell})\frac{\partial f_{x_0}}{\partial \bar z_\ell}(0)\right).$$

Formulas (3.3), (3.23), (3.31), (5.5) and (5.44)–(5.47) yield

$$(5.48)\qquad \big((\Delta^E)^2 f\big)(x_0)$$
$$= -\left(\frac{\partial^2}{\partial Z_m^2}\Delta^E f\right)(0)$$
$$= \frac{2}{3}\operatorname{Ric}(e_i, e_q)(\nabla_{e_i}^E\nabla_{e_q}^E f)(0) + \left(\frac{\partial^2}{\partial Z_m^2}\nabla_{e_i}^E\nabla_{e_i}^E f\right)(0) - \frac{\partial^2}{\partial Z_m^2}(\Gamma_{ii}^{l}\nabla_{e_l}^E f)(0)$$
$$= 16\frac{\partial^4 f_{x_0}}{\partial z_i \partial z_q \partial \bar z_i \partial \bar z_q}(0) - \frac{16}{3}\operatorname{Ric}_{m\bar\ell}\frac{\partial^2 f_{x_0}}{\partial z_\ell \partial \bar z_m}(0) - 4[R^E_{k\bar\ell}, [R^E_{\ell\bar k}, f(x_0)]]$$
$$+\frac{8}{3}\left[R^E_{m\bar q;\bar m}, \frac{\partial f_{x_0}}{\partial z_q}(0)\right] - \frac{8}{3}\left[R^E_{q\bar m;m}, \frac{\partial f_{x_0}}{\partial \bar z_q}(0)\right]$$
$$-\frac{4}{3}\left((\operatorname{Ric}_{m\bar\ell;\bar m} + \operatorname{Ric}_{m\bar m;\bar\ell})\frac{\partial f_{x_0}}{\partial z_\ell}(0) + (\operatorname{Ric}_{\ell\bar m;m} + \operatorname{Ric}_{m\bar m;\ell})\frac{\partial f_{x_0}}{\partial \bar z_\ell}(0)\right).$$

By Lemma 5.2, the discussion after (5.17), formulas (5.17), (5.21), (5.23), (5.27) and (5.28), we have

$$(5.49)\qquad \pi^2 Q_{4,x_0}(f)(0,0) = 2K_{41}f(x_0) + K_{42}f(x_0) + f(x_0)K_{42} + K_{2f} + K_{3f}$$
$$+ \boldsymbol{b}_{\mathbb{C}f} + \boldsymbol{b}_{Ef2} - \frac{1}{3}R_{\ell\bar k k\bar q}\frac{\partial^2 f_{x_0}}{\partial z_q \partial \bar z_\ell}(0) + \frac{1}{2}\frac{\partial^4 f_{x_0}}{\partial z_i \partial z_q \partial \bar z_i \partial \bar z_q}(0).$$

Note that $[R^E_{k\bar\ell}, [R^E_{\ell\bar k}, f(x_0)]] = R^E_{k\bar\ell}R^E_{\ell\bar k}f(x_0) - 2R^E_{k\bar\ell}f(x_0)R^E_{\ell\bar k} + f(x_0)R^E_{k\bar\ell}R^E_{\ell\bar k}$, so by (3.8), (5.12), (5.48) and (5.49), we get (5.8). The proof of Theorem 5.1 is completed.

5.3. Composition of Berezin–Toeplitz operators: Proofs of Theorems 0.2 and 0.3.

Proof of Theorem 0.2. By Lemma 2.2, we deduce as in the proof of Lemma 2.2, that for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/4$, we have (cf. [27], (4.79), [24], (7.4.6))

$$p^{-n}(T_{f,p}\circ T_{g,p})_{x_0}(Z,Z') \cong \sum_{r=0}^{k}\big(Q_{r,x_0}(f,g)\mathscr{P}_{x_0}\big)(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}+\mathcal{O}(p^{-\frac{k+1}{2}}), \tag{5.50}$$

where

$$Q_{r,x_0}(f,g)=\sum_{r_1+r_2=r}\mathscr{K}[Q_{r_1,x_0}(f),Q_{r_2,x_0}(g)]\in \operatorname{End}(E)_{x_0}[Z,Z'], \tag{5.51}$$

is a polynomial in Z, Z' with the same parity as r.

The existence of the expansion (0.15) and the expression of $\boldsymbol{b}_{0,f,g}$ follow from (2.12), (5.50) and (5.51); we get also

$$\boldsymbol{b}_{r,f,g}(x_0)=Q_{2r,x_0}(f,g)(0,0). \tag{5.52}$$

By (5.51),

$$Q_{2,x_0}(f,g)=\mathscr{K}[f(x_0),Q_{2,x_0}(g)]+\mathscr{K}[Q_{1,x_0}(f),Q_{1,x_0}(g)] + \mathscr{K}[Q_{2,x_0}(f),g(x_0)]. \tag{5.53}$$

Formulas (1.8), (5.13)–(5.14) yield

$$\begin{aligned}\mathscr{K}[Q_{2,x_0}(f),g(x_0)]\mathscr{P}&=\big(Q_{2,x_0}(f)\mathscr{P}\big)\mathscr{P}g(x_0)\\&=\Big(\mathscr{P}\sum_{|\alpha|=2}\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\mathscr{P}-\mathscr{L}^{-1}\mathcal{O}_2\mathscr{P}f(x_0)\Big)g(x_0),\\ \mathscr{K}[f(x_0),Q_{2,x_0}(g)]\mathscr{P}&=f(x_0)\mathscr{P}\big(Q_{2,x_0}(g)\mathscr{P}\big)\\&=f(x_0)\Big(\mathscr{P}\sum_{|\alpha|=2}\frac{\partial^\alpha g_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\mathscr{P}-g(x_0)\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp\Big).\end{aligned} \tag{5.54}$$

Using (0.9), (4.8), (4.9), (4.13), (5.5) and (5.54), we obtain

$$\begin{aligned}\mathscr{K}[Q_{2,x_0}(f),g(x_0)](0,0)&=\Big(-\frac{1}{4\pi}(\Delta^E f)(x_0)+\frac{1}{2}(\boldsymbol{b}_1 f)(x_0)\Big)g(x_0),\\ \mathscr{K}[f(x_0),Q_{2,x_0}(g)](0,0)&=f(x_0)\Big(-\frac{1}{4\pi}(\Delta^E g)(x_0)+\frac{1}{2}(g\boldsymbol{b}_1)(x_0)\Big).\end{aligned} \tag{5.55}$$

By (1.8), (2.11) and (2.21), we have (cf. [24], (7.4.14)),

$$Q_{1,x_0}(f)(Z,Z')=\mathscr{K}\Big[1,\frac{\partial f_x}{\partial Z_q}(0)Z_q\Big](Z,Z')=\frac{\partial f_{x_0}}{\partial z_i}(0)z_i+\frac{\partial f_{x_0}}{\partial \bar z_i}(0)\bar z_i'. \tag{5.56}$$

Thus from (4.13), (5.55), we get as in [24], (7.4.15) (cf. also [24], (7.1.11)),

$$\mathscr{K}[Q_{1,x_0}(f), Q_{1,x_0}(g)](0,0) = \sum_{i=1}^{n} \frac{1}{\pi} \frac{\partial f_{x_0}}{\partial \bar{z}_i}(0) \frac{\partial g_{x_0}}{\partial z_i}(0). \tag{5.57}$$

Now, (5.52), (5.53), (5.55) and (5.57) imply the formula for $\boldsymbol{b}_{1,f,g}$ given in (0.16).

We prove next (0.17). It suffices to consider $f, g \in \mathscr{C}^\infty(X, \mathbb{R})$, which we henceforth assume. By (2.9), we have $Q_{r,x_0}(1) = J_{r,x_0}$. Hence, taking $f = 1$ in (5.51), we get

$$\begin{aligned} Q_{4,x}(g) = \mathscr{K}[1, Q_{4,x}(g)] + \mathscr{K}[J_{2,x}, Q_{2,x}(g)] + \mathscr{K}[J_{3,x}, Q_{1,x}(g)] \\ + g(x)\mathscr{K}[J_{4,x}, 1]. \end{aligned} \tag{5.58}$$

Taking $g = 1$ in (5.51) yields

$$\begin{aligned} Q_{4,x}(f) = f(x)\mathscr{K}[1, J_{4,x}] + \mathscr{K}[Q_{1,x}(f), J_{3,x}] + \mathscr{K}[Q_{2,x}(f), J_{2,x}] \\ + \mathscr{K}[Q_{4,x}(f), 1]. \end{aligned} \tag{5.59}$$

By (2.12) and (5.51), we get

$$\begin{aligned} Q_{4,x_0}(f,g) = \mathscr{K}[f(x_0), Q_{4,x_0}(g)] + \mathscr{K}[Q_{1,x_0}(f), Q_{3,x_0}(g)] \\ + \mathscr{K}[Q_{2,x_0}(f), Q_{2,x_0}(g)] + \mathscr{K}[Q_{3,x_0}(f), Q_{1,x_0}(g)] \\ + \mathscr{K}[Q_{4,x_0}(f), g(x_0)]. \end{aligned} \tag{5.60}$$

Set

$$\begin{aligned} \tilde{Q}_{3,x_0}(g) &= \mathscr{K}\left[1, \frac{\partial g_{x_0}}{\partial Z_q}(0) Z_q J_{2,x_0}\right] + \mathscr{K}\left[J_{2,x_0}, \frac{\partial g_{x_0}}{\partial Z_q}(0) Z_q\right] \\ &\quad + \mathscr{K}\left[1, \sum_{|\alpha|=3} \frac{\partial^\alpha g_{x_0}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!}\right], \\ \boldsymbol{I}_{4,f,g} &= \mathscr{K}\left[\mathscr{K}\left[1, \sum_{|\alpha|=2} \frac{\partial^\alpha g_{x_0}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!}\right], \mathscr{K}\left[1, \sum_{|\alpha|=2} \frac{\partial^\alpha g_{x_0}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!}\right]\right]. \end{aligned} \tag{5.61}$$

Note that by (2.13) and (2.19), we have

$$J_{3,x_0} = \mathscr{K}[1, J_{3,x_0}] + \mathscr{K}[J_{3,x_0}, 1] \tag{5.62}$$

and (2.11), (2.21) together with (5.62) imply

$$Q_{3,x_0}(g) = g(x_0) J_{3,x_0} + \tilde{Q}_{3,x_0}(g). \tag{5.63}$$

Since $g \in \mathscr{C}^\infty(X, \mathbb{R})$, (5.13) entails

$$Q_{2,x_0}(g) = g(x_0) J_{2,x_0} + \mathscr{K}\left[1, \sum_{|\alpha|=2} \frac{\partial^\alpha g_{x_0}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!}\right]. \tag{5.64}$$

By Remark 4.3, (5.18), (5.58)–(5.64), we get

$$Q_{4,x_0}(f,g) = f(x_0)Q_{4,x_0}(g) + g(x_0)Q_{4,x_0}(f) - f(x_0)g(x_0)J_{4,x_0} + \mathscr{K}[Q_{1,x_0}(f), \tilde{Q}_{3,x_0}(g)] + \mathscr{K}[\tilde{Q}_{3,x_0}(f), Q_{1,x_0}(g)] + \boldsymbol{I}_{4,f,g}. \tag{5.65}$$

By (4.13) and (5.15), we get

$$\begin{aligned}(\boldsymbol{I}_{4,f,g}\mathscr{P})(0,0) &= \left\{\mathscr{P}\sum_{|\beta|=2}\frac{\partial^\beta f_{x_0}}{\partial Z^\beta}(0)\frac{Z^\beta}{\beta!}\left(\sum_{|\alpha|=2}\frac{\partial^\alpha g_{x_0}}{\partial z^\alpha}(0)\frac{z^\alpha}{\alpha!} + \frac{1}{\pi}\frac{\partial^2 g_{x_0}}{\partial z_i\partial\bar{z}_i}(0)\right)\mathscr{P}\right\}(0,0)\\ &= \frac{1}{\pi^2}\left(\frac{1}{2}\frac{\partial^2 f_{x_0}}{\partial\bar{z}_i\partial\bar{z}_q}(0)\frac{\partial^2 g_{x_0}}{\partial z_i\partial z_q}(0) + \frac{\partial^2 f_{x_0}}{\partial z_q\partial\bar{z}_q}(0)\frac{\partial^2 g_{x_0}}{\partial z_i\partial\bar{z}_i}(0)\right).\end{aligned} \tag{5.66}$$

By Remark 2.3, $Q_{1,x_0}(f)$, $Q_{3,x_0}(g)$ are self-adjoint for $f,g \in \mathscr{C}^\infty(X,\mathbb{R})$, thus by (5.63),

$$\mathscr{K}[\tilde{Q}_{3,x_0}(f), Q_{1,x_0}(g)] = \mathscr{K}[Q_{1,x_0}(g), \tilde{Q}_{3,x_0}(f)]^*. \tag{5.67}$$

Thus we only need to compute the fourth term in (5.65).

An examination of (4.4) shows that in each term of the sum giving $\mathcal{O}_2$, the total degree in b^+, z equals the total degree in b, $\bar{z}$. Hence Remark 4.3, (1.8), (2.19), (4.8), (4.9), (4.13), (5.25) and (5.56) yield

$$\begin{aligned}\mathscr{K}&\left[Q_{1,x_0}(f), \mathscr{K}\left[1, \frac{\partial g_{x_0}}{\partial Z_q}(0)Z_q J_{2,x_0}\right]\right]\mathscr{P}(0,0)\\ &= \left(\mathscr{P}\frac{\partial f_{x_0}}{\partial\bar{z}_u}(0)\bar{z}_u\frac{\partial g_{x_0}}{\partial z_v}(0)z_v\mathscr{F}_{2,x_0}\right)(0,0)\\ &= \frac{1}{2\pi^2}\frac{\partial f_{x_0}}{\partial\bar{z}_u}(0)\frac{\partial g_{x_0}}{\partial z_v}(0)\left[\delta_{uv}(R_{s\bar{s}q\bar{q}} + R^E_{q\bar{q}}) - \frac{4}{3}R_{u\bar{k}k\bar{v}} - R^E_{u\bar{v}}\right].\end{aligned} \tag{5.68}$$

Using (2.19), and the formula $(\mathscr{P}^\perp z_i\mathscr{P})(Z,0) = (\mathscr{P}\bar{z}_i\mathscr{P})(Z,0) = 0$ (cf. (1.6)), we get

$$\begin{aligned}&\left(\mathscr{K}\left[J_{2,x_0}, \frac{\partial g_{x_0}}{\partial Z_q}(0)Z_q\right]\mathscr{P}\right)(Z,0)\\ &= \left(\mathscr{F}_{2,x_0}\frac{\partial g_{x_0}}{\partial Z_q}(0)Z_q\mathscr{P}\right)(Z,0)\\ &= \left(-\mathscr{L}^{-1}\mathcal{O}_2\mathscr{P}\frac{\partial g_{x_0}}{\partial z_q}(0)z_q\mathscr{P} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp\frac{\partial g_{x_0}}{\partial\bar{z}_q}(0)\bar{z}_q\mathscr{P}\right)(Z,0).\end{aligned} \tag{5.69}$$

By Remark 4.3, (1.7), (1.8), (5.56) and (5.69), we obtain as in (5.68)

$$(5.70)\qquad \mathscr{K}\left[Q_{1,x_0}(f),\mathscr{K}\left[J_{2,x_0},\frac{\partial g_{x_0}}{\partial Z_q}(0)Z_q\right]\right]\mathscr{P}(0,0)$$
$$=\left(\mathscr{P}\frac{\partial f_{x_0}}{\partial \bar z_u}(0)\bar z_u\mathscr{K}\left[J_{2,x_0},\frac{\partial g_{x_0}}{\partial Z_q}(0)Z_q\right]\mathscr{P}\right)(0,0)$$
$$=-\left(\mathscr{P}\frac{\partial f_{x_0}}{\partial \bar z_u}(0)\bar z_u\mathscr{L}^{-1}\mathscr{O}_2\mathscr{P}\frac{\partial g_{x_0}}{\partial z_q}(0)z_q\mathscr{P}\right)(0,0)=0,$$

since by (1.6) and (4.11) we have $(\mathscr{P}\bar z_u b^\alpha z^\beta\mathscr{P})(Z,0)=0$ for $|\alpha|\geqq 1$.

Note that for homogeneous degree 3 polynomials H in Z the analogue of formula (4.12) holds for $(H\mathscr{P})(Z,0)$. Using this analogue together with (1.6) and (4.11) we obtain

$$(5.71)\qquad (\mathscr{P}H\mathscr{P})(Z,0)=\left(\sum_{|\alpha|=3}\frac{\partial^3 H}{\partial z^\alpha}\frac{z^\alpha}{\alpha!}+\frac{z_q}{\pi}\frac{\partial^3 H}{\partial z_q\partial z_i\partial \bar z_i}\right)\mathscr{P}(Z,0).$$

Finally, (1.8), (4.13), (5.56), (5.71) and the equality $\mathscr{P}(0,0)=1$ imply

$$(5.72)\qquad \mathscr{K}\left[Q_{1,x_0}(f),\mathscr{K}\left[1,\sum_{|\alpha|=3}\frac{\partial^\alpha g_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}\right]\right](0,0)=\frac{1}{\pi^2}\frac{\partial f_{x_0}}{\partial \bar z_u}(0)\frac{\partial^3 g_{x_0}}{\partial z_u\partial z_i\partial \bar z_i}(0).$$

By Lemma 3.1, (3.23), (3.25) and (5.44) for $E=\mathbb{C}$, we get

$$(5.73)\qquad \frac{\partial}{\partial z_q}(\Delta g)(0)=-4\frac{\partial^3 g_{x_0}}{\partial z_q\partial z_i\partial \bar z_i}(0)+\frac{4}{3}\,\mathrm{Ric}_{q\bar\ell}\frac{\partial g_{x_0}}{\partial z_\ell}(0).$$

Lemma 3.1, (5.61) and (5.68)–(5.73) entail

$$(5.74)\qquad \pi^2\mathscr{K}[Q_{1,x_0}(f),\tilde Q_{3,x_0}(g)](0,0)$$
$$=\frac{\partial f_{x_0}}{\partial \bar z_u}(0)\frac{\partial^3 g_{x_0}}{\partial z_u\partial z_i\partial \bar z_i}(0)$$
$$+\frac{1}{2}\frac{\partial f_{x_0}}{\partial \bar z_u}(0)\frac{\partial g_{x_0}}{\partial z_v}(0)\left[\delta_{uv}(R_{s\bar s q\bar q}+R^E_{q\bar q})-\frac{4}{3}R_{u\bar k k\bar v}-R^E_{u\bar v}\right]$$
$$=-\frac{1}{8}\langle\bar\partial f,\partial\Delta g\rangle+\frac{1}{4}\langle\bar\partial f,\partial g\rangle(R_{s\bar s q\bar q}+R^E_{q\bar q})-\frac{1}{8}\langle\bar\partial f\wedge\partial g,R^E\rangle_\omega.$$

From (5.67) and (5.74), we have

$$(5.75)\qquad \pi^2\mathscr{K}[\tilde Q_{3,x_0}(f),Q_{1,x_0}(g)](0,0)$$
$$=-\frac{1}{8}\langle\bar\partial\Delta f,\partial g\rangle+\frac{1}{4}\langle\bar\partial f,\partial g\rangle(R_{s\bar s q\bar q}+R^E_{q\bar q})-\frac{1}{8}\langle\bar\partial f\wedge\partial g,R^E\rangle_\omega.$$

By (0.9), (5.5), (5.65), (5.66), (5.74) and (5.75), we get (0.17). The proof of Theorem 0.2 is completed. □

Proof of Theorem 0.3. The existence of the expansion (0.18) and formula $C_0(f,g) = fg$ were established in [27], Theorem 1.1 (cf. also [24], Theorem 7.4.1) in general symplectic settings.

By (0.18), (2.10), (5.12), (5.50) and (5.52), we obtain (cf. also [24], (7.4.9)),

$$C_1(f,g) = \big(Q_{2,x}(f,g) - Q_{2,x}(fg)\big)(0,0) = \boldsymbol{b}_{1,f,g} - \boldsymbol{b}_{1,fg}. \tag{5.76}$$

Hence (0.5), (0.13), (0.16) and (5.76) yield the formula for $C_1(f,g)$ given in (0.20). Moreover, (0.18), (5.12) and (5.52) imply the formula for $C_2(f,g)$ from (0.20).

We will prove (0.21) now. Let $\{e_i\}$ be an orthonormal frame of (TX, g^{TX}), and $\{w_i\}$ be an orthonormal frame of $T^{(1,0)}X$. Let $\square = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$ be the Kodaira Laplacian on $\Lambda(T^*X)\otimes_{\mathbb{R}}\mathbb{C}$, and let Δ be the Bochner Laplacian on $\Lambda(T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ associated with the connection $\nabla^{\Lambda(T^*X)}$ on $\Lambda(T^*X)$ induced by ∇^{TX} (cf. (0.5)). Let $R^{\Lambda(T^{*(1,0)}X)}$ be the curvature of the holomorphic Hermitian connection on $\Lambda(T^{*(1,0)}X)$. By the Lichnerowicz formula [24], Remark 1.4.8, and (5.4), we have

$$\begin{aligned} R^{\Lambda(T^{*(1,0)}X)} &= -\langle R^{TX} w_l, \overline{w}_k\rangle w^l \wedge i_{w_k},\\ 2\square &= \Delta - R^{\Lambda(T^{*(1,0)}X)}(w_q,\overline{w}_q) + (2R^{\Lambda(T^{*(1,0)}X)} + \mathrm{Ric})(w_l,\overline{w}_k)\overline{w}^k\wedge i_{\overline{w}_l}. \end{aligned}\tag{5.77}$$

Since (X,ω,J) is Kähler, $\square$ commutes with the operators ∂, $\bar\partial$, d (cf. [24], Corollary 1.4.13), and (5.77) shows that $2\square f = \Delta f$ for any $f\in\mathscr{C}^\infty(X)$. From Lemma 3.1, (5.4) and (5.77), we have for any $f\in\mathscr{C}^\infty(X)$:

$$\begin{aligned} \Delta\partial f &= \partial\Delta f - \mathrm{Ric}(\cdot,\overline{w}_k)w_k(f),\\ \Delta\bar\partial f &= \bar\partial\Delta f - \mathrm{Ric}(\cdot,w_k)\overline{w}_k(f),\\ \Delta df &= d\Delta f - \mathrm{Ric}(\cdot,e_j)e_j(f). \end{aligned}\tag{5.78}$$

Thus (0.5), (5.78) yield for any $f,g\in\mathscr{C}^\infty(X)$:

$$\begin{aligned} \Delta(fg) &= g\Delta f + f\Delta g - 2\langle df, dg\rangle,\\ \Delta\langle\partial f,\bar\partial g\rangle &= \langle\Delta\partial f,\bar\partial g\rangle + \langle\partial f,\Delta\bar\partial g\rangle - 2\langle\nabla^{T^*X}\partial f,\nabla^{T^*X}\bar\partial g\rangle\\ &= \langle\partial\Delta f,\bar\partial g\rangle + \langle\partial f,\bar\partial\Delta g\rangle - 2\langle\nabla^{T^*X}\partial f,\nabla^{T^*X}\bar\partial g\rangle\\ &\quad - 2\,\mathrm{Ric}(w_m,\overline{w}_q)\overline{w}_m(g)w_q(f). \end{aligned}\tag{5.79}$$

Using (0.5), (5.78) and (5.79), we infer

$$\begin{aligned} \Delta^2(fg) &= f\Delta^2 g + g\Delta^2 f + 2(\Delta f)\Delta g - 4\langle d\Delta f, dg\rangle - 4\langle df, d\Delta g\rangle\\ &\quad + 4\langle\nabla^{T^*X}df,\nabla^{T^*X}dg\rangle + 4\,\mathrm{Ric}(e_i,e_j)e_i(g)e_j(f). \end{aligned}\tag{5.80}$$

We examine now closely the expression of $\pi^2 C_2(f,g)$ given by (0.20). Using (0.14), (0.17), (0.20), we see that the term of differential order 0 in f, g from the expression of

$\pi^2 C_2(f,g)$ is zero, and the term of total differential order 2 in f, g, disregarding the term involving R^E, in $\pi^2 C_2(f,g)$ is

$$(5.81)\qquad C_{22} = \frac{\sqrt{-1}}{8}\langle \mathrm{Ric}_\omega, -f\partial\bar{\partial}g - g\partial\bar{\partial}f + \partial\bar{\partial}(fg)\rangle.$$

The term of total differential order 4 in f, g in the expression of $\pi^2 C_2(f,g)$ is

$$(5.82)\qquad C_{24} = \frac{1}{32}f\Delta^2 g + \frac{1}{32}g\Delta^2 f - \frac{1}{8}\langle\bar{\partial}f, \partial\Delta g\rangle - \frac{1}{8}\langle\bar{\partial}\Delta f, \partial g\rangle + \frac{1}{16}(\Delta f)\Delta g$$
$$+ \frac{1}{8}\langle D^{0,1}\bar{\partial}f, D^{1,0}\partial g\rangle - \frac{1}{32}\Delta^2(fg) - \frac{1}{8}\Delta\langle\partial f, \bar{\partial}g\rangle.$$

By (5.79), (5.80), (5.82) and by the formula $\langle D^{1,0}\bar{\partial}f, D^{0,1}\partial g\rangle = \langle D^{0,1}\partial f, D^{1,0}\bar{\partial}g\rangle$, we get

$$(5.83)\qquad C_{24} = \frac{1}{8}\langle D^{1,0}\partial f, D^{0,1}\bar{\partial}g\rangle + \frac{\sqrt{-1}}{8}\langle \mathrm{Ric}_\omega, \partial f\wedge\bar{\partial}g - \partial g\wedge\bar{\partial}f\rangle.$$

Finally, by inspecting (0.14), (0.17), (0.20), we see that the term involving R^E in the expression of $\pi^2 C_2(f,g)$ is

$$(5.84)\qquad \frac{\sqrt{-1}}{8}\big(-f\Delta g - g\Delta f + \Delta(fg)\big)R^E_\Lambda + \frac{1}{4}\langle g\partial\bar{\partial}f + f\partial\bar{\partial}g - \partial\bar{\partial}(fg), R^E\rangle_\omega$$
$$+ \frac{\sqrt{-1}}{4}\langle\bar{\partial}f, \partial g\rangle R^E_\Lambda - \frac{1}{4}\langle\bar{\partial}f\wedge\partial g, R^E\rangle_\omega + \frac{\sqrt{-1}}{4}\langle\partial f, \bar{\partial}g\rangle R^E_\Lambda.$$

Combining (5.81), (5.83) and (5.84), we get (0.21). The proof of Theorem 0.3 is completed. □

6. Donaldson's *Q*-operator

In this section we study the asymptotics of the sequence of operators introduced by Donaldson [10]. We suppose henceforth that $E = \mathbb{C}$. Set $\mathrm{Vol}(X, dv_X) := \int_X dv_X$. Following [10], §4, set

$$(6.1)\qquad K_p(x,x') := |P_p(x,x')|^2_{h^{L^p}_x\otimes h^{L^{p*}}_{x'}}, \quad R_p := \big(\dim H^0(X,L^p)\big)/\mathrm{Vol}(X, dv_X).$$

Let K_p, Q_{K_p} be the integral operators associated to K_p, defined for $f\in\mathscr{C}^\infty(X)$ by

$$(6.2)\qquad (K_p f)(x) := \int_X K_p(x,y)f(y)\,dv_X(y), \quad Q_{K_p}(f) = \frac{1}{R_p}K_p f.$$

Recall that, just as the Bergman kernel appears when comparing a Kähler metric ω to its algebraic approximations ω_p (i.e. pull-backs of the Fubini–Study metrics by the Kodaira embeddings), the operators Q_{K_p} appear when one relates infinitesimal deformations of the metric ω to the corresponding deformations of the approximations ω_p. The asymptotics of the operator K_p were obtained in [21], Theorem 26,[1)] and used in [16]. The following result refines [21], Theorem 26, and is applied in the recent paper [17].

Theorem 6.1. *For every $m \in \mathbb{N}$, there exists $C > 0$ such that for any $f \in \mathscr{C}^\infty(X)$, $p \in \mathbb{N}^*$,*

$$\left| \frac{1}{p^n} K_p f - f + \frac{1}{8\pi p}(-\boldsymbol{r} f + 2\Delta f) \right|_{\mathscr{C}^m(X)} \leqq Cp^{-3/2} |f|_{\mathscr{C}^{m+3}(X)} \text{ or } Cp^{-2}|f|_{\mathscr{C}^{m+4}(X)}. \tag{6.3}$$

Proof. By (2.9) with $Z = 0$, (2.13), (2.19) and (2.21), we get

$$\begin{aligned} &\left| \left(\frac{1}{p^{2n}} K_{p,x_0}(0, Z')\kappa_{x_0}(Z') - \left(1 + \sum_{r=2}^{k} p^{-r/2} J'_r(0, \sqrt{p}Z')\right) e^{-\pi p|Z'|^2} \right) \right|_{\mathscr{C}^m(X)} \\ &\qquad \leqq Cp^{-(k+1)/2}(1 + |\sqrt{p}Z'|)^N \exp(-C_0\sqrt{p}|Z'|) + \mathscr{O}(p^{-\infty}), \end{aligned} \tag{6.4}$$

with

$$J'_2(0, Z') = (J_2 + \overline{J_2})(0, Z'). \tag{6.5}$$

Now we have the analogue of [21], (32),

$$\begin{aligned} &\left| p^{-n} K_p f - p^n \int_{|Z'| \leqq \varepsilon} \left(1 + \sum_{r=2}^{k} p^{-r/2} J'_r(0, \sqrt{p}Z')\right) \right. \\ &\qquad \left. \times\, e^{-\pi p|Z'|^2} f_{x_0}(Z')\, dv_{T_{x_0}X}(Z') \right|_{\mathscr{C}^m(X)} \\ &\qquad \leqq Cp^{-(k+1)/2} |f|_{\mathscr{C}^m(X)}. \end{aligned} \tag{6.6}$$

But as in the proof of [2], Theorem 2.29 (2), we get

$$\left| p^n \int_{|Z'| \leqq \varepsilon} J'_r(0, \sqrt{p}Z') e^{-\pi p|Z'|^2} f_{x_0}(Z')\, dv_{T_{x_0}X}(Z') \right|_{\mathscr{C}^m(X)} \leqq C|f|_{\mathscr{C}^m(X)}. \tag{6.7}$$

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1) Note that in the present context [21], Theorem 26, should be modified as follows:

$$\lim_{p\to\infty} Q_{K_p} f = \frac{\mathrm{Vol}(X, v)}{\mathrm{Vol}(X, dv_X)} \eta f \quad \text{in } \mathscr{C}^m(X),$$

or

$$\left| Q_{K_p} f - \frac{\mathrm{Vol}(X, v)}{\mathrm{Vol}(X, dv_X)} \eta f \right|_{\mathscr{C}^m(X)} \leqq Cp^{-1/2} |f|_{\mathscr{C}^{m+1}(X)} \text{ or } Cp^{-1}|f|_{\mathscr{C}^{m+2}(X)},$$

since the right-hand side of the second equation of [21], (33), and [21], (34), should read as convergence in $\mathscr{C}^m(X)$ without the speed, or $Cp^{-1/2}|f|_{\mathscr{C}^{m+1}(X)}$ or $Cp^{-1}|f|_{\mathscr{C}^{m+2}(X)}$.

Moreover

$$(6.8)\qquad \left| p^n \int_{|Z'|\leqq\varepsilon} e^{-\pi p|Z'|^2} f_{x_0}(Z')\, dv_{T_{x_0}X}(Z') - f(x_0) + \frac{1}{4\pi p}(\Delta f)(x_0) \right|_{\mathscr{C}^m(X)} \leqq Cp^{-3/2}|f|_{\mathscr{C}^{m+3}(X)} \quad \text{or} \quad Cp^{-2}|f|_{\mathscr{C}^{m+4}(X)}.$$

Finally, by (4.9) (cf. also [24], (4.1.110)), we have

$$(6.9)\qquad \int_{Z'\in\mathbb{C}^n} \overline{J_2}(0,Z')|\mathscr{P}|^2(0,Z')\, dZ' = \int_{Z'\in\mathbb{C}^n} \mathscr{P}(0,Z')J_2(Z',0)\mathscr{P}(Z',0)\, dZ' = (\mathscr{P}J_2\mathscr{P})(0,0) = -(\mathscr{P}\mathscr{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp})(0,0) = \frac{1}{16\pi}\boldsymbol{r}.$$

Thus

$$(6.10)\qquad \left| p^n \int_{|Z'|\leqq\varepsilon} J_2'(0,\sqrt{p}Z')e^{-\pi p|Z'|^2} f_{x_0}(Z')\, dv_{T_{x_0}X}(Z') - \left(\frac{\boldsymbol{r}}{8\pi}f\right)(x_0) \right|_{\mathscr{C}^m(X)} \leqq Cp^{-1/2}|f|_{\mathscr{C}^{m+1}(X)} \quad \text{or} \quad Cp^{-1}|f|_{\mathscr{C}^{m+2}(X)}.$$

The proof of Theorem 6.1 is completed. □

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Eingegangen 27. September 2010

J. reine angew. Math. **662** (2012), 57—58
DOI 10.1515/CRELLE.2011.187

Journal für die reine und angewandte Mathematik

Addendum to

"Berezin–Toeplitz quantization on Kähler manifolds"

(J. reine angew. Math. 662)

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In our paper [1], we compute the term $\boldsymbol{b}_2$ from the asymptotic expansion of the Bergman kernel. Actually, in Theorem 0.1 of this paper we consider the expansion of a Toeplitz operator with symbol f. By taking $f = 1$ we obtain the Bergman kernel.

Here we wish to simplify our formula for $\boldsymbol{b}_2$. The Bianchi identity reads $[\nabla^E, R^E] = 0$. Take the derivative of

$$[\nabla^E, R^E]\left(\widetilde{\frac{\partial}{\partial z_i}}, \widetilde{\frac{\partial}{\partial z_k}}, \widetilde{\frac{\partial}{\partial \bar{z}_k}}\right) = 0,$$

and use [1], (3.10), (3.20), (3.31), which imply

$$\left[\widetilde{\frac{\partial}{\partial z_i}}, \widetilde{\frac{\partial}{\partial \bar{z}_k}}\right] = -\frac{1}{2} R^{TX}_{x_0}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_k}\right)\mathcal{R} + \mathcal{O}(|Z|^2), \quad \left[\widetilde{\frac{\partial}{\partial z_i}}, \widetilde{\frac{\partial}{\partial z_k}}\right] = \mathcal{O}(|Z|^2).$$

We conclude that

$$R^E_{m\bar{k};k\bar{m}} = R^E_{k\bar{k};m\bar{m}}.$$

The term

$$\frac{1}{4}(-R^E_{k\bar{k};m\bar{m}} + 3R^E_{m\bar{k};k\bar{m}})$$

in [1], (0.9), can be thus replaced by $\frac{1}{2} R^E_{k\bar{k};m\bar{m}}$. Equivalently, by using [1], (5.5), one can replace

$$+\frac{\sqrt{-1}}{32}\Delta^E R^E_\Lambda + \frac{3}{16}\bar{\partial}^{E*}\nabla^{1,0*}R^E$$

in [1], (0.8), by $-\frac{\sqrt{-1}}{16}\Delta^E R^E_\Lambda$. Thus the term $\boldsymbol{b}_{2E}$ in [1], (0.8), (0.9), equals

$$\begin{aligned}(1)\qquad \boldsymbol{b}_{2E} &= R^E_{q\bar{q}}R_{k\bar{k}m\bar{m}} - R^E_{m\bar{q}}R_{k\bar{k}q\bar{m}} + \frac{1}{2}(R^E_{q\bar{q}}R^E_{m\bar{m}} - R^E_{m\bar{q}}R^E_{q\bar{m}}) + \frac{1}{2}R^E_{k\bar{k};m\bar{m}} \\ &= \frac{\sqrt{-1}}{16}(\boldsymbol{r}R^E_\Lambda - 2\langle \mathrm{Ric}_\omega, R^E\rangle_\omega - \Delta^E R^E_\Lambda) - \frac{1}{8}(R^E_\Lambda)^2 + \frac{1}{8}\langle R^E, R^E\rangle_\omega.\end{aligned}$$

References

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Eingegangen 16. November 2011