# Transversal Index and $L^2$-index for Manifolds with Boundary

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*Dedicated to Jeff Cheeger for his 65th birthday*

**Abstract.** For manifolds with boundary, we present a self-contained proof of Braverman's result which gives an alternative interpretation of the transversal index through certain kind of $L^2$-indices.

**Mathematics Subject Classification (2000).** Primary 58J20; Secondary 53D50.

**Keywords.** Index theory, Dirac operator, geometric quantization.

## 1. Introduction

In her ICM 2006 plenary lecture [15], Michèle Vergne formulated a conjecture on "quantization commutes with reduction" for non-compact symplectic manifolds. This conjecture extends the original Guillemin-Sternberg geometric quantization conjecture on compact symplectic manifolds to the non-compact setting. Vergne's conjecture [15] is stated in terms of the indices of transversally elliptic symbols on possibly non-compact manifolds in the sense of Atiyah [1] and Paradan [9], and these indices coincide with the indices of Spin$^c$ Dirac operators for compact manifolds. For a survey on the Guillemin-Sternberg conjecture, see [14].

In [7], [8], we established an extended version of Vergne's conjecture, in the sense that we did not make any extra assumptions besides the properness of the associated moment map. One important step of our approach is to establish an alternative interpretation of the transversal index appearing in the Vergne conjecture by using the Atiyah-Patodi-Singer type index. In this step, we used Braverman's interpretation of the transversal index for manifolds with boundary through certain kind of $L^2$-indices [2, §5]. The purpose of this note is to give a self-contained proof of Braverman's result.

Let $M$ be an even-dimensional compact oriented Spin$^c$-manifold with non-empty boundary $\partial M$. Let $n = \dim M$. Let $E$ be a complex vector bundle over $M$.

The work of X.M. was partially supported by Institut Universitaire de France. The work of W.Z. was partially supported by MOEC and NNSFC.

Let $G$ be a compact connected Lie group. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ its dual, and let $G$ act on $\mathfrak{g}$ by $\mathrm{Ad}_G$-action.

Let $\Lambda^*_+ \subset \mathfrak{g}^*$ be the set of dominant weights. For $\gamma \in \Lambda^*_+$, we denote by $V^G_\gamma$ the irreducible $G$-representation with highest weight $\gamma$. Then $V^G_\gamma$, $\gamma \in \Lambda^*_+$, form a $\mathbb{Z}$-basis of the representation ring $R(G)$.

We assume that $G$ acts on the left on $M$ and that this action lifts on $E$ and on the spin$^c$ structure of the tangent bundle $\pi : TM \to M$.

Let $g^{TM}$ be a $G$-invariant Riemannian metric on $TM$, and we identify $TM$ and $T^*M$ via $g^{TM}$. For any $K \in \mathfrak{g}$, let $K^M$ be the vector field generated by $K$ on $M$. Following [1, p. 7] (cf. [9, §3]), set

$$T_G M = \big\{(x,v) \in TM : x \in M, v \in T_x M \text{ such that} \\ \big\langle v, K^M(x)\big\rangle = 0 \text{ for all } K \in \mathfrak{g}\big\}. \quad (1.1)$$

Let $\Psi : M \to \mathfrak{g}$ be a $G$-equivariant map. Let $\Psi^M$ denote the vector field on $M$ such that

$$\Psi^M(x) := (\Psi(x))^M(x) \quad \text{for any } x \in M, \quad (1.2)$$

where $(\Psi(x))^M$ is the vector field over $M$ generated by $\Psi(x) \in \mathfrak{g}$.

We make the fundamental assumption that $\Psi^M$ is nowhere zero on $\partial M$.

Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the bundle of spinors associated to the spin$^c$-structure on $TM$ and $g^{TM}$ (cf. [5, Appendix D]). For $V \in TM$, let $c(V)$ be the Clifford action of $V$ on $S(TM)$ which exchanges the $\mathbb{Z}_2$-grading $S_\pm(TM)$ of $S(TM)$. Let $\sigma^M_{E,\Psi} \in \mathrm{Hom}(\pi^*(S_+(TM) \otimes E), \pi^*(S_-(TM) \otimes E))$ denote the symbol defined by

$$\sigma^M_{E,\Psi}(x,v) = \pi^*\left(\sqrt{-1}c(v + \Psi^M) \otimes \mathrm{Id}_E\right)\big|_{(x,v)} \quad \text{for } x \in M,\ v \in T_x M. \quad (1.3)$$

Since $\Psi^M$ is nowhere zero on $\partial M$, the subset $\{(x,v) \in T_G M : \sigma^M_{E,\Psi}(x,v)$ is non-invertible$\}$ of $T_G M$ is contained in a compact subset of $T_G\widehat{M}$ (where $\widehat{M} = M \setminus \partial M$ is the interior of $M$). Thus, $\sigma^M_{E,\Psi}$ defines a $G$-transversally elliptic symbol on $T_G\widehat{M}$ in the sense of Atiyah [1, §1, §3] and Paradan [9, §3], [10, §3], which in turn determines a transversal index in the formal representation ring $R[G]$ of $G$,

$$\mathrm{Ind}\left(\sigma^M_{E,\Psi}\right) = \bigoplus\nolimits_{\gamma \in \Lambda^*_+} \mathrm{Ind}_\gamma\left(\sigma^M_{E,\Psi}\right) \cdot V^G_\gamma \in R[G], \quad (1.4)$$

obtained by embedding $M$ into a compact $G$-manifold of the same dimension as that of $M$ (cf. [1, §1]). For any $\gamma \in \Lambda^*_+$, $\mathrm{Ind}_\gamma(\sigma^M_{E,\Psi}) \in \mathbb{Z}$ is the multiplicity of $V^G_\gamma$ in the transversal index $\mathrm{Ind}(\sigma^M_{E,\Psi})$ which depends on the homotopy class of $\Psi$ such that $\Psi^M$ is nowhere zero on $\partial M$, and which does not depend on $g^{TM}$. Moreover, the character of $\mathrm{Ind}(\sigma^M_{E,\Psi})$ is a distribution on $G$. Note that the set of $\gamma \in \Lambda^*_+$ such that $\mathrm{Ind}_\gamma(\sigma^M_{E,\Psi}) \neq 0$ could be infinite.

To compute $\mathrm{Ind}(\sigma^M_{E,\Psi})$, we deform $\Psi : M \to \mathfrak{g}$ inside $M$ (leaving $\Psi|_{\partial M}$ unchanged) to a $G$-equivariant map $\Psi' : M \to \mathfrak{g}$ with product structure near $\partial M$, then by the homotopy invariance of the transversal index (cf. [1, Theorems

2.6, 3.7], [9, §3]), $\operatorname{Ind}(\sigma^M_{E,\Psi}) = \operatorname{Ind}(\sigma^M_{E,\Psi'})$. Thus we can and we will assume that $\Psi : M \to \mathfrak{g}$ has a product structure near $\partial M$.

Let $g^{TM}, h^{S(TM)}, h^E$ be metrics on $TM, S(TM), E$ and let $\nabla^{S(TM)}$ be the canonically induced Clifford connection on $(S(TM), h^{S(TM)})$ and $\nabla^E$ be a Hermitian connection on $(E, h^E)$. We assume that the metrics and connections involved are $G$-invariant, and have a product structure near the boundary $\partial M$, and the $G$-action on objects such as $E, S(TM)$ near $\partial M$ is the product of the $G$-action on their restrictions to $\partial M$ and the identity in the normal direction to $\partial M$.

We attach now an infinite cylinder $\partial M \times (-\infty, 0]$ to $M$ along the boundary $\partial M$ and extend trivially all objects on $M$ to $\widetilde{M} = M \cup (\partial M \times (-\infty, 0])$. We decorate the extended objects on $\widetilde{M}$ by a "$\sim$".

Let $f$ be a $G$-invariant smooth real function on $\widetilde{M}$ such that there exists a smooth function $\varrho : (-\infty, 0] \to \mathbb{R}$ (for example, $\varrho(u) = e^{-2u}$) verifying that for $(y, x_n) \in \partial M \times (-\infty, 0]$,

$$f(y, x_n) = \varrho(x_n), \tag{1.5}$$

$$\lim_{x_n \to -\infty} \varrho(x_n) = +\infty \quad \text{and} \quad \lim_{x_n \to -\infty} \frac{\varrho^2}{|\varrho'| + \varrho}(x_n) = +\infty. \tag{1.6}$$

Then $f$ is an admissible function on $\widetilde{M}$ for the triple $(S(T\widetilde{M}) \otimes \widetilde{E}, \nabla^{S(T\widetilde{M}) \otimes \widetilde{E}}, \widetilde{\Psi})$ in the sense of [2, Definition 2.6] (cf. Remark 2.2).

Let $D^{\widetilde{E}}_f$ be the operator acting on $\mathscr{C}^\infty_0(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ to be defined by (2.6). Let $D^{\widetilde{E}}_{\pm,f}$ be the restrictions of $D^{\widetilde{E}}_f$ to the spaces associated to $S_\pm(T\widetilde{M}) \otimes \widetilde{E}$.

The purpose of this note is to give a self-contained proof of the following result of Braverman [2, Theorems 2.9, 5.5].

**Theorem 1.1.**

a) *For any $\gamma \in \Lambda^*_+$, the multiplicity of $V^G_\gamma$ in $\operatorname{Ker}(D^{\widetilde{E}}_f)$ is finite.*

b) *The following identity holds:*

$$\operatorname{Ker}(D^{\widetilde{E}}_{+,f}) - \operatorname{Ker}(D^{\widetilde{E}}_{-,f}) = \operatorname{Ind}(\sigma^M_{E,\Psi}) \in R[G]. \tag{1.7}$$

*Equivalently, let $\operatorname{Ind}_\gamma(D^{\widetilde{E}}_{+,f})$ be the multiplicity of $V^G_\gamma$ in $\operatorname{Ker}(D^{\widetilde{E}}_{+,f}) - \operatorname{Ker}(D^{\widetilde{E}}_{-,f})$, then*

$$\operatorname{Ind}_\gamma(D^{\widetilde{E}}_{+,f}) = \operatorname{Ind}_\gamma(\sigma^M_{E,\Psi}). \tag{1.8}$$

*In particular, $\operatorname{Ind}_\gamma(D^{\widetilde{E}}_{+,f})$ does not depend on $g^{TM}, h^{S(TM)}, h^E, \nabla^{S(TM)}, \nabla^E$, $f$ and it depends only on the homotopy class of $\Psi$ such that $\Psi^M$ is nowhere zero on $\partial M$.*

Certainly all argument here works for any Clifford module without the Spin$^c$ assumption on $M$.

This paper is organized as follows: In Section 2, we establish Theorem 1.1a). In Section 3, we obtain (1.7).

## 2. $L^2$-index

This section is organized as follows: In Section 2.1, we recall the definition of Spin$^c$ Dirac operators. In Section 2.2, we explain the self-adjoint extension of $D_f^{\widetilde{E}}$. In Section 2.3, we prove Theorem 1.1a).

We will use the notation and assumption in Introduction.

### 2.1. Spin$^c$ Dirac operator

We recall first our set-up. The manifold $M$ is a compact $G$-manifold with boundary $\partial M$, and $\widetilde{M}$ is an oriented $G$-Spin$^c$ manifold such that $M \subset \widetilde{M}$.

Fix $\varepsilon_1 > 0$. We assume that there exists a neighborhood $\partial M \times (-\infty, \varepsilon_1]$ of $\partial M$ in $\widetilde{M}$, where we identify $\partial M \times \{0\}$ to $\partial M$, such that $\widetilde{M} = M \cup (\partial M \times (-\infty, 0])$.

Let $S(T\widetilde{M}) = S_+(T\widetilde{M}) \oplus S_-(T\widetilde{M})$ be the bundle of spinors associated to the spin$^c$-structure on $T\widetilde{M}$ and a $G$-invariant Riemannian metric $g^{T\widetilde{M}}$.

Let $\widetilde{E}$ be a $G$-complex vector bundle over $\widetilde{M}$. Let $h^{\widetilde{E}}$ be a $G$-invariant Hermitian metric on $\widetilde{E}$, $\nabla^{\widetilde{E}}$ a $G$-invariant Hermitian connection on $(\widetilde{E}, h^{\widetilde{E}})$. Let $h^{S(T\widetilde{M})}$ be the $G$-invariant Hermitian metric on $S(T\widetilde{M})$ induced by $g^{T\widetilde{M}}$ and a $G$-invariant metric on the line bundle defining the spin$^c$ structure (cf. [5, Appendix D]). Let $h^{S(T\widetilde{M})\otimes\widetilde{E}}$ be the metric on $S(T\widetilde{M}) \otimes \widetilde{E}$ induced by the metrics on $S(T\widetilde{M})$ and on $\widetilde{E}$.

Let $\nabla^{S(T\widetilde{M})}$ be the Clifford connection on $S(T\widetilde{M})$ induced by the Levi-Civita connection $\nabla^{T\widetilde{M}}$ of $g^{T\widetilde{M}}$ and a $G$-invariant Hermitian connection on the line bundle defining the spin$^c$ structure. Let $\nabla^{S(T\widetilde{M})\otimes\widetilde{E}}$ be the Hermitian connection on $S(T\widetilde{M}) \otimes \widetilde{E}$ obtained by the tensor product of the connections $\nabla^{S(T\widetilde{M})}$ and $\nabla^{\widetilde{E}}$. Let $\widetilde{\Psi} : \widetilde{M} \to \mathfrak{g}$ be a $G$-equivariant map.

Let $g^{T\partial M}$ be the Riemannian metric on $\partial M$ induced by $g^{T\widetilde{M}}$. We denote the restriction of the objects on $\widetilde{M}$ to $M$ by canceling the superscript "$\sim$".

We assume that for $(y, x_n) \in \partial M \times (-\infty, \varepsilon_1]$, we have

$$\begin{aligned}
&\widetilde{\Psi}(y, x_n) = \Psi(y, 0) \in \mathfrak{g}, \quad g^{T\widetilde{M}}_{(y,x_n)} = g^{T\partial M}_y + (dx_n)^2,\\
&(S(T\widetilde{M}), h^{S(T\widetilde{M})}, \nabla^{S(T\widetilde{M})})|_{\partial M\times(-\infty,\varepsilon_1]} = \pi_1^*((S(TM), h^{S(TM)}, \nabla^{S(TM)})|_{\partial M}),\\
&(\widetilde{E}, h^{\widetilde{E}}, \nabla^{\widetilde{E}})|_{\partial M\times(-\infty,\varepsilon_1]} = \pi_1^*((E, h^E, \nabla^E)|_{\partial M}),
\end{aligned} \tag{2.1}$$

with $\pi_1 : \partial M \times (-\infty, \varepsilon_1] \to \partial M$ the natural projection. Moreover, the $G$-action on objects such as $\widetilde{M}$, $\widetilde{E}, S(T\widetilde{M})$ on $\partial M \times (-\infty, \varepsilon_1]$ is the product of the $G$-action on their restriction to $\partial M$ and the identity in the direction $(-\infty, \varepsilon_1]$.

Let $\mathscr{C}^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ be the space of smooth sections of $S(T\widetilde{M}) \otimes \widetilde{E}$ on $\widetilde{M}$, and let $\mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ be the subspace of smooth sections with compact support. Let $dv_{\widetilde{M}}$ be the Riemannian volume form on $(\widetilde{M}, g^{T\widetilde{M}})$. For

$s \in \mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$, the $L^2$-norm $\|s\|_{\widetilde{M},0}$ is defined by

$$\|s\|^2_{\widetilde{M},0} = \int_{\widetilde{M}} |s|^2(x) dv_{\widetilde{M}}(x). \tag{2.2}$$

Let $\langle\cdot,\cdot\rangle_{\widetilde{M}}$ be the Hermitian product on $\mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ corresponding to $\|\cdot\|^2_{\widetilde{M},0}$. Let $L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ be the $L^2$-completion of $(\mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E}), \|\cdot\|_{\widetilde{M},0})$.

Let $\{e_i\}$ be an orthonormal frame of $T\widetilde{M}$. The Spin$^c$-Dirac operator $D^{\widetilde{E}}$ on $\mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ is defined by (cf. [5, Appendix D])

$$D^{\widetilde{E}} = \sum_{i=1}^{n} c(e_i) \nabla_{e_i}^{S(T\widetilde{M})\otimes\widetilde{E}}. \tag{2.3}$$

Then $D^{\widetilde{E}}$ is $G$-equivariant and formally self-adjoint.

Let $e_n$ be the inward unit normal vector field perpendicular to $\partial M$. Let $e_1, \ldots, e_{n-1}$ be an oriented orthonormal frame of $T\partial M$ so that $e_1, \ldots, e_{n-1}$, $e_n$ is an oriented orthonormal frame of $TM|_{\partial M}$. Set

$$D^E_{\partial M} = -\sum_{j=1}^{n-1} c(e_n)c(e_j) \nabla_{e_j}^{(S(TM)\otimes E)|_{\partial M}}. \tag{2.4}$$

Then $D^E_{\partial M}$ is the Dirac operator on $((S(TM)\otimes E)|_{\partial M}, \nabla^{(S(TM)\otimes E)|_{\partial M}})$. By (2.1), (2.3) and (2.4), we have on $\partial M \times (-\infty, \varepsilon_1]$,

$$D^{\widetilde{E}} = c(e_n) D^E_{\partial M} + c(e_n)\frac{\partial}{\partial x_n}. \tag{2.5}$$

## 2.2. Self-adjoint extension of $D_f^{\widetilde{E}}$

Let $f$ be a $G$-invariant smooth real function on $\widetilde{M}$. In this subsection we need not assume that $f$ verifies (1.5) and (1.6). Let $D_f^{\widetilde{E}}$ be the operator acting on $\mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ defined by (cf. (1.2), (2.3))

$$D_f^{\widetilde{E}} = D^{\widetilde{E}} + \sqrt{-1} f\, c(\widetilde{\Psi}^{\widetilde{M}}). \tag{2.6}$$

By definition, the graph of the minimal extension $(D_f^{\widetilde{E}})_{\min}$ of $D_f^{\widetilde{E}}$ is the closure of the graph of $D_f^{\widetilde{E}}$, i.e.,

$$\begin{aligned}\mathrm{Dom}\left((D_f^{\widetilde{E}})_{\min}\right) = \Big\{ s \in L^2\left(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E}\right) :\ &\text{there exists a sequence} \\ s_k \in \mathscr{C}_0^\infty\left(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E}\right)\ &\text{such that} \\ \lim_{k\to+\infty} s_k = s,\ \text{and} \lim_{k\to+\infty} D_f^{\widetilde{E}} s_k \text{ exists in } L^2&\left(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E}\right) \Big\},\end{aligned} \tag{2.7}$$

and for $s \in \mathrm{Dom}((D_f^{\widetilde{E}})_{\min})$, $(D_f^{\widetilde{E}})_{\min} s$ is defined by $\lim_{k\to+\infty} D_f^{\widetilde{E}} s_k$ in (2.7).

Since the Riemannian manifold $(\widetilde{M}, g^{T\widetilde{M}})$ is complete, by [4] (cf. also [6, §3.1, §3.3]), the minimal and the maximal extensions of $D_f^{\widetilde{E}}$ coincide, and form a self-adjoint operator. We still denote by $D_f^{\widetilde{E}}$, $(D_f^{\widetilde{E}})^2$ the corresponding maximal extensions, whose domains are, by definition,

$$\begin{aligned} \mathrm{Dom}\left(D_f^{\widetilde{E}}\right) &= \left\{s :\ s,\ D_f^{\widetilde{E}} s \in L^2\left(\widetilde{M}, S\left(T\widetilde{M}\right) \otimes \widetilde{E}\right)\right\}, \\ \mathrm{Dom}\left((D_f^{\widetilde{E}})^2\right) &= \left\{s :\ s,\ \left(D_f^{\widetilde{E}}\right)^2 s \in L^2\left(\widetilde{M}, S\left(T\widetilde{M}\right) \otimes \widetilde{E}\right)\right\}. \end{aligned} \tag{2.8}$$

Note that $\mathrm{Dom}(D_f^{\widetilde{E}})$ is a Hilbert space endowed with the graph-norm $(\|D_f^{\widetilde{E}} s\|^2_{\widetilde{M},0} + \|s\|^2_{\widetilde{M},0})^{1/2}$, for $s \in \mathrm{Dom}(D_f^{\widetilde{E}})$.

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth even function such that $\varphi(u) = 1$ for $|u| < 1/2$, and $\varphi(u) = 0$ for $|u| > 1$. For $k \geqslant 1$, let $\varphi_k : \widetilde{M} \to \mathbb{R}$ be defined by

$$\varphi_k = 1 \ \text{ on } M, \quad \varphi_k(y, x_n) = \varphi\left(x_n/k\right) \ \text{ for } (y, x_n) \in \partial M \times (-\infty, 0]. \tag{2.9}$$

Then each $\varphi_k$ is $G$-invariant, smooth, and has a compact support on $\widetilde{M}$.

If $s \in \mathrm{Dom}((D_f^{\widetilde{E}})^2)$, then by the basic elliptic estimate, $\varphi_k s$ lies in the local Sobolev space of second order on $\widetilde{M}$. From (2.5), (2.9), there exists $C > 0$ such that for any $k \geqslant 1$, we have

$$\begin{aligned} \mathrm{Re}\left\langle \varphi_k^2 (D_f^{\widetilde{E}})^2 s, s\right\rangle_{\widetilde{M}} &= \mathrm{Re}\left\langle D_f^{\widetilde{E}}\left(\varphi_k^2 D_f^{\widetilde{E}} s\right) - 2\varphi_k c\left((d\varphi_k)^*\right) D_f^{\widetilde{E}} s, s\right\rangle_{\widetilde{M}} \\ &= \left\|\varphi_k D_f^{\widetilde{E}} s\right\|^2_{\widetilde{M},0} + 2\,\mathrm{Re}\left\langle \varphi_k D_f^{\widetilde{E}} s, c\left((d\varphi_k)^*\right) s\right\rangle_{\widetilde{M}} \\ &\geqslant \frac{1}{2}\left\|\varphi_k D_f^{\widetilde{E}} s\right\|^2_{\widetilde{M},0} - \frac{C}{2k}\|s\|^2_{\widetilde{M},0}, \end{aligned} \tag{2.10}$$

where $(d\varphi_k)^* \in T\widetilde{M}$ denotes the metric dual of $d\varphi_k$ with respect to $g^{T\widetilde{M}}$.

By taking $k \to \infty$ in (2.10), we get $D_f^{\widetilde{E}} s \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$. Thus $s \in \mathrm{Dom}((D_f^{\widetilde{E}})^2)$ implies $s \in \mathrm{Dom}(D_f^{\widetilde{E}})$. From this, we obtain as in [6, §3.3] that the maximal extension of $(D_f^{\widetilde{E}})^2$ is self-adjoint, and coincides also with the minimal extension of the original operator $(D_f^{\widetilde{E}})^2$. For $s \in \mathrm{Dom}((D_f^{\widetilde{E}})^2)$, we get from (2.10), by letting $k \to \infty$,

$$\left\|\left(1 + (D_f^{\widetilde{E}})^2\right)^{\frac{1}{2}} s\right\|^2_{\widetilde{M},0} = \left\langle\left(1 + (D_f^{\widetilde{E}})^2\right) s, s\right\rangle_{\widetilde{M}} = \left\|D_f^{\widetilde{E}} s\right\|^2_{\widetilde{M},0} + \|s\|^2_{\widetilde{M},0}. \tag{2.11}$$

Finally, from the von Neumann Lemma (cf. [6, Lemma C.1.3]) about self-adjoint operators, $1 + (D_f^{\widetilde{E}})^2 : \mathrm{Dom}((D_f^{\widetilde{E}})^2) \to L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ is bijective and has bounded inverse.

Relation (2.11) shows also that $\mathrm{Dom}((1 + (D_f^{\widetilde{E}})^2)^{\frac{1}{2}}) = \mathrm{Dom}(D_f^{\widetilde{E}})$. From the above discussion, we see that $(1 + (D_f^{\widetilde{E}})^2)^{-1} : L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E}) \to \mathrm{Dom}((D_f^{\widetilde{E}})^2)$

and $(1+(D_f^{\widetilde{E}})^2)^{-\frac{1}{2}} : L^2(\widetilde{M}, S(T\widetilde{M})\otimes\widetilde{E}) \to \mathrm{Dom}(D_f^{\widetilde{E}})$ are bounded bijective linear operators. Moreover,

$$D_f^{\widetilde{E}}\left(1+(D_f^{\widetilde{E}})^2\right)^{-1/2} = \left(1+(D_f^{\widetilde{E}})^2\right)^{-1/2} D_f^{\widetilde{E}} \quad \text{on } \mathrm{Dom}(D_f^{\widetilde{E}}). \tag{2.12}$$

### 2.3. $L^2$-index of $D_f^{\widetilde{E}}$

For $\gamma \in \Lambda_+^*$, recall that $V_\gamma^G$ is the irreducible representation of $G$ with highest weight $\gamma$. For $V, W$ two $G$-vector spaces, let $\mathrm{Hom}_G(V,W)$ denote the linear space of $G$-equivariant homomorphisms. By the Peter-Weyl theorem, we have the Hilbert space direct sum decomposition

$$L^2\left(\widetilde{M}, S(T\widetilde{M})\otimes\widetilde{E}\right) = \bigoplus_{\gamma\in\Lambda_+^*} L^2\left(\widetilde{M}, S(T\widetilde{M})\otimes\widetilde{E}\right)^\gamma, \tag{2.13}$$

where each $L^2\left(\widetilde{M}, S(T\widetilde{M})\otimes\widetilde{E}\right)^\gamma$ is a multiple of $V_\gamma^G$.

Let $D_f^{\widetilde{E}}(\gamma)$ be the restriction of $D_f^{\widetilde{E}}$ to the $\gamma$-component. Let $D_{\pm,f}^{\widetilde{E}}(\gamma)$ be the restrictions of $D_f^{\widetilde{E}}(\gamma)$ to the spaces associated to $S_\pm(T\widetilde{M})\otimes\widetilde{E}$. Then by (2.8), we have

$$\mathrm{Dom}\left(D_f^{\widetilde{E}}(\gamma)\right) = \mathrm{Dom}\left(D_f^{\widetilde{E}}\right) \cap L^2\left(\widetilde{M}, S\left(T\widetilde{M}\right)\otimes\widetilde{E}\right)^\gamma. \tag{2.14}$$

Clearly, $\mathrm{Ker}\left(D_f^{\widetilde{E}}\right)$ is closed in $L^2\left(\widetilde{M}, S(T\widetilde{M})\otimes\widetilde{E}\right)$, thus it is a Hilbert space with norm $\|\ \ \|_{\widetilde{M},0}$.

The following result is a reformulation of [2, Theorem 2.9].

**Theorem 2.1.** *Assume that* (2.1) *holds and that* $\widetilde{\Psi}^{\widetilde{M}}$ *is nowhere zero on* $\partial M$*, and that* $f$ *is a* $G$*-invariant real function on* $\widetilde{M}$ *verifying* (1.5) *and* (1.6)*. Then for any* $\gamma \in \Lambda_+^*$*,*

$$D_f^{\widetilde{E}}(\gamma) : \mathrm{Dom}\left(D_f^{\widetilde{E}}(\gamma)\right) \to L^2\left(\widetilde{M}, S\left(T\widetilde{M}\right)\otimes\widetilde{E}\right)^\gamma$$

*is a Fredholm operator, and*

$$\mathrm{Ker}\left(D_f^{\widetilde{E}}(\gamma)\right) = \mathrm{Hom}_G\left(V_\gamma^G, \mathrm{Ker}(D_f^{\widetilde{E}})\right)\otimes V_\gamma^G, \tag{2.15a}$$

$$\mathrm{Ker}(D_f^{\widetilde{E}}) = \bigoplus_{\gamma\in\Lambda_+^*} \mathrm{Ker}(D_f^{\widetilde{E}}(\gamma)). \tag{2.15b}$$

*Proof.* Let $\mathfrak{g}$ be equipped with an $\mathrm{Ad}_G$-invariant metric. Let $V_1, \ldots, V_{\dim G}$ be an orthonormal basis of $\mathfrak{g}$. Then one has

$$\widetilde{\Psi}(x) = \sum_{i=1}^{\dim G} \widetilde{\Psi}_i(x) V_i \quad \text{for } x \in \widetilde{M}, \tag{2.16}$$

where $\widetilde{\Psi}_i : \widetilde{M} \to \mathbb{R}$, $1 \leqslant i \leqslant \dim G$, are bounded smooth functions (together with their derivatives). From (2.16), one gets

$$\widetilde{\Psi}^{\widetilde{M}}(x) = \sum_{i=1}^{\dim G} \widetilde{\Psi}_i(x) V_i^{\widetilde{M}}(x). \tag{2.17}$$

From (2.3) and (2.6), we have

$$\begin{aligned}\left(D_f^{\widetilde{E}}\right)^2 = \left(D^{\widetilde{E}}\right)^2 + \sqrt{-1} \sum_{i=1}^{n} c\left(e_i\right) c\left(\nabla_{e_i}^{T\widetilde{M}}\left(f \widetilde{\Psi}^{\widetilde{M}}\right)\right) \\ - 2\sqrt{-1} \nabla_{f\widetilde{\Psi}^{\widetilde{M}}}^{S(T\widetilde{M})\otimes\widetilde{E}} + f^2 \left|\widetilde{\Psi}^{\widetilde{M}}\right|^2.\end{aligned} \tag{2.18}$$

For any $K \in \mathfrak{g}$, let $L_K$ denote the Lie derivative of $K$ acting on $\mathscr{C}^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$, and we set

$$\mu^{S\otimes\widetilde{E}}(K) := \nabla_{K^{\widetilde{M}}}^{S(T\widetilde{M})\otimes\widetilde{E}} - L_K \in \mathscr{C}^\infty\left(\widetilde{M}, \mathrm{End}\left(S(T\widetilde{M}) \otimes \widetilde{E}\right)\right). \tag{2.19}$$

By (2.1), for $K \in \mathfrak{g}$ fixed, $\mu^{S\otimes\widetilde{E}}(K)$ is bounded on $\widetilde{M}$. Set

$$B_f = \sqrt{-1} \sum_{i=1}^{n} c\left(e_i\right) c\left(\nabla_{e_i}^{T\widetilde{M}}\left(f \widetilde{\Psi}^{\widetilde{M}}\right)\right) - 2\sqrt{-1} f \sum_{i=1}^{\dim G} \widetilde{\Psi}_i \mu^{S\otimes\widetilde{E}}(V_i). \tag{2.20}$$

Let $B_f^*$ be the adjoint of $B_f$ with respect to $h^{S(T\widetilde{M})\otimes\widetilde{E}}$.

From (2.18)–(2.20), we have

$$\left(D_f^{\widetilde{E}}\right)^2 = \left(D^{\widetilde{E}}\right)^2 + B_f - 2\sqrt{-1} f \sum_{i=1}^{\dim G} \widetilde{\Psi}_i L_{V_i} + f^2 \left|\widetilde{\Psi}^{\widetilde{M}}\right|^2. \tag{2.21}$$

From (1.5), (2.1) and (2.20), we get on $\partial M \times (-\infty, 0]$,

$$\begin{aligned}B_f =& \sqrt{-1} f \Big( \sum_{i=1}^{n-1} c(e_i) c\left(\nabla_{e_i}^{T\widetilde{M}} \widetilde{\Psi}^{\widetilde{M}}\right) - 2 \sum_{i=1}^{\dim G} \widetilde{\Psi}_i\, \mu^{S\otimes\widetilde{E}}(V_i) \Big) \\ &+ \sqrt{-1} \frac{\partial f}{\partial x_n} c(e_n) c\left(\widetilde{\Psi}^{\widetilde{M}}\right).\end{aligned} \tag{2.22}$$

Since $\widetilde{\Psi}^{\widetilde{M}}$, $\nabla_{e_i}^{T\widetilde{M}} \widetilde{\Psi}^{\widetilde{M}} \in T\partial M$ (for $1 \leq i \leq n-1$), $\widetilde{\Psi}_j$ are constant in $x_n$ on $\partial M \times (-\infty, 0]$, (2.22) implies that there exists $C_0 > 0$ such that the following pointwise estimate on $\widetilde{M}$ holds:

$$\frac{1}{2}(B_f + B_f^*) \geqslant -C_0\left(|f| + |df|\right). \tag{2.23}$$

Now, we fix $\gamma \in \Lambda_+^*$.

For $K \in \mathfrak{g}$, we denote by $L_K(\gamma)$ the Lie derivative $L_K$ acting on $V_\gamma^G$. Let $\|L_K(\gamma)\|$ be the operator norm of $L_K(\gamma)$ with any (fixed) $G$-invariant Hermitian norm on $V_\gamma^G$. Then the Lie derivative $L_K$ acting on $L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$ acts only on the factor $V_\gamma^G$ (cf. (2.13)), and coincides with the linear bounded operator $L_K(\gamma)$. Thus its operator norm is $\|L_K(\gamma)\|$.

From (2.21) and (2.23), there exists $C_0 > 0$ such that for $s \in \mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$, we have

$$\left\langle \left(D_f^{\widetilde{E}}(\gamma)\right)^2 s, s \right\rangle_{\widetilde{M}} \geqslant \left\|D^{\widetilde{E}} s\right\|_{\widetilde{M},0}^2 + \left\|f \left|\widetilde{\Psi}^{\widetilde{M}}\right| s\right\|_{\widetilde{M},0}^2 - C_0 \left\langle (|f| + |df|)\, s, s\right\rangle_{\widetilde{M}} - \|s\|_{\widetilde{M},0} \sum_{i=1}^{\dim G} \|L_{V_i}(\gamma)\| \left\|f \widetilde{\Psi}_i s\right\|_{\widetilde{M},0}. \tag{2.24}$$

Recall that $\widetilde{\Psi}^{\widetilde{M}}$ is nowhere zero on $\partial M \times (-\infty, 0]$ and is constant in $x_n$. Moreover $\widetilde{\Psi}_i$ is constant in $x_n$ on $\partial M \times (-\infty, 0]$. Set

$$Q_1 = \inf_{y \in \partial M} \left|\widetilde{\Psi}^{\widetilde{M}}\right|^2 (y, 0) > 0, \quad Q_2 = \sup_{y \in \partial M} \left|\widetilde{\Psi}\right|^2 (y, 0). \tag{2.25}$$

By (1.6), there exists $C_\gamma > 0$ such that on $\partial M \times (-\infty, -C_\gamma]$, the following pointwise estimate holds:

$$\frac{1}{4} f^2 \left|\widetilde{\Psi}^{\widetilde{M}}\right|^2 > C_0(|f| + |df|) + \frac{Q_2}{Q_1} \sum_{i=1}^{\dim G} \|L_{V_i}(\gamma)\|^2. \tag{2.26}$$

By (2.16), the last term of (2.24) can be controlled by $\frac{Q_1}{4Q_2}\left\|f|\widetilde{\Psi}|s\right\|_{\widetilde{M},0}^2 + \frac{Q_2}{Q_1}\sum_{i=1}^{\dim G} \|L_{V_i}(\gamma)\|^2 \|s\|_{\widetilde{M},0}^2$. Thus by (2.24) and (2.26), there exists $C'_\gamma > 0$ such that for $s \in \mathscr{C}_0^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$,

$$\left\|D_f^{\widetilde{E}}(\gamma) s\right\|_{\widetilde{M},0}^2 \geqslant \left\|D^{\widetilde{E}} s\right\|_{\widetilde{M},0}^2 + \frac{1}{2} Q_1 \|f s\|_{\widetilde{M},0}^2 - C'_\gamma \int_{M \cup \partial M \times [-C_\gamma, 0]} |s|^2 dv_{\widetilde{M}}. \tag{2.27}$$

From (2.27), we can adapt the argument in [6, §3.1]) to know that $D_f^{\widetilde{E}}(\gamma)$ is a Fredholm operator. Here we will show that, by the argument in [12, Prop. 8.2.8], its spectrum is discrete.

We claim that the operator

$$\left(1 + (D_f^{\widetilde{E}}(\gamma))^2\right)^{-1/2} : L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma \to L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma \tag{2.28}$$

is compact.

Equivalently, we need to prove that for any sequence $\omega_k \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$, $\|\omega_k\|_{\widetilde{M},0} = 1$, the sequence $s_k = \left(1 + (D_f^{\widetilde{E}}(\gamma))^2\right)^{-1/2} \omega_k$ has a convergent subsequence in $L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$.

Take $l > 2C'_\gamma$, for $\varphi_l$ in (2.9), by (2.11) and (2.12), there exists $C > 0$ such that for any $k \in \mathbb{N}$, we have

$$\left\|D^{\widetilde{E}}(\varphi_l s_k)\right\|_{\widetilde{M},0} + \|\varphi_l s_k\|_{\widetilde{M},0} \leqslant C. \tag{2.29}$$

By Gårding's inequality and Rellich's theorem, (2.29) and the fact that $\{\varphi_l s_k\}$'s have a common compact support, we can select a convergent subsequence of

$\{\varphi_l s_k\}_k$ in $L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$. Now by a diagonal argument, we have a subsequence $\{s_{m_k}\}_k$ of $\{s_k\}_k$ such that for any $l \in \mathbb{N}$, $s_{m_k}|_{M\cup(\partial M\times[-l,0])}$, the restriction of $s_{m_k}$ to $M \cup (\partial M \times [-l, 0])$, converges in $L^2$-norm.

By (2.27), $\{f s_{m_k}\}$ is uniformly bounded in $L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$, by (1.6), we know $\{s_{m_k}\}$ converges in $L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$.

Thus $\left(1 + (D_f^{\widetilde{E}}(\gamma))^2\right)^{-1/2}$ is compact, and the spectrum of $\left(D_f^{\widetilde{E}}(\gamma)\right)^2$ is discrete.

As $D_f^{\widetilde{E}}$ is $G$-invariant, from (2.13), we get (2.15a). The equation (2.15b) is a consequence of Peter-Weyl theorem (cf. [3, Ch. 2, Theorem 5.7]). The proof of Theorem 2.1 is completed. □

**Remark 2.2.** *Let $v$ be the function on $\widetilde{M}$ defined by $v = |\widetilde{\Psi}^{\widetilde{M}}| + |\widetilde{\Psi}| + |\nabla^{T\widetilde{M}}\widetilde{\Psi}^{\widetilde{M}}| + |\mu^{S\otimes\widetilde{E}}| + 1$. Recall that a $G$-invariant real function $f$ on $\widetilde{M}$ is an admissible function for the triple $(S(T\widetilde{M}) \otimes \widetilde{E}, \nabla^{S(T\widetilde{M})\otimes\widetilde{E}}, \widetilde{\Psi})$ in the sense of* [2, Definition 2.6] *if and only if*

$$\lim_{x_n\to-\infty} \frac{f^2|\widetilde{\Psi}^{\widetilde{M}}|^2}{|df||\widetilde{\Psi}^{\widetilde{M}}| + fv + 1}(y, x_n) = +\infty \quad \textit{uniformly for } y \in \partial M. \tag{2.30}$$

*Since $\widetilde{\Psi}^{\widetilde{M}}$, $\nabla_{e_i}^{T\widetilde{M}}\widetilde{\Psi}^{\widetilde{M}} \in T\partial M$, $\widetilde{\Psi}_i$ are constant in $x_n$ on $\partial M \times (-\infty, 0]$, if* (1.5) *holds, then* (2.30) *is equivalent to* (1.6).

**Definition 2.3.** *For each $\gamma \in \Lambda_+^*$, the index of $D_{+,f}^{\widetilde{E}}(\gamma)$, thought of as a virtual $G$-representation, is defined by*

$$\operatorname{Ind}\left(D_{+,f}^{\widetilde{E}}(\gamma)\right) := \operatorname{Ker}\left(D_{+,f}^{\widetilde{E}}(\gamma)\right) - \operatorname{Ker}\left(D_{-,f}^{\widetilde{E}}(\gamma)\right). \tag{2.31}$$

Let $\operatorname{Ind}_\gamma(D_{+,f}^{\widetilde{E}})$ be the multiplicity of $V_\gamma^G$ in $\operatorname{Ind}\left(D_{+,f}^{\widetilde{E}}(\gamma)\right)$. By Theorem 2.1, we have

$$\operatorname{Ker}\left(D_{+,f}^{\widetilde{E}}\right) - \operatorname{Ker}\left(D_{-,f}^{\widetilde{E}}\right) = \bigoplus_{\gamma\in\Lambda_+^*} \operatorname{Ind}_\gamma\left(D_{+,f}^{\widetilde{E}}\right) \cdot V_\gamma^G \in R[G]. \tag{2.32}$$

**Lemma 2.4.** *Under the condition in Theorem* 2.1*, if $f_1$ is a $G$-invariant function on $\widetilde{M}$ such that one of the following two conditions holds:*

a) *$f = f_1$ outside a compact subset of $\widetilde{M}$,*
b) *$f_1 = e^{-2x_n} f$ on $\partial M \times (-\infty, 0]$.*

*Then for any $\gamma \in \Lambda_+^*$,*

$$\operatorname{Ind}_\gamma(D_{+,f}^{\widetilde{E}}) = \operatorname{Ind}_\gamma(D_{+,f_1}^{\widetilde{E}}). \tag{2.33}$$

*Proof.* Note that $C_0$ in (2.23) does not depend on $f$.

If $f = f_1$ outside a compact subset of $\widetilde{M}$, let $f_t = (1-t)f + tf_1$ for $t \in [0, 1]$. Then for $\gamma \in \Lambda_+^*$, there exists $C_\gamma > 0$ such that (2.26) holds, thus (2.27) for

$D^{\widetilde{E}}_{f_t}$ holds uniformly on $t \in [0,1]$. Now $D^{\widetilde{E}}_f(\gamma)\left(1+(D^{\widetilde{E}}_f(\gamma))^2\right)^{-1/2}$ is a continuous family of bounded Fredholm operators, thus $\mathrm{Ind}_\gamma(D^{\widetilde{E}}_{+,f_t})$ does not depend on $t \in [0,1]$. In particular (2.33) holds.

If $f_1 = e^{-2x_n} f$ on $\partial M \times (-\infty, 0]$, set $f_t = e^{-2tx_n} f$ for $t \in [0,1]$. Then again for $\gamma \in \Lambda^*_+$, there exists $C_\gamma > 0$ such that (2.26) holds uniformly on $t \in [0,1]$, thus (2.27) for $D^{\widetilde{E}}_{f_t}$ holds uniformly on $t \in [0,1]$. We conclude as above (2.33) holds. □

The following lemma will be used in the proof of Theorem 3.1.

**Lemma 2.5.** *If* $(1+|f|)^{-1/2}\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$ *and*

$$\left(D^{\widetilde{E}} + \sqrt{-1} f c(\widetilde{\Psi}^{\widetilde{M}})\right)\omega = 0 \tag{2.34}$$

*in the sense of distribution, then for any* $m \in \mathbb{N}$,

$$f^m\omega,\ (1-\varphi_1) f^m D^E_{\partial M}\omega,\ (1-\varphi_1) f^m \frac{\partial \omega}{\partial x_n} \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma. \tag{2.35}$$

*Proof.* We assume that $\omega$ verifies (2.34) and $(1+|f|)^{-1/2}\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$. Then by the ellipticity of $D^{\widetilde{E}}$, $\omega \in \mathscr{C}^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$.

Take $C_f > 0$ such that $f$ is strictly positive on $\partial M \times (-\infty, -C_f]$. For $k > 1$, $m \in \mathbb{R}$, let $\varphi_{k,m}$ be a $G$-invariant smooth function on $\widetilde{M}$ such that

$$\varphi_{k,m} = \varphi_k f^m \quad \text{ on } \partial M \times (-\infty, -C_f]. \tag{2.36}$$

Then by (2.5), (2.6) and (2.34), on $\mathcal{V}_1 := \partial M \times (-\infty, -C_f]$, we get

$$D^{\widetilde{E}}(\varphi^2_{k,m}\omega) = \varphi_k f^{2m}\Big[2c(e_n)\Big(\varphi'_k + m\varphi_k f^{-1}\frac{\partial f}{\partial x_n}\Big) - \sqrt{-1}\varphi_k f c(\widetilde{\Psi}^{\widetilde{M}})\Big]\omega. \tag{2.37}$$

As in (2.2), we denote by $\| \ \|_{\mathcal{V}_1,0}$, $\langle\,,\,\rangle_{\mathcal{V}_1}$ the $L^2$-norm and Hermitian product on $L^2(\mathcal{V}_1, S(T\widetilde{M}) \otimes \widetilde{E})$. Note that $dv_{\widetilde{M}} = -dx_n \wedge dv_{\partial M}$ on $\mathcal{V}_1$, thus by (2.21), (2.23), (2.34) and the argument around (2.24), there exists $C > 0$ such that for any $k > 1$, $m \in \mathbb{R}$, we have

$$\begin{aligned}
0 = \Big\langle (D^{\widetilde{E}}_f)^2\omega, \varphi^2_{k,m}\omega\Big\rangle_{\mathcal{V}_1} &\geqslant \mathrm{Re}\,\Big\langle D^{\widetilde{E}}\omega, D^{\widetilde{E}}(\varphi^2_{k,m}\omega)\Big\rangle_{\mathcal{V}_1} \\
&+ \int_{\partial M \times \{-C_f\}} \mathrm{Re}\,\Big\langle c(-e_n) D^{\widetilde{E}}\omega, \varphi^2_{k,m}\omega\Big\rangle\, dv_{\partial M} - C\|f^{m+1/2}\varphi_k\omega\|^2_{\mathcal{V}_1,0} \\
&+ \left\langle \Big( - C_0(|f| + |df|) + f^2\left|\widetilde{\Psi}^{\widetilde{M}}\right|^2\Big)\omega, \varphi^2_{k,m}\omega\right\rangle_{\mathcal{V}_1}.
\end{aligned} \tag{2.38}$$

Here we estimate the Lie derivative term in (2.21) by the term $\|\cdot\|^2_{\mathcal{V}_1,0}$ in (2.38).

Recall that on $\partial M \times (-\infty, 0]$, $\widetilde{\Psi}^{\widetilde{M}}(y, x_n)$ is nowhere zero and constant in $x_n$. By (1.5), (1.6), (2.34), (2.37) and the assumption that $(1+|f|)^{-1/2}\omega \in$

$L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$, there exists $C > 0$ such that for any $k \in \mathbb{N}$,

$$\begin{aligned} \operatorname{Re} \left\langle D^{\widetilde{E}}\omega, D^{\widetilde{E}}(\varphi_{k,-1}^2\,\omega) \right\rangle_{\mathcal{V}_1} &= \operatorname{Re} \left\langle -\sqrt{-1} f c(\widetilde{\Psi}^{\widetilde{M}})\omega, D^{\widetilde{E}}(\varphi_{k,-1}^2\,\omega) \right\rangle_{\mathcal{V}_1} \\ &\geqslant \frac{1}{2} \left\| \varphi_k \left| \widetilde{\Psi}^{\widetilde{M}} \right| \omega \right\|_{\mathcal{V}_1,0}^2 - C. \end{aligned} \tag{2.39}$$

From (1.5), (1.6), (2.39) and $(1+|f|)^{-1/2}\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$, when $k \to \infty$, we get $\left|\widetilde{\Psi}^{\widetilde{M}}\right| \omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ from (2.38) by taking $m = -1$. This implies $\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$.

By repeating the above process, we know that $f^m\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$ for any $m \in \mathbb{N}$.

For $k, m \in \mathbb{N}$, by (2.5), as in (2.38), we get

$$\begin{aligned} \left\langle (D^{\widetilde{E}})^2(\varphi_{k,m}\omega), \varphi_{k,m}\omega \right\rangle_{\mathcal{V}_1} &= \left\| \varphi_{k,m} D^E_{\partial M}\omega \right\|_{\mathcal{V}_1,0}^2 \\ &+ \int_{\partial M \times \{-C_f\}} \left\langle \tfrac{\partial}{\partial x_n}(\varphi_{k,m}\omega), \varphi_{k,m}\omega \right\rangle dv_{\partial M} + \left\| \tfrac{\partial}{\partial x_n}(\varphi_{k,m}\omega) \right\|_{\mathcal{V}_1,0}^2. \end{aligned} \tag{2.40}$$

Note that the following pointwise estimate holds:

$$\left| \tfrac{\partial}{\partial x_n}(\varphi_{k,m}\omega) \right|^2 \geqslant \frac{1}{2} \left| \varphi_{k,m} \tfrac{\partial \omega}{\partial x_n} \right|^2 - \left| \tfrac{\partial \varphi_{k,m}}{\partial x_n}\omega \right|^2. \tag{2.41}$$

From (2.34), (2.36) and $f^m\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$, we know that the left-hand side of (2.40) is uniformly bounded on $k > 1$, and $\frac{\partial \varphi_{k,m}}{\partial x_n}\omega \to m f^{m-1} \frac{\partial f}{\partial x_n}\omega$ in $L^2(\mathcal{V}_1, (S(T\widetilde{M}) \otimes \widetilde{E})|_{\mathcal{V}_1})$ as $k \to +\infty$. From (2.40), (2.41), for any $m \in \mathbb{N}$, when $k \to +\infty$, $f^m D^E_{\partial M}\omega$, $f^m \frac{\partial \omega}{\partial x_n} \in L^2(\mathcal{V}_1, (S(T\widetilde{M}) \otimes \widetilde{E})|_{\mathcal{V}_1})$.

The proof of Lemma 2.5 is completed. □

## 3. Transversal index and $L^2$-index

We use the notation in Introduction and Section 2.1.

Let $f$ be a $G$-invariant function on $\widetilde{M}$ verifying (1.5) and (1.6), and we assume that $\Psi^M$ is nowhere zero on $\partial M$.

The purpose of this section is to give a self-contained proof of Braverman's following result [2, Theorem 5.5] for manifolds with boundary, which identifies the transversal index in (1.4) and the $L^2$-index appearing in (2.32).

**Theorem 3.1.** *For any $\gamma \in \Lambda_+^*$, the following identity holds:*

$$\operatorname{Ind}_\gamma \left( D^{\widetilde{E}}_{+,f} \right) = \operatorname{Ind}_\gamma \left( \sigma^M_{E,\Psi} \right). \tag{3.1}$$

*Proof.* To prove (3.1), by Lemma 2.4a), we can and we will assume that there exists $C_1 > 0$ such that $f > C_1$ on $\widetilde{M}$.

Set $U = M \cup \partial M \times [-1,0]$ and $\widehat{U} = M \cup \partial M \times (-1,0]$ the interior of $U$. Let $\rho : (-1,0] \to (-\infty,0]$ be a strictly increasing smooth function such that

$$\rho(t_n) = t_n \text{ for } t_n \in [-1/4,0], \;\; \rho(t_n) = \log(1+t_n) \text{ for } t_n \in (-1,-1/2]. \tag{3.2}$$

We define the diffeomorphism $\tau : \widehat{U} \to \widetilde{M}$ by

$$\tau(x) = x \text{ for } x \in M, \quad \tau(y,t_n) = (y,\rho(t_n)) \text{ for } (y,t_n) \in \partial M \times (-1,0]. \tag{3.3}$$

Let $g^{T\widehat{U}} := \tau^* g^{T\widetilde{M}}$ be the induced metric on $T\widehat{U}$, then on $\partial M \times (-1,0] \subset \widehat{U}$,

$$g^{T\widehat{U}}(y,t_n) = g_y^{T\partial M} + (\rho'(t_n))^2 (dt_n)^2. \tag{3.4}$$

We define the metric $g^{TU}$ on $TU$ by $g^{TM}$ on $M$ and

$$g^{TU}(y,t_n) = g_y^{T\partial M} + (dt_n)^2 \quad \text{ for } (y,t_n) \in \partial M \times (-1,0]. \tag{3.5}$$

From (2.1), the pull-back of $(S(T\widetilde{M}), h^{S(T\widetilde{M})}, \nabla^{S(T\widetilde{M})})$, $(\widetilde{E}, h^{\widetilde{E}}, \nabla^{\widetilde{E}})$ on $\widehat{U}$ are still the pull-back of the corresponding objects on $\partial M$. Thus they extend naturally to $U$, and we denote them by $(S(TU), h^{S(TU)}, \nabla^{S(TU)})$, $(E, h^E, \nabla^E)$. Moreover, the $G$-action on $\partial M \times [-1,0] \subset U$ is induced by the $G$-action on $\partial M$, and the induced map $\Psi : U \to \mathfrak{g}$ is still constant in $t_n$ on $\partial M \times [-1,0]$.

Since $\Psi^M$ is nowhere zero on $U \setminus M$, by the additivity of the transversal index (cf. [1, Theorem 3.7, §6] and [9, Prop. 4.1]), one has

$$\operatorname{Ind}\left(\sigma^M_{E,\Psi}\right) = \operatorname{Ind}\left(\sigma^U_{E,\Psi}\right) \in R[G]. \tag{3.6}$$

Let $L^2(U, S(TU) \otimes E)$ be the space of $L^2$-sections of $S(TU) \otimes E$ on $U$ with norm $\| \;\; \|_0$ associated to $g^{TU}, h^{S(TU)}, h^E$ as in (2.2), and $H^k(U, S(TU) \otimes E)$ the corresponding $k^{\text{th}}$ Sobolev space.

We adapt now the idea of [2, §14.2–§14.5] to deform the transversal elliptic symbol $\sigma^U_{E,\Psi}$ in (1.3) and to identify $\operatorname{Ind}_\gamma(D^{\widetilde{E}}_{+,f})$ to $\operatorname{Ind}_\gamma(\sigma^U_{E,\Psi})$.

Let $\vartheta : [0,\infty) \to [1,\infty)$ be a smooth function such that $\vartheta(t) = 1$ for $t \leqslant 1$ and $\vartheta(t) = t$ for $t \geqslant 2$. Let $\vartheta_M : U \to [0,1]$ be a smooth function such that $\vartheta_M \equiv 1$ on $M$ and $\vartheta_M \equiv 0$ on $\partial M \times [-1,-1/2]$. We still denote by $\pi : TU \to U$ the natural projection. Consider the symbol for $x \in U, \xi \in T_x U$,

$$\sigma(x,\xi) := \sqrt{-1}\Big\{ \vartheta_M(x) \vartheta(|\xi|_{g^{TU}})^{-1} c(\xi) + c(\Psi^U) \Big\} \otimes \operatorname{Id}_{\pi^* E} \,. \tag{3.7}$$

Then $\sigma$ is a transversally elliptic symbol of order 0 and is homotopic to $\sigma^U_{E,\Psi}$ on $T_G U$.

As the manifold $U$ with boundary is compact, by [11, Proposition 2.4], there exists a $G$-vector bundle $F$ over $U$ such that the bundle $S_+(TU) \otimes E \oplus F$ is a trivial $G$-vector bundle on $U$. The map $\sqrt{-1}c(\Psi^U) \otimes \operatorname{Id}_E + \operatorname{Id}_F$ defines an isomorphism of the restriction of $S_+(TU) \otimes E \oplus F$ and $S_-(TU) \otimes E \oplus F$ on $\partial M \times [-1,0) = U \setminus M$, and so a trivialization of $S_-(TU) \otimes E \oplus F$ on $U \setminus M$.

Let $\jmath : U \to N$ be a $G$-equivariant embedding of the manifold $U$ with boundary into a smooth compact $G$-manifold $N$ with the same dimension (for example, we can take $N$ as the double of $U$). Then by extending $S_\pm(TU) \otimes E \oplus F$ on $U$ as trivial $G$-vector bundles on $N \setminus M$, we get $G$-vector bundles $\widetilde{\mathcal{E}}_\pm$ over $N$. Moreover we have a natural map $\widetilde{c} : \widetilde{\mathcal{E}}_+ \to \widetilde{\mathcal{E}}_-$ whose restriction to $N \setminus M$ is invertible and

$$\widetilde{c} = \sqrt{-1}c(\Psi^U) \otimes \mathrm{Id}_E + \mathrm{Id}_F \quad \text{on } U. \tag{3.8}$$

We still denote by $\vartheta_M$ the extension of $\vartheta_M$ on $N$ by taking $\vartheta_M = 0$ on $N \setminus U$, and $\pi : TN \to N$ the natural projection. Then the symbol

$$\widetilde{\sigma}_N(x,\xi) := \sqrt{-1}\vartheta_M(x)\vartheta(|\xi|_{g^{TU}})^{-1}c(\xi) \otimes \mathrm{Id}_{\pi^*E} + \pi^*\widetilde{c}(x) : \pi^*\widetilde{\mathcal{E}}_+ \to \pi^*\widetilde{\mathcal{E}}_- \tag{3.9}$$

for $x \in N, \xi \in T_{G,x}N$, is a transversally elliptic symbol of order 0 on $T_GN$ which extends the transversally elliptic symbol $\widetilde{\sigma} = \sigma + \mathrm{Id}_{\pi^*F}$ on $T_GU$.

The excision theorem [1, Theorem 3.7] tells us that the index $\mathrm{Ind}(\widetilde{\sigma}_N)$ of the transversally elliptic symbol $\widetilde{\sigma}_N$ depends only on $\sigma^U_{E,\Psi}$, but not on the choice of $\jmath$, $\widetilde{\sigma}_N$, and its character is a distribution on $G$. By the definition of $\mathrm{Ind}(\sigma^U_{E,\Psi})$ (cf. [1, §1], [9, §3]), we have

$$\mathrm{Ind}(\sigma^U_{E,\Psi}) := \mathrm{Ind}(\widetilde{\sigma}_N). \tag{3.10}$$

We construct now a particular zeroth-order transversally elliptic operator $P$ on $N$ whose symbol is homotopic to $\widetilde{\sigma}_N$.

Let $g^{TN}$ be a $G$-invariant Riemannian metric on $N$ which extends $g^{TU}$ on $U$, and $h^{\widetilde{\mathcal{E}}} = h^{\widetilde{\mathcal{E}}_+} \oplus h^{\widetilde{\mathcal{E}}_-}$ be a $G$-invariant Hermitian metric on $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_+ \oplus \widetilde{\mathcal{E}}_-$ on $N$ which extends $h^{S(TU)\otimes E}$ on $U$. The metrics $g^{TN}$, $h^{\widetilde{\mathcal{E}}}$ induce a norm $\| \ \|_0$ on $L^2(N,\widetilde{\mathcal{E}})$, the space of $L^2$-sections of $\widetilde{\mathcal{E}}$ on $N$, as in (2.2). Let $\mathcal{A} : \mathscr{C}^\infty(N,\widetilde{\mathcal{E}}_+) \to \mathscr{C}^\infty(N,\widetilde{\mathcal{E}}_+)$ be an invertible positive-definite self-adjoint $G$-invariant second-order differential operator, whose principal symbol is $\sigma(\mathcal{A})(x,\xi) = |\xi|^2_{g^{TN}} \mathrm{Id}_{\widetilde{\mathcal{E}}_+}$.

Recall that we assume that $f > C_1 > 0$ on $\widetilde{M}$. Set

$$\widetilde{f}(x) = \frac{1}{f \circ \tau(x)} \quad \text{if } x \in \widehat{U}; \quad \widetilde{f}(x) = 0 \quad \text{if } x \in N \setminus U. \tag{3.11}$$

By (1.5) and (1.6), $\widetilde{f}$ is a $G$-invariant $\mathscr{C}^0$ function on $N$.

We denote by $D^E_{\widehat{U}}$ on $\widehat{U}$ the operator $D^{\widetilde{E}}$ under the map $\tau : \widehat{U} \to \widetilde{M}$. By (2.5) and (3.3), we get

$$D^E_{\widehat{U}} = c(e_n)D^E_{\partial M} + \frac{1}{\rho'(t_n)}c(e_n)\frac{\partial}{\partial t_n} \quad \text{on } \partial M \times (-1,0], \tag{3.12}$$

and the restriction of $D^E_{\widehat{U}}$ to $M$ is the Spin$^c$ Dirac operator $D^E$ on $M$ associated to $g^{TM}, h^{S(TM)}, h^E$. Set

$$P = \widetilde{c} + \widetilde{f} D^E_{\widehat{U}} \mathcal{A}^{-1/2}. \tag{3.13}$$

By (3.11) and (3.12), $P$ is a well-defined pseudo-differential operator on $N$.

For $\xi \in TN$, we define $\widehat{c}(\xi) = c(\xi)$ on $M$, and

$$\widehat{c}(\xi) = c(\xi) \ \text{ for } \ \xi \in T\partial M|_{\partial M \times (-1,0]},$$
$$\widehat{c}\left(\frac{\partial}{\partial t_n}\right)_{(y,t_n)} = \frac{1}{\rho'(t_n)}c(e_n) \ \text{ for } \ (y,t_n) \in \partial M \times (-1,0].$$

Then by (3.12), under the identification of $T^*N$ and $TN$ by using the metric $g^{TN}$, the principal symbol of $P$ is

$$\sigma(P)(x,\xi) = \pi^*\widetilde{c} + \sqrt{-1}\widetilde{f}\,\vartheta(|\xi|_{g^{TN}})^{-1}\widehat{c}(\xi), \quad \text{for } x \in N,\ \xi \in T_xN. \tag{3.14}$$

By (3.9) and (3.14), for $t \in [0,1]$, $x \in N$, $\xi \in T_{G,x}N$, we have

$$\begin{aligned}(t\widetilde{\sigma}_N + (1-t)\sigma(P))(x,\xi)&\\ = \sqrt{-1}\vartheta(|\xi|_{g^{TN}})^{-1}&\left(t\,\vartheta_M(x)c(\xi) + (1-t)\widetilde{f}\,\widehat{c}(\xi)\right) + \pi^*\widetilde{c}(x).\end{aligned} \tag{3.15}$$

Thus for $x \in N \setminus U$, $(t\widetilde{\sigma}_N + (1-t)\sigma(P))(x,\xi)$ is $\pi^*\widetilde{c}(x)$ which is invertible, and for $\xi \in T_{G,x}U$, as $\langle \xi, \Psi^U\rangle = 0$, we know if $\Psi^U(x) \neq 0$, by (3.8), $\widetilde{c}(x)$ is invertible, thus $(t\widetilde{\sigma}_N + (1-t)\sigma(P))(x,\xi)$ is invertible. We conclude finally that for any $t \in [0,1]$,

$$\begin{aligned}\{(x,\xi) \in T_GN : (t\widetilde{\sigma}_N + (1-t)\sigma(P))(x,\xi) \text{ is not invertible}\}&\\ = \{(x,0) \in T_GM : \Psi^U(x) = 0\}&\end{aligned} \tag{3.16}$$

is compact. In particular, $\sigma(P)$ is a transversally elliptic symbol and homotopic to $\widetilde{\sigma}_N$, thus $P$ is a zeroth-order transversally elliptic operator on $N$, and by (3.10) and the homotopy invariance of the transversal index (cf. [1, Theorem 3.7, §6] and [9, Prop. 4.1]),

$$\mathrm{Ind}(\sigma^U_{E,\Psi}) = \mathrm{Ind}(\widetilde{\sigma}_N) = \mathrm{Ind}(P). \tag{3.17}$$

For $t \in [0,1]$, consider the family of operators

$$P_t = (1-t)\widetilde{c} + t\,\widetilde{c}\,\mathcal{A}^{-1/2} + \widetilde{f}\,D^E_{\widehat{U}}\,\mathcal{A}^{-1/2} : L^2(N,\widetilde{\mathcal{E}}_+) \to L^2(N,\widetilde{\mathcal{E}}_-). \tag{3.18}$$

For $\gamma \in \Lambda^*_+$, we denote by $P_t(\gamma)$ the restriction of $P_t$ to $L^2(N,\widetilde{\mathcal{E}}_+)^\gamma$, the $\gamma$-component of $L^2(N,\widetilde{\mathcal{E}}_+)$. For any $t<1$, the operator $P_t$ is a transversally elliptic operator depending continuously on $t$, thus $P_t(\gamma)$ is Fredholm for any $\gamma \in \Lambda^*_+$, and as $P_0 = P$, from (1.4), (3.17),

$$\mathrm{Ind}(P_t(\gamma)) = \mathrm{Ind}(P_0(\gamma)) = \mathrm{Ind}_\gamma(\sigma^U_{E,\Psi}) \cdot V^G_\gamma \quad \text{for } t<1. \tag{3.19}$$

Since $P_t$ is a family of bounded operators which depends continuously on $t$, to show (3.19) holds for $t=1$, we only need to prove that the operator $P_1(\gamma)$ is Fredholm for any $\gamma \in \Lambda^*_+$.

From (3.18), we have

$$P_1 = \widetilde{c}\,\mathcal{A}^{-1/2} + \widetilde{f}\,D^E_{\widehat{U}}\mathcal{A}^{-1/2} : L^2(N,\widetilde{\mathcal{E}}_+) \to L^2(N,\widetilde{\mathcal{E}}_-). \tag{3.20}$$

Thus $s \in \mathrm{Ker}(P_1)$ if and only if $\omega := \mathcal{A}^{-1/2}s \in H^1(N,\widetilde{\mathcal{E}}_+)$, the $1^{\text{st}}$ Sobolev space on $N$ with values in $\widetilde{\mathcal{E}}_+$ associated to $g^{TN}, h^{\widetilde{\mathcal{E}}_+}$, satisfies that

$$\left(\widetilde{c} + \widetilde{f}\,D^E_{\widehat{U}}\right)\omega = 0. \tag{3.21}$$

As $\widetilde{f} \equiv 0$ and $\widetilde{c}$ is invertible on $N \setminus U$, from (3.21), $\omega = 0$ on $N \setminus U$. Thus (3.21) holds if and only if $\operatorname{supp}(\omega) \subset U$ and (3.21) holds on $U$. By (3.8) and (3.11), (3.21) is equivalent to $\omega \in H^1(U, S_+(TU) \otimes E)$, $\omega = 0$ on $\partial U$, and

$$\left( D^E_{\widehat{U}} + \sqrt{-1} f \circ \tau \ c\left(\Psi^U\right) \right) \omega = 0. \tag{3.22}$$

**Lemma 3.2.** *Assume that* (*cf.* (1.5))

$$\lim_{x_n \to -\infty} \varrho(x_n) e^{x_n} = +\infty. \tag{3.23}$$

*If $\omega$ verifies* (3.22), *then $\omega \in L^2(U, S(TU) \otimes E)^\gamma$ if and only if $\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})^\gamma$. In this case, for any $m \in \mathbb{N}$, $f^m \omega \in L^2(U, S(TU) \otimes E)^\gamma$, and $\omega \in H^1(U, S(TU) \otimes E)$, $\omega = 0$ on $\partial U$.*

*Proof.* By (2.1), (3.12) and the discussion after (3.5), (3.22) is equivalent to (2.34) in the sense of distribution.

Let $dv_{\widehat{U}}$, $dv_U$ be the Riemannian volume forms on $(\widehat{U}, g^{T\widehat{U}})$, $(U, g^{TU})$, respectively. Then by (3.4) and (3.5), we have

$$dv_{\widehat{U}}(y, t_n) = \rho'(t_n)\, dv_U(y, t_n) \quad \text{ on } \partial M \times (-1, 0]. \tag{3.24}$$

Note that on $(-1, -1/2]$, $\rho'(t_n) = \frac{1}{1+t_n}$.

By (3.24), if $\omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$, then $\omega \in L^2(U, S(TU) \otimes E)$.

Now assume that $\omega$ verifies (3.22) and $\omega \in L^2(U, S(TU) \otimes E)^\gamma$. Then by the ellipticity of $D^{\widetilde{E}}$, $\omega \in \mathscr{C}^\infty(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$.

By (3.23), on $\partial M \times (-1, -1/2] \subset \widehat{U}$, $\lim_{t_n \to -1} (f \circ \tau)(t_n)(t_n + 1) = +\infty$. Thus from (3.24),

$$\begin{aligned} &\int_{\partial M \times (-\infty, \log(1/2)]} (1 + |f|)^{-1} |\omega|^2 dv_{\widetilde{M}} \\ &\qquad = \int_{\partial M \times (-1, -1/2]} (1 + |f \circ \tau|)^{-1} |\omega|^2 (t_n + 1)^{-1} dv_U < +\infty. \end{aligned} \tag{3.25}$$

This means $(1 + |f|)^{-1/2} \omega \in L^2(\widetilde{M}, S(T\widetilde{M}) \otimes \widetilde{E})$. Now from Lemma 2.5 and (3.24), we get $\omega, f\omega \in H^1(U, S(TU) \otimes E)$. Thus the restrictions of $\omega$, $f\omega$ on $\partial U$ are well defined. But $f = \infty$ on $\partial U$, thus $\omega = 0$ on $\partial U$. □

From Lemma 2.4, we can assume that $f$ is a strictly positive $G$-invariant smooth function on $\widetilde{M}$ verifying (1.5), (1.6) and (3.23).

From Lemma 3.2, (2.14), (3.21) and (3.22), we know that $\operatorname{Ker}(P_1(\gamma))$ is isomorphic to $\operatorname{Ker}(D^{\widetilde{E}}_{+,f}(\gamma))$, in the same way, $\operatorname{Coker}(P_1(\gamma))$ is isomorphic to $\operatorname{Ker}(D^{\widetilde{E}}_{-,f}(\gamma))$. But we have proved that $D^{\widetilde{E}}_f(\gamma)$ is a Fredholm operator, thus $P_1(\gamma)$ is a Fredholm operator and (3.19) holds for $t = 1$. The proof of Theorem 3.1 is completed. □

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