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The spin^c Dirac operator on high tensor powers of a line bundle

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Abstract. We study the asymptotic of the spectrum of the spin^c Dirac operator on high tensor powers of a line bundle. As application, we get a simple proof of the main result of Guillemin–Uribe [13, Theorem 2], which was originally proved by using the analysis of Toeplitz operators of Boutet de Monvel and Guillemin [10].

1. Introduction

Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a hermitian line bundle L over X endowed with a hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$, where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L). Let E be a hermitian vector bundle on X.

Let g^{TX} be a riemannian metric on X. Let $J_0 : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation $\omega(u,v)=g^{TX}(J_0u,v)$. Let J be an almost complex structure which is compatible with g^{TX} and ω. Then one can construct canonically a spin^c Dirac operator D_k acting on $\Omega^{0,\bullet}(X,L^k\otimes E)=\bigoplus_{q=0}^n\Omega^{0,q}(X,L^k\otimes E)$, the direct sum of spaces of $(0,q)$–forms with values in $L^k\otimes E$. Let

$$\lambda=\inf_{u\in T_x^{(1,0)}X,\, x\in X} R_x^L(u,\overline{u})/|u|^2_{g^{TX}}>0\ .$$

One of our main results is as follows:

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Theorem 1.1. *There exists $C > 0$ such that for $k \in \mathbb{N}$, the spectrum of D_k^2 is contained in the set $\{0\} \cup (2k\lambda - C, +\infty)$. Set $D_k^- = D_k \restriction_{\Omega^{0,odd}}$, then for k large enough, we have*

$$\operatorname{Ker} D_k^- = \{0\}. \tag{1.1}$$

We recover with (1.1) the vanishing result of [6, Theorem 2.3], [11, Theorem 3.2]. Another interesting application is to describe the asymptotics of the spectrum of the metric Laplacian $\Delta_k = (\nabla^{L^k\otimes E})^* \nabla^{L^k\otimes E}$ acting on $\mathcal{C}^\infty(X, L^k \otimes E)$. Introduce the smooth function $\tau(x) = \sum_j R^L(w_j, \overline{w}_j) > 0$, $x \in X$, where $\{w_j\}_{j=1}^n$ is an orthonormal basis of $T_x^{(1,0)}X$.

Corollary 1.2. *The spectrum of the Schrödinger operator $\Delta_k^\# = \Delta_k - k\tau$ is contained in the union $(-a, a) \cup (2k\lambda - b, +\infty)$, where a and b are positive constants independent of k. For k large enough, the number d_k of eigenvalues on the interval $(-a, a)$ satisfies $d_k = \langle \operatorname{ch}(L^k \otimes E) \operatorname{Td}(X), [X]\rangle$. In particular $d_k \sim k^n(\operatorname{rank} E) \operatorname{vol}_\omega(X)$.*

In the case E is a trivial line bundle, Corollary 1.2 is the main result of Guillemin and Uribe [13, Theorem 2] [1]. The idea in [6], [11], [12], [13] is that one first reduces the problem to a problem on the unitary circle bundle of L^*, then one applies Melin inequality [14, Theorem 22.3.2] to show that $\Delta_k^\#$ is semi–bounded from below. In order to prove [13, Theorem 2], they apply the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [10]. For the interesting applications of [13, Theorem 2], we refer the reader to Borthwick and Uribe [6], [8], [9]. For the related topic on geometric quantization, see [16], [21]. Our proof is based on a direct application of Lichnerowicz formula.

This paper is organized as follows. In Sect. 2, we recall the construction of the spin^c Dirac operator and prove our main technical result, Theorem 2.5. In Sect. 3, we prove Theorem 1.1 and Corollary 1.2. In Sect. 4, we generalize our result to the L_2 case. In particular, we obtain a new proof of [12, Theorem 2.6].

2. The Lichnerowicz formula

Let (X, ω) be a compact symplectic manifold. Let (L, h^L), (E, h^E) be two hermitian complex vector bundles endowed with hermitian connections ∇^L and ∇^E respectively. Let R^L and R^E be their curvatures. We assume $\operatorname{rank} L = 1$ and $R^L = -2\pi\sqrt{-1}\omega$. Let g^{TX} be an arbitrary riemannian metric on TX. Let J be an almost complex structure which is compatible

[1] In [13], they only knew $d_k \sim k^n \operatorname{vol}_\omega(X)$. When $J_0 = J$, Borthwick and Uribe [6, p. 854] got the precise value d_k, for large enough k, in this case.

with g^{TX} and ω (For the existence of J, we refer to [17, p.61]). Then J defines canonically an orientation of X. Let $J_0 : TX \longrightarrow TX$ be the skew–adjoint linear map defined by

$$\omega(u,v) = g^{TX}(J_0 u, v), \ \text{ for } \ u,v \in TX. \tag{2.1}$$

Then J commutes with J_0.

Let $TX^c = TX \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the tangent bundle. The almost complex structure J induces a splitting $TX^c = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Accordingly, we have a decomposition of the complexified cotangent bundle: $T^*X^c = T^{(1,0)\,*}X \oplus T^{(0,1)\,*}X$. The exterior algebra bundle decomposes as $\Lambda T^*X^c = \oplus_{p,q}\Lambda^{p,q}$, where $\Lambda^{p,q} := \Lambda^{p,q}T^*X^c = \Lambda^p(T^{(1,0)\,*}X) \otimes \Lambda^q(T^{(0,1)\,*}X)$.

Let ∇^{TX} be the Levi–Civita connection of the metric g^{TX}, and let $\nabla^{1,0}$ and $\nabla^{0,1}$ be the canonical hermitian connections on $T^{(1,0)}X$ and $T^{(0,1)}X$ respectively:

$$\begin{aligned}\nabla^{1,0} &= \tfrac{1}{4}(1-\sqrt{-1}J)\,\nabla^{TX}\,(1-\sqrt{-1}J)\,,\\ \nabla^{0,1} &= \tfrac{1}{4}(1+\sqrt{-1}J)\,\nabla^{TX}\,(1+\sqrt{-1}J)\,.\end{aligned}$$

Set $A_2 = \nabla^{TX} - \left(\nabla^{1,0} \oplus \nabla^{0,1}\right) \in T^*X \otimes \mathrm{End}(TX)$ which satisfies $J\,A_2 = -A_2\,J$.

Let us recall some basic facts about the spin^c Dirac operator on an almost complex manifold [15, Appendix D]. The fundamental $\mathbb{Z}_2$ spinor bundle induced by J is given by $\Lambda^{0,\bullet} = \Lambda^{\text{even}}(T^{(0,1)\,*}X) \oplus \Lambda^{\text{odd}}(T^{(0,1)\,*}X)$. For any $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0} \in T^{(0,1)*}X$ be the metric dual of $v_{1,0}$. Then $\mathbf{c}(v) = \sqrt{2}(\overline{v}^*_{1,0} \wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda^{0,\bullet}$, where $\wedge$ and i denote the exterior and interior product respectively.

Formally, we may think

$$\Lambda^{0,\bullet} = S\,(TX) \otimes \left(\det T^{(1,0)}X\right)^{1/2}\,,$$

where $S\,(TX)$ is the spinor bundle of the possibly non–existent spin structure on TX, and $\left(\det T^{(1,0)}X\right)^{1/2}$ is the possibly non–existent square root of $\det T^{(1,0)}X$.

Moreover, by [15, pp.397–398], ∇^{TX} induces canonically a Clifford connection on $\Lambda^{0,\bullet}$. Formally, let $\nabla^{S(TX)}$ be the Clifford connection on $S(TX)$ induced by ∇^{TX}, and let $\nabla^{\det}$ be the connection on $(\det T^{(1,0)}X)^{1/2}$ induced by $\nabla^{1,0}$. Then

$$\nabla^{\text{Cliff}} = \nabla^{S(TX)} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\det}\,.$$

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$. Then

$$e_{2j} = \tfrac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j-1} = \tfrac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)\,, \tag{2.2}$$
$$j = 1, \dots, n\,,$$

form an orthonormal frame of TX. Let $\{w^j\}_{j=1}^n$ be the dual frame of $\{w_j\}_{j=1}^n$. Let Γ be the connection form of $\nabla^{1,0} \oplus \nabla^{0,1}$ in local coordinates. Then $\nabla^{TX} = d + \Gamma + A_2$. By [15, Theorem 4.14, p.110], the Clifford connection ∇^{Cliff} on $\Lambda^{0,\bullet}$ has the following local form:

$$\begin{aligned} \nabla^{\text{Cliff}} &= d + \tfrac{1}{4} \sum_{i,j} \big\langle (\Gamma + A_2) e_i, e_j \big\rangle \mathbf{c}(e_i)\mathbf{c}(e_j) \\ &= d + \sum_{l,m} \Big\{ \big\langle \Gamma w_l, \overline{w}_m \big\rangle\, \overline{w}^l \wedge i_{\overline{w}_m} + \\ &\qquad \tfrac{1}{2} \big\langle A_2 w_l, w_m \big\rangle\, i_{\overline{w}_l}\, i_{\overline{w}_m} + \tfrac{1}{2} \big\langle A_2 \overline{w}_l, \overline{w}_m \big\rangle\, \overline{w}^l \wedge \overline{w}^m \wedge \Big\}\,. \end{aligned} \tag{2.3}$$

Let $\nabla^{L^k \otimes E}$ be the connection on $L^k \otimes E$ induced by ∇^L, ∇^E. Let $\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E}$ be the connection on $\Lambda^{0,\bullet} \otimes L^k \otimes E$,

$$\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} = \nabla^{\text{Cliff}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{L^k \otimes E}. \tag{2.4}$$

Along the fibers of $\Lambda^{0,\bullet} \otimes L^k \otimes E$, we consider the pointwise scalar product $\langle \cdot, \cdot \rangle$ induced by g^{TX}, h^L and h^E. Let dv_X be the riemannian volume form of (TX, g^{TX}). The L_2–scalar product on $\Omega^{0,\bullet}(X, L^k \otimes E)$, the space of smooth sections of $\Lambda^{0,\bullet} \otimes L^k \otimes E$, is given by

$$(s_1, s_2) = \int_X \langle s_1(x), s_2(x) \rangle\, dv_X(x)\,. \tag{2.5}$$

We denote the corresponding norm with $\|\cdot\|$.

Definition 2.1. *The* spin$^{\mathrm{c}}$ *Dirac operator* D_k *is defined by*

$$D_k = \sum_{j=1}^{2n} \mathbf{c}(e_j) \nabla_{e_j}^{\Lambda^{0,\bullet} \otimes L^k \otimes E} : \Omega^{0,\bullet}(X, L^k \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^k \otimes E)\,. \tag{2.6}$$

D_k is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^k \otimes E)$, which interchanges $\Omega^{0,\text{even}}(X, L^k \otimes E)$ and $\Omega^{0,\text{odd}}(X, L^k \otimes E)$. We denote

$$D_k^+ = D_k \restriction_{\Omega^{0,\text{even}}}\,, \quad D_k^- = D_k \restriction_{\Omega^{0,\text{odd}}}\,. \tag{2.7}$$

Let $R^{T^{(1,0)}X}$ be the curvature of $\big(T^{(1,0)}X, \nabla^{1,0}\big)$. Let

$$\begin{aligned} \omega_d &= -\sum_{l,m} R^L(w_l, \overline{w}_m)\, \overline{w}^m \wedge i_{\overline{w}_l}\,, \\ \tau(x) &= \sum_j R^L(w_j, \overline{w}_j)\,. \end{aligned} \tag{2.8}$$

Remark that by (2.1), at $x \in X$, there exists $\{w_i\}_{i=1}^n$ an orthogonal basis of $T^{(1,0)}X$, such that $J_0 = \sqrt{-1}\,\operatorname{diag}(a_1(x), \cdots, a_n(x)) \in \operatorname{End}(T^{(1,0)}X)$, and $a_i(x) > 0$ for $i \in \{1, \cdots, n\}$. So

$$\begin{aligned} \omega_d &= -2\pi \sum_l a_l(x)\, \overline{w}^l \wedge i_{\overline{w}_l}\,, \\ \tau(x) &= 2\pi \sum_l a_l(x)\,. \end{aligned} \tag{2.9}$$

The following Lichnerowicz formula is crucial for us.

Theorem 2.2. *The square of the Dirac operator satisfies the equation:*

$$D_k^2 = \Big(\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}\Big)^* \nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} - 2k\omega_d - k\tau + \tfrac{1}{4}K + \mathbf{c}(R), \tag{2.10}$$

where K is the scalar curvature of (TX, g^{TX}), and

$$\mathbf{c}(R) = \sum_{l<m} \Big(R^E + \tfrac{1}{2}\operatorname{Tr}\Big[R^{T^{(1,0)}X}\Big]\Big)(e_l, e_m)\, \mathbf{c}(e_l)\, \mathbf{c}(e_m)\,.$$

Proof. By Lichnerowicz formula [3, Theorem 3.52], we know that

$$\begin{aligned} D_k^2 = &\Big(\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}\Big)^* \nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} \\ &+\tfrac{1}{4}K + \mathbf{c}(R) + k\sum_{l<m} R^L(e_l, e_m)\, \mathbf{c}(e_l)\, \mathbf{c}(e_m)\,. \end{aligned} \tag{2.11}$$

Now, we identify R^L with a purely imaginary antisymmetric matrix $-2\pi\sqrt{-1}J_0 \in \operatorname{End}(TX)$ by (2.1). As $J_0 \in \operatorname{End}(T^{(1,0)}X)$, by [3, Lemma 3.29], we get (2.10). $\square$

Remark 2.3. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a Clifford module. Then it was observed by Braverman [11, § 9] that, with the same proof of [3, Proposition 3.35], there exists a vector bundle W on X such that $\mathcal{E} = \Lambda^{0,\bullet} \otimes W$ as a $\mathbb{Z}_2$–graded Clifford module.

As a simple consequence of Theorem 2.2, we recover the statement on the drift of spectrum of the metric Laplacian first proved by Guillemin–Uribe [13, Theorem 1], (see also [6, Theorem 2.1], [11, Theorem 4.4]), by passing to the circle bundle of L^* and applying Melin's inequality [14, Theorems 22.3.2–3].

Corollary 2.4. *There exists $C>0$ such that for $k\in\mathbb{N}$, the metric Laplacian $\Delta_k=\left(\nabla^{L^k\otimes E}\right)^*\nabla^{L^k\otimes E}$ on $\mathcal{C}^\infty(X,L^k\otimes E)$ satisfies :*

$$\Delta_k-k\tau\geqslant -C\,. \tag{2.12}$$

Proof. By (2.10), $s\in\mathcal{C}^\infty(X,L^k\otimes E)$,

$$\begin{aligned}\|D_ks\|^2&=\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}s\|^2-k(\tau(x)s,s)\\&\quad+\left(\left(\tfrac14K+\mathbf{c}(R)\right)s,s\right)\,.\end{aligned}\tag{2.13}$$

From (2.3), we infer that

$$\left\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}s\right\|^2=\left\|\nabla^{L^k\otimes E}s\right\|^2+\left\|\sum_{l,m}\left\langle A_2\overline{w}_l,\overline{w}_m\right\rangle\overline{w}^l\wedge\overline{w}^m\wedge s\right\|^2.$$

and therefore there exists a constant $C>0$ not depending on k such that

$$\begin{aligned}0\leqslant\|D_ks\|^2&\leqslant\left\|\nabla^{L^k\otimes E}s\right\|^2-k(\tau(x)s,s)+C\|s\|^2\\&=\left((\Delta_k-k\tau(x))s,s\right)+C\|s\|^2\,.\end{aligned}$$

□

The following is our main technical result.

Theorem 2.5. *There exists $C>0$ such that for any $k\in\mathbb{N}$ and any $s\in\Omega^{>0}(X,L^k\otimes E)=\bigoplus_{q\geqslant1}\Omega^{0,q}(X,L^k\otimes E)$,*

$$\|D_ks\|^2\geqslant(2k\lambda-C)\|s\|^2\,. \tag{2.14}$$

Proof. By (2.10), for $s\in\Omega^{0,\bullet}(X,L^k\otimes E)$,

$$\begin{aligned}\|D_ks\|^2&=\{\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}s\|^2-k(\tau(x)s,s)\}-2k(\omega_ds,s)\\&\quad+\left(\left(\tfrac14K+\mathbf{c}(R)\right)s,s\right)\,.\end{aligned}\tag{2.15}$$

We consider now $s\in\mathcal{C}^\infty(X,L^k\otimes E')$, where $E'=E\otimes\Lambda^{0,\bullet}$. Estimate (2.12) becomes

$$\left\|\nabla^{L^k\otimes E'}s\right\|^2-k(\tau(x)s,s)\geqslant -C\|s\|^2\,. \tag{2.16}$$

If $s\in\Omega^{>0}(X,L^k\otimes E)$, the second term of (2.15), $-2k(\omega_ds,s)$ is bounded below by $2k\lambda\|s\|^2$. While the third term of (2.15) is $O(\|s\|^2)$. The proof of (2.14) is completed. □

3. Applications of Theorem 2.5

Proof of Theorem 1.1. By (2.14), we get immediately (1.1). For the rest, we use the trick of the proof of Mckean–Singer formula.

Let $\mathcal{H}_\mu$ be the spectral space of D_k^2 corresponding to the interval $(0,\mu)$. Let $\mathcal{H}_\mu^+$, $\mathcal{H}_\mu^-$ be the intersections of $\mathcal{H}_\mu$ with the spaces of forms of even and odd degree respectively. Then $\mathcal{H}_\mu = \mathcal{H}_\mu^+ \oplus \mathcal{H}_\mu^-$. Since D_k^+ commutes with the spectral projection, we have a well defined operator $D_k^+ : \mathcal{H}_\mu^+ \longrightarrow \mathcal{H}_\mu^-$ which is obviously injective. But estimate (2.14) implies that $\mathcal{H}_\mu^- = 0$ for every $\mu < 2k\lambda - C$, hence also $\mathcal{H}_\mu^+ = 0$, for this range of μ. Thus $\mathcal{H}_\mu = 0$, for $0 < \mu < 2k\lambda - C$. The proof of our theorem is completed. □

Proof of Corollary 1.2. Let $P_k : \Omega^{0,\bullet}(X, L^k \otimes E) \longrightarrow \mathcal{C}^\infty(X, L^k \otimes E)$ be the orthogonal projection. For $s \in \Omega^{0,\bullet}(X, L^k \otimes E)$, we will denote $s_0 = P_k s$ its $0-$degree component. We will estimate $\Delta_k^\#$ on $P_k(\operatorname{Ker} D_k^+)$ and $(\operatorname{Ker} D_k^+)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E)$.

In the sequel we denote with C all positive constants independent of k, although there may be different constants for different estimates. From (2.13), there exists $C > 0$ such that for $s \in \mathcal{C}^\infty(X, L^k \otimes E)$,

$$\big|\|D_k s\|^2 - (\Delta_k^\# s, s)\big| \leqslant C\|s\|^2 . \tag{3.1}$$

Theorem 1.1 and (3.1) show that there exists $b > 0$ such that for $k \in \mathbb{N}$,

$$(\Delta_k^\# s, s) \geqslant (2k\lambda - b)\|s\|^2 , \quad \text{for } s \in \mathcal{C}^\infty(X, L^k \otimes E) \cap (\operatorname{Ker} D_k^+)^\perp . \tag{3.2}$$

We focus now on elements from $P_k(\operatorname{Ker} D_k^+)$, and assume $s \in \operatorname{Ker} D_k$. Set $s' = s - s_0 \in \Omega^{>0}(X, L^k \otimes E)$. By (2.15), (2.16),

$$-2k(\omega_d s, s) \leqslant C\|s\|^2 . \tag{3.3}$$

We obtain thus [6, Theorem 2.3] (see also [7], [11, Theorem 3.13]) for $k \gg 1$,

$$\|s'\| \leqslant Ck^{-1/2}\|s_0\| . \tag{3.4}$$

(from (3.4), they got $\operatorname{Ker} D_k^- = 0$ for $k \gg 1$ as $s_0 = 0$ if $s \in \operatorname{Ker} D_k^-$). In view of (2.15) and (3.4),

$$\|\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} s\|^2 - k(\tau(x)s_0, s_0) \leqslant C\|s_0\|^2 . \tag{3.5}$$

By (2.3),

$$\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} s = \nabla^{L^k \otimes E} s_0 + A_2' s_2 + \alpha , \tag{3.6}$$

where s_2 is the component of degree 2 of s, A'_2 is a contraction operator comming from the middle term of (2.3), and $\alpha \in \Omega^{>0}(X, L^k \otimes E)$. By (3.5), (3.6), we know

$$\left\|\nabla^{L^k\otimes E} s_0 + A'_2 s_2\right\|^2 - k(\tau(x)s_0, s_0) \leqslant C\|s_0\|^2\,, \tag{3.7}$$

and by (3.4), (3.7),

$$\left\|\nabla^{L^k\otimes E} s_0\right\|^2 \leqslant Ck\|s_0\|^2\,, \tag{3.8}$$

By (3.4) and (3.8), we get

$$\begin{aligned}\left\|\nabla^{L^k\otimes E} s_0 + A'_2 s_2\right\|^2 &\geqslant \left\|\nabla^{L^k\otimes E} s_0\right\|^2 - 2\left\|\nabla^{L^k\otimes E} s_0\right\|\left\|A'_2 s_2\right\| \\ &\geqslant \left\|\nabla^{L^k\otimes E} s_0\right\|^2 - C\|s_0\|^2.\end{aligned} \tag{3.9}$$

Thus, (3.7) and (3.9) yield

$$\left\|\nabla^{L^k\otimes E} s_0\right\|^2 - k(\tau(x)s_0, s_0) \leqslant C\|s_0\|^2\,. \tag{3.10}$$

By (2.12) and (3.10), there exists a constant $a > 0$ such that

$$\left|\left(\Delta^{\#}_k s, s\right)\right| \leqslant a\|s\|^2\,, \quad s \in P_k(\operatorname{Ker} D^+_k)\,. \tag{3.11}$$

By (3.4), we know that for $k \gg 1$, $P_k : \operatorname{Ker} D^+_k \longrightarrow P_k(\operatorname{Ker} D^+_k)$ is bijective, and

$$\mathcal{C}^\infty(X, L^k\otimes E) = P_k(\operatorname{Ker} D^+_k) \oplus (\operatorname{Ker} D^+_k)^\perp \cap \mathcal{C}^\infty(X, L^k\otimes E)\,. \tag{3.12}$$

The proof is now reduced to a direct application of the minimax principle for the operator $\Delta^{\#}_k$. It is clear that (3.2) and (3.11) still hold for elements in the Sobolev space $W^1(X, L^k\otimes E)$, which is the domain of the quadratic form $Q_k(f) = \left\|\nabla^{L^k\otimes E} f\right\|^2 - k(\tau(x)f, f)$ associated to $\Delta^{\#}_k$. Let $\mu^k_1 \leqslant \mu^k_2 \leqslant \cdots \leqslant \mu^k_j \leqslant \cdots$ ($j \in \mathbb{N}$) be the eigenvalues of $\Delta^{\#}_k$. Then, by the minimax principle [18, pp.76–78],

$$\mu^k_j = \min_{F\subset \operatorname{Dom} Q_k}\ \max_{f\in F,\, \|f\|=1} Q_k(f)\,. \tag{3.13}$$

where F runs over the subspaces of dimension j of $\operatorname{Dom} Q_k$.

By (3.11) and (3.13), we know $\mu^k_j \leqslant a$, for $j \leqslant \dim \operatorname{Ker} D^+_k$. Moreover, any subspace $F \subset \operatorname{Dom} Q_k$ with $\dim F \geqslant \dim \operatorname{Ker} D^+_k + 1$ contains an element $0 \neq f \in F \cap (\operatorname{Ker} D^+_k)^\perp$. By (3.2), (3.13), we obtain $\mu^k_j \geqslant 2k\lambda - b$, for $j \geqslant \dim \operatorname{Ker} D^+_k + 1$.

By Theorem 1.1 and Atiyah–Singer theorem [2],

$$\dim \operatorname{Ker} D^+_k = \operatorname{index} D^+_k = \langle \operatorname{ch}(L^k\otimes E)\operatorname{Td}(X), [X]\rangle \tag{3.14}$$

where $\mathrm{Td}(X)$ is the Todd class of an almost complex structure compatible with ω. The index is a polynomial in k of degree n and of leading term $k^n(\mathrm{rank}\, E)\,\mathrm{vol}_\omega(X)$, where $\mathrm{vol}_\omega(X)$ is the symplectic volume of X.

The proof of our corollary is completed. □

Remark 3.1. If (X, ω) is Kähler and if L, E are holomorphic vector bundles, then $D_k = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ where $\overline{\partial} = \overline{\partial}^{L^k \otimes E}$. D_k^2 preserves the $\mathbb{Z}$–grading of $\Omega^{0,\bullet}$. By using the Bochner–Kodaira–Nakano formula, Bismut and Vasserot [4, Theorem 1.1] proved Theorem 2.5. As $\overline{\partial} : (\mathrm{Ker}\, D_k^+)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E) \longrightarrow \Omega^{0,1}(X, L^k \otimes E)$ is injective, we infer

$$\begin{aligned} & 2\big\|\overline{\partial} s\big\|^2 \geqslant (2k\lambda - C)\|s\|^2, \\ & \qquad \text{for } s \in (\mathrm{Ker}\, D_k^+)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E). \end{aligned} \tag{3.15}$$

By Lichnerowicz formula [4, (21)], $2\overline{\partial}^*\overline{\partial} = \Delta_k^\# + \frac{1}{4}K + \mathbf{c}(R)$ on $\mathcal{C}^\infty(X, L^k \otimes E)$, and Corollary 1.2 follows immediately. This observation motivated our work.

Remark 3.2. As in [5], we assume that (L, h^L, ∇^L) is a positive Hermitian vector bundle, i.e. the curvature R^L is an $\mathrm{End}(L)$–valued $(1,1)$–form, and for any $u \in T^{(1,0)}X \smallsetminus \{0\}$, $s \in L \smallsetminus \{0\}$, $\langle R^L(u, \overline{u})s, \overline{s}\rangle > 0$. Let $S^k(L)$ be the k^{th} symmetric tensor power of L. Then if we replace L^k in § 2, 3 by $S^k(L)$, or by the irreducible representations of L, which are associated with the weight ka (where a is a given weight), when k tends to $+\infty$, the techniques used in our paper still apply.

4. Covering manifolds

We extend in this section our results to covering manifolds.

4.1. Covering manifolds, von Neumann dimension

We present here some generalities about elliptic operators on covering manifolds and Γ–dimension. For details, the reader is referred to [1, § 4], [19, § 1,§ 3].

Let $\widetilde{X}$ be a paracompact smooth manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ having a compact quotient $X = \widetilde{X}/\Gamma$. Let $g^{T\widetilde{X}}$ be a Γ–invariant metric on $T\widetilde{X}$. Let $p : \widetilde{X} \longrightarrow X$ be the projection.

For a Γ–invariant hermitian vector bundle $(\widetilde{F}, h^{\widetilde{F}})$, we denote by $\mathcal{C}_c^\infty(\widetilde{X}, \widetilde{F})$ the space of compactly supported sections. Then $g^{T\widetilde{X}}$, $h^{\widetilde{F}}$ define an L_2–scalar product on $\mathcal{C}_c^\infty(\widetilde{X}, \widetilde{F})$ as in (2.5). The corresponding L_2 space is denoted by $L_2(\widetilde{X}, \widetilde{F})$.

We have a decomposition $L_2(\widetilde{X},\widetilde{F}) \cong L_2\Gamma \otimes \mathcal{H}$ where $\mathcal{H} = L_2(U,\widetilde{F})$ is the L_2 space over the relatively compact fundamental domain U of the Γ action. This makes $L_2(\widetilde{X},\widetilde{F})$ into a free Hilbert Γ–module. Since Γ acts by left translations l_γ on $L_2\Gamma$, we obtain a unitary action of Γ on $L_2(\widetilde{X},\widetilde{F})$ by left translations $L_\gamma = l_\gamma \otimes \mathrm{Id}$. We will consider in the sequel closed Γ–invariant subspaces of $L_2(\widetilde{X},\widetilde{F})$ for this action, called (projective) Γ–modules.

Let $\mathcal{A}_\Gamma$ be the von Neumann algebra which consists of all bounded linear operators in $L_2\Gamma \otimes \mathcal{H}$ which commute to the action of Γ. Let $\mathcal{R}_\Gamma$ be the von Neumann algebra of all bounded operators on $L_2\Gamma$ which commute with all l_γ. Then $\mathcal{R}_\Gamma$ is generated by all right translations. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Then $\mathcal{A}_\Gamma = \mathcal{R}_\Gamma \otimes \mathcal{B}(\mathcal{H})$.

If we consider the orthonormal basis $(\delta_\gamma)_\gamma$ in $L_2\Gamma$, where δ_γ is the Dirac delta function at $\gamma \in \Gamma$, then the matrix of any operator $A \in \mathcal{R}_\Gamma$ has the property that all its diagonal elements are equal. Therefore we define a natural trace on $\mathcal{R}_\Gamma$ as the diagonal element, that is, $\mathrm{tr}_\Gamma A = (A\delta_e, \delta_e)$ where e is the neutral element. Let Tr be the usual trace on $\mathcal{B}(\mathcal{H})$, then we define a trace on $\mathcal{A}_\Gamma$ by $\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr}$.

For any closed Γ–invariant space $V \subset L_2\Gamma \otimes \mathcal{H}$ i.e. for any Γ–module, the projection $P_V \in \mathcal{A}_\Gamma$ and we define $\dim_\Gamma V = \mathrm{Tr}_\Gamma P_V$. In general the Γ–dimension is an element of $[0,\infty]$. We also need the following fact [20, p.398].

Proposition 4.1. *Let $A : V_1 \longrightarrow V_2$ be a bounded linear operator between two Γ–modules, commuting with the action of Γ. Then* $\mathrm{Ker}\, A = 0$ *implies* $\dim_\Gamma V_1 \leqslant \dim_\Gamma V_2$.

Consider an elliptic, Γ–invariant, formally self–adjoint differential operator $\widetilde{P}$ defined in the first instance on $\mathcal{C}^\infty_c(\widetilde{X},\widetilde{F})$. By a theorem of Atiyah [1, Proposition 3.1], the minimal extension of $\widetilde{P}$ (i.e. the operator closure of $\widetilde{P}$) and the maximal extension of $\widetilde{P}$ (i.e. $\widetilde{P}^*$) coincide. Hence

Lemma 4.2 (Atiyah). *$\widetilde{P}$ defined on $\mathcal{C}^\infty_c(\widetilde{X},\widetilde{F})$ is essentially self–adjoint.*

Therefore $\widetilde{P}$ has a unique self–adjoint extension, namely its closure. From now on, we always work with this extension of $\widetilde{P}$, which we will denote with the same symbol.

Then the self–adjoint extension $\widetilde{P}$, as well as its spectral projections commute with the action of Γ. In particular, the spectral spaces are Γ–modules. For a Borel set $B \subset \mathbb{R}$, we denote by $E(B,\widetilde{P})$ the spectral projection corresponding to the subset B, and for $\mu \in \mathbb{R}$, set $E_\mu(\widetilde{P}) = E\big((-\infty,\mu],\widetilde{P}\big)$. We introduce now a quantitative characteristic of the spectrum, namely the von Neumann spectrum distribution function. For $\mu \in \mathbb{R}$, set

$$N_\Gamma(\mu,\widetilde{P}) := \mathrm{Tr}_\Gamma\, E_\mu(\widetilde{P}) = \dim_\Gamma \mathrm{Range}\, E_\mu(\widetilde{P})\,.$$

It is non–decreasing and the spectrum of $\widetilde{P}$ coincides with the points of growth of $N_\Gamma(\mu,\widetilde{P})$. If $\widetilde{P}$ is semi–bounded from below, we have $\operatorname{Range} E_\mu(\widetilde{P}) \subset \operatorname{Dom} \widetilde{P}^m$ for $m \in \mathbb{N}$. Using the uniform Sobolev spaces [19, pp. 511–512], it is easily seen that $\operatorname{Range} E_\mu(\widetilde{P}) \subset \mathcal{C}^\infty(\widetilde{X},\widetilde{F})$, so that $E_\mu(\widetilde{P}) : L_2(\widetilde{X},\widetilde{F}) \longrightarrow \mathcal{C}^\infty(\widetilde{X},\widetilde{F})$ is linear continuous. Let $K_\mu(\widetilde{x},\widetilde{y})$ be the kernel of $E_\mu(\widetilde{P})$ with respect to the riemannian volume $dv_{\widetilde{X}}$ of $g^{T\widetilde{X}}$. By Schwartz kernel theorem, $K_\mu(\widetilde{x},\widetilde{y})$ is smooth. By [1, Lemma 4.16],

$$N_\Gamma(\mu,\widetilde{P}) = \operatorname{Tr}_\Gamma E_\mu(\widetilde{P}) = \int_U \operatorname{Tr} K_\mu(\widetilde{x},\widetilde{x})\, dv_{\widetilde{X}} < +\infty.$$

4.2. The spin^{c} *Dirac operator on a covering manifold*

Assume that there exists a Γ–invariant pre–quantum line bundle $\widetilde{L}$ on $\widetilde{X}$ and a Γ–invariant connection $\nabla^{\widetilde{L}}$ such that $\widetilde{\omega} = \frac{\sqrt{-1}}{2\pi}(\nabla^{\widetilde{L}})^2$ is non–degenerate. Let $(\widetilde{E},h^{\widetilde{E}})$ be a Γ–invariant hermitian vector bundle. Let $\nabla^{\widetilde{E}}$ be a Γ-invariant hermitian connection on $\widetilde{E}$. Let $\widetilde{J}$ be an Γ-invariant almost complex structure on $T\widetilde{X}$ such that $\widetilde{J}$ is compatible with $\widetilde{\omega}$ and $g^{T\widetilde{X}}$. Let $\widetilde{J}_0 \in \operatorname{End}(TX)$ be defined by

$$\widetilde{\omega}(u,v) = g^{T\widetilde{X}}(\widetilde{J}_0 u, v), \;\text{ for }\; u,v \in T\widetilde{X}.$$

Then $\widetilde{J}$ commutes with $\widetilde{J}_0$ and $\widetilde{J}_0, g^{T\widetilde{X}}, \widetilde{\omega}, \widetilde{J}$ are the pull-back of the corresponding objects in Sect. 2 by $p : \widetilde{X} \to X$.

We use in the sequel the same notation as in Sect. 2 for the corresponding objects on X. Following Sect. 2, we introduce the Γ–invariant spin^{c} Dirac operator $\widetilde{D}_k$ on $\Omega^{0,\bullet}(\widetilde{X},\widetilde{L}^k \otimes \widetilde{E})$ and the Γ–invariant Laplacian $\widetilde{\Delta}_k = \big(\nabla^{\widetilde{L}^k\otimes\widetilde{E}}\big)^* \nabla^{\widetilde{L}^k\otimes\widetilde{E}}$ on $\mathcal{C}^\infty(\widetilde{X},\widetilde{L}^k \otimes \widetilde{E})$. Let $\widetilde{D}^+_k$ and $\widetilde{D}^-_k$ be the restrictions of $\widetilde{D}_k$ to $L_2^{0,\text{even}}(\widetilde{X},\widetilde{L}^k \otimes \widetilde{E})$ and $L_2^{0,\text{odd}}(\widetilde{X},\widetilde{L}^k \otimes \widetilde{E})$, respectively.

Proposition 4.3. *There exists $C>0$ such that for $k \in \mathbb{N}$, $\widetilde{\Delta}_k - k\cdot\tau\circ p \geqslant -C$ on $L_2(\widetilde{X},\widetilde{L}^k \otimes \widetilde{E})$.*

Proof. By applying Lichnerowicz formula (2.10) for $s \in \mathcal{C}^\infty_c(\widetilde{X},\widetilde{L}^k \otimes \widetilde{E})$, we obtain as in the proof of Corollary 2.4, that there exists $C>0$ such that $\big((\widetilde{\Delta}_k - k\cdot\tau\circ p)s,s\big) \geqslant -C\|s\|^2$. By Lemma 4.2, this is valid for any $s \in \operatorname{Dom}(\widetilde{\Delta}_k - k\cdot\tau\circ p)$. $\square$

In the same vein, we can generalize Theorem 2.5.

Theorem 4.4. *There exists $C>0$ such that for $k \in \mathbb{N}$ and any $s \in \operatorname{Dom}(\widetilde{D}_k)$ with vanishing degree zero component,*

$$\|\widetilde{D}_k s\|^2 \geqslant (2k\lambda - C)\,\|s\|^2\,. \tag{4.1}$$

As an immediate application of the estimate (4.1) for the Dirac operator and Remark 2.3, we get the following asymptotic vanishing theorem which is the main result in [12, Theorem 2.6].

Corollary 4.5. $\operatorname{Ker} \widetilde{D}_k^- = \{0\}$ *for large enough* k.

We have also an analogue of Theorem 1.1.

Corollary 4.6. *There exists* $C > 0$ *such that for* $k \in \mathbb{N}$, *the spectrum of* $\widetilde{D}_k^2$ *is contained in the set* $\{0\} \cup (2k\lambda - C, +\infty)$.

Proof. The proof of Theorem 1.1 does not use the fact that the spectrum is discrete. Therefore it applies in this context, too. □

We study now the spectrum of the Γ–invariant Schrödinger operator $\widetilde{\Delta}_k - k \cdot \tau \circ p$.

Corollary 4.7. *The spectrum of the Schrödinger operator* $\widetilde{\Delta}_k^{\#} = \widetilde{\Delta}_k - k \cdot \tau \circ p$ *is contained in the union* $(-a, a) \cup (2k\lambda - b, +\infty)$ *, where* a *and* b *are positive constants independent of* k. *For large enough* k, *the* Γ*–dimension* d_k *of the spectral space* $E\big((-a,a), \widetilde{\Delta}_k^{\#}\big)$ *corresponding to* $(-a,a)$ *satisfies* $d_k = \langle \operatorname{ch}(L^k \otimes E) \operatorname{Td}(X), [X] \rangle$. *In particular* $d_k \sim k^n (\operatorname{rank} E) \operatorname{vol}_\omega(X)$.

Proof. By repeating the proof of Corollary 1.2, we get estimates (3.2) and (3.11) for smooth elements with compact support. Lemma 4.2 yields then

$$\big|\big(\widetilde{\Delta}_k^{\#} s, s\big)\big| \leqslant a \|s_0\|^2 \,, \quad s \in \operatorname{Dom}(\widetilde{\Delta}_k^{\#}) \cap P_k(\operatorname{Ker} \widetilde{D}_k^+)\,, \tag{4.2a}$$

$$(\widetilde{\Delta}_k^{\#} s, s) \geqslant (2k\lambda - b)\|s\|^2 \,, \quad s \in \operatorname{Dom}(\widetilde{\Delta}_k^{\#}) \cap (\operatorname{Ker} \widetilde{D}_k^+)^{\perp} \,. \tag{4.2b}$$

Recall that P_k represents the projection $L_2^{0,\bullet}(\widetilde{X}, \widetilde{L}^k \otimes \widetilde{E}) \longrightarrow L_2^{0,0}(\widetilde{X}, \widetilde{L}^k \otimes \widetilde{E})$. Since the curvatures of all our bundles are Γ–invariant, estimate (3.4) extends to the covering context with the same proof. In particular, $P_k : \operatorname{Ker} \widetilde{D}_k^+ \longrightarrow P_k(\operatorname{Ker} \widetilde{D}_k^+)$ is bijective, $P_k \restriction_{\operatorname{Ker} \widetilde{D}_k^+}$ and its inverse are bounded. So $P_k(\operatorname{Ker} \widetilde{D}_k^+)$ is closed. By Proposition 4.1,

$$\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+ = \dim_\Gamma P_k(\operatorname{Ker} \widetilde{D}_k^+)\,. \tag{4.3}$$

As in (3.12), we have

$$\operatorname{Dom}(\widetilde{\Delta}_k^{\#}) = P_k(\operatorname{Ker} \widetilde{D}_k^+) \oplus (\operatorname{Ker} \widetilde{D}_k^+)^{\perp} \cap \operatorname{Dom}(\widetilde{\Delta}_k^{\#})\,. \tag{4.4}$$

We use now a suitable form of the minimax principle from [20, Lemma 2.4]:

$$\begin{aligned} N_\Gamma(\mu, \widetilde{\Delta}_k^{\#}) = \sup\{\dim_\Gamma V \ :\ & V \subset \operatorname{Dom} \widetilde{\Delta}_k^{\#} \\ & \big(\widetilde{\Delta}_k^{\#} f, f\big) \leqslant \mu \|f\|^2 \,, \forall\, f \in V\} \end{aligned} \tag{4.5}$$

where V runs over the Γ–modules of $L_2(\widetilde{X}, \widetilde{L}^k \otimes \widetilde{E})$.
By (4.1), (4.2a) and (4.5), we get

$$N_\Gamma(a, \widetilde{\Delta}_k^{\#}) \geqslant \dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+ \,. \tag{4.6}$$

Let us consider $\nu < 2k\lambda - b$. We prove that

$$N_\Gamma(\nu, \widetilde{\Delta}_k^{\#}) \leqslant \dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+ \,. \tag{4.7}$$

Let $V \subset \operatorname{Dom}(\widetilde{\Delta}_k^{\#})$ be an arbitrary Γ–module with $\big(\widetilde{\Delta}_k^{\#} u, u\big) \leqslant \nu \|u\|^2$. If $\dim_\Gamma V > \dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$, by Proposition 4.1 and (4.4), there exists $0 \neq v \in V \cap (\operatorname{Ker} D_k^+)^\perp$, which in view of (4.2b) is a contradiction. Therefore $\dim_\Gamma V \leqslant \dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$. By (4.5), we get (4.7).

By (4.6) and (4.7), we know that the function $N_\Gamma(\nu, \widetilde{\Delta}_k^{\#})$ is constant in the interval $\nu \in [a, 2k\lambda - b)$ and equal to $\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$. Enlarging a bit a if necessary, we see that the spectrum of $\widetilde{\Delta}_k^{\#}$ is indeed contained in $(-a, a) \cup (2k\lambda - b, +\infty)$, and the Γ–dimension d_k of the spectral space $E\big((-a,a), \widetilde{\Delta}_k^{\#}\big)$ equals $\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$.

By Corollary 4.5, $\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+ = \operatorname{index}_\Gamma \widetilde{D}_k^+$. Moreover, Atiyah's L_2 index theorem [1, Theorem 3.8] shows that $\operatorname{index}_\Gamma \widetilde{D}_k^+ = \operatorname{index} D_k^+$.

By (3.14), the proof is achieved. □

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