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Toeplitz Operators on Symplectic Manifolds

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Abstract We study the Berezin-Toeplitz quantization on symplectic manifolds making use of the full off-diagonal asymptotic expansion of the Bergman kernel. We give also a characterization of Toeplitz operators in terms of their asymptotic expansion. The semi-classical limit properties of the Berezin-Toeplitz quantization for non-compact manifolds and orbifolds are also established.

Keywords Toeplitz operator · Berezin-Toeplitz quantization · Bergman kernel · spin^c Dirac operator

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1 Introduction

Quantization is a procedure that leads from a classical dynamical system to an associated algebra whose behavior reduces to that of the given classical system in an

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appropriate limit. In the usual case, the limit involves Planck's constant $\hbar$ approaching zero. The aim of the geometric quantization theory [2–5, 21, 25, 35] is to relate the classical observables (smooth functions) on a phase space (a symplectic manifold) to the quantum observables (bounded linear operators) on the quantum space (sections of a line bundle). One particular way to quantize the phase space is the Berezin-Toeplitz quantization, which we briefly describe.

Let us consider a compact Kähler manifold X with Kähler form ω. On X we are given a holomorphic Hermitian line bundle (L, h^L) endowed with the Chern connection ∇^L with curvature R^L. We assume that the prequantization condition $\frac{\sqrt{-1}}{2\pi} R^L = \omega$ is fulfilled. For any $p \in \mathbb{N}$ let $L^p := L^{\otimes p}$ be the p^{th} tensor power of L, $L^2(X, L^p)$ be the space of L^2-sections of L^p with norm induced by h^L and ω, and $P_p : L^2(X, L^p) \to H^0(X, L^p)$ be the orthogonal projection on the space of holomorphic sections. To any function $f \in \mathcal{C}^\infty(X)$ we associate a sequence of linear operators

$$T_{f,p} : L^2(X, L^p) \to L^2(X, L^p), \quad T_{f,p} = P_p f P_p, \tag{1.1}$$

where for simplicity we denote by f the operator of multiplication with f. Then as $p \to \infty$, the following properties hold:

$$\begin{aligned} &\lim_{p\to\infty} \|T_{f,p}\| = \|f\|_\infty := \sup_{x\in X} |f(x)|, \\ &[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \end{aligned} \tag{1.2}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on $(X, 2\pi\omega)$ (cf. (4.77)) and $\|\cdot\|$ is the operator norm. Thus, the Poisson algebra $(\mathcal{C}^\infty(X), \{\cdot,\cdot\})$ is approximated by the operator algebras of Toeplitz operators in the norm sense as $p \to \infty$; the role of the Planck constant is played by $\hbar = 1/p$. This is the so-called semi-classical limit process.

The relations (1.2) were proved first in some special cases: in [24] for Riemannian surfaces, in [19] for $\mathbb{C}^n$ and in [9] for bounded symmetric domains in $\mathbb{C}^n$, by using explicit calculations. Then, Bordemann et al. [8] treated the case of a compact Kähler manifold using the theory of Toeplitz structures (generalized Szegö operators) by Boutet de Monvel and Guillemin [11]. Moreover, Schlichenmaier [34] (cf. also [17, 23]) continued this train of thought and showed that for any $f, g \in \mathcal{C}^\infty(X)$, the product $T_{f,p} T_{g,p}$ has an asymptotic expansion

$$T_{f,p} T_{g,p} = \sum_{k=0}^{\infty} T_{C_k(f,g)} p^{-k} + \mathcal{O}(p^{-\infty}) \tag{1.3}$$

in the sense of (4.5), where C_k are bidifferential operators, satisfying $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \sqrt{-1}\{f, g\}$. As a consequence, one constructs geometrically an associative star product, defined by setting for any $f, g \in \mathcal{C}^\infty(X)$,

$$f * g := \sum_{k=0}^{\infty} C_k(f, g) \hbar^k \in \mathcal{C}^\infty(X)[[\hbar]]. \tag{1.4}$$

For previous work on Berezin-Toeplitz star products in special cases see [13–16, 31].

The articles [8, 17, 23, 34] rely on the method and results of Boutet de Monvel, Guillemin and Sjöstrand [11, 12]. They perform the analysis on the principal bundle associated to L, i.e., the circle bundle Y of the dual bundle L^* of L. Actually $Y = \partial D$, where $D := \{v \in L^* : |v|_{h^{L^*}} < 1\}$, which is a strictly pseudoconvex domain, due to the positivity of (L, h^L) (this fact is a basic observation due to Grauert).

Let us endow Y with the volume form $d\theta \wedge \varrho^*\omega^n$, where $\varrho : Y \to X$ is the bundle projection. Consider the space $L^2(Y)$ and for each $p \in \mathbb{Z}$ the subspace $L^2(Y)_p$ of functions on Y transforming under the S^1-action on Y according to the rule $\varphi(e^{i\theta} y) = e^{ip\theta}\varphi(y)$. There is a canonical isometry $L^2(Y)_p \cong L^2(X, L^p)$ which together with the Fourier decomposition $L^2(Y) \cong \bigoplus_{p\in\mathbb{Z}} L^2(Y)_p$ (the latter is a Hilbert space direct sum) delivers a canonical isometry $L^2(Y) \cong \bigoplus_{p\in\mathbb{Z}} L^2(X, L^p)$.

Let $\overline{\partial}_b$ denote the tangential Cauchy-Riemann operator on Y. A function $\varphi \in L^2(Y)$ is called Cauchy-Riemann (CR for short) if it satisfies the tangential Cauchy-Riemann equations $\overline{\partial}_b\varphi = 0$ (in the sense of distributions). Let $\mathscr{H}^2(Y) \subset L^2(Y)$ be the space of CR functions (Hardy space). For every $p \in \mathbb{N}$ let us denote $\mathscr{H}^2_p(Y) = L^2(Y)_p \cap \mathscr{H}^2(Y)$. Then we have the Hilbert sum decomposition $\mathscr{H}^2(Y) = \bigoplus_{p\in\mathbb{N}} \mathscr{H}^2_p(Y)$. Moreover, $\mathscr{H}^2_p(Y)$ is identified through the canonical isometry $L^2(Y)_p \cong L^2(X, L^p)$ to the subspace $H^0(X, L^p)$. Thus, $\mathscr{H}^2(Y) \cong \bigoplus_{p\in\mathbb{N}} H^0(X, L^p)$.

Therefore, in order to study the Bergman projections P_p, one can replace the family $\{P_p\}_{p\in\mathbb{N}}$ with the orthogonal projection $\boldsymbol{S} : L^2(Y) \to \bigoplus_{p\in\mathbb{N}} \mathscr{H}^p(Y)$, called Szegö projection. The key result is that $\boldsymbol{S}$ is a Fourier integral operator of order 0 of Hermite type (Boutet de Monvel-Sjöstrand [12]) and this allows to apply the theory of Fourier integral operators to obtain the properties of Toeplitz structures.

In the framework of Toeplitz structures, Guillemin [22] (cf. also [10] for related results) constructed a star product on compact symplectic manifolds by replacing the CR functions with functions annihilated by a first order pseudodifferential operator D_b on the circle bundle of L^* introduced in [11]. The operator D_b has the same microlocal structure as the tangential Cauchy-Riemann operator $\overline{\partial}_b$ and it is derived actually by first constructing the Szegö kernel.

In this article, we propose a different approach to the study of Berezin-Toeplitz quantization and Toeplitz operators. This consists in applying the off-diagonal asymptotic expansion as $p \to \infty$ of the Bergman kernel $P_p(x, x')$, which is the Schwartz kernel of the Bergman projection P_p.

We can actually treat the case of symplectic manifolds. Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Let (L, h^L) be a Hermitian line bundle on X endowed with a Hermitian connection ∇^L. The curvature of this connection is given by $R^L = (\nabla^L)^2$. We will assume throughout the article that (L, h^L, ∇^L) satisfies the *prequantization condition*:

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{1.5}$$

(L, h^L, ∇^L) is called a *prequantum line bundle*. Due to the analogy to the complex manifolds the bundle L will be also called positive. We also consider a twisting Hermitian vector bundle (E, h^E) on X with Hermitian connection ∇^E.

Let J be an almost complex structure on TX such that ω is compatible with J and $\omega(\cdot, J\cdot) > 0$. Let g^{TX} be a Riemannian metric on TX compatible with J.

A natural geometric generalization of the operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ acting on $\Omega^{0,\bullet}(X, L^p)$ is the *spin^c Dirac operator* D_p acting on $\Omega^{0,\bullet}(X, L^p \otimes E)$ (cf. Definition 3.1) associated to $J, g^{TX}, \nabla^L, \nabla^E$.

We refer to the orthogonal projection P_p from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto $\operatorname{Ker}(D_p)$ as the *Bergman projection* of D_p. The Schwartz kernel $P_p(\cdot,\cdot)$ of P_p is called *Bergman kernel* of D_p (cf. Definition 3.2). For $f \in \mathcal{C}^\infty(X, \operatorname{End}(E))$, we define the *Berezin-Toeplitz quantization* of f as in (1.1) by

$$T_{f,p} := P_p f P_p \in \operatorname{End}(L^2(X, \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E)). \tag{1.6}$$

Dai, Liu and Ma [20] proved the asymptotic expansion as $p \to \infty$ of the Bergman kernel $P_p(x, x')$ of D_p on the symplectic manifold (X, ω) by working directly on the base manifold. The main idea of their proof is that the positivity of the bundle L implies the existence of a spectral gap of the square of the spin^c Dirac operator, which in turn insures that the problem can be localized and transferred to the tangent space of a point of the manifold.

We are thus lead to study the model operator $\mathscr{L}$ on $\mathbb{C}^n$, its Bergman projection $\mathscr{P}$ and Bergman kernel $\mathscr{P}(Z, Z')$. The strategy of our approach is to first study the calculus of kernels of the type $(F\mathscr{P})(Z, Z')$ on $\mathbb{C}^n$, where $F \in \mathbb{C}[Z, Z']$ is a polynomial.

Using this calculus, the asymptotic expansion as $p \to \infty$ of the Bergman kernel of D_p from [20] and the Taylor series expansion of the sections f and $g \in \mathcal{C}^\infty(X, \operatorname{End}(E))$, we find the asymptotic expansion of the kernel of $T_{f,p}$ (cf. Lemma 4.6), and we establish that this kind of asymptotic expansion is also a sufficient condition for a family of operators to be a Toeplitz operator (cf. Theorem 4.9). In this way, we conclude from the asymptotic expansion of $T_{f,p}T_{g,p}$ that $T_{f,p}T_{g,p}$ is a Toeplitz operator in the sense of Definition 4.1.

The following result is one of our main results in this article.

Theorem 1.1 *Let (X, J, ω) be a compact symplectic manifold, (L, h^L, ∇^L), (E, h^E, ∇^E) be Hermitian vector bundles as above, and g^{TX} be an J-compatible Riemannian metric on TX.*

Let $f, g \in \mathcal{C}^\infty(X, \operatorname{End}(E))$. Then the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits the asymptotic expansion in the sense of (4.5):

$$T_{f,p}T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{1.7}$$

where C_r are bidifferential operators and $C_r(f, g) \in \mathcal{C}^\infty(X, \operatorname{End}(E))$ and $C_0(f, g) = fg$.

If $f, g \in \mathcal{C}^\infty(X)$, we have

$$C_1(f, g) - C_1(g, f) = \sqrt{-1}\{f, g\} \operatorname{Id}_E, \tag{1.8}$$

and therefore

$$[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}). \tag{1.9}$$

In conclusion, the set of Toeplitz operators forms an algebra. Moreover, the Berezin-Toeplitz quantization has the correct semi-classical behavior (cf. Theorem 4.19). In particular, when (X, J, ω) is a compact Kähler manifold and $E = \mathbb{C}$, $g^{TX} = \omega(\cdot, J\cdot)$, these results give a new proof of (1.2)–(1.4) (cf. Remark 5.1). Some related results were also announced in [10].

Note that we have established the off-diagonal asymptotic expansion of the Bergman kernel for certain non-compact manifolds [28, §3.5] (e.g., quasi-projective manifolds) and for orbifolds [20, §5.2]. By combining these results and the method in this article, we carry the Berezin-Toeplitz quantization over to these cases (cf. Theorems 5.3, 6.13, 6.16).

As explained as above, an interesting corollary of our results is a canonical geometric construction of associated star products (1.4) in several cases. We refer to Fedosov's book [21] for a construction of formal star products on symplectic manifolds and to Pflaum [32] for the generalization to orbifolds. Related results appear in [18, 33].

We refer the readers to our book [30] for a comprehensive study of the Berezin-Toeplitz quantization along the lines of the present article.

For the reader's convenience, we conclude the introduction with a brief outline of the article. We begin in Sect. 2 by explaining the formal calculus on $\mathbb{C}^n$ for the model operator $\mathcal{L}$. In Sect. 3, we recall the definition of the spin^c Dirac operator and the asymptotic expansion of the Bergman kernel obtained in [20]. In Sect. 4, we establish the characterization of Toeplitz operators in terms of their kernel. As a consequence, we establish that the set of Toeplitz operators forms an algebra. Finally, in Sects. 5 and 6, we study the Berezin-Toeplitz quantization for non-compact manifolds and orbifolds.

We will use the following notations throughout. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $B = (B_1, \dots, B_n) \in \mathbb{C}^n$, we set

$$|\alpha| = \sum_{j=1}^{n} \alpha_j, \qquad \alpha! = \prod_j (\alpha_j!), \qquad B^\alpha = \prod_j B_j^{\alpha_j}.$$

2 Kernel Calculus on $\mathbb{C}^n$

In this Section we explain the formal calculus on $\mathbb{C}^n$ for our model operator $\mathcal{L}$, and we derive the properties of the calculus of the kernels $(F\mathcal{P})(Z, Z')$, where $F \in \mathbb{C}[Z, Z']$ and $\mathcal{P}(Z, Z')$ is the kernel of the projection on the null space of the model operator $\mathcal{L}$. This calculus is the main ingredient of our approach.

Let us consider the canonical coordinates $(Z_1, \dots, Z_{2n})$ on the real vector space $\mathbb{R}^{2n}$. On the complex vector space $\mathbb{C}^n$ we consider the complex coordinates $(z_1, \dots, z_n)$. The two sets of coordinates are linked by the relation $z_j = Z_{2j-1} + \sqrt{-1} Z_{2j}$, $j = 1, \dots, n$.

We consider the L^2-norm $\|\cdot\|_{L^2} = (\int_{\mathbb{R}^{2n}} |\cdot|^2 dZ)^{1/2}$ on $\mathbb{R}^{2n}$, where $dZ = dZ_1 \cdots dZ_{2n}$ is the standard Euclidean volume form.

Let $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. We define the differential operators:

$$\begin{aligned} &b_i = -2\tfrac{\partial}{\partial z_i} + \frac{1}{2} a_i \overline{z}_i, \qquad b_i^+ = 2\tfrac{\partial}{\partial \overline{z}_i} + \frac{1}{2} a_i z_i, \\ &b = (b_1, \dots, b_n). \end{aligned} \tag{2.1}$$

Then b_i^+ is the adjoint of b_i on $(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2})$. Set

$$\mathcal{L} = \sum_i b_i b_i^+. \tag{2.2}$$

Then $\mathcal{L}$ acts as a densely defined self-adjoint operator on $(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2})$.

Theorem 2.1 *The spectrum of $\mathcal{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathcal{L}) = \left\{ 2 \sum_{i=1}^n \alpha_i a_i : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\} \tag{2.3}$$

and an orthogonal basis of the eigenspace of $2\sum_{i=1}^n \alpha_i a_i$ is given by

$$b^\alpha \left(z^\beta \exp\left(-\frac{1}{4} \sum_{i=1}^n a_i |z_i|^2 \right) \right), \quad \textit{with } \beta \in \mathbb{N}^n. \tag{2.4}$$

In particular, an orthonormal basis of $\operatorname{Ker}(\mathcal{L})$ *is*

$$\varphi_\beta(z) = \left(\frac{a^\beta}{(2\pi)^n 2^{|\beta|} \beta!} \prod_{i=1}^n a_i \right)^{1/2} z^\beta \exp\left(-\frac{1}{4} \sum_{j=1}^n a_j |z_j|^2 \right), \quad \beta \in \mathbb{N}^n. \tag{2.5}$$

For a proof we refer to [28, Theorem 1.15] (cf. also [30, Theorem 4.1.20]). Let $\mathscr{P}(Z, Z')$ denote the kernel of the orthogonal projection $\mathscr{P} : L^2(\mathbb{R}^{2n}) \longrightarrow \operatorname{Ker}(\mathcal{L})$ with respect to dZ. We call $\mathscr{P}(\cdot, \cdot)$ the Bergman kernel of $\mathcal{L}$.

It is easy to see that $\mathscr{P}(Z, Z') = \sum_\beta \varphi_\beta(z) \overline{\varphi_\beta(z')}$. We infer the following formula for the kernel $\mathscr{P}(Z, Z')$:

$$\mathscr{P}(Z, Z') = \prod_{i=1}^n \frac{a_i}{2\pi} \exp\left(-\frac{1}{4} \sum_{i=1}^n a_i \left(|z_i|^2 + |z_i'|^2 - 2 z_i \overline{z}_i' \right) \right). \tag{2.6}$$

In the calculations involving the kernel $\mathscr{P}(\cdot, \cdot)$, we prefer however to use the orthogonal decomposition of $L^2(\mathbb{R}^{2n})$ given in Theorem 2.1 and the fact that $\mathscr{P}$ is an orthogonal projection, rather than integrating against the expression (2.6) of $\mathscr{P}(\cdot, \cdot)$. This point of view helps simplify a lot the computations and understand better the operations. As an example, if $\varphi(Z) = b^\alpha z^\beta \exp(-\frac{1}{4} \sum_{j=1}^n a_j |z_j|^2)$ with $\alpha, \beta \in \mathbb{N}^n$, then Theorem 2.1 implies immediately that

$$(\mathscr{P}\varphi)(Z) = \begin{cases} z^\beta \exp(-\frac{1}{4}\sum_{j=1}^n a_j |z_j|^2) & \text{if } |\alpha| = 0, \\ 0 & \text{if } |\alpha| > 0. \end{cases} \tag{2.7}$$

In the rest of this section, all operators are defined by their kernels with respect to dZ. In this way, if F is a polynomial on Z, Z', then $F\mathscr{P}$ is an operator on $L^2(\mathbb{R}^{2n})$ with kernel $F(Z,Z')\mathscr{P}(Z,Z')$ with respect to dZ.

We will add a subscript z or z' when we need to specify the operator is acting on the variables Z or Z'.

Lemma 2.2 *For any polynomial $F(Z,Z') \in \mathbb{C}[Z,Z']$, there exist polynomials $F_\alpha \in \mathbb{C}[z,Z']$ and $F_{\alpha,0} \in \mathbb{C}[z,\overline{z}']$, $(\alpha \in \mathbb{N}^n)$ such that*

$$(F\mathscr{P})(Z,Z') = \sum_\alpha b_z^\alpha (F_\alpha \mathscr{P})(Z,Z'), \tag{2.8}$$

$$((F\mathscr{P}) \circ \mathscr{P})(Z,Z') = \sum_\alpha b_z^\alpha F_{\alpha,0}(z,\overline{z}')\mathscr{P}(Z,Z'). \tag{2.9}$$

Moreover, $|\alpha| + \deg F_\alpha$, $|\alpha| + \deg F_{\alpha,0}$ have the same parity with the degree of F in Z, Z'. In particular, $F_{0,0}(z,\overline{z}')$ is a polynomial in z, $\overline{z}'$ and its degree has the same parity with $\deg F$.

For any polynomials $F, G \in \mathbb{C}[Z,Z']$ there exist polynomial $\mathscr{K}[F,G] \in \mathbb{C}[Z,Z']$ such that

$$((F\mathscr{P}) \circ (G\mathscr{P}))(Z,Z') = \mathscr{K}[F,G](Z,Z')\mathscr{P}(Z,Z'). \tag{2.10}$$

Proof Note that from (2.1) and (2.6), for any polynomial $g(z,\overline{z}) \in \mathbb{C}[z,\overline{z}]$, we get

$$\begin{aligned} b_{j,z}\mathscr{P}(Z,Z') &= a_j(\overline{z}_j - \overline{z}'_j)\mathscr{P}(Z,Z'), \\ [g(z,\overline{z}), b_{j,z}] &= 2\frac{\partial}{\partial z_j} g(z,\overline{z}). \end{aligned} \tag{2.11}$$

Let $F(Z,Z') \in \mathbb{C}[Z,Z']$. Using repeatedly (2.11) we can replace $\overline{z}$ in the expression of $F(Z,Z')$ by a combination of $b_{j,z}$ and $\overline{z}'$ and (2.8) follows. We deduce from (2.7) and (2.8) that there exists $F_0 \in \mathbb{C}[z,Z']$ such that

$$(\mathscr{P} \circ (F\mathscr{P}))(Z,Z') = (F_0\mathscr{P})(Z,Z'). \tag{2.12}$$

We apply now (2.12) for $\overline{F}$ instead of F and take the adjoint of the so obtained equality. Since $\mathscr{P}$ is self-adjoint, this implies the existence of a polynomial F' in $Z, \overline{z}'$ such that

$$((F\mathscr{P}) \circ \mathscr{P})(Z,Z') = F'(Z,\overline{z}')\mathscr{P}(Z,Z').$$

The latter formula together with (2.8) imply (2.9). Finally, (2.10) results from (2.8) and (2.9). □

Example 2.3 We illustrate how Lemma 2.2 works. Observe that (2.11) entails

$$\overline{z}_j\mathscr{P}(Z,Z') = \frac{b_{j,z}}{a_j}\mathscr{P}(Z,Z') + \overline{z}'_j\mathscr{P}(Z,Z'). \tag{2.13}$$

Moreover, specializing (2.11) for $g(z,\overline{z}) = z_i$ we get

$$z_i b_{j,z}\mathscr{P}(Z,Z') = b_{j,z}(z_i\mathscr{P})(Z,Z') + 2\delta_{ij}\mathscr{P}(Z,Z'). \tag{2.14}$$

Formulas (2.13) and (2.14) give

$$z_i\overline{z}_j\mathscr{P}(Z,Z') = \frac{1}{a_j}b_{j,z}z_i\mathscr{P}(Z,Z') + \frac{2}{a_j}\delta_{ij}\mathscr{P}(Z,Z') + z_i\overline{z}'_j\mathscr{P}(Z,Z'). \tag{2.15}$$

Using the preceding formula we calculate further some examples for the expression $\mathscr{K}[F,G]$ introduced (2.10). Indeed, equations (2.7), (2.13) and (2.15) imply that

$$\begin{aligned}
&\mathscr{K}[1,\overline{z}_j]\mathscr{P} = \mathscr{P}\circ(\overline{z}_j\mathscr{P}) = \overline{z}'_j\mathscr{P}, \quad \mathscr{K}[1,z_j]\mathscr{P} = \mathscr{P}\circ(z_j\mathscr{P}) = z_j\mathscr{P},\\
&\mathscr{K}[z_i,\overline{z}_j]\mathscr{P} = (z_i\mathscr{P})\circ(\overline{z}_j\mathscr{P}) = z_i\mathscr{P}\circ(\overline{z}_j\mathscr{P}) = z_i\overline{z}'_j\mathscr{P},\\
&\mathscr{K}[\overline{z}_i,z_j]\mathscr{P} = (\overline{z}_i\mathscr{P})\circ(z_j\mathscr{P}) = \overline{z}_i\mathscr{P}\circ(z_j\mathscr{P}) = \overline{z}_i z_j\mathscr{P},\\
&\mathscr{K}[z'_i,\overline{z}_j]\mathscr{P} = (z'_i\mathscr{P})\circ(\overline{z}_j\mathscr{P}) = \mathscr{P}\circ(z_i\overline{z}_j\mathscr{P}) = \frac{2}{a_j}\delta_{ij}\mathscr{P} + z_i\overline{z}'_j\mathscr{P},\\
&\mathscr{K}[\overline{z}'_i,z_j]\mathscr{P} = (\overline{z}'_i\mathscr{P})\circ(z_j\mathscr{P}) = \mathscr{P}\circ(\overline{z}_i z_j\mathscr{P}) = \frac{2}{a_j}\delta_{ij}\mathscr{P} + \overline{z}'_i z_j\mathscr{P}.
\end{aligned} \tag{2.16}$$

Thus, we get:

$$\begin{aligned}
&\mathscr{K}[1,\overline{z}_j] = \overline{z}'_j, \qquad \mathscr{K}[1,z_j] = z_j,\\
&\mathscr{K}[z_i,\overline{z}_j] = z_i\overline{z}'_j, \qquad \mathscr{K}[\overline{z}_i,z_j] = \overline{z}_i z_j,\\
&\mathscr{K}[\overline{z}'_i,z_j] = \mathscr{K}[z'_j,\overline{z}_i] = \frac{2}{a_j}\delta_{ij} + \overline{z}'_i z_j.
\end{aligned} \tag{2.17}$$

Notation 2.4 To simplify our calculations, we introduce the following notation. For any polynomial $F \in \mathbb{C}[Z,Z']$ we denote by $(F\mathscr{P})_p$ the operator defined by the kernel $p^n(F\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')$, that is,

$$((F\mathscr{P})_p\varphi)(Z) = \int_{\mathbb{R}^{2n}} p^n(F\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')\varphi(Z')dZ', \quad \text{for any } \varphi \in L^2(\mathbb{R}^{2n}). \tag{2.18}$$

Let $F, G \in \mathbb{C}[Z,Z']$. By a change of variables we obtain

$$((F\mathscr{P})_p\circ(G\mathscr{P})_p)(Z,Z') = p^n((F\mathscr{P})\circ(G\mathscr{P}))(\sqrt{p}Z,\sqrt{p}Z'). \tag{2.19}$$

3 Bergman Kernels on Symplectic Manifolds

This section is organized as follows. We recall the definition of the spin^c Dirac operator in Sect. 3.1, and in Sect. 3.2, we explain the asymptotic expansion of the Bergman kernel.

3.1 The spin^c Dirac Operator

Let X be a compact manifold of real dimension $2n$ with almost complex structure J. Let g^{TX} be a Riemannian metric on X compatible with J, i.e., $g^{TX}(J\cdot, J\cdot) = g^{TX}(\cdot,\cdot)$.

The almost complex structure J induces a splitting of the complexification of the tangent bundle, $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $P^{(1,0)} = \frac{1}{2}(1-\sqrt{-1}J)$ and $P^{(0,1)}$ be the natural projections from $TX \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)}X$ and $T^{(0,1)}X$. Accordingly, we have a decomposition of the complexified cotangent bundle: $T^*X \otimes_{\mathbb{R}} \mathbb{C} = T^{*(1,0)}X \oplus T^{*(0,1)}X$. The exterior algebra bundle decomposes as $\Lambda(T^*X)\otimes_{\mathbb{R}}\mathbb{C} = \oplus_{p,q}\Lambda^{p,q}(T^*X)$, where $\Lambda^{p,q}(T^*X) := \Lambda^p(T^{*(1,0)}X)\otimes\Lambda^q(T^{*(0,1)}X)$.

Let ∇^{TX} be the Levi–Civita connection of (TX, g^{TX}) with associated curvature R^{TX}. Let $\nabla^X J \in T^*X \otimes \mathrm{End}(TX)$ be the covariant derivative of J induced by ∇^{TX}. Set

$$\begin{aligned} \nabla^{T^{(1,0)}X} &= P^{(1,0)}\nabla^{TX}P^{(1,0)}, & \nabla^{T^{(0,1)}X} &= P^{(0,1)}\nabla^{TX}P^{(0,1)},\\ {}^0\nabla^{TX} &= \nabla^{T^{(1,0)}X}\oplus\nabla^{T^{(0,1)}X}, & A_2 &= \nabla^{TX} - {}^0\nabla^{TX}. \end{aligned} \tag{3.1}$$

Then $\nabla^{T^{(1,0)}X}$ and $\nabla^{T^{(0,1)}X}$ are the canonical Hermitian connections on $T^{(1,0)}X$ and $T^{(0,1)}X$ respectively with curvatures $R^{T^{(1,0)}X}$ and $R^{T^{(0,1)}X}$. Moreover, ${}^0\nabla^{TX}$ is an Euclidean connection on TX. The tensor $A_2 \in T^*X\otimes\mathrm{End}(TX)$ satisfies

$$A_2 = \frac{1}{2}J(\nabla^X J), \qquad JA_2 = -A_2J. \tag{3.2}$$

For any $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then

$$\mathbf{c}(v) = \sqrt{2}(\overline{v}^*_{1,0}\wedge - i_{v_{0,1}}) \tag{3.3}$$

defines the Clifford action of v on $\Lambda^{0,\bullet} = \Lambda^{\mathrm{even}}(T^{*(0,1)}X)\oplus\Lambda^{\mathrm{odd}}(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product respectively.

The connection $\nabla^{T^{(1,0)}X}$ on $T^{(1,0)}X$ induces naturally a Hermitian connection $\nabla^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet} = \Lambda^\bullet(T^{*(0,1)}X)$ which preserves the natural $\mathbb{Z}$-grading on $\Lambda^{0,\bullet}$. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$. Let $\{w^j\}_{j=1}^n$ be the dual frame of $\{w_j\}_{j=1}^n$. Then

$$e_{2j-1} = \tfrac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad\text{and}\quad e_{2j} = \tfrac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j), \quad j = 1,\dots,n, \tag{3.4}$$

form an orthonormal frame of TX. Set

$$\begin{aligned}\mathbf{c}(A_2) &= \frac{1}{4}\sum_{i,j}\langle A_2 e_i, e_j\rangle \mathbf{c}(e_i)\mathbf{c}(e_j)\\ &= \frac{1}{2}\sum_{l,m}(\langle A_2 w_l, w_m\rangle i_{\overline{w}_l} i_{\overline{w}_m} + \langle A_2\overline{w}_l, \overline{w}_m\rangle \overline{w}^l\wedge\overline{w}^m\wedge), \qquad (3.5)\\ \nabla^{\mathrm{Cliff}} &= \nabla^{\Lambda^{0,\bullet}} + \mathbf{c}(A_2).\end{aligned}$$

The connection ∇^{Cliff} is the Clifford connection on $\Lambda^{0,\bullet}$ induced canonically by ∇^{TX} (cf. [27, §2]). (Note that in the definition of the Clifford connection in [27, (2.3)], one should add the term " $+\frac{1}{2}\operatorname{Tr}|_{T^{(0,1)}X}\Gamma$ " in the right-hand side of the first line, and the second line should read " $= d + \sum_{lm}\{\langle\Gamma w_l, \overline{w}_m\rangle\overline{w}^m\wedge i_{\overline{w}_l} +$ ".)

Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and curvature R^E. Let (L, h^L) be a Hermitian line bundle over X endowed with a Hermitian connection ∇^L with curvature $R^L = (\nabla^L)^2$. We assume that (L, ∇^L) satisfies the *prequantization condition*, that is

$$\omega(\cdot, J\cdot) > 0, \qquad \omega(J\cdot, J\cdot) = \omega(\cdot,\cdot), \quad \text{where } \omega := \frac{\sqrt{-1}}{2\pi}R^L. \qquad (3.6)$$

This implies in particular that ω is a symplectic form on X.

We relate g^{TX} with ω by means of the skew–adjoint linear map $\boldsymbol{J}: TX \longrightarrow TX$ which satisfies the relation

$$\omega(u, v) = g^{TX}(\boldsymbol{J}u, v) \quad \text{for } u, v \in TX. \qquad (3.7)$$

Then J commutes with $\boldsymbol{J}$, and $J = \boldsymbol{J}(-\boldsymbol{J}^2)^{-\frac{1}{2}}$.

We denote

$$E_p := \Lambda^{0,\bullet}\otimes L^p \otimes E. \qquad (3.8)$$

Along the fibers of E_p, we consider the pointwise Hermitian product $\langle\cdot,\cdot\rangle$ induced by g^{TX}, h^L and h^E. Let dv_X be the Riemannian volume form of (TX, g^{TX}). The L^2-Hermitian product on the space $\Omega^{0,\bullet}(X, L^p\otimes E)$ of smooth sections of E_p is given by

$$\langle s_1, s_2\rangle = \int_X \langle s_1(x), s_2(x)\rangle dv_X(x). \qquad (3.9)$$

We denote the corresponding norm with $\|\cdot\|_{L^2}$ and with $L^2(X, E_p)$ the completion of $\Omega^{0,\bullet}(X, L^p\otimes E)$ with respect to this norm.

Let $\nabla^{L^p\otimes E}$ be the connection on $L^p\otimes E$ induced by ∇^L and ∇^E. Let ∇^{E_p} be the connection on E_p induced by ∇^{Cliff}, $\nabla^{L^p\otimes E}$:

$$\nabla^{E_p} = \nabla^{\mathrm{Cliff}}\otimes \mathrm{Id} + \mathrm{Id}\otimes\nabla^{L^p\otimes E}. \qquad (3.10)$$

Definition 3.1 The *spin^c Dirac operator* D_p is defined by

$$D_p = \sum_{j=1}^{2n} \mathbf{c}(e_j)\nabla^{E_p}_{e_j} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E). \tag{3.11}$$

D_p is a formally self-adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^p \otimes E)$, which interchanges $\Omega^{0,\mathrm{even}}(X, L^p \otimes E)$ and $\Omega^{0,\mathrm{odd}}(X, L^p \otimes E)$ (cf. [30, §1.3]).

Definition 3.2 The orthogonal projection

$$P_p : L^2(X, E_p) \longrightarrow \mathrm{Ker}(D_p) \tag{3.12}$$

is called the *Bergman projection*. Let π_1 and π_2 be the projections of $X \times X$ on the first and second factor. Since P_p is a smoothing operator, the Schwartz kernel theorem [36, p. 296], [30, Th. B.2.7] shows that the Schwartz kernel of P_p is smooth, i.e., there exists a section $P_p(\cdot,\cdot) \in \mathcal{C}^\infty(X \times X, \pi_1^*(E_p) \otimes \pi_2^*(E_p^*))$ such that for any $s \in L^2(X, E_p)$ we have

$$(P_p s)(x) = \int_X P_p(x, x')s(x')dv_X(x'). \tag{3.13}$$

The smooth kernel $P_p(\cdot,\cdot)$ is called the *Bergman kernel* of D_p. Observe that $P_p(x, x)$ is an element of $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$.

We wish to describe the kernel and spectrum of D_p in the sequel. For any operator A, we denote by $\mathrm{Spec}(A)$ the spectrum of A.

Recall that $\{w_i\}$ is an orthonormal frame of $(T^{(1,0)}X, g^{TX})$. Set

$$\begin{aligned} \omega_d &= -\sum_{l,m} R^L(w_l, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l}, \\ \tau(x) &= \sum_j R^L(w_j, \overline{w}_j) = -\pi \,\mathrm{Tr}|_{TX}[J\boldsymbol{J}], \\ \mu_0 &= \inf\{R^L_x(u, \overline{u})/|u|^2_{g^{TX}} : u \in T^{(1,0)}_x X,\ x \in X\} > 0. \end{aligned} \tag{3.14}$$

The following result was proved in [27, Theorems 1.1, 2.5] as an application of the Lichnerowicz formula [6, Theorem 3.52] (cf. also [30, Theorem 1.3.5]) for D_p^2.

Theorem 3.3 *There exists $C > 0$ such that for any $p \in \mathbb{N}$, $s \in \Omega^{0,>0}(X, L^p \otimes E) := \bigoplus_{k>0} \Omega^{0,k}(X, L^p \otimes E)$,*

$$\|D_p s\|^2_{L^2} \geqslant (2p\mu_0 - C)\|s\|^2_{L^2}. \tag{3.15}$$

Moreover,

$$\mathrm{Spec}(D_p^2) \subset \{0\} \cup [2p\mu_0 - C, +\infty[. \tag{3.16}$$

3.2 Off-Diagonal Asymptotic Expansion of Bergman Kernel

The existence of the spectral gap expressed in Theorem 3.3 allows us to *localize* the behavior of the Bergman kernel.

Let a^X be the injectivity radius of (X, g^{TX}). We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. Then the exponential map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ onto $B^X(x,\varepsilon)$ for $\varepsilon \leqslant a^X$. From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ via the exponential map for $\varepsilon \leqslant a^X$. Throughout what follows, ε runs in the fixed interval $]0, a^X/4[$.

Let $\mathbf{f}: \mathbb{R} \to [0,1]$ be a smooth even function such that $\mathbf{f}(v) = 1$ for $|v| \leqslant \varepsilon/2$, and $\mathbf{f}(v) = 0$ for $|v| \geqslant \varepsilon$. Set

$$F(a) = \left(\int_{-\infty}^{+\infty} \mathbf{f}(v)dv \right)^{-1} \int_{-\infty}^{+\infty} e^{iva}\mathbf{f}(v)dv. \tag{3.17}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$. By [20, Proposition 4.1], we have the *far off-diagonal* behavior of the Bergman kernel:

Proposition 3.4 *For any $l, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $C_{l,m,\varepsilon} > 0$ such that for any $p \geqslant 1$, $x, x' \in X$, the following estimate holds*:

$$|F(D_p)(x,x') - P_p(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m,\varepsilon}\, p^{-l}. \tag{3.18}$$

Especially, for $d(x,x') > \varepsilon$,

$$|P_p(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m,\varepsilon}\, p^{-l}. \tag{3.19}$$

The $\mathscr{C}^m$ norm in (3.18) and (3.19) is induced by ∇^L, ∇^E, h^L, h^E and g^{TX}.

We consider the orthogonal projection:

$$I_{\mathbb{C}\otimes E} : \boldsymbol{E} := \Lambda(T^{*(0,1)}X) \otimes E \longrightarrow \mathbb{C} \otimes E. \tag{3.20}$$

Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Let $\nabla^{\mathrm{End}(\boldsymbol{E})}$ be the connection on $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)$ induced by ∇^{Cliff} and ∇^E.

Let us elaborate on the identifications we use in the sequel, which we state as a Lemma.

Lemma 3.5 *Let $x_0 \in X$ be fixed and consider the diffeomorphism $B^{T_{x_0}X}(0,4\varepsilon) \ni Z \to \exp_{x_0}^X(Z) \in B^X(x_0,4\varepsilon)$. We denote the pull-back of the vector bundles L, E and E_p via this diffeomorphism by the same symbols.*

(i) *There exist trivializations of L, E and E_p over $B^{T_{x_0}X}(0,4\varepsilon)$ given by unit frames which are parallel with respect to ∇^L, ∇^E and ∇^{E_p} along the curves $\gamma_Z : [0,1] \to B^{T_{x_0}X}(0,4\varepsilon)$ defined for every $Z \in B^{T_{x_0}X}(0,4\varepsilon)$ by $\gamma_Z(u) = \exp_{x_0}^X(uZ)$.*

(ii) *With the previous trivializations,* $P_p(x,x')$ *induces a smooth section* $B^{T_{x_0}X}(0,4\varepsilon) \ni Z, Z' \mapsto P_{p,x_0}(Z,Z')$ *of* $\pi^*(\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E))$ *over* $TX \times_X TX$*, which depends smoothly on* x_0.

(iii) $\nabla^{\mathrm{End}(\boldsymbol{E})}$ *induces naturally a* $\mathcal{C}^m$*-norm with respect to the parameter* $x_0 \in X$.

(iv) *If* dv_{TX} *is the Riemannian volume form on* $(T_{x_0}X, g^{T_{x_0}X})$*, there exists a smooth positive function* $\kappa_{x_0} : T_{x_0}X \to \mathbb{R}$, $Z \mapsto \kappa_{x_0}(Z)$ *defined by*

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1, \tag{3.21}$$

where the subscript x_0 *of* $\kappa_{x_0}(Z)$ *indicates the base point* $x_0 \in X$.

(v) *By* (3.7), $\boldsymbol{J}$ *is an element of* $\mathrm{End}(T^{(1,0)}X)$*. Consequently, we can diagonalize* $\boldsymbol{J}_{x_0}$*, i.e., choose an orthonormal basis* $\{w_j\}_{j=1}^n$ *of* $T_{x_0}^{(1,0)}X$ *such that*

$$\boldsymbol{J}_{x_0}\omega_j = \frac{\sqrt{-1}}{2\pi} a_j(x_0) w_j, \quad \textit{for all } j = 1, 2, \dots, n, \tag{3.22}$$

where $0 < a_1(x_0) \leqslant a_2(x_0) \leqslant \cdots \leqslant a_n(x_0)$*. Then* $\{e_j\}_{j=1}^{2n}$ *defined in* (3.4) *forms an orthonormal basis of* $T_{x_0}X$*. The diffeomorphism*

$$\mathbb{R}^{2n} \ni (Z_1, \dots, Z_{2n}) \longmapsto \sum_i Z_i e_i \in T_{x_0}X \tag{3.23}$$

induces coordinates on $T_{x_0}X$*, which we use throughout the article. In these coordinates we have* $e_j = \partial/\partial Z_j$.

Let ∇_U denote the ordinary differentiation operator on $T_{x_0}X$ in the direction U. We introduce the *model operator* $\mathscr{L}$ on $T_{x_0}X \simeq \mathbb{R}^{2n}$ by setting

$$\begin{aligned} \nabla_{0,U} &:= \nabla_U + \frac{1}{2} R^L_{x_0}(Z,U), \quad \text{for } U \in T_{x_0}X, \\ \mathscr{L} &:= -\sum_j (\nabla_{0,e_j})^2 - \tau(x_0). \end{aligned} \tag{3.24}$$

By (3.14) and (3.22), $\tau(x_0) = \sum_j a_j(x_0)$. The operator $\mathscr{L}$ defined in (3.24) coincides with the operator $\mathscr{L}$ given by (2.1) and (2.2), with $a_j = a_j(x_0)$ for $1 \leqslant j \leqslant n$.

We denote by $\det_{\mathbb{C}}$ for the determinant function on the complex bundle $T^{(1,0)}X$ and set $|\boldsymbol{J}_{x_0}| = (-\boldsymbol{J}_{x_0}^2)^{1/2}$. The Bergman kernel $T_{x_0}X \ni Z, Z' \mapsto \mathscr{P}(Z,Z')$ of $\mathscr{L}$ has the following form in view of (2.6):

$$\begin{aligned} \mathscr{P}(Z,Z') &= \det\nolimits_{\mathbb{C}}(|\boldsymbol{J}_{x_0}|) \\ &\quad \times \exp\left(-\frac{\pi}{2}\big\langle |\boldsymbol{J}_{x_0}|(Z-Z'), (Z-Z')\big\rangle - \pi\sqrt{-1}\big\langle \boldsymbol{J}_{x_0}Z, Z'\big\rangle \right). \end{aligned} \tag{3.25}$$

By [20, Theorem 4.18′] we have the *off diagonal expansion* of the Bergman kernel:

Theorem 3.6 *Let* $\varepsilon \in]0, a^X/4[$*. For every* $x_0 \in X$ *and* $r \in \mathbb{N}$ *there exist polynomials* $J_{r,x_0}(Z,Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$*, in* Z, Z' *with the same parity as* r *and with*

$\deg J_{r,x_0} \leqslant 3r$, *whose coefficients are polynomials in* R^{TX}, $R^{T^{(1,0)}X}$, R^E (*and* R^L) *and their derivatives of order* $\leqslant r-1$ (*resp.* $\leqslant r$) *and reciprocals of linear combinations of eigenvalues of* $\boldsymbol{J}$ *at* x_0, *such that by setting*

$$\mathcal{P}^{(r)}_{x_0}(Z,Z') = J_{r,x_0}(Z,Z')\mathcal{P}(Z,Z'), \quad J_{0,x_0}(Z,Z') = I_{\mathbb{C}\otimes E}, \tag{3.26}$$

the following statement holds: *There exists* $C''>0$ *such that for every* $k,m,m'\in\mathbb{N}$, *there exist* $N\in\mathbb{N}$ *and* $C>0$ *such that the following estimate holds*

$$\begin{aligned}&\bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\bigg(\frac{1}{p^n}P_p(Z,Z')\\ &\qquad -\sum_{r=0}^{k}\mathcal{P}^{(r)}(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-1/2}(Z)\kappa^{-1/2}(Z')p^{-r/2}\bigg)\bigg|_{\mathcal{C}^{m'}(X)}\\ &\quad\leqslant Cp^{-(k+1-m)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\\ &\qquad\times\exp(-\sqrt{C''\mu_0}\sqrt{p}|Z-Z'|)+\mathcal{O}(p^{-\infty}),\end{aligned} \tag{3.27}$$

for any $\alpha,\alpha'\in\mathbb{N}^n$, *with* $|\alpha|+|\alpha'|\leqslant m$, *any* $Z,Z'\in T_{x_0}X$ *with* $|Z|,|Z'|\leqslant\varepsilon$ *and any* $x_0\in X$, $p\geqslant 1$.

Here $\mathcal{C}^{m'}(X)$ is the $\mathcal{C}^{m'}$-norm for the parameter $x_0\in X$. We say that a term $G_p=\mathcal{O}(p^{-\infty})$ if for any $l,l_1\in\mathbb{N}$, there exists $C_{l,l_1}>0$ such that the $\mathcal{C}^{l_1}$-norm of G_p is dominated by $C_{l,l_1}p^{-l}$.

Remark 3.7 Set $\boldsymbol{E}^+ := \oplus_j\Lambda^{2j}(T^{*(0,1)}X)\otimes E$ and $\boldsymbol{E}^- := \oplus_j\Lambda^{2j+1}(T^{*(0,1)}X)\otimes E$; $E_p^- := \boldsymbol{E}^-\otimes L^p$ and $E_p^+ := \boldsymbol{E}^+\otimes L^p$. By Theorem 3.3 and because D_p^2 preserves the $\mathbb{Z}_2$-grading of $\Omega^{0,\bullet}(X,L^p\otimes E)$, P_p is the orthogonal projection from $\mathcal{C}^\infty(X,E_p^+)$ onto $\operatorname{Ker}(D_p)$ for p large enough. Thus, $P_p(x,x)\in\operatorname{End}(\boldsymbol{E}^+)_x$ and $J_r(Z,Z')\in\operatorname{End}(\boldsymbol{E}^+)_{x_0}$ for p large enough.

Let $\nabla^X\boldsymbol{J}\in T^*X\otimes\operatorname{End}(TX)$ be the covariant derivative of $\boldsymbol{J}$ induced by ∇^{TX}. We denote by $\mathcal{R}=\sum_i Z_ie_i=Z$ the radial vector field on $\mathbb{R}^{2n}$.

For $s\in\mathcal{C}^\infty(T_{x_0}X,\boldsymbol{E}_{x_0})$, set

$$\|s\|_{0,0}^2=\int_{\mathbb{R}^{2n}}|s(Z)|^2_{h^{\Lambda(T^{*(0,1)}X)\otimes E}_{x_0}}dv_{TX}(Z). \tag{3.28}$$

We adopt the convention that all tensors will be evaluated at the base point $x_0\in X$, and most of the time, we will omit the subscript x_0. From (3.14) and (3.24), let us set

$$\begin{aligned}\mathcal{L}_2^0 &:= \mathcal{L}-2\omega_d=\sum_j(b_jb_j^+ + 2a_j\overline{w}^j\wedge i_{\overline{w}_j}),\\ \mathcal{O}_1 &:= -\frac{2}{3}\partial_j(R^L(e_k,e_i))_{x_0}Z_jZ_k\nabla_{0,e_i}-\frac{1}{3}\partial_i(R^L(e_j,e_i))_{x_0}Z_j\\ &\quad -\pi\sqrt{-1}\Big\langle(\nabla^X_{\mathcal{R}}\boldsymbol{J})_{x_0}e_l,e_m\Big\rangle c(e_l)c(e_m).\end{aligned} \tag{3.29}$$

Let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n}, \boldsymbol{E}_{x_0}), \|\cdot\|_{0,0})$ onto $N = \mathrm{Ker}(\mathscr{L}_2^0)$, and $P^N(Z, Z')$ its smooth kernel with respect to $dv_{TX}(Z')$. Let $P^{N^\perp} = \mathrm{Id} - P^N$. Since $a_j > 0$ we get from (3.29) that

$$P^N(Z, Z') = \mathscr{P}(Z, Z') I_{\mathbb{C}\otimes E}. \tag{3.30}$$

By [20, Theorem 4.6, (4.107), (4.115) and (4.117)] (or proceeding as in [28, (1.111)], or by [29, Theorem 2.2]), we obtain:

Theorem 3.8 *The following identity holds*:

$$\mathscr{P}_{x_0}^{(1)} = -P^{N^\perp}(\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N - P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}. \tag{3.31}$$

Remark 3.9 It is interesting to observe the role of $\mathcal{O}_1$ in different geometric situations. Firstly, if (X, J, ω) is Kähler, $\boldsymbol{J} = J$ and L, E are holomorphic vector bundles, we have $\mathcal{O}_1 = 0$. Secondly, if (X, J, ω) is symplectic and E is trivial, we do not need the precise formula of $\mathcal{O}_1$ for the proof of Lemma 4.7, but just the information that $\mathcal{O}_1$ acts as the identity on E. Thirdly, to compute the coefficient $J_{2,x_0}(0,0)$ in (3.26) as in [29, Theorem 2.1], we need certainly the precise formula of $\mathcal{O}_1$ given in (3.29).

Finally, for the proof of Theorem 1.1, the precise formulas for $\mathcal{O}_1$ or $\mathscr{P}_{x_0}^{(1)}$ are not needed (cf. Remark 4.8 and formulas (4.86), (4.87)).

4 Berezin-Toeplitz Quantization on Symplectic Manifolds

We give a brief summary of this section. We begin in Sect. 4.1 by establishing the asymptotic expansion for the kernel of Toeplitz operators. In Sect. 4.2, we show that the asymptotic expansion is also a sufficient condition for a family of operators to be Toeplitz. Finally, in Sect. 4.3, we conclude that set of Toeplitz operators forms an algebra.

4.1 Asymptotic Expansion of Toeplitz Operators

In this section, we define the Toeplitz operators and deduce the asymptotic expansion of their Schwartz kernels.

We use the same setting and notations as in Sect. 3. Let (X, J, ω) be a compact symplectic manifold of real dimension $2n$, a Hermitian line bundle (L, h^L) over X endowed with a Hermitian connection ∇^L with curvature $R^L = (\nabla^L)^2$ satisfying the prequantization condition (3.6). Let g^{TX} be an arbitrary Riemannian metric on X compatible with the almost complex structure J. We consider a Hermitian vector bundle (E, h^E) on X with Hermitian connection ∇^E, and the space $(L^2(X, E_p), \langle\cdot,\cdot\rangle)$ introduced in (3.9).

A section $g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ defines a vector bundle morphism $\mathrm{Id}_{\Lambda(T^{*(0,1)}X)\otimes L^p} \otimes g$ of $E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$, which we still denote by g.

Definition 4.1 A *Toeplitz operator* is a sequence $\{T_p\} = \{T_p\}_{p\in\mathbb{N}}$ of linear operators

$$T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p), \tag{4.1}$$

with the properties:

(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_p T_p P_p. \tag{4.2}$$

(ii) There exist a sequence $g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that for all $k \geqslant 0$ there exists $C_k > 0$ with

$$\left\| T_p - P_p\left(\sum_{l=0}^{k} p^{-l} g_l\right) P_p \right\| \leqslant C_k p^{-k-1}, \tag{4.3}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators.

The full symbol of $\{T_p\}$ is the formal series $\sum_{l=0}^{\infty} \hbar^l g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))[[\hbar]]$ and the *principal symbol* of $\{T_p\}$ is g_0. If each T_p is self-adjoint, $\{T_p\}$ is called self-adjoint.

We express (4.3) symbolically by

$$T_p = P_p\left(\sum_{l=0}^{k} p^{-l} g_l\right) P_p + \mathscr{O}(p^{-k-1}). \tag{4.4}$$

If (4.3) holds for any $k \in \mathbb{N}$, then we write

$$T_p = P_p\left(\sum_{l=0}^{\infty} p^{-l} g_l\right) P_p + \mathscr{O}(p^{-\infty}). \tag{4.5}$$

An important particular case is when $g_l = 0$ for $l \geqslant 1$. We set $g_0 = f$. We denote then

$$T_{f,p} : L^2(X, E_p) \longrightarrow L^2(X, E_p), \quad T_{f,p} = P_p f P_p. \tag{4.6}$$

The Schwartz kernel of $T_{f,p}$ is given by

$$T_{f,p}(x, x') = \int_X P_p(x, x'') f(x'') P_p(x'', x') dv_X(x''). \tag{4.7}$$

Let us remark that if $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ is self-adjoint, i.e., $f(x) = f(x)^*$ for all $x \in X$, then the operators $\mathrm{Id}_{\Lambda(T^{*(0,1)}X)\otimes L^p} \otimes f$ and $T_{f,p}$ are self-adjoint.

The map which associates to a section $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ the bounded operator $T_{f,p}$ on $L^2(X, E_p)$ is called the *Berezin-Toeplitz quantization*.

We examine now the asymptotic expansion of the kernel of the Toeplitz operators $T_{f,p}$. The first observation is that outside the diagonal of $X \times X$, the kernel of $T_{f,p}$ has the growth $\mathscr{O}(p^{-\infty})$.

Lemma 4.2 *For every $\varepsilon > 0$ and every $l, m \in \mathbb{N}$, there exists $C_{l,m,\varepsilon} > 0$ such that*

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m,\varepsilon}\, p^{-l} \tag{4.8}$$

for all $p \geqslant 1$ and all $(x,x') \in X \times X$ with $d(x,x') > \varepsilon$, where the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E and h^L, h^E, g^{TX}.

Proof Due to (3.19), (4.8) holds if we replace $T_{f,p}$ by P_p. Moreover, from (3.27), for any $m \in \mathbb{N}$, there exist $C_m > 0$, $M_m > 0$ such that $|P_p(x,x')|_{\mathscr{C}^m(X\times X)} < Cp^{M_m}$ for all $(x,x') \in X \times X$. These two facts and formula (4.7) imply the lemma. □

We concentrate next on a neighborhood of the diagonal in order to obtain the asymptotic expansion of the kernel $T_{f,p}(x,x')$.

We adhere to the identifications made in Lemma 3.5. We also identify in the sequel $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ with a family $f_{x_0}(Z) \in \mathrm{End}(E_{x_0})$ (with parameter $x_0 \in X$) of functions in Z in normal coordinates near x_0. In general, for functions in the normal coordinates, we will add a subscript x_0 to indicate the base point $x_0 \in X$.

Let $\{\varXi_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\varXi_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)$ with smooth kernel $\varXi_p(x,y)$ with respect to $dv_X(y)$.

Recall that $\pi : TX \times_X TX \to X$ is the natural projection from the fiberwise product of TX on X. Under our trivialization, $\varXi_p(x,y)$ induces a smooth section $\varXi_{p,x_0}(Z,Z')$ of $\pi^*(\mathrm{End}(\varLambda(T^{*(0,1)}X) \otimes E))$ over $TX \times_X TX$ with $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < a_X$. Recall also that $\mathscr{P}_{x_0} = \mathscr{P}$ was defined in (2.6).

Consider the following condition for $\{\varXi_p\}_{p\in\mathbb{N}}$.

Condition 4.3 Let $k \in \mathbb{N}$. There exists a family $\{Q_{r,x_0}\}_{0\leqslant r\leqslant k, x_0\in X}$ such that

(a) $Q_{r,x_0} \in \mathrm{End}(\varLambda(T^{*(0,1)}X) \otimes E)_{x_0}[Z,Z']$,
(b) $\{Q_{r,x_0}\}_{r\in\mathbb{N}, x_0\in X}$ is smooth with respect to the parameter $x_0 \in X$,
(c) there exist constants $\varepsilon' \in]0, a_X]$ and $C_0 > 0$ with the following property: for every $l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $M > 0$ such that for every $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$ and $p \in \mathbb{N}$ the following estimate holds (in the sense of (3.27)):

$$\begin{aligned}
&\left| p^{-n}\varXi_{p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} \right|_{\mathscr{C}^l(X)} \\
&\quad\leqslant C_{k,l}\, p^{-\frac{k+1}{2}}(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \\
&\qquad \times \exp(-\sqrt{C_0 p}|Z - Z'|) + \mathscr{O}(p^{-\infty}).
\end{aligned} \tag{4.9}$$

Notation 4.4 Assume that $\{\varXi_p\}_{p\in\mathbb{N}}$ is subject to the Condition 4.3. Then we write

$$p^{-n}\varXi_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathscr{O}(p^{-\frac{k+1}{2}}). \tag{4.10}$$

The family $\{J_{r,x_0}\}_{r\in\mathbb{N}, x_0\in X}$ of polynomials $J_{r,x_0}(Z,Z') \in \mathrm{End}(\varLambda(T^{*(0,1)}X) \otimes E)_{x_0}$ was defined in Theorem 3.6. Moreover, $J_{r,x_0}(Z,Z')$ have the same parity as

r, $\deg J_{r,x_0} \leqslant 3r$, and

$$J_{0,x_0} = I_{\mathbb{C}\otimes E}. \tag{4.11}$$

Lemma 4.5 *For any $k \in \mathbb{N}$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < 2\varepsilon$, we have*

$$p^{-n} P_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{4.12}$$

in the sense of Notation 4.4.

Proof Theorem 3.6 shows that for any $k, m' \in \mathbb{N}$, there exist $M \in \mathbb{N}$, $C > 0$ such that

$$\begin{aligned}
&\left| p^{-n} P_{p,x_0}(Z, Z')\kappa_{x_0}^{\frac{1}{2}}(Z)\kappa_{x_0}^{\frac{1}{2}}(Z') - \sum_{r=0}^{k} (J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} \right|_{\mathscr{C}^{m'}(X)} \\
&\quad \leqslant C p^{-(k+1)/2}(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \\
&\qquad \times \exp(-\sqrt{C''\mu_0}\sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}),
\end{aligned} \tag{4.13}$$

for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant 2\varepsilon$. Hence (4.13) immediately entails (4.12). □

Lemma 4.6 *Let $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$. There exists a family $\{Q_{r,x_0}(f)\}_{r\in\mathbb{N}, x_0\in X}$ such that*

(a) *$Q_{r,x_0}(f) \in \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}[Z, Z']$ are polynomials with the same parity as r,*
(b) *$\{Q_{r,x_0}(f)\}_{r\in\mathbb{N}, x_0\in X}$ is smooth with respect to $x_0 \in X$,*
(c) *for every $k \in \mathbb{N}$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/2$ we have*

$$p^{-n} T_{f,p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}(f)\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{4.14}$$

in the sense of Notation 4.4.

$Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\left[J_{r_1,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} J_{r_2,x_0} \right]. \tag{4.15}$$

Especially,

$$Q_{0,x_0}(f) = f(x_0) I_{\mathbb{C}\otimes E}. \tag{4.16}$$

We have used here the notations (2.10) and (3.20).

Proof From (4.7) and (4.8), we know that for $|Z|, |Z'| < \varepsilon/2$, $T_{f,p,x_0}(Z, Z')$ is determined up to terms of order $\mathscr{O}(p^{-\infty})$ by the behavior of f in $B^X(x_0, \varepsilon)$. Let

$\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\rho(v) = 1 \quad \text{if } |v| < 2; \qquad \rho(v) = 0 \quad \text{if } |v| > 4. \tag{4.17}$$

For $|Z|, |Z'| < \varepsilon/2$, we get

$$\begin{aligned} T_{f,p,x_0}(Z, Z') = \int_{T_{x_0}X} & P_{p,x_0}(Z, Z'')\rho(2|Z''|/\varepsilon) f_{x_0}(Z'') P_{p,x_0}(Z'', Z') \\ & \times \kappa_{x_0}(Z'') dv_{TX}(Z'') + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{4.18}$$

We consider the Taylor expansion of f_{x_0}:

$$\begin{aligned} f_{x_0}(Z) &= \sum_{|\alpha| \leqslant k} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0) \frac{Z^\alpha}{\alpha!} + \mathscr{O}(|Z|^{k+1}) \\ &= \sum_{|\alpha| \leqslant k} p^{-|\alpha|/2} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0) \frac{(\sqrt{p} Z)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}} \mathscr{O}(|\sqrt{p} Z|^{k+1}). \end{aligned} \tag{4.19}$$

We multiply now the expansions given in (4.19) and (4.13) and obtain the expansion of

$$\kappa_{x_0}^{1/2}(Z) P_{p,x_0}(Z, Z'')(\kappa_{x_0} f_{x_0})(Z'') P_{p,x_0}(Z'', Z') \kappa_{x_0}^{1/2}(Z')$$

which we substitute in (4.18). We integrate then on $T_{x_0}X$ by using the change of variable $\sqrt{p} Z'' = W$ and conclude (4.14) and (4.15) by using formulas (2.10) and (2.19). From (4.11) and (4.15), we get

$$Q_{0,x_0}(f) = \mathscr{K}[1, f_{x_0}(0)] I_{\mathbb{C}\otimes E} = f_{x_0}(0) I_{\mathbb{C}\otimes E} = f(x_0) I_{\mathbb{C}\otimes E}. \tag{4.20}$$

The proof of Lemma 4.6 is complete. □

As an example, we compute $Q_{1,x_0}(f)$.

Lemma 4.7 *$Q_{1,x_0}(f)$ appearing in (4.14) is given by*

$$Q_{1,x_0}(f) = f(x_0) J_{1,x_0} + \mathscr{K}\left[J_{0,x_0}, \frac{\partial f_{x_0}}{\partial Z_j}(0) Z_j J_{0,x_0} \right]. \tag{4.21}$$

Proof At first, by taking $f = 1$ in (4.15), we get

$$J_{1,x_0} = \mathscr{K}[J_{0,x_0}, J_{1,x_0}] + \mathscr{K}[J_{1,x_0}, J_{0,x_0}]. \tag{4.22}$$

The operator $\mathscr{O}_1$ defined in (3.29) (considered as a differential operator with coefficients in $\mathrm{End}(\Lambda(T^{*(1,0)}X) \otimes E)_{x_0}$) acts as the identity on the E-component. Thus, from (3.31) and (2.10), we obtain

$$\mathscr{K}[J_{1,x_0}, f(x_0) J_{0,x_0}] = f(x_0) \mathscr{K}[J_{1,x_0}, J_{0,x_0}]. \tag{4.23}$$

From (4.15), (4.22) and (4.23), we get (4.21). □

Remark 4.8 If $f \in \mathcal{C}^\infty(X)$ we get (4.23), thus also (4.21), without using the precise formulas of $\mathcal{O}_1$ or J_1.

4.2 A Criterion for Toeplitz Operators

We will prove next a useful criterion which ensures that a given family is a Toeplitz operator.

Theorem 4.9 *Let* $\{T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)\}$ *be a family of bounded linear operators which satisfies the following three conditions*:

(i) *For any* $p \in \mathbb{N}$, $P_p T_p P_p = T_p$.
(ii) *For any* $\varepsilon_0 > 0$ *and any* $l \in \mathbb{N}$, *there exists* $C_{l,\varepsilon_0} > 0$ *such that for all* $p \geqslant 1$ *and all* $(x, x') \in X \times X$ *with* $d(x, x') > \varepsilon_0$,

$$|T_p(x, x')| \leqslant C_{l,\varepsilon_0} p^{-l}. \tag{4.24}$$

(iii) *There exists a family of polynomials* $\{\mathcal{Q}_{r,x_0} \in \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}[Z, Z']\}_{x_0 \in X}$ *such that*:
  (a) *each* $\mathcal{Q}_{r,x_0}$ *has the same parity as* r,
  (b) *the family is smooth in* $x_0 \in X$ *and*
  (c) *there exists* $0 < \varepsilon' < a_X/4$ *such that for every* $x_0 \in X$, *every* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| < \varepsilon'$ *and every* $k \in \mathbb{N}$ *we have*

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{4.25}$$

  in the sense of (4.10) *and* (4.9).

Then $\{T_p\}$ *is a Toeplitz operator.*

Remark 4.10 By Lemmas 4.2 and 4.6, and by (4.2), (4.3) and the Sobolev inequality (cf. [20, (4.14)]), it follows that every Toeplitz operator in the sense of Definition 4.1 verifies the conditions (i), (ii), (iii) of Theorem 4.9.

We start the proof of Theorem 4.9. Let T_p^* be the adjoint of T_p. By writing

$$T_p = \frac{1}{2}(T_p + T_p^*) + \sqrt{-1}\frac{1}{2\sqrt{-1}}(T_p - T_p^*), \tag{4.26}$$

we may and will assume from now on that T_p is self-adjoint.

We will define inductively the sequence $(g_l)_{l \geqslant 0}$, $g_l \in \mathcal{C}^\infty(X, \operatorname{End}(E))$ such that

$$T_p = \sum_{l=0}^{m} P_p g_l p^{-l} P_p + \mathcal{O}(p^{-m-1}), \quad \text{for every } m \geqslant 0. \tag{4.27}$$

Moreover, we can make these g_l's to be self-adjoint.

Let us start with the case $m=0$ of (4.27). For an arbitrary but fixed $x_0 \in X$, we set

$$g_0(x_0) = \mathcal{Q}_{0,x_0}(0,0)|_{\mathbb{C}\otimes E} \in \mathrm{End}(E_{x_0}). \tag{4.28}$$

We will show that

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z,Z') \cong \mathcal{O}(p^{-1}), \tag{4.29}$$

which implies the case $m=0$ of (4.27), namely,

$$T_p = P_p g_0 P_p + \mathcal{O}(p^{-1}). \tag{4.30}$$

The proof of (4.29)–(4.30) will be done in Propositions 4.11 and 4.17.

Proposition 4.11 *In the conditions of Theorem 4.9 we have* $\mathcal{Q}_{0,x_0}(Z,Z') = \mathcal{Q}_{0,x_0}(0,0) \in \mathrm{End}(E_{x_0}) \circ I_{\mathbb{C}\otimes E}$ *for all* $x_0 \in X$ *and all* $Z, Z' \in T_{x_0}X$.

Proof The proof is divided in the series of Lemmas 4.12, 4.13, 4.14, 4.15 and 4.16. Our first observation is as follows.

Lemma 4.12 $\mathcal{Q}_{0,x_0} \in \mathrm{End}(E_{x_0}) \circ I_{\mathbb{C}\otimes E}[Z,Z']$, *and* $\mathcal{Q}_{0,x_0}$ *is a polynomial in* $z, \overline{z}'$.

Proof Indeed, by (4.25)

$$p^{-n}T_{p,x_0}(Z,Z') \cong (\mathcal{Q}_{0,x_0}\mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1/2}). \tag{4.31}$$

Moreover, by (4.11) and (4.12), we have

$$\begin{aligned} &p^{-n}(P_p T_p P_p)_{x_0}(Z,Z') \\ &\quad \cong ((\mathcal{P}J_0)\circ(\mathcal{Q}_0\mathcal{P})\circ(\mathcal{P}J_0))_{x_0}(\sqrt{p}Z,\sqrt{p}Z') + \mathcal{O}(p^{-1/2}). \end{aligned} \tag{4.32}$$

Since $P_pT_pP_p = T_p$, we deduce from (4.31), (4.11) and (4.32) that

$$\mathcal{Q}_{0,x_0}\mathcal{P}_{x_0} = I_{\mathbb{C}\otimes E}\mathcal{P}_{x_0} \circ (\mathcal{Q}_{0,x_0}\mathcal{P}_{x_0}) \circ \mathcal{P}_{x_0} I_{\mathbb{C}\otimes E}, \tag{4.33}$$

hence $\mathcal{Q}_{0,x_0} \in \mathrm{End}(E_{x_0}) \circ I_{\mathbb{C}\otimes E}[z,\overline{z}']$ by (2.8) and (2.12). □

For simplicity we denote in the rest of the proof $F_x = \mathcal{Q}_{0,x}|_{\mathbb{C}\otimes E} \in \mathrm{End}(E_x)$. Let $F_x = \sum_{i\geqslant 0} F_x^{(i)}$ be the decomposition of F_x in homogeneous polynomials $F_x^{(i)}$ of degree i. We will show that $F_x^{(i)}$ vanish identically for $i>0$, that is,

$$F_x^{(i)}(z,\overline{z}') = 0 \quad \text{for all } i>0 \text{ and } z,z' \in \mathbb{C}^n. \tag{4.34}$$

The first step is to prove

$$F_x^{(i)}(0,\overline{z}') = 0 \quad \text{for all } i>0 \text{ and all } z' \in \mathbb{C}^n. \tag{4.35}$$

Let us remark that since T_p are self-adjoint we have

$$F_x^{(i)}(z,\overline{z}') = (F_x^{(i)}(z',\overline{z}))^*. \tag{4.36}$$

Consider $\varepsilon' > 0$ as in hypothesis (iii) (c) of Theorem 4.9. For $Z' \in \mathbb{R}^{2n} \simeq T_x X$ with $|Z'| < \varepsilon'$ and $y = \exp_x^X(Z')$, set

$$\begin{aligned} F^{(i)}(x, y) &= F_x^{(i)}(0, \overline{z}') \in \mathrm{End}(E_x), \\ \widetilde{F}^{(i)}(x, y) &= (F^{(i)}(y, x))^* \in \mathrm{End}(E_y). \end{aligned} \tag{4.37}$$

$F^{(i)}$ and $\widetilde{F}^{(i)}$ define smooth sections on a neighborhood of the diagonal of $X \times X$. Clearly, the $\widetilde{F}^{(i)}(x, y)$'s need not be polynomials in z and $\overline{z}'$.

Since we wish to define global operators induced by these kernels, we use a cut-off function in the neighborhood of the diagonal. Pick a smooth function $\eta \in \mathcal{C}^\infty(\mathbb{R})$, such that $\eta(u) = 1$ for $|u| \leqslant \varepsilon'/2$ and $\eta(u) = 0$ for $|u| \geqslant \varepsilon'$.

We denote by $F^{(i)} P_p$ and $P_p \widetilde{F}^{(i)}$ the operators defined by the kernels

$$\eta(d(x, y)) F^{(i)}(x, y) P_p(x, y) \quad \text{and} \quad \eta(d(x, y)) P_p(x, y) \widetilde{F}^{(i)}(x, y)$$

with respect to $dv_X(y)$. Set

$$\mathcal{T}_p = T_p - \sum_{i \leqslant \deg F_x} (F^{(i)} P_p) p^{i/2}. \tag{4.38}$$

The operators $\mathcal{T}_p$ extend naturally to bounded operators on $L^2(X, E_p)$.

From (4.25) and (4.38) we deduce that for all $k \geqslant 1$ and $|Z'| \leqslant \varepsilon'$, we have the following expansion in the normal coordinates around $x_0 \in X$ (which has to be understood , in the sense of (4.10)):

$$p^{-n} \mathcal{T}_{p,x_0}(0, Z') \cong \sum_{r=1}^{k} (R_{r,x_0} \mathcal{P}_{x_0})(0, \sqrt{p} Z') p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{4.39}$$

for some polynomials R_{r,x_0} of the same parity as r. For simplicity let us define similarly to (4.37) the kernel

$$R_{r,p}(x, y) = p^n (R_{r,x} \mathcal{P}_x)(0, \sqrt{p} Z') \kappa_x^{-1/2}(Z') \eta(d(x, y)), \tag{4.40}$$

where $y = \exp_x^X(Z')$, and denote by $R_{r,p}$ the operator defined by this kernel.

Lemma 4.13 *There exists $C > 0$ such that for every $p > p_0$ and $s \in L^2(X, E_p)$ we have*

$$\|\mathcal{T}_p s\|_{L^2} \leqslant C p^{-1/2} \|s\|_{L^2}, \tag{4.41}$$

$$\|\mathcal{T}_p^* s\|_{L^2} \leqslant C p^{-1/2} \|s\|_{L^2}. \tag{4.42}$$

Proof In order to use (4.39) we write

$$\|\mathcal{T}_p s\|_{L^2} \leqslant \left\| (\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p}) s \right\|_{L^2} + \left\| \sum_{r=1}^{k} p^{-r/2} R_{r,p} s \right\|_{L^2}. \tag{4.43}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left\| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right) s \right\|_{L^2}^2 \\
&\quad \leqslant \int_X \left(\int_X \left| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right)(x,y) \right| dv_X(y) \right) \\
&\qquad \times \left(\int_X \left| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right)(x,y) \right| |s(y)|^2 dv_X(y) \right) dv_X(x).
\end{aligned} \tag{4.44}$$

We split then the inner integrals into integrals over $B^X(x,\varepsilon')$ and $X \smallsetminus B^X(x,\varepsilon')$ and use the fact that the kernel of $\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p}$ has the growth $\mathscr{O}(p^{-\infty})$ outside the diagonal. Indeed, this follows by (4.24), the definition of the operators $F^{(i)} P_p$ in (4.38) (using the cut-off function η), and the definition (4.40) of $R_{r,p}$ (which involves $\mathscr{P}$). We get for example, uniformly in $x \in X$,

$$\begin{aligned}
&\int_X \left| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right)(x,y) \right| |s(y)|^2 dv_X(y) \\
&\quad = \int_{B^X(x,\varepsilon')} \left| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right)(x,y) \right| |s(y)|^2 dv_X(y) \\
&\qquad + \mathscr{O}(p^{-\infty}) \int_{X \smallsetminus B^X(x,\varepsilon')} |s(y)|^2 dv_X(y).
\end{aligned} \tag{4.45}$$

By (4.9) and (4.39) applied for k sufficiently large, which we fix from now on, we obtain

$$\begin{aligned}
&\int_{B^X(x,\varepsilon')} \left| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right)(x,y) \right| |s(y)|^2 dv_X(y) \\
&\quad = \mathscr{O}(p^{-1}) \int_{B^X(x,\varepsilon')} |s(y)|^2 dv_X(y).
\end{aligned} \tag{4.46}$$

In the same vein we obtain

$$\int_X \left| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right)(x,y) \right| dv_X(y) = \mathscr{O}(p^{-1}) + \mathscr{O}(p^{-\infty}). \tag{4.47}$$

Combining (4.44)–(4.47) we infer

$$\left\| \left(\mathcal{T}_p - \sum_{r=1}^{k} p^{-r/2} R_{r,p} \right) s \right\|_{L^2} \leqslant C p^{-1} \|s\|_{L^2}, \quad s \in L^2(X, E_p). \tag{4.48}$$

A similar proof as for (4.48) delivers for $s \in L^2(X, E_p)$

$$\left\| R_{r,p} s \right\|_{L^2} \leqslant C \|s\|_{L^2}, \tag{4.49}$$

which implies

$$\left\| \sum_{r=1}^{k} p^{-r/2} R_{r,p} s \right\|_{L^2} \leqslant C p^{-1/2} \|s\|_{L^2}, \quad \text{for } s \in L^2(X, E_p), \tag{4.50}$$

for some constant $C > 0$. Relations (4.48) and (4.50) entail (4.41), which is equivalent to (4.42), by taking the adjoint. □

Let us consider the Taylor development of $\widetilde{F}^{(i)}$ in normal coordinates around x with $y = \exp_x^X(Z')$:

$$\widetilde{F}^{(i)}(x, y) = \sum_{|\alpha| \leqslant k} \frac{\partial^\alpha \widetilde{F}^{(i)}}{\partial Z'^\alpha}(x, 0) \frac{(\sqrt{p} Z')^\alpha}{\alpha!} p^{-|\alpha|/2} + \mathcal{O}(|Z'|^{k+1}). \tag{4.51}$$

The next step in the proof of Proposition 4.11 is the following.

Lemma 4.14 *For every $j > 0$ we have*

$$\frac{\partial^\alpha \widetilde{F}^{(i)}}{\partial Z'^\alpha}(x, 0) = 0, \quad \textit{for } i - |\alpha| \geqslant j > 0. \tag{4.52}$$

Proof The definition (4.38) of $\mathcal{T}_p$ shows that

$$\mathcal{T}_p^* = T_p - \sum_{i \leqslant \deg F_x} p^{i/2} (P_p \widetilde{F}^{(i)}). \tag{4.53}$$

Let us develop the sum in the right-hand side. Combining the Taylor development (4.51) with the expansion (4.12) of the Bergman kernel we obtain:

$$\begin{aligned} p^{-n} \sum_i (P_p \widetilde{F}^{(i)})_{x_0}(0, Z') p^{i/2} &\\ \cong \sum_i \sum_{|\alpha|, r \leqslant k} \big(J_{r,x_0} \mathcal{P}_{x_0}\big)(0, \sqrt{p} Z') &\frac{\partial^\alpha \widetilde{F}^{(i)}}{\partial Z'^\alpha}(x_0, 0) \frac{(\sqrt{p} Z')^\alpha}{\alpha!} p^{(i-|\alpha|-r)/2} \\ + \mathcal{O}(p^{(\deg F - k - 1)/2}), & \end{aligned} \tag{4.54}$$

where $k \geqslant \deg F_x + 1$. Having in mind (4.42), this is only possible if for every $j > 0$ the coefficients of $p^{j/2}$ in the right-hand side of (4.54) vanish. Thus, we have for every $j > 0$:

$$\sum_{l=j}^{\deg F_x} \sum_{|\alpha| + r = l - j} J_{r,x_0}(0, \sqrt{p} Z') \frac{\partial^\alpha \widetilde{F}^{(l)}}{\partial Z'^\alpha}(x_0, 0) \frac{(\sqrt{p} Z')^\alpha}{\alpha!} = 0. \tag{4.55}$$

From (4.55), we will prove by recurrence that for any $j > 0$, (4.52) holds. As the first step of the recurrence let us take $j = \deg F_x$ in (4.55). Since $J_{0,x_0} = I_{\mathbb{C}\otimes E}$ (see (4.11)), we get immediately $\widetilde{F}^{(\deg F_x)}(x_0, 0) = 0$. Hence (4.52) holds for $j = \deg F_x$.

Assume that (4.52) holds for $j > j_0 > 0$. Then for $j = j_0$, the coefficient with $r > 0$ in (4.55) is zero. Since $J_{0,x_0} = I_{\mathbb{C}\otimes E}$, (4.55) reads

$$\sum_\alpha \frac{\partial^\alpha \widetilde{F}^{(j_0+|\alpha|)}}{\partial Z'^\alpha}(x_0, 0)\frac{(\sqrt{p}Z')^\alpha}{\alpha!} = 0, \tag{4.56}$$

which entails (4.52) for $j = j_0$. The proof of (4.52) is complete. □

Lemma 4.15 *For $i > 0$, we have*

$$\frac{\partial^\alpha F_x^{(i)}}{\partial \overline{z}'^\alpha}(0, 0) = 0, \quad |\alpha| \leqslant i. \tag{4.57}$$

Therefore $F_x^{(i)}(0, \overline{z}') = 0$ for all $i > 0$ and $z' \in \mathbb{C}^n$ i.e., (4.35) holds true. Moreover,

$$F_x^{(i)}(z, 0) = 0 \quad \textit{for all } i > 0 \textit{ and all } z \in \mathbb{C}^n. \tag{4.58}$$

Proof Let us start with some preliminary observations.

In view of (4.42), (4.52) and (4.54), a comparison the coefficient of p^0 in (4.31) and (4.53) yields

$$\widetilde{F}^{(i)}(x, Z') = F_x^{(i)}(0, \overline{z}') + \mathscr{O}(|Z'|^{i+1}). \tag{4.59}$$

Using the definition (4.37) of $\widetilde{F}^{(i)}(x, Z')$, and taking the adjoint of (4.59) we get

$$F^{(i)}(Z', x) = (F_x^{(i)}(0, \overline{z}'))^* + \mathscr{O}(|Z'|^{i+1}), \tag{4.60}$$

which implies

$$\frac{\partial^\alpha}{\partial z^\alpha} F^{(i)}(\cdot, x)|_x = \left(\left(\frac{\partial^\alpha}{\partial \overline{z}'^\alpha} F_x^{(i)}\right)(0, \overline{z}')\right)^*, \quad \text{for } |\alpha| \leqslant i, \tag{4.61}$$

so in order to prove the Lemma it suffices to show that

$$\frac{\partial^\alpha}{\partial z^\alpha} F^{(i)}(\cdot, x)|_x = 0, \quad \text{for } |\alpha| \leqslant i. \tag{4.62}$$

We prove this by induction over $|\alpha|$. For $|\alpha| = 0$, it is obvious that $F^{(i)}(0, x) = 0$, since $F^{(i)}(\cdot, x)$ is a homogeneous polynomial of degree $i > 0$. For the induction step let $j_X : X \to X \times X$ be the diagonal injection. By Lemma 4.12 and the definition (4.37) of $F^{(i)}(x, y)$,

$$\frac{\partial}{\partial z'_j} F^{(i)}(x, y) = 0, \quad \text{near } j_X(X), \tag{4.63}$$

where $y = \exp_x^X(Z')$. Assume now that $\alpha \in \mathbb{N}^n$ and (4.62) holds for $|\alpha| - 1$. Consider j with $\alpha_j > 0$ and set $\alpha' = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_n)$.

Taking the derivative of (4.37) and using the induction hypothesis and (4.63), we have

$$\frac{\partial^\alpha}{\partial z^\alpha} F^{(i)}(\cdot, x)\Big|_x = \frac{\partial}{\partial z_j} j_X^* \Big(\frac{\partial^{\alpha'}}{\partial z^{\alpha'}} F^{(i)} \Big)\Big|_x - \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial}{\partial z'_j} F^{(i)}(\cdot, \cdot)\Big|_{0,0} = 0. \tag{4.64}$$

Thus, (4.57) is proved. The identity (4.35) follows too, since it is equivalent to (4.57). Furthermore, (4.58) results from (4.35) and (4.36). This finishes the proof of Lemma 4.15. □

Lemma 4.16 *We have* $F_x^{(i)}(z, \overline{z}') = 0$ *for all* $i > 0$ *and* $z, z' \in \mathbb{C}^n$.

Proof Let us consider the operator

$$\frac{1}{\sqrt{p}} P_p (\nabla^{E_p}_{X,x} T_p) P_p \quad \text{with } X \in \mathscr{C}^\infty(X, TX),\ X(x_0) = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}. \tag{4.65}$$

The leading term of its asymptotic expansion (4.10) is

$$\Big(\frac{\partial}{\partial z_j} F_{x_0} \Big)(\sqrt{p} z, \sqrt{p}\overline{z}') \mathscr{P}_{x_0}(\sqrt{p} Z, \sqrt{p} Z'). \tag{4.66}$$

By (4.35) and (4.58), $(\frac{\partial}{\partial z_j} F_{x_0})(z, \overline{z}')$ is an odd polynomial in z,$\overline{z}'$ whose constant term vanishes. We reiterate the arguments from (4.38)–(4.61) by replacing the operator T_p with the operator (4.65); we get for $i > 0$,

$$\frac{\partial}{\partial z_j} F_x^{(i)}(0, \overline{z}') = 0. \tag{4.67}$$

By (4.36) and (4.67),

$$\frac{\partial}{\partial \overline{z}'_j} F_x^{(i)}(z, 0) = 0. \tag{4.68}$$

By continuing this process, we show that for all $i > 0, \alpha \in \mathbb{Z}^n$, $z, z' \in \mathbb{C}^n$,

$$\frac{\partial^\alpha}{\partial z^\alpha} F_x^{(i)}(0, \overline{z}') = \frac{\partial^\alpha}{\partial \overline{z}'^\alpha} F_x^{(i)}(z, 0) = 0. \tag{4.69}$$

Thus, the Lemma is proved and (4.34) holds true. □

The Lemma 4.16 finishes the proof of Proposition 4.11. □

We come now to the proof of the first induction step leading to (4.27).

Proposition 4.17 *We have* $p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \cong \mathscr{O}(p^{-1})$ (*in the sense of Notation* 4.4). *Consequently,* $T_p = P_p g_0 P_p + \mathscr{O}(p^{-1})$ (*i.e.,* *relation* (4.30)) *holds true in the sense of* (4.4) .

Proof Let us compare the asymptotic expansion of T_p and $T_{g_0,p} = P_p g_0 P_p$. Using the Notation 4.4, the expansion (4.14) (for $k = 1$) reads

$$p^{-n} T_{g_0,p,x_0}(Z, Z') \\ \cong (g_0(x_0) I_{\mathbb{C}\otimes E} \mathscr{P}_{x_0} + Q_{1,x_0}(g_0) \mathscr{P}_{x_0} p^{-1/2})(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1}), \quad (4.70)$$

since $\mathcal{Q}_{0,x_0}(g_0) = g_0(x_0) I_{\mathbb{C}\otimes E}$ by (4.16). The expansion (4.25) (also for $k = 1$) takes the form

$$p^{-n} T_{p,x_0} \cong (g_0(x_0) I_{\mathbb{C}\otimes E} \mathscr{P}_{x_0} + \mathcal{Q}_{1,x_0} \mathscr{P}_{x_0} p^{-1/2})(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1}), \quad (4.71)$$

where we have used Proposition 4.11 and the definition (4.28) of g_0. Thus, subtracting (4.70) from (4.71) we obtain

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \\ \cong \big((\mathcal{Q}_{1,x_0} - Q_{1,x_0}(g_0)) \mathscr{P}_{x_0}\big)(\sqrt{p}Z, \sqrt{p}Z') p^{-1/2} + \mathcal{O}(p^{-1}). \quad (4.72)$$

Thus, it suffices to prove the following.

Lemma 4.18

$$F_{1,x} := \mathcal{Q}_{1,x} - Q_{1,x}(g_0) \equiv 0. \quad (4.73)$$

Proof We note first that $F_{1,x}$ is an odd polynomial in z and $\overline{z}'$; we verify this statement as in Lemma 4.12. Thus, the constant term of $F_{1,x}$ vanishes. To show that the rest of the terms vanish, we consider the decomposition $F_{1,x} = \sum_{i\geqslant 0} F_{1,x}^{(i)}$ in homogeneous polynomials $F_{1,x}^{(i)}$ of degree i. To prove (4.73) it suffices to show that

$$F_{1,x}^{(i)}(z, \overline{z}') = 0 \quad \text{for all } i > 0 \text{ and } z, z' \in \mathbb{C}^n. \quad (4.74)$$

The proof of (4.74) is similar to that of (4.34). Namely, we define as in (4.37) the operator $F_1^{(i)}$, by replacing $F_x^{(i)}(0, \overline{z}')$ by $F_{1,x}^{(i)}(0, \overline{z}')$, and we set (analogously to (4.38))

$$\mathcal{T}_{p,1} = T_p - P_p g_0 P_p - \sum_{i\leqslant \deg F_1} (F_1^{(i)} P_p) p^{(i-1)/2}. \quad (4.75)$$

Due to (4.14) and (4.25), there exist polynomials $\widetilde{R}_{r,x_0} \in \mathbb{C}[Z, Z']$ of the same parity as r such that the following expansion in the normal coordinates around $x_0 \in X$ holds for $k \geqslant 2$ and $|Z'| \leqslant \varepsilon'/2$:

$$p^{-n} \mathcal{T}_{p,1,x_0}(0, Z') \cong \sum_{r=2}^{k} (\widetilde{R}_{r,x_0} \mathscr{P}_{x_0})(0, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-(k+1)/2}), \quad (4.76)$$

This is the analogue of (4.39). Now we can repeat with obvious modifications the proof of (4.34) and obtain the analogue of (4.34) with F_x replaced by $F_{1,x}$. This completes the proof of Lemma 4.18. □

Lemma 4.18 and the expansion (4.72) imply immediately Proposition 4.17. □

Proof of Theorem 4.9 Proposition 4.17 shows that the asymptotic expansion (4.27) of T_p holds for $m=0$. Moreover, if T_p is self-adjoint, then from (4.70), (4.71), g_0 is also self-adjoint. We show inductively that (4.27) holds for every $m\in\mathbb{N}$. To prove (4.27) for $m=1$ let us consider the operator $p(T_p-P_pg_0P_p)$. We have to show now that $p(T_p-T_{g_0,p})$ satisfies the hypotheses of Theorem 4.9. The first two conditions are easily verified. To prove the third, just subtract the asymptotics of $T_{p,x_0}(Z,Z')$ (given by (4.25)) and $T_{g_0,p,x_0}(Z,Z')$ (given by (4.14)). Taking into account Proposition 4.11 and (4.73) the coefficients of p^0 and $p^{-1/2}$ in the difference vanish, which yields the desired conclusion.

Propositions 4.11 and 4.17 applied to $p(T_p-P_pg_0P_p)$ yield $g_1\in\mathscr{C}^\infty(X,\mathrm{End}(E))$ such that (4.27) holds true for $m=1$.

We continue in this way the induction process to get (4.27) for any m. This completes the proof of Theorem 4.9. □

4.3 Algebra of Toeplitz Operators

The Poisson bracket $\{\cdot,\cdot\}$ on $(X,2\pi\omega)$ is defined as follows. For $f,g\in\mathscr{C}^\infty(X)$, let ξ_f be the Hamiltonian vector field generated by f, which is defined by $2\pi i_{\xi_f}\omega=df$. Then

$$\{f,g\}:=\xi_f(dg). \tag{4.77}$$

One of our main goals is to show that Theorem 1.1 holds, thus the set of Toeplitz operators is closed under the composition of operators, so forms an algebra.

Proof of Theorem 1.1 Firstly, it is obvious that $P_pT_{f,p}T_{g,p}P_p=T_{f,p}T_{g,p}$. Lemmas 4.2 and 4.6 imply $T_{f,p}T_{g,p}$ verifies (4.24). Like in (4.18), we have for $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|<\varepsilon/4$:

$$\begin{aligned}(T_{f,p}T_{g,p})_{x_0}(Z,Z')=\int_{T_{x_0}X}&T_{f,p,x_0}(Z,Z'')\rho(4|Z''|/\varepsilon)T_{g,p,x_0}(Z'',Z')\\ &\times\kappa_{x_0}(Z'')dv_{TX}(Z'')+\mathscr{O}(p^{-\infty}).\end{aligned} \tag{4.78}$$

By Lemma 4.6 and (4.78), we deduce as in the proof of Lemma 4.6, that for $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|<\varepsilon/4$, we have

$$p^{-n}(T_{f,p}T_{g,p})_{x_0}(Z,Z')\cong\sum_{r=0}^{k}(Q_{r,x_0}(f,g)\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}+\mathscr{O}(p^{-\frac{k+1}{2}}), \tag{4.79}$$

and with the notation (2.10),

$$Q_{r,x_0}(f,g)=\sum_{r_1+r_2=r}\mathscr{K}[Q_{r_1,x_0}(f),Q_{r_2,x_0}(g)]. \tag{4.80}$$

Thus, $T_{f,p}T_{g,p}$ is a Toeplitz operator by Theorem 4.9. Moreover, it follows from the proofs of Lemma 4.6 and Theorem 4.9 that $g_l = C_l(f,g)$, where C_l are bidifferential operators.

Recall that we denote by $I_{\mathbb{C}\otimes E} : \Lambda(T^{*(0,1)}X)\otimes E \to \mathbb{C}\otimes E$ the natural projection. From (2.10), (4.16) and (4.80), we get

$$\begin{aligned} C_0(f,g)(x) &= I_{\mathbb{C}\otimes E} Q_{0,x}(f,g)|_{\mathbb{C}\otimes E} = I_{\mathbb{C}\otimes E}\mathcal{K}[Q_{0,x}(f), Q_{0,x}(g)]|_{\mathbb{C}\otimes E} \\ &= f(x)g(x). \end{aligned} \tag{4.81}$$

By the proof of Theorem 4.9 (cf. Proposition 4.11, Lemma 4.18 and (4.28)), we get

$$\begin{aligned} &Q_{1,x}(f,g) = Q_{1,x}(C_0(f,g)), \\ &C_1(f,g) = I_{\mathbb{C}\otimes E}(Q_{2,x}(f,g) - Q_{2,x}(C_0(f,g)))(0,0)|_{\mathbb{C}\otimes E}. \end{aligned} \tag{4.82}$$

Moreover, by (4.16) and (4.80), we get

$$\begin{aligned} Q_{2,x}(f,g) = \mathcal{K}[f(x)I_{\mathbb{C}\otimes E}, Q_{2,x}(g)] + \mathcal{K}[Q_{1,x}(f), Q_{1,x}(g)] \\ + \mathcal{K}[Q_{2,x}(f), g(x)I_{\mathbb{C}\otimes E}]. \end{aligned} \tag{4.83}$$

Now $T_{f,p}P_p = P_pT_{f,p}$ implies $Q_{r,x}(f,1) = Q_{r,x}(1,f)$, so we get from (4.83):

$$\begin{aligned} &\mathcal{K}[J_{0,x}, Q_{2,x}(f)] - \mathcal{K}[Q_{2,x}(f), J_{0,x}] \\ &\quad = \mathcal{K}[Q_{1,x}(f), J_{1,x}] - \mathcal{K}[J_{1,x}, Q_{1,x}(f)] \\ &\qquad + \mathcal{K}[f(x)J_{0,x}, J_{2,x}] - \mathcal{K}[J_{2,x}, f(x)J_{0,x}]. \end{aligned} \tag{4.84}$$

Assume now that $f, g \in \mathcal{C}^\infty(X)$. By (4.82), (4.83) and (4.84), we get

$$\begin{aligned} &C_1(f,g)(x) - C_1(g,f)(x) \\ &\quad = I_{\mathbb{C}\otimes E}[\mathcal{K}[Q_{1,x}(f), Q_{1,x}(g)] - \mathcal{K}[Q_{1,x}(g), Q_{1,x}(f)] \\ &\qquad + f(x)(\mathcal{K}[Q_{1,x}(g), J_{1,x}] - \mathcal{K}[J_{1,x}, Q_{1,x}(g)]) \\ &\qquad - g(x)(\mathcal{K}[Q_{1,x}(f), J_{1,x}] - \mathcal{K}[J_{1,x}, Q_{1,x}(f)])]|_{\mathbb{C}\otimes E}. \end{aligned} \tag{4.85}$$

By Lemma 4.7, Remark 4.8, we have

$$\begin{aligned} &\mathcal{K}[Q_{1,x}(f), Q_{1,x}(g)] \\ &\quad = \mathcal{K}\left[\mathcal{K}\left[1, \frac{\partial f_x}{\partial Z_j}(0)Z_j\right], \mathcal{K}\left[1, \frac{\partial g_x}{\partial Z_j}(0)Z_j\right]\right] \\ &\qquad + \mathcal{K}[f(x)J_1, Q_{1,x}(g)] + \mathcal{K}[Q_{1,x}(f), g(x)J_1] - \mathcal{K}[f(x)J_1, g(x)J_1]. \end{aligned} \tag{4.86}$$

From (4.11), (4.85) and (4.86), we get

$$C_1(f,g)(x) - C_1(g,f)(x)$$

$$= \mathcal{K}\left[\mathcal{K}\left[1, \frac{\partial f_x}{\partial Z_j}(0)Z_j\right], \mathcal{K}\left[1, \frac{\partial g_x}{\partial Z_j}(0)Z_j\right]\right]$$
$$- \mathcal{K}\left[\mathcal{K}\left[1, \frac{\partial g_x}{\partial Z_j}(0)Z_j\right], \mathcal{K}\left[1, \frac{\partial f_x}{\partial Z_j}(0)Z_j\right]\right]. \tag{4.87}$$

From (2.17) we get

$$\mathcal{K}\left[1, \frac{\partial f_x}{\partial Z_j}(0)Z_j\right] = \frac{\partial f_x}{\partial z_i}(0)z_i + \frac{\partial f_x}{\partial \overline{z}_i}(0)\overline{z}'_i. \tag{4.88}$$

Plugging (4.88) into (4.87) and using (2.17) we finally obtain:

$$C_1(f,g)(x) - C_1(g,f)(x) = \sum_{i=1}^{n} \frac{2}{a_i}\left[\frac{\partial f_x}{\partial \overline{z}_i}(0)\frac{\partial g_x}{\partial z_i}(0) - \frac{\partial f_x}{\partial z_i}(0)\frac{\partial g_x}{\partial \overline{z}_i}(0)\right]\mathrm{Id}_E$$
$$= \sqrt{-1}\{f,g\}\,\mathrm{Id}_E\,. \tag{4.89}$$

This finishes the proof of Theorem 1.1. □

The next result and Theorem 1.1 show that the Berezin-Toeplitz quantization has the correct semi-classical behavior.

Theorem 4.19 *For* $f \in \mathcal{C}^\infty(X, \mathrm{End}(E))$, *the norm of* $T_{f,p}$ *satisfies*

$$\lim_{p\to\infty} \|T_{f,p}\| = \|f\|_\infty := \sup_{0\neq u\in E_x, x\in X} |f(x)(u)|_{h^E}/|u|_{h^E}\,. \tag{4.90}$$

Proof Take a point $x_0 \in X$ and $u_0 \in E_{x_0}$ with $|u_0|_{h^E} = 1$ such that $|f(x_0)(u_0)| = \|f\|_\infty$. Recall that in Sect. 4.1, we trivialize the bundles L, E in our normal coordinates near x_0, and e_L is the unit frame of L which trivialize L. Moreover, in this normal coordinates, u_0 is a trivial section of E. Considering the sequence of sections $S^p_{x_0} = p^{-n/2}P_p(e_L^{\otimes p} \otimes u_0)$, we have by (3.27),

$$\|T_{f,p}S^p_{x_0} - f(x_0)S^p_{x_0}\|_{L^2} \leqslant \frac{C}{\sqrt{p}}\|S^p_{x_0}\|_{L^2}. \tag{4.91}$$

If f is a real function, then $df(x_0) = 0$, so we can improve the constant $\frac{C}{\sqrt{p}}$ in (4.91) to $\frac{C}{p}$. The proof of (4.90) is complete. □

Remark 4.20 For $E = \mathbb{C}$, Theorem 1.1 shows that we can associate to $f, g \in \mathcal{C}^\infty(X)$ a formal power series $\sum_{l=0}^\infty \hbar^l C_l(f,g) \in \mathcal{C}^\infty(X)[[\hbar]]$, where C_l are bidifferential operators. Therefore, we have constructed in a canonical way an associative star-product $f * g = \sum_{l=0}^\infty \hbar^l C_l(f,g)$, called the *Berezin-Toeplitz star-product*.

5 Berezin-Toeplitz Quantizations on Non-Compact Manifolds

In this section, we extend our results to non-compact manifolds. We consider for simplicity only complex manifolds, that is, we suppose that (X, J) is a complex manifold with complex structure J and E, L are holomorphic vector bundles on X with

$\mathrm{rk}(L)=1$. We assume that ∇^E, ∇^L are the holomorphic Hermitian (i.e., Chern) connections on (E,h^E), (L,h^L). Let g^{TX} be any Riemannian metric on TX compatible with J. Since g^{TX} is not necessarily Kähler, the endomorphism $\boldsymbol{J}$ defined in (3.7) does not satisfy $\boldsymbol{J}\neq J$ in general. Set

$$\Theta(X,Y)=g^{TX}(JX,Y). \tag{5.1}$$

Then the 2-form Θ need not be closed.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X,L^p\otimes E)$ with the Hermitian product induced by g^{TX}, h^L, h^E as in (3.9). Set

$$\begin{aligned} D_p&=\sqrt{2}\big(\overline{\partial}^{L^p\otimes E}+\overline{\partial}^{L^p\otimes E,*}\big),\\ \square_p&=\overline{\partial}^{L^p\otimes E}\overline{\partial}^{L^p\otimes E,*}+\overline{\partial}^{L^p\otimes E,*}\overline{\partial}^{L^p\otimes E}. \end{aligned} \tag{5.2}$$

Then $\square_p$ is the Kodaira-Laplacian which preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X,L^p\otimes E)$ and

$$D_p^2=2\square_p. \tag{5.3}$$

Note that D_p is not a spin^c Dirac operator on $\Omega^{0,\bullet}(X,L^p\otimes E)$.

The space of holomorphic sections of $L^p\otimes E$ which are L^2 with respect to the norm given by (3.9) is denoted by $H^0_{(2)}(X,L^p\otimes E)$. Let $P_p(x,x')$, $(x,x'\in X)$ be the Schwartz kernel of the orthogonal projection P_p, from the space of L^2 sections of $L^p\otimes E$ onto $H^0_{(2)}(X,L^p\otimes E)$, with respect to the Riemannian volume form $dv_X(x')$ associated to (X,g^{TX}). Then $P_p(x,x')$ is smooth by the ellipticity of the Kodaira Laplacian and the Schwartz kernel theorem (cf. also [30, Remark 1.3.3]).

Remark 5.1 If $\boldsymbol{J}=J$, then (X,J,Θ) is Kähler and D_p in (3.11) and (5.2) coincide. Assume moreover X is compact. Then by the Kodaira vanishing theorem and the Dolbeault isomorphism we have

$$H^0(X,L^p\otimes E)=\mathrm{Ker}(D_p), \tag{5.4}$$

for p large enough. Thus, if (X,J,Θ) is a compact Kähler manifold and $\boldsymbol{J}=J$, $E=\mathbb{C}$, Theorems 1.1, 4.19 recover the main results of Bordemann et al. [8, 17, 23, 34].

We denote by $R^{\det}$ the curvature of the holomorphic Hermitian connection $\nabla^{\det}$ on $K_X^*=\det(T^{(1,0)}X)$.

For a $(1,1)$-form Ω, we write $\Omega>0$ (resp. $\geqslant 0$) if $\Omega(\cdot,J\cdot)>0$ (resp. $\geqslant 0$).

The following result, obtained in [28, Theorem 3.11], extends the asymptotic expansion of the Bergman kernel to non-compact manifolds.

Theorem 5.2 *Suppose that (X,g^{TX}) is a complete Hermitian manifold and there exist $\varepsilon>0$, $C>0$ such that*:

$$\sqrt{-1}R^L>\varepsilon\Theta,\qquad \sqrt{-1}(R^{\det}+R^E)>-C\Theta\,\mathrm{Id}_E,\qquad |\partial\Theta|_{g^{TX}}<C, \tag{5.5}$$

then the kernel $P_p(x,x')$ *has a full off–diagonal asymptotic expansion analogous to that of Theorem* 3.6 *uniformly for any* $x,x'\in K$, *a compact set of* X. *If* $L=K_X:=\det(T^{*(1;0)}X)$ *is the canonical line bundle on* X, *the first two conditions in* (5.5) *are to be replaced by*

$$h^L \text{ is induced by } \Theta \text{ and } \sqrt{-1}R^{\det}<-\varepsilon\Theta,\ \sqrt{-1}R^E>-C\Theta\,\mathrm{Id}_E.$$

The idea of the proof is that (5.5) together with the Bochner-Kodaira-Nakano formula imply the existence of the spectral gap for $\square_p$ acting on $L^2(X,L^p\otimes E)$ as in (3.16).

Let $\mathcal{C}^\infty_{const}(X,\mathrm{End}(E))$ denote the algebra of smooth sections of X which are constant map outside a compact set. For any $f\in\mathcal{C}^\infty_{const}(X,\mathrm{End}(E))$, we consider the Toeplitz operator $(T_{f,p})_{p\in\mathbb{N}}$ as in (4.6):

$$T_{f,p}:L^2(X,L^p\otimes E)\longrightarrow L^2(X,L^p\otimes E),\quad T_{f,p}=P_p\,f\,P_p. \tag{5.6}$$

The following result generalizes Theorems 1.1 and 4.19 to non-compact manifolds.

Theorem 5.3 *Assume that* (X,g^{TX}) *is a complete Hermitian manifold*, (L,h^L) *and* (E,h^E) *are holomorphic vector bundles satisfying the hypotheses of Theorem* 5.2 *with* $\mathrm{rk}(L)=1$. *Let* $f,g\in\mathcal{C}^\infty_{const}(X,\mathrm{End}(E))$. *Then the following assertions hold*:

(i) *The product of the two corresponding Toeplitz operators admits the asymptotic expansion* (1.7) *in the sense of* (4.5), *where* C_r *are bidifferential operators, especially,* $\mathrm{supp}(C_r(f,g))\subset\mathrm{supp}(f)\cap\mathrm{supp}(g)$, *and* $C_0(f,g)=fg$.
(ii) *If* $f,g\in\mathcal{C}^\infty_{const}(X)$, *then* (1.9) *holds.*
(iii) *Relation* (4.90) *also holds for any* $f\in\mathcal{C}^\infty_{const}(X,\mathrm{End}(E))$.

Proof The most important observation here is that the spectral gap property (3.16) and a similar argument as in Proposition 3.4 deliver

$$F(D_p)s=P_ps,\qquad \|F(D_p)-P_p\|=\mathcal{O}(p^{-\infty}), \tag{5.7}$$

for p large enough and each $s\in H^0_{(2)}(X,L^p\otimes E)$. Moreover, by the proof of Proposition 3.4, for any compact set K, and any $l,m\in\mathbb{N}$, $\varepsilon>0$, there exists $C_{l,m,\varepsilon}>0$ such that

$$|F(D_p)(x,x')-P_p(x,x')|_{\mathcal{C}^m(K\times K)}\leqslant C_{l,m,\varepsilon}\,p^{-l}, \tag{5.8}$$

for $p\geqslant1$, $x,x'\in K$. By the finite propagation speed for solutions of hyperbolic equations [36, §2.8], [30, Appendix D.2] (cf. also [20, Proposition 4.1]), $F(D_p)(x,\cdot)$ only depends on the restriction of D_p to $B^X(x,\varepsilon)$ and is zero outside $B^X(x,\varepsilon)$.

For $g\in\mathcal{C}^\infty_0(X,\mathrm{End}(E))$, let $(F(D_p)gF(D_p))(x,x')$ be the smooth kernel of $F(D_p)gF(D_p)$ with respect to $dv_X(x')$. Then for any relative compact open set U in X such that $\mathrm{supp}(g)\subset U$, we have from (5.7) and (5.8),

$$\begin{aligned}&T_{g,p}-F(D_p)gF(D_p)=\mathcal{O}(p^{-\infty}),\\&T_{g,p}(x,x')-(F(D_p)gF(D_p))(x,x')=\mathcal{O}(p^{-\infty})\quad\text{on } U\times U.\end{aligned} \tag{5.9}$$

Now we fix $f, g \in \mathcal{C}_0^\infty(X, \mathrm{End}(E))$. Let U be relative compact open sets in X such that $\mathrm{supp}(f) \cup \mathrm{supp}(g) \subset U$ and $d(x,y) > 2\varepsilon$ for any $x \in \mathrm{supp}(f) \cup \mathrm{supp}(g)$, $y \in X \setminus U$. From (5.7), we have

$$T_{f,p}T_{g,p} = P_p F(D_p) f P_p g F(D_p) P_p. \tag{5.10}$$

Let $(F(D_p) f P_p g F(D_p))(x,x')$, be the smooth kernel of $F(D_p) f P_p g F(D_p)$ with respect to $dv_X(x')$. Then the support of $(F(D_p) f P_p g F(D_p))(\cdot,\cdot)$ is contained in $U \times U$. If we fix $x_0 \in U$, it follows from (5.8) that the kernel of $F(D_p) f P_p g F(D_p)$ has exactly the same asymptotic expansion as in the compact case. More precisely, as in (4.79), we have

$$\begin{aligned} & p^{-n}(F(D_p) f P_p g F(D_p))_{x_0}(Z,Z') \\ & \quad \cong \sum_{r=0}^{k} (Q_{r,x_0}(f,g)\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \end{aligned} \tag{5.11}$$

with the same local formula for $Q_{r,x_0}(f,g)$ given in (4.80).

But since all formal computations are local, $Q_{r,x_0}(f,g)$ are the same as in the compact case, i.e., polynomials with coefficients bidifferential operators acting on f and g.

Thus, we know from (5.9) that there exist $(g_l)_{l\geqslant 0}$, where $g_l \in \mathcal{C}_0^\infty(X, \mathrm{End}(E))$, $\mathrm{supp}(g_l) \subset \mathrm{supp}(f) \cap \mathrm{supp}(g)$ such that for any $k \geqslant 1$, $s \in L^2(X, E_p)$,

$$\left\| F(D_p) f P_p g F(D_p) s - \sum_{l=0}^{k} F(D_p) P_p g_l p^{-l} P_p F(D_p) s \right\|_{L^2} \leqslant \frac{C}{p^{k+1}} \|s\|_{L^2}. \tag{5.12}$$

(5.10) and (5.12) imply that

$$\left\| T_{f,p}T_{g,p} - \sum_{l=0}^{k} P_p g_l p^{-l} P_p \right\| \leqslant C_k p^{-k-1}. \tag{5.13}$$

Therefore, (i) is proved. With the asymptotic expansion at hand, we have just to repeat the proofs given in the compact case in order to verify assertions (ii) and (iii). More precisely, (ii) follows exactly in the same way as in the proof of (1.9) given in Theorem 1.1. Finally, to derive assertion (iii), we apply verbatim the proof of Theorem 4.19. This completes the proof of Theorem 5.3. □

Example 5.4 Theorem 5.3 holds for every quasi-projective manifold with L the restriction of the hyperplane line bundle associated to some arbitrary projective embedding and E the trivial bundle. By definition, a quasi-projective manifold X has the form $X = Y \setminus Z$, where Y and Z are projective varieties, and $Z \subset Y$ contains the singular set of Y. Let us consider a holomorphic embedding $Y \subset \mathbb{CP}^m$, the hyperplane line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^m$, and set $L = \mathcal{O}(1)|_X$.

By Hironaka's theorem of resolution of singularities there exists a projective manifold $\widetilde{Y}$ and a holomorphic map $\pi : \widetilde{Y} \longrightarrow Y$ (a composition of a finite succession of blow-ups with smooth centers) such that $\pi : \widetilde{Y} \setminus \pi^{-1}(Z) \longrightarrow Y \setminus Z$ is biholomorphic and $\pi^{-1}(Z)$ is a divisor with normal crossings.

In this situation it is shown in [28, §3.6], [30, §6.2] that there exist a complete Kähler metric g^{TX} on $\widetilde{Y} \setminus \pi^{-1}(Z) \simeq X$, called the generalized Poincaré metric, and a metric h^L on $\pi^* L \simeq L$ satisfying the hypotheses of Theorem 5.2 (with E trivial).

Remark 5.5 It is appropriate to remark that the results of Sect. 4.1-4.3 learn that we can associate to any $f, g \in \mathcal{C}^\infty(X, \mathrm{End}(E))$ a formal power series $\sum_{l=0}^\infty \hbar^l C_l(f, g) \in \mathcal{C}^\infty(X, \mathrm{End}(E))[[\hbar]]$, where C_l are bidifferential operators. This follows from the fact that the construction in Sect. 4.3 is local. However, the problem we addressed in this section is which Hilbert space the Toeplitz operators act on in the case of a non-compact manifold. Theorem 5.6 shows that the space of holomorphic L^2-sections $H^0_{(2)}(X, L^p \otimes E)$ of $L^p \otimes E$, is a suitable Hilbert space which allows the Berezin-Toeplitz quantization of the algebra $\mathcal{C}^\infty_{const}(X, \mathrm{End}(E))$.

6 Berezin-Toeplitz Quantization on Orbifolds

In this section we establish the theory of Berezin-Toeplitz quantization on symplectic orbifolds, especially we show that set of Toeplitz operators forms an algebra. For convenience of exposition, we explain the results in detail in the Kähler orbifold case. In [30, §5.4] we find more complete explanations and references for Sects. 6.1 and 6.2. For related topics about orbifolds we refer to [1].

This section is organized as follows. In Sect. 6.1 we recall the basic definitions about orbifolds. In Sect. 6.2 we explain the asymptotic expansion of Bergman kernel on complex orbifolds [20, §5.2], which we apply in Sect. 6.3 to derive the Berezin-Toeplitz quantization on Kähler orbifolds. Finally, we state in Sect. 6.4 the corresponding version for symplectic orbifolds.

6.1 Basic Definitions on Orbifolds

We define at first a category $\mathcal{M}_s$ as follows : The objects of $\mathcal{M}_s$ are the class of pairs (G, M) where M is a connected smooth manifold and G is a finite group acting effectively on M (i.e., if $g \in G$ such that $gx = x$ for any $x \in M$, then g is the unit element of G). If (G, M) and (G', M') are two objects, then a morphism $\Phi : (G, M) \to (G', M')$ is a family of open embeddings $\varphi : M \to M'$ satisfying:

(i) For each $\varphi \in \Phi$, there is an injective group homomorphism $\lambda_\varphi : G \to G'$ that makes φ be λ_φ-equivariant.
(ii) For $g \in G'$, $\varphi \in \Phi$, we define $g\varphi : M \to M'$ by $(g\varphi)(x) = g\varphi(x)$ for $x \in M$. If $(g\varphi)(M) \cap \varphi(M) \neq \emptyset$, then $g \in \lambda_\varphi(G)$.
(iii) For $\varphi \in \Phi$, we have $\Phi = \{g\varphi, g \in G'\}$.

Definition 6.1 Let X be a paracompact Hausdorff space. An m-dimensional *orbifold chart* on X consists of a connected open set U of X, an object $(G_U, \widetilde{U})$ of $\mathcal{M}_s$ with $\dim \widetilde{U} = m$, and a ramified covering $\tau_U : \widetilde{U} \to U$ which is G_U-invariant and induces a homeomorphism $U \simeq \widetilde{U}/G_U$. We denote the chart by $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$.

An m-dimensional *orbifold atlas* $\mathcal{V}$ on X consists of a family of m-dimensional orbifold charts $\mathcal{V}(U) = ((G_U, \widetilde{U}) \xrightarrow{\tau_U} U)$ satisfying the following conditions:

(i) The open sets $U \subset X$ form a covering $\mathcal{U}$ with the property:

$$\begin{aligned} &\text{For any } U, U' \in \mathcal{U} \text{ and } x \in U \cap U', \\ &\quad \text{there is } U'' \in \mathcal{U} \text{ such that } x \in U'' \subset U \cap U'. \end{aligned} \tag{6.1}$$

(ii) for any $U, V \in \mathcal{U}, U \subset V$ there exists a morphism $\varphi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$, which covers the inclusion $U \subset V$ and satisfies $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$ for any $U, V, W \in \mathcal{U}$, with $U \subset V \subset W$.

It is easy to see that there exists a unique maximal orbifold atlas $\mathcal{V}_{max}$ containing $\mathcal{V}$; $\mathcal{V}_{max}$ consists of all orbifold charts $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, which are locally isomorphic to charts from $\mathcal{V}$ in the neighborhood of each point of U. A maximal orbifold atlas $\mathcal{V}_{max}$ is called an *orbifold structure* and the pair $(X, \mathcal{V}_{max})$ is called an orbifold. As usual, once we have an orbifold atlas $\mathcal{V}$ on X we denote the orbifold by $(X, \mathcal{V})$, since $\mathcal{V}$ determines uniquely $\mathcal{V}_{max}$.

Note that if $\mathcal{U}'$ is a refinement of $\mathcal{U}$ satisfying (6.1), then there is an orbifold atlas $\mathcal{V}'$ such that $\mathcal{V} \cup \mathcal{V}'$ is an orbifold atlas, hence $\mathcal{V} \cup \mathcal{V}' \subset \mathcal{V}_{max}$. This shows that we may choose $\mathcal{U}$ arbitrarily fine.

Let $(X, \mathcal{V})$ be an orbifold. For each $x \in X$, we can choose a small neighborhood $(G_x, \widetilde{U}_x) \to U_x$ such that $x \in \widetilde{U}_x$ is a fixed point of G_x (it follows from the definition that such a G_x is unique up to isomorphisms for each $x \in X$). We denote by $|G_x|$ the cardinal of G_x. If $|G_x| = 1$, then X has a smooth manifold structure in the neighborhood of x, which is called a smooth point of X. If $|G_x| > 1$, then X is not a smooth manifold in the neighborhood of x, which is called a singular point of X. We denote by $X_{sing} = \{x \in X; |G_x| > 1\}$ the singular set of X, and $X_{reg} = \{x \in X; |G_x| = 1\}$ the regular set of X.

It is useful to note that on an orbifold $(X, \mathcal{V})$ we can construct partitions of unity. First, let us call a function on X smooth, if its lift to any chart of the orbifold atlas $\mathcal{V}$ is smooth in the usual sense. Then the definition and construction of a smooth partition of unity associated to a locally finite covering carries over easily from the manifold case. The point is to construct smooth G_U-invariant functions with compact support on $(G_U, \widetilde{U})$.

In Definition 6.1 we can replace $\mathcal{M}_s$ by a category of manifolds with an additional structure such as orientation, Riemannian metric, almost-complex structure or complex structure. We impose that the morphisms (and the groups) preserve the specified structure. So we can define oriented, Riemannian, almost-complex or complex orbifolds.

Let $(X, \mathcal{V})$ be an arbitrary orbifold. By the above definition, a *Riemannian metric* on X is a Riemannian metric g^{TX} on X_{reg} such that the lift of g^{TX} to any chart of the orbifold atlas $\mathcal{V}$ can be extended to a smooth Riemannian metric. Certainly, for any $(G_U, \widetilde{U}) \in \mathcal{V}$, we can always construct a G_U-invariant Riemannian metric on $\widetilde{U}$. By a partition of unity argument, we see that there exist Riemannian metrics on the orbifold $(X, \mathcal{V})$.

Definition 6.2 An orbifold vector bundle E over an orbifold $(X,\mathcal{V})$ is defined as follows: E is an orbifold and for $U\in\mathcal{U}$, $(G^E_U,\widetilde{p}_U:\widetilde{E}_U\to\widetilde{U})$ is a G^E_U-equivariant vector bundle and $(G^E_U,\widetilde{E}_U)$ (resp. $(G_U=G^E_U/K^E_U,\widetilde{U})$, $K^E_U=\mathrm{Ker}(G^E_U\to \mathrm{Diffeo}(\widetilde{U})))$ is the orbifold structure of E (resp. X). If G^E_U acts effectively on $\widetilde{U}$ for $U\in\mathcal{U}$, i.e., $K^E_U=\{1\}$, we call E a proper orbifold vector bundle.

Note that any structure on X or E is locally G_x or $G^E_{U_x}$-equivariant.

Remark 6.3 Let E be an orbifold vector bundle on $(X,\mathcal{V})$. For $U\in\mathcal{U}$, let $\widetilde{E^{\mathrm{pr}}_U}$ be the maximal K^E_U-invariant sub-bundle of $\widetilde{E}_U$ on $\widetilde{U}$. Then $(G_U,\widetilde{E^{\mathrm{pr}}_U})$ defines a proper orbifold vector bundle on $(X,\mathcal{V})$, denoted by E^{pr}.

The (proper) orbifold tangent bundle TX on an orbifold X is defined by $(G_U,T\widetilde{U}\to\widetilde{U})$, for $U\in\mathcal{U}$. In the same vein we introduce the cotangent bundle T^*X. We can form tensor products of bundles by taking the tensor products of their local expressions in the charts of an orbifold atlas. Note that a Riemannian metric on X induces a section of $T^*X\otimes T^*X$ over X which is a positive definite bilinear form on T_xX at each point $x\in X$.

Let $E\to X$ be an orbifold vector bundle and $k\in\mathbb{N}\cup\{\infty\}$. A section $s:X\to E$ is called $\mathscr{C}^k$ if for each $U\in\mathcal{U}$, $s|_U$ is covered by a G^E_U-invariant $\mathscr{C}^k$ section $\widetilde{s}_U:\widetilde{U}\to\widetilde{E}_U$. We denote by $\mathscr{C}^k(X,E)$ the space of $\mathscr{C}^k$ sections of E on X.

If X is oriented, we define the integral $\int_X\alpha$ for a form α over X (i.e., a section of $\Lambda(T^*X)$ over X) as follows. If $\mathrm{supp}(\alpha)\subset U\in\mathcal{U}$ set

$$\int_X\alpha:=\frac{1}{|G_U|}\int_{\widetilde{U}}\widetilde{\alpha}_U. \tag{6.2}$$

It is easy to see that the definition is independent of the chart. For general α we extend the definition by using a partition of unity.

If X is an oriented Riemannian orbifold, there exists a canonical volume element dv_X on X, which is a section of $\Lambda^m(T^*X)$, $m=\dim X$. Hence, we can also integrate functions on X.

Assume now that the Riemannian orbifold $(X,\mathcal{V})$ is compact. For $x,y\in X$, put

$$\begin{aligned} d(x,y)=\mathrm{Inf}_\gamma\Big\{&\textstyle\sum_i\int_{t_{i-1}}^{t_i}|\frac{\partial}{\partial t}\widetilde{\gamma}_i(t)|dt\;\Big|\;\gamma:[0,1]\to X,\gamma(0)=x,\gamma(1)=y,\\ &\text{such that there exist } t_0=0<t_1<\cdots<t_k=1,\gamma([t_{i-1},t_i])\subset U_i,\\ &U_i\in\mathcal{U},\text{ and a } \mathscr{C}^\infty \text{ map } \widetilde{\gamma}_i:[t_{i-1},t_i]\to\widetilde{U}_i \text{ that covers } \gamma|_{[t_{i-1},t_i]}\Big\}.\end{aligned}$$

Then (X,d) is a metric space. For $x\in X$, set $d(x,X_{sing}):=\inf_{y\in X_{sing}}d(x,y)$.

Let us discuss briefly kernels and operators on orbifolds. For any open set $U\subset X$ and orbifold chart $(G_U,\widetilde{U})\xrightarrow{\tau_U}U$, we will add a superscript $\sim$ to indicate the corresponding objects on $\widetilde{U}$. Assume that $\widetilde{\mathcal{K}}(\widetilde{x},\widetilde{x}')\in\mathscr{C}^\infty(\widetilde{U}\times\widetilde{U},\pi_1^*\widetilde{E}\otimes\pi_2^*\widetilde{E}^*)$ verifies

$$(g,1)\widetilde{\mathcal{K}}(g^{-1}\widetilde{x},\widetilde{x}')=(1,g^{-1})\widetilde{\mathcal{K}}(\widetilde{x},g\widetilde{x}')\quad\text{for any } g\in G_U, \tag{6.3}$$

where (g_1, g_2) acts on $\widetilde{E}_{\widetilde{x}} \times \widetilde{E}^*_{\widetilde{x}'}$ by $(g_1, g_2)(\xi_1, \xi_2) = (g_1\xi_1, g_2\xi_2)$.

We define the operator $\widetilde{\mathcal{K}} : \mathcal{C}_0^\infty(\widetilde{U}, \widetilde{E}) \to \mathcal{C}^\infty(\widetilde{U}, \widetilde{E})$ by

$$(\widetilde{\mathcal{K}}\widetilde{s})(\widetilde{x}) = \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}')\widetilde{s}(\widetilde{x}')dv_{\widetilde{U}}(\widetilde{x}') \quad \text{for } \widetilde{s} \in \mathcal{C}_0^\infty(\widetilde{U}, \widetilde{E}). \tag{6.4}$$

For $\widetilde{s} \in \mathcal{C}^\infty(\widetilde{U}, \widetilde{E})$ and $g \in G_U$, g acts on $\mathcal{C}^\infty(\widetilde{U}, \widetilde{E})$ by: $(g \cdot \widetilde{s})(\widetilde{x}) := g \cdot \widetilde{s}(g^{-1}\widetilde{x})$. We can then identify an element $s \in \mathcal{C}^\infty(U, E)$ with an element $\widetilde{s} \in \mathcal{C}^\infty(\widetilde{U}, \widetilde{E})$ verifying $g \cdot \widetilde{s} = \widetilde{s}$ for any $g \in G_U$.

With this identification, we define the operator $\mathcal{K} : \mathcal{C}_0^\infty(U, E) \to \mathcal{C}^\infty(U, E)$ by

$$(\mathcal{K}s)(x) = \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}')\widetilde{s}(\widetilde{x}')dv_{\widetilde{U}}(\widetilde{x}') \quad \text{for } s \in \mathcal{C}_0^\infty(U, E), \tag{6.5}$$

where $\widetilde{x} \in \tau_U^{-1}(x)$. Then the smooth kernel $\mathcal{K}(x, x')$ of the operator $\mathcal{K}$ with respect to dv_X is

$$\mathcal{K}(x, x') = \sum_{g \in G_U} (g, 1)\widetilde{\mathcal{K}}(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{6.6}$$

Indeed, if $s \in \mathcal{C}_0^\infty(U, E)$, by (6.3) and (6.5), we have

$$\begin{aligned}(\mathcal{K}s)(x) &= \frac{1}{|G_U|} \sum_{g \in G_U} \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}')g \cdot \widetilde{s}(g^{-1}\widetilde{x}')(\widetilde{x}')dv_{\widetilde{U}}(\widetilde{x}') \\ &= \frac{1}{|G_U|} \sum_{g \in G_U} \int_{\widetilde{U}} (g, 1)\widetilde{\mathcal{K}}(g^{-1}\widetilde{x}, \widetilde{x}')s(\widetilde{x}')dv_{\widetilde{U}}(\widetilde{x}') \\ &= \int_U \sum_{g \in G_U} (g, 1)\widetilde{\mathcal{K}}(g^{-1}\widetilde{x}, \widetilde{x}')s(x')dv_X(x'). \end{aligned} \tag{6.7}$$

Let $\mathcal{K}_1, \mathcal{K}_2$ be two operators as above and assume that the kernel of one of $\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2$ has compact support. By (6.2), (6.3) and (6.5), the kernel of $\mathcal{K}_1 \circ \mathcal{K}_2$ is given by

$$(\mathcal{K}_1 \circ \mathcal{K}_2)(x, x') = \sum_{g \in G_U} (g, 1)(\widetilde{\mathcal{K}}_1 \circ \widetilde{\mathcal{K}}_2)(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{6.8}$$

6.2 Bergman Kernel on Kähler Orbifolds

Let X be a compact complex orbifold of complex dimension n with complex structure J. Let E be a holomorphic orbifold vector bundle on X.

Let $\mathcal{O}_X$ be the sheaf over X of local G_U-invariant holomorphic functions over $\widetilde{U}$, for $U \in \mathcal{U}$. The local G_U^E -invariant holomorphic sections of $\widetilde{E} \to \widetilde{U}$ define a sheaf $\mathcal{O}_X(E)$ over X. Let $H^\bullet(X, \mathcal{O}_X(E)$ be the cohomology of the sheaf $\mathcal{O}_X(E)$ over X.

Notice that by Definition, we have

$$\mathcal{O}_X(E) = \mathcal{O}_X(E^{\mathrm{pr}}). \tag{6.9}$$

Thus, without lost generality, we may and will assume that E is a proper orbifold vector bundle on X.

Consider a section $s \in \mathcal{C}^\infty(X, E)$ and a local section $\widetilde{s} \in \mathcal{C}^\infty(\widetilde{U}, \widetilde{E}_U)$ covering s. Then $\overline{\partial}^{\widetilde{E}_U}\widetilde{s}$ covers a section of $T^{*(0,1)}X \otimes E$ over U, denoted $\overline{\partial}^E s|_U$. The family of sections $\{\overline{\partial}^E s|_U : U \in \mathcal{U}\}$ patch together to define a global section $\overline{\partial}^E s$ of $T^{*(0,1)}X \otimes E$ over X. In a similar manner we define $\overline{\partial}^E \alpha$ for a $\mathcal{C}^\infty$ section α of $\Lambda(T^{*(0,1)}X) \otimes E$ over X. We obtain thus the Dolbeault complex $(\Omega^{0,\bullet}(X, E), \overline{\partial}^E)$:

$$0 \longrightarrow \Omega^{0,0}(X, E) \xrightarrow{\overline{\partial}^E} \cdots \xrightarrow{\overline{\partial}^E} \Omega^{0,n}(X, E) \longrightarrow 0. \tag{6.10}$$

From the abstract de Rham theorem there exists a canonical isomorphism

$$H^\bullet(\Omega^{0,\bullet}(X, E), \overline{\partial}^E) \simeq H^\bullet(X, \mathcal{O}_X(E)). \tag{6.11}$$

In the sequel, we also denote $H^\bullet(X, \mathcal{O}_X(E))$ by $H^\bullet(X, E)$.

We consider a complex orbifold (X, J) endowed with the complex structure J. Let g^{TX} be a Riemannian metric on TX compatible with J. There is then an associated $(1, 1)$-form Θ given by $\Theta(U, V) = g^{TX}(JU, V)$. The metric g^{TX} is called a Kähler metric and the orbifold (X, J) is called a *Kähler orbifold* if Θ is a closed form, that is, $d\Theta = 0$. In this case Θ is a symplectic form, called Kähler form. We will denote the Kähler orbifold by (X, J, Θ) or shortly by (X, Θ).

Let (L, h^L) be a holomorphic Hermitian proper orbifold line bundle on an orbifold X, and let (E, h^E) be a holomorphic Hermitian proper orbifold vector bundle on X.

We assume that the associated curvature R^L of (L, h^L) verifies (3.6), i.e., (L, h^L) is a positive proper orbifold line bundle on X. This implies that $\omega := \frac{\sqrt{-1}}{\pi} R^L$ is a Kähler form on X, (X, ω) is a Kähler orbifold and (L, h^L, ∇^L) is a prequantum line bundle on (X, ω).

Note that the existence of a positive line bundle L on a compact complex orbifold X implies that the Kodaira map associated to high powers of L gives a holomorphic embedding of X in the projective space. This is the generalization due to Baily of the Kodaira embedding theorem (see e.g. [30, Theorem 5.4.20]).

Let $g^{TX} = \omega(\cdot, J\cdot)$ be the Riemannian metric on X induced by $\omega = \frac{\sqrt{-1}}{2\pi} R^L$.

Using the Hermitian product along the fibers of L^p, E, $\Lambda(T^{*(1,0)}X)$, the Riemannian volume form dv_X and the definition (6.2) of the integral on an orbifold, we introduce an L^2-Hermitian product on $\Omega^{0,\bullet}(X, L^p \otimes E)$ similar to (3.9). This allows to define the formal adjoint $\overline{\partial}^{L^p\otimes E,*}$ of $\overline{\partial}^{L^p\otimes E}$ and as in (5.2), the operators D_p and $\square_p$. Then D_p^2 preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p \otimes E)$. We note that Hodge theory extends to compact orbifolds and delivers a canonical isomorphism

$$H^q(X, L^p \otimes E) \simeq \operatorname{Ker}(D_p^2|_{\Omega^{0,q}}). \tag{6.12}$$

By the same proof as in [27, Theorems 1.1, 2.5], [7, Theorem 1], we get vanishing results and the spectral gap property.

Theorem 6.4 *Let (X,ω) be a compact Kähler orbifold, (L,h^L) be a prequantum holomorphic Hermitian proper orbifold line bundle on (X,ω) and (E,h^E) be an arbitrary holomorphic Hermitian proper orbifold vector bundle on X.*

Then there exists $C>0$ such that the Dirac operator D_p satisfies for any $p\in\mathbb{N}$

$$\operatorname{Spec}(D_p^2)\subset\{0\}\cup]4\pi p-C,+\infty[, \tag{6.13}$$

and $D_p^2|_{\Omega^{0,>0}}$ is invertible for p large enough. Consequently, we have the Kodaira-Serre vanishing theorem, namely, for p large enough,

$$H^q(X,L^p\otimes E)=0, \quad \textit{for every } q>0. \tag{6.14}$$

In view of Theorem 6.4 and of the isomorphism (6.12), we can define for $p>C(2\pi)^{-1}$ the Bergman kernel

$$P_p(\cdot,\cdot)\in\mathcal{C}^\infty(X\times X,\pi_1^*(L^p\otimes E)\otimes\pi_2^*((L^p\otimes E)^*))$$

like in Definition 3.2. Namely, the Bergman kernel is the smooth kernel with respect to the Riemannian volume form $dv_X(x')$ of the orthogonal projection (Bergman projection) P_p from $\mathcal{C}^\infty(X,L^p\otimes E)$ onto $H^0(X,L^p\otimes E)$.

From now on, we assume $p>C(2\pi)^{-1}$. Let $d_p=\dim H^0(X,L^p\otimes E)$ and consider an arbitrary orthonormal basis $\{S_i^p\}_{i=1}^{d_p}$ of $H^0(X,L^p\otimes E)$ with respect to the Hermitian product (3.9) and (6.2). In fact, in the local coordinate above, $\widetilde{S}_i^p(\widetilde{z})$ are G_x-invariant on $\widetilde{U}_x$, and

$$P_p(y,y')=\sum_{i=1}^{d_p}\widetilde{S}_i^p(\widetilde{y})\otimes(\widetilde{S}_i^p(\widetilde{y}'))^*, \tag{6.15}$$

where we use $\widetilde{y}$ to denote the point in $\widetilde{U}_x$ representing $y\in U_x$.

The spectral gap property (6.13) shows that we have the analogue of Proposition 3.4, with the same F as given in (3.17):

$$|P_p(x,x')-F(D_p)(x,x')|_{\mathcal{C}^m(X\times X)}\leqslant C_{l,m,\varepsilon}\,p^{-l}. \tag{6.16}$$

As pointed out in [26], the property of the finite propagation speed of solutions of hyperbolic equations still holds on an orbifold (see the proof in [30, Appendix D.2]). Thus, $F(D_p)(x,x')=0$ for every for $x,x'\in X$ satisfying $d(x,x')\geqslant\varepsilon$. Likewise, given $x\in X$, $F(D_p)(x,\cdot)$ only depends on the restriction of D_p to $B^X(x,\varepsilon)$. Thus the problem of the asymptotic expansion of $P_p(x,\cdot)$ is local.

We recall that for every open set $U\subset X$ and orbifold chart $(G_U,\widetilde{U})\xrightarrow{\tau_U}U$, we add a superscript $\widetilde{\ }$ to indicate the corresponding objects on $\widetilde{U}$. Let $\partial U=\overline{U}\setminus U$, $U_1=\{x\in U, d(x,\partial U)<\varepsilon\}$. Then $F(\widetilde{D}_p)(\widetilde{x},\widetilde{x}')$ is well defined for $\widetilde{x},\widetilde{x}'\in\widetilde{U}_1=\tau_U^{-1}(U_1)$. Since $g\cdot F(\widetilde{D}_p)=F(\widetilde{D}_p)g$, we get

$$(g,1)F(\widetilde{D}_p)(g^{-1}\widetilde{x},\widetilde{x}')=(1,g^{-1})F(\widetilde{D}_p)(\widetilde{x},g\widetilde{x}'), \tag{6.17}$$

for every $g \in G_U$, $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_1$. Formula (6.6) shows that for every $x, x' \in U_1$ and $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_1$ representing x, x', we have

$$F(D_p)(x,x') = \sum_{g\in G_U} (g,1) F(\widetilde{D}_p)(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{6.18}$$

For $\widetilde{x}_0 \in \widetilde{U}_2 := \{x \in \widetilde{U}, d(x, \partial\widetilde{U}) < 2\varepsilon\}$, and $\widetilde{Z}, \widetilde{Z}' \in T_{\widetilde{x}_0}X$ with $|\widetilde{Z}|, |\widetilde{Z}'| \leqslant \varepsilon$, the kernel $F(\widetilde{D}_p)(\widetilde{Z}, \widetilde{Z}')$ has an asymptotic expansion as in Theorem 3.6 by the same argument as in Proposition 3.4. In the present situation $\boldsymbol{J} = J$, so that $a_j = 2\pi$ and the kernel $\mathscr{P}$ defined in (3.25) takes the form

$$\mathscr{P}(\widetilde{Z}, \widetilde{Z}') = \exp\Big(-\frac{\pi}{2}\sum_i \big(|\widetilde{z}_i|^2 + |\widetilde{z}'_i|^2 - 2\widetilde{z}_i\overline{\widetilde{z}}'_i\big)\Big). \tag{6.19}$$

6.3 Berezin-Toeplitz Quantization on Kähler Orbifolds

We apply now the results of Sect. 6.2 to establish the Berezin-Toeplitz quantization on Kähler orbifolds. We use the notations and assumptions of that section.

Since we consider the holomorphic case, we denote directly by P_p the orthogonal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $H^0(X, L^p \otimes E)$ and we replace in (4.1) the space $L^2(X, E_p)$ with $L^2(X, L^p \otimes E)$. Thus, we have the following definition.

Definition 6.5 A *Toeplitz operator* is a family $\{T_p\}$ of linear operators

$$T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E), \tag{6.20}$$

verifying (4.2) and (4.3).

For any section $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the *Berezin-Toeplitz quantization* of f is defined by

$$T_{f,p} : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E), \quad T_{f,p} = P_p f P_p. \tag{6.21}$$

Now, by the same argument as in Lemma 4.2, we get

Lemma 6.6 *For any $\varepsilon > 0$ and any $l, m \in \mathbb{N}$ there exists $C_{l,m,\varepsilon} > 0$ such that*

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m,\varepsilon}\, p^{-l} \tag{6.22}$$

for all $p \geqslant 1$ and all $(x,x') \in X \times X$ with $d(x,x') > \varepsilon$, where the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E and h^L, h^E, g^{TX}.

As in Sect. 4 we obtain next the asymptotic expansion of the kernel $T_{f,p}(x,x')$ in a neighborhood of the diagonal.

We need to introduce the appropriate analogue of the Condition 4.3 in the orbifold case, in order to take into account the group action associated to an orbifold chart. Let $\{\Xi_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\Xi_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)$ with smooth kernel $\Xi_p(x,y)$ with respect to $dv_X(y)$.

Condition 6.7 Let $k \in \mathbb{N}$. Assume that for every open set $U \in \mathcal{U}$ and every orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, there exists a sequence of kernels $\{\widetilde{\Xi}_{p,U}(\widetilde{x}, \widetilde{x}')\}_{p\in\mathbb{N}}$ and a family $\{Q_{r,x_0}\}_{0\leqslant r\leqslant k, x_0\in X}$ such that

(a) $Q_{r,x_0} \in \operatorname{End}(E)_{x_0}[\widetilde{Z}, \widetilde{Z}']$,
(b) $\{Q_{r,x_0}\}_{r\in\mathbb{N}, x_0\in X}$ is smooth with respect to the parameter $x_0 \in X$,
(c) for every fixed $\varepsilon'' > 0$ and every $\widetilde{x}, \widetilde{x}' \in \widetilde{U}$ the following holds

$$\begin{aligned} &(g,1)\widetilde{\Xi}_{p,U}(g^{-1}\widetilde{x}, \widetilde{x}') = (1, g^{-1})\widetilde{\Xi}_{p,U}(\widetilde{x}, g\widetilde{x}') \\ &\quad \text{for any } g \in G_U \text{ (cf. (6.17))}, \\ &\widetilde{\Xi}_{p,U}(\widetilde{x}, \widetilde{x}') = \mathscr{O}(p^{-\infty}) \quad \text{for } d(x, x') > \varepsilon'', \\ &\Xi_p(x, x') = \sum_{g\in G_U} (g,1)\widetilde{\Xi}_{p,U}(g^{-1}\widetilde{x}, \widetilde{x}') + \mathscr{O}(p^{-\infty}), \end{aligned} \tag{6.23}$$

and moreover, for every relatively compact open subset $\widetilde{V} \subset \widetilde{U}$, the relation

$$\begin{aligned} &p^{-n}\,\widetilde{\Xi}_{p,U,\widetilde{x}_0}(\widetilde{Z}, \widetilde{Z}') \cong \sum_{r=0}^{k} (Q_{r,\widetilde{x}_0}\mathscr{P}_{\widetilde{x}_0})(\sqrt{p}\widetilde{Z}, \sqrt{p}\widetilde{Z}')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}) \\ &\quad \text{for } \widetilde{x}_0 \in \widetilde{V}, \end{aligned} \tag{6.24}$$

holds in the sense of (4.10).

Notation 6.8 If the sequence $\{\Xi_p\}_{p\in\mathbb{N}}$ satisfies Condition 6.7, we write

$$p^{-n}\,\Xi_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}). \tag{6.25}$$

Note that although the Notations 6.8 and 4.4 are formally similar, they have different meaning.

Lemma 6.9 *The smooth family* $Q_{r,x_0} \in \operatorname{End}(E)_{x_0}[\widetilde{Z}, \widetilde{Z}']$ *in Condition* 6.7 *is uniquely determined by* Ξ_p.

Proof Clearly, for $W \subset U$, the restriction of $\widetilde{\Xi}_{p,U}$ to $\widetilde{W} \times \widetilde{W}$ verifies (6.23), thus we can take $\widetilde{\Xi}_{p,W} = \widetilde{\Xi}_{p,U}|_{\widetilde{W}\times\widetilde{W}}$. Since G_U acts freely on $\tau_U^{-1}(U_{reg}) \subset \widetilde{U}$, we deduce from (6.23) and (6.24) that

$$\Xi_{p,x_0}(Z, Z') = \widetilde{\Xi}_{p,U,\widetilde{x}_0}(\widetilde{Z}, \widetilde{Z}') + \mathscr{O}(p^{-\infty}), \tag{6.26}$$

for every $x_0 \in U_{reg}$ and $|\widetilde{Z}|, |\widetilde{Z}'|$ small enough. We infer from (6.24) and (6.26) that $Q_{r,x_0} \in \operatorname{End}(E)_{x_0}[\widetilde{Z}, \widetilde{Z}']$ is uniquely determined for $x_0 \in X_{reg}$. Since Q_{r,x_0} depends smoothly on x_0, its lift to $\widetilde{U}$ is smooth. Since the set $\tau_U^{-1}(U_{reg})$ is dense in $\widetilde{U}$, we see that the smooth family Q_{r,x_0} is uniquely determined by Ξ_p. □

Lemma 6.10 *There exist polynomials* $J_{r,x_0}, Q_{r,x_0}(f) \in \mathrm{End}(E)_{x_0}[\widetilde{Z}, \widetilde{Z}']$ *such that Lemmas* 4.2, 4.5, 4.6 *and* 4.7 *still hold under the notation* (6.25). *Moreover,*

$$J_{0,x_0} = \mathrm{Id}_E, \qquad J_{1,x_0} = 0. \tag{6.27}$$

Proof The analogues of Proposition 3.4, Theorem 3.6 for the current situation and (6.17), (6.18) show that Lemmas 4.2 and 4.5 still hold under the notation (6.25). Since in our case ω is a Kähler form with respect to the complex structure J and $\boldsymbol{J} = J$, we have $\mathcal{O}_1 = 0$ (cf. (3.29) and Remark 3.9). Hence (3.31) entails (6.27). Moreover, (6.16) implies

$$T_{f,p}(x, x') = \int_X F(D_p)(x, x'')f(x'')F(D_p)(x'', x')dv_X(x'') + \mathcal{O}(p^{-\infty}). \tag{6.28}$$

Therefore, we deduce from (6.8), (6.17), (6.18) and (6.28) that Lemmas 4.6 and 4.7 still hold under the notation (6.25). □

We will prove next a useful criterion (an analogue of Theorem 4.9) which ensures that a given family is a Toeplitz operator.

Theorem 6.11 *Let* $\{T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)\}$ *be a family of bounded linear operators which satisfies the following three conditions*:

(i) *For any* $p \in \mathbb{N}$, $P_p T_p P_p = T_p$.
(ii) *For any* $\varepsilon_0 > 0$ *and any* $l \in \mathbb{N}$, *there exists* $C_{l,\varepsilon_0} > 0$ *such that for all* $p \geqslant 1$ *and all* $(x, x') \in X \times X$ *with* $d(x, x') > \varepsilon_0$,

$$|T_p(x, x')| \leqslant C_{l,\varepsilon_0} p^{-l}. \tag{6.29}$$

(iii) *There exists a family of polynomials* $\{\mathcal{Q}_{r,x_0} \in \mathrm{End}(E)_{x_0}[Z, Z']\}_{x_0 \in X}$ *such that*:
  (a) *each* $\mathcal{Q}_{r,x_0}$ *has the same parity as* r,
  (b) *the family is smooth in* $x_0 \in X$ *and*
  (c) *there exists* $0 < \varepsilon' < a_X/4$ *such that for every* $x_0 \in X$, *every* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| < \varepsilon'$ *and every* $k \in \mathbb{N}$, *we have*

$$p^{-n}T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0}\mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}) \tag{6.30}$$

  in the sense of (6.25).

Then $\{T_p\}$ *is a Toeplitz operator.*

Proof As explained in (4.26), we can assume that T_p is self-adjoint. We will define inductively the sequence $(g_l)_{l\geqslant 0}$, $g_l \in \mathcal{C}^\infty(X, \mathrm{End}(E))$ such that

$$T_p = \sum_{l=0}^{m} P_p g_l p^{-l} P_p + \mathcal{O}(p^{-m-1}) \quad \text{for every } m \geqslant 0, \tag{6.31}$$

using the same procedure as in (4.27). Moreover, we can take these g_l's to be self-adjoint. For $x_0 \in X$, we set

$$g_0(x_0) = \mathcal{Q}_{0,x_0}(0,0) \in \operatorname{End}(E_{x_0}). \tag{6.32}$$

We will show that

$$T_p = P_p g_0 P_p + \mathcal{O}(p^{-1}). \tag{6.33}$$

We need to establish the following analogue of Proposition 4.11.

Proposition 6.12 *In the conditions of Theorem 6.11, we have* $\mathcal{Q}_{0,x_0}(Z,Z') = \mathcal{Q}_{0,x_0}(0,0) \in \operatorname{End}(E_{x_0})$ *for all* $x_0 \in X$ *and all* $Z, Z' \in T_{x_0}X$.

Proof The key observation is the following. Let $\{\varXi_p\}_{p\in\mathbb{N}}$, $\{Q_{0,x_0}\}_{x_0\in X}$ and $\{\varXi'_p\}_{p\in\mathbb{N}}$, $\{Q'_{0,x_0}\}_{x_0\in X}$ two pairs satisfying Condition 6.7 for $k=0$. Then (6.8), (6.23) and (6.24) imply that

$$\begin{aligned} &p^{-n}(\varXi_p \circ \varXi'_p)_{x_0}(Z,Z') \\ &\quad\cong ((Q_{0,x_0}\mathscr{P}_{x_0}) \circ (Q'_{0,x_0}\mathscr{P}_{x_0}))(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-\frac{1}{2}}), \end{aligned} \tag{6.34}$$

in the sense of Notation 6.8 and (2.10).

We modify now the proof of Lemma 4.12. Formula (6.30) for $k=0$ gives

$$p^{-n}T_{p,x_0}(Z,Z') \cong (\mathcal{Q}_{0,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1/2}). \tag{6.35}$$

Moreover, the analogue of Lemma 4.5 shows that

$$p^{-n}P_{p,x_0}(Z,Z') \cong (J_{0,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-1/2}). \tag{6.36}$$

By (6.35) and (6.36) we can apply the observation at the beginning for $\varXi_p = T_p$ and $\varXi'_p = P_p$ to obtain

$$\begin{aligned} &p^{-n}(P_pT_pP_p)_{x_0}(Z,Z') \\ &\quad\cong ((\mathscr{P}J_0) \circ (\mathcal{Q}_0\mathscr{P}) \circ (\mathscr{P}J_0))_{x_0}(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1/2}). \end{aligned} \tag{6.37}$$

Using the same argument as in the proof of Lemma 4.12 (note also that $J_{0,x_0} = \operatorname{Id}_E$ by (6.27)) we see that $\mathcal{Q}_{0,x_0}$ is a polynomial in $z, \overline{z}'$.

Now, we need to establish the analogue of (4.34). We define $F^{(i)}(\widetilde{x},\widetilde{y})$, $\widetilde{F}^{(i)}(\widetilde{x},\widetilde{y})$ as in (4.37). Then from (6.17), (6.23), we know that for $g \in G_U$, $\widetilde{x}, \widetilde{y} \in \widetilde{U}_1$,

$$g \cdot F^{(i)}(g^{-1}\widetilde{x}, \widetilde{y}) = F^{(i)}(\widetilde{x}, g\widetilde{y}). \tag{6.38}$$

We denote by $F^{(i)}F(D_p)$ and $F(D_p)\widetilde{F}^{(i)}$ the operators defined by the kernels

$$\eta(d(x,y))F^{(i)}(\widetilde{x},\widetilde{y})F(\widetilde{D}_p)(\widetilde{x},\widetilde{y}) \quad \text{and} \quad \eta(d(x,y))F(\widetilde{D}_p)(\widetilde{x},\widetilde{y})\widetilde{F}^{(i)}(\widetilde{x},\widetilde{y})$$

as in (6.5) and (6.6). Set

$$\mathcal{T}_p = T_p - \sum_{i \leqslant \deg F_x} (F^{(i)} F(D_p)) p^{i/2}. \tag{6.39}$$

Now using (6.39) instead of (4.38), by (6.5) and the proof of Proposition 4.11, we get the analogue of (4.34) and hence Proposition 6.12. □

We go on with the proof of Theorem 6.11. Applying Proposition 6.12 and the proof of Proposition 4.17, we obtain (6.33).

Finally, we deduce (6.31) according to the pattern set down in the proof of Theorem 4.9. This completes the proof of Theorem 6.11. □

We can therefore show that the set of Toeplitz operators on a compact orbifold is closed under the composition of operators, so forms an algebra.

Theorem 6.13 *Let (X, ω) be a compact Kähler orbifold and (L, h^L) be a holomorphic Hermitian proper orbifold line bundle satisfying the prequantization condition (1.5). Let (E, h^E) be an arbitrary holomorphic Hermitian proper orbifold vector bundle on X.*

Consider $f, g \in \mathcal{C}^\infty(X, \mathrm{End}(E))$. Then the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits an asymptotic expansion in the sense of (1.7), where $C_r(f, g) \in \mathcal{C}^\infty(X, \mathrm{End}(E))$ and C_r are bidifferential operators defined locally as in (1.7) on each covering $\widetilde{U}$ of an orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$. In particular $C_0(f, g) = fg$.

If $f, g \in \mathcal{C}^\infty(X)$, then (1.9) holds.

Relation (4.90) also holds for any $f \in \mathcal{C}^\infty(X, \mathrm{End}(E))$.

Proof Notice that by using (6.34) we have

$$(T_{f,p} T_{g,p})(x, x') = \int_X (F(D_p) f F(D_p))(x, x'')(F(D_p) g F(D_p))(x'', x') dv_X(x'') + \mathcal{O}(p^{-\infty}). \tag{6.40}$$

From (6.8), (6.40) and the proof of Theorem 1.1, we get Theorem 6.13. □

Remark 6.14 As in Remark 4.20, Theorem 6.13 shows that on every compact Kähler orbifold X admitting a prequantum line bundle (L, h^L), we can define in a canonical way an associative star-product $f * g = \sum_{l=0}^{\infty} \hbar^l C_l(f, g) \in \mathcal{C}^\infty(X)[[\hbar]]$ for every $f, g \in \mathcal{C}^\infty(X)$, called the *Berezin-Toeplitz star-product*. Moreover, $C_l(f, g)$ are bidifferential operators defined locally as in the smooth case.

6.4 Symplectic Orbifolds

In this section we state the result for symplectic orbifolds.

We work on a compact symplectic orbifold (X, ω) of real dimension $2n$. Assume that there exists a proper orbifold Hermitian line bundle L over X endowed with a

Hermitian connection ∇^L with the prequantization property $\frac{\sqrt{-1}}{2\pi}R^L = \omega$. This implies in particular that there exist $k \in \mathbb{N}$ such that L^k is a line bundle in the usual sense. Let (E, h^E) be a proper orbifold Hermitian vector bundle on X equipped with a Hermitian connection ∇^E.

Let J be an almost complex structure on TX such that (3.6) holds. We endow X with a Riemannian metric g^{TX} compatible with J.

Then the construction in Sect. 3.1 goes through, especially, we can define the spin^c Dirac operator $D_p : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E)$. The orthogonal projection $P_p : L^2(X, E_p) \longrightarrow \mathrm{Ker}(D_p)$ with $E_p := \Lambda^{0,\bullet} \otimes L^p \otimes E$ is called the Bergman projection. The smooth kernel $P_p(\cdot,\cdot)$ of P_p with respect to the Riemannian volume form dv_X, is called the Bergman kernel of D_p.

We define the Toeplitz operator $T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)$ as in Definition 4.1 by using the orthogonal projection P_p defined above. Especially $T_{f,p} = P_p f P_p$ for $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$.

By the argument in Sect. 6.2 we see that Theorem 3.3 and Proposition 3.4 still hold:

Theorem 6.15 *Assume that (X, J, ω) is a compact symplectic orbifold endowed with a prequantum proper line bundle (L, h^L, ∇^L). We endow X with a Riemannian metric g^{TX} compatible with J. Let (E, h^E) be a proper orbifold Hermitian vector bundle on X with Hermitian connection ∇^E. Then*

(i) *the associated Dirac operator D_p has a spectral gap* (3.16), *and*
(ii) *$P_p(x, x') = \mathscr{O}(p^{-\infty})$ for $d(x, x') > \varepsilon > 0$ in the sense of* (3.19).

Now by combining the argument in Sects. 3.2 and 6.3, we get the following extension of Theorem 1.1.

Theorem 6.16 *Let us make the same assumptions as in Theorem 6.15. Then for every $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits an asymptotic expansion in the sense of* (1.7), *where $C_r(f, g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and C_r are bidifferential operators defined locally as in* (1.7) *on each covering $\widetilde{U}$ of an orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$. In particular, $C_0(f, g) = fg$.*

If $f, g \in \mathscr{C}^\infty(X)$, then (1.9) *holds.*

Relation (4.90) *also holds for any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$.*

As before, for the given data $X, J, g^{TX}, L, h^L, \nabla^L$ from Theorem 6.15 and $E = \mathbb{C}$, Theorem 6.16 implies a canonical construction of the (associative) Berezin-Toeplitz star-product $f * g = \sum_{l=0}^{\infty} \hbar^l C_l(f, g) \in \mathscr{C}^\infty(X)[[\hbar]]$ for every $f, g \in \mathscr{C}^\infty(X)$.

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