Ann. Inst. Fourier, Grenoble
50, 5 (2000), 1539–1588

SUBMERSIONS AND EQUIVARIANT QUILLEN METRICS

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Introduction.

Let ξ be a Hermitian vector bundle on a compact Hermitian complex manifold X. Let $\lambda(\xi)$ be the inverse of the determinant of the cohomology of ξ. Quillen defined first a metric on $\lambda(\xi)$ in the case that X is a Riemann surface. Quillen metric is the product of the L^2 metric on $\lambda(\xi)$ by the analytic torsion of Ray-Singer of ξ. The analytic torsion of Ray-Singer [RS] is the regularized determinant of the Kodaira Laplacian on ξ. In [BGS3], Bismut, Gillet, and Soulé have extended it to complex manifolds. They have established the anomaly formulas for Quillen metrics, which tell us the variation of Quillen metric on the metrics on ξ and TX by using some Bott-Chern classes.

Later, Bismut and Köhler [BKö] (refer also [BGS2], [GS1] in the special case) have extended the analytic torsion of Ray-Singer to the analytic torsion forms T for a holomorphic submersion. In particular, the equation on $(\bar{\partial}\partial/2i\pi)T$ gives a refinement of the Grothendiek-Riemann-Roch Theorem. They have also established the corresponding anomaly formulas.

In [GS1], Gillet and Soulé had conjectured an arithmetic Riemann-Roch Theorem in Arakelov geometry. In [GS2], they have proved it for the first Chern class. The analytic torsion forms are contained in their definition of direct image.

Let $i:Y \to X$ be an immersion of compact complex manifolds. Let η be a holomorphic vector bundle on Y, and let (ξ, v) be a complex of

Key words: Characteristic classes – Index theory and fixed points theory.
Math. classification: 32L10 – 57R20 – 58J20 – 58J52.

holomorphic vector bundles which provides a resolution of $i_*\eta$. Then by [KM], the line $\lambda^{-1}(\eta)\otimes\lambda(\xi)$ has a nonzero canonical section σ. In [BL], Bismut and Lebeau have given a formula for the Quillen norm of σ in terms of Bott-Chern currents on X and of a genus R introduced by Gillet and Soulé [GS1].

In [BerB], Bismut and Berthomieu solved a similar problem. In fact, let $\pi: M \to B$ be a submersion of compact complex manifolds. Let ξ be a holomorphic vector bundle on M. Let $R^\bullet\pi_*\xi$ be the direct image of ξ. Then, by [KM], the line $\lambda(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)$ has a nonzero canonical section σ. In [BerB], they have given a formula for the Quillen norm of σ in terms of Bott-Chern classes on M and the analytic torsion forms of π.

Now, let G be a compact Lie group acting holomorphically on X and ξ. Then Bismut [B5] defined $\lambda_G(\xi)$ the inverse of the equivariant determinant of the cohomology of ξ on X. He also defined an equivariant Quillen metric on $\lambda_G(\xi)$ which is a central function on G (refer also §1a)). In [B5], Bismut computed the equivariant Quillen metric of the nonzero canonical section of $\lambda_G^{-1}(\eta)\otimes\lambda_G(\xi)$ for a G-equivariant immersion $i: Y\to X$. In this way, he has generalized the result of [BL] to the equivariant case. In [B4], he also conjectured an equivariant arithmetic Riemann-Roch Theorem in Arakelov geometry. Recently, using the result of [B5], Köhler and Roessler [KRo] have proved a version of this conjecture.

In this paper, we shall extend the result of Bismut and Berthomieu to the G-equivariant case. This completes the picture on the G-equivariant case.

Let $\pi: M\to B$ be a submersion of compact complex manifolds with fibre X. Let ξ be a holomorphic vector bundle on M. Let G be a compact Lie group acting holomorphically on M and B, and commuting with π, whose actions lift holomorphically on ξ.

Let $R^\bullet\pi_*\xi$ be the direct image of ξ. We assume that the $R^k\pi_*\xi$ $(0\le k\le \dim X)$ are locally free.

Let σ be the canonical section of $\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)$.

Let h^{TM}, h^{TB} be G-invariant Kähler metrics on TM and TB. Let h^{TX} be the metric induced by h^{TM} on TX. Let h^ξ be a G-invariant Hermitian metric on ξ. Let ω^M be the Kähler form of h^{TM}.

Let $H(X,\xi_{|X})$ be the cohomology of $\xi_{|X}$. By identifying $H(X,\xi_{|X})$ to the corresponding fiberwise harmonic forms in Dolbeault complex $(\Omega(X,\xi_{|X}),\overline{\partial}^X)$, the $\mathbb{Z}$-graded vector bundle $H(X,\xi_{|X})$ is naturally equipped with a L^2-metric $h^{H(X,\xi|_X)}$ associated to h^{TX}, h^ξ.

Let $\| \ \|_{\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)}$ be the G-equivariant Quillen metric on the line $\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)$ attached to the metrics $h^{TM}, h^\xi, h^{TB}, h^{H(X,\xi|_X)}$ on TM, ξ, TB, $R^\bullet\pi_*\xi$. The purpose of this paper is to calculate the G-equivariant Quillen metric $\|\sigma\|_{\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)}$.

For $g\in G$, let $\mathrm{Td}_g(TM, g^{TM})$ be the Chern-Weil Todd form on $M^g=\{x\in M\,;\ gx=x\}$ associated to the holomorphic hermitian connection on (TM, h^{TM}) [B5, §2 (a)], which appears in the Lefschetz formulas of Atiyah-Bott [ABo]. Other Chern-Weil forms will be denoted in a similar way. In particular, the forms $\mathrm{ch}_g(\xi, h^\xi)$ on M^g are the Chern-Weil representative of the g-Chern character form of (ξ, h^ξ).

In this paper, by an extension of [BKö], we first construct the equivariant analytic torsion forms $T_g(\omega^M, h^\xi)$ on $B^g=\{x\in B\,;\ gx=x\}$, such that

$$\frac{\bar\partial\partial}{2i\pi}T_g(\omega^M,h^\xi)=\mathrm{ch}_g\big(H(X,\xi|_X),h^{H(X,\xi|_X)}\big)-\int_{X^g}\mathrm{Td}_g(TX,h^{TX})\,\mathrm{ch}_g(\xi,h^\xi). \tag{0.1}$$

We also establish the corresponding anomaly formulas. The equivariant analytic torsion forms will play a role in the higher degree version of Köhler and Roessler's Theorem. Notice that in [K], Köhler defined the equivariant analytic torsion forms for (possibly non-Kähler) torus fibrations and proved curvature and anomaly formulas for them.

Let $\widetilde{\mathrm{Td}}_g(TM,TB,h^{TM},h^{TB})\in P^{M^g}/P^{M^g,0}$ be the Bott-Chern class, constructed in [BGS1], such that

$$\frac{\bar\partial\partial}{2i\pi}\widetilde{\mathrm{Td}}_g(TM,TB,h^{TM},h^{TB})=\mathrm{Td}_g(TM,h^{TM})-\pi^*\big(\mathrm{Td}_g(TB,h^{TB})\big)\,\mathrm{Td}_g(TX,h^{TX}). \tag{0.2}$$

The main result of this paper is the following extension of [BerB, Thm. 3.1]. Namely, we prove in Theorem 3.1 the formula

$$\log\big(\|\sigma\|^2_{\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)}\big)(g)=-\int_{B^g}\mathrm{Td}_g(TB,h^{TB})\,T_g(\omega^M,h^\xi)+\int_{M^g}\widetilde{\mathrm{Td}}_g(TM,TB,h^{TM},h^{TB})\,\mathrm{ch}_g(\xi,h^\xi). \tag{0.3}$$

We apply the methods and techniques in [BerB] and [B5], with necessary equivariant extensions, to prove Theorem 3.1. The local index theory [B1] and finite propagation speed of the solution of the hyperbolic equation [CP], [T] will also play an important role as in [BerB] and [B5].

This paper is organized as follows. In Section 1, we recall the construction of the equivariant Quillen metrics [B5]. In Section 2, we construct the equivariant analytic torsion forms, and we prove the corresponding anomaly formulas. In Section 3, we extend the result of [BerB] to the equivariant case. In Section 4, we state eight intermediate results which we need for the proof of Theorem 3.1, and we prove Theorem 3.1. In Sections 5–9, we prove the eight intermediate results.

Throughout, we use the superconnection formalism of Quillen. In particular, Tr_s is our notation for the supertrace. The reader is referred for more details to [B5], [BGS1], [BerB].

1. Equivariant Quillen metrics.

This section is organized as follows. In a), we recall the construction of the equivariant Quillen metrics of [B5, §1]. In b), we indicate the characteristic classes which we will often use.

a) Equivariant Quillen metrics [B5].

Let X be a compact complex manifold of complex dimension ℓ. Let ξ be a holomorphic vector bundle on X. Let $H(X,\xi)$ be the cohomology groups of the sheaf $\mathcal{O}_X(\xi)$ of holomorphic sections of ξ over X.

Let G be a compact Lie group. We assume that G acts on X by holomorphic diffeomorphisms and that the action of G lifts to a linear holomorphic action on ξ.

Let $E = \bigoplus_{i=0}^{\dim X} E^i$ be the vector space of $\mathcal{C}^\infty$ sections of

$$\Lambda(T^{*(0,1)}X) \otimes \xi = \bigoplus_{i=0}^{\dim X} \Lambda^i(T^{*(0,1)}X) \otimes \xi$$

over X. Let $\overline{\partial}^X$ be the Dolbeault operator acting on E. Then G acts on the Dolbeault complex $(E, \overline{\partial}^X)$ by chain homomorphisms, and we have an identification of G-vector spaces

$$H(E, \overline{\partial}^X) \simeq H(X,\xi). \tag{1.1}$$

Let h^{TX}, h^{ξ} be G-invariant Hermitian metrics on TX, ξ. Let dv_X be the Riemannian volume form on X associated to h^{TX}. Let $*$ be the Hodge operator attached to the metric h^{TX}. Let $\langle\ \rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the Hermitian product induced by h^{TX}, h^{ξ} on $\Lambda(T^{*(0,1)}X)\otimes\xi$. If $s, s' \in E$, set

$$\begin{aligned}\langle s, s'\rangle &= \Big(\frac{1}{2\pi}\Big)^{\dim X}\int_X \langle s, s'\rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}\, dv_X \\ &= \Big(\frac{1}{2\pi}\Big)^{\dim X}\int_X \langle s\wedge *s'\rangle_{h^{\xi}}.\end{aligned} \tag{1.2}$$

Let $\bar\partial^{X*}$ be the formal adjoint of $\bar\partial^X$ with respect to the Hermitian product (1.2). Set

$$D^X = \bar\partial^X + \bar\partial^{X*}, \quad K(X,\xi) = \operatorname{Ker} D^X. \tag{1.3}$$

By Hodge theory,

$$K(X,\xi) \simeq H(X,\xi). \tag{1.4}$$

Clearly, for $g \in G$, g commutes with D^X, so (1.4) is an identification of G-spaces.

Clearly $K(X,\xi)$ inherits a G-invariant metric from $\langle\ \rangle$. Let $h^{H(X,\xi)}$ be the corresponding metric on $H(X,\xi)$.

Let $\widehat{G}$ be the set of equivalence classes of complex irreducible representations of G. Let F^i $(0 \le i \le k)$ be finite dimensional complex G-vector spaces. We consider $F = \bigoplus_{i=0}^k F^i$ as a natural $\mathbb{Z}$-graded G-vector space. Let $h^F = \bigoplus_{i=0}^k h^{F^i}$ be a G-invariant metric on $F = \bigoplus_{i=0}^k F^i$. Then we have the isotypical decomposition

$$F = \bigoplus_{W\in\widehat{G}} \operatorname{Hom}_G(W,F)\otimes W,$$

and this decomposition is orthogonal with respect to h^F. Set

$$\det(F,G) = \bigoplus_{W\in\widehat{G}} \bigotimes_{i=0}^{k} \big(\det(\operatorname{Hom}_G(W,F^i)\otimes W)\big)^{(-1)^i}. \tag{1.5}$$

For $W \in \widehat{G}$, let $\chi(W)$ be the character of the representation. Set

$$\lambda_W(\xi) = \bigotimes_{i=0}^{\dim X} \big(\det(\operatorname{Hom}_G(W, H^i(X,\xi))\otimes W)\big)^{(-1)^{i+1}}. \tag{1.6}$$

Put

$$\lambda_G(\xi) = \bigoplus_{W\in\widehat{G}} \lambda_W(\xi). \tag{1.7}$$

In the sequel, $\lambda_G(\xi)$ will be called the inverse of the equivariant determinant of the cohomology of ξ. So $\lambda_G(\xi)$ is a direct sum of complex lines.

Let $|\ |_{\lambda_W(\xi)}$ be the L^2-metric on $\lambda_W(\xi)$ induced by $h^{H(X,\xi)}$. Set

$$\log\big(|\ |^2_{\lambda_G(\xi)}\big) = \sum_{W\in\widehat{G}} \log\big(|\ |^2_{\lambda_W(\xi)}\big)\,\frac{\chi(W)}{\dim W}\cdot \tag{1.8}$$

The formal symbol $|\ |_{\lambda_G(\xi)}$ will be called the (equivariant) L_2 metric on $\lambda_G(\xi)$. In effect, it is a product of metrics on $\lambda_G(\xi) = \bigoplus_{W\in\widehat{G}} \lambda_W(\xi)$.

Take $g \in G$. Set

$$X^g = \big\{x \in X\ ;\ gx = x\big\}. \tag{1.9}$$

Then X^g is a compact complex totally geodesic submanifold of X.

Let P be the orthogonal projection operator from E on $K(X,\xi)$ with respect to the Hermitian product (1.2). Set $P^\perp = 1 - P$. Let N be the number operator of E, *i.e.* N acts by multiplication by i on E^i. Then by standard heat equation methods, we know that for any $g \in G$, $k \in \mathbb{N}$, there exist a_j $(-\ell \le j \le k)$ such that as $t \to 0$,

$$\mathrm{Tr}_s\big[gN\exp(-tD^{X,2})\big] = \sum_{j=-\ell}^{k} a_j\, t^j + O(t^{k+1}). \tag{1.10}$$

Definition 1.1. — *For $s \in \mathbb{C}$, $\mathrm{Re}(s) > \dim X$, set*

$$\theta^X(g)(s) = -\mathrm{Tr}_s\big[gN(D^{X,2})^{-s}P^\perp\big]. \tag{1.11}$$

By (1.10), $\theta^X(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$.

Definition 1.2. — *For $g \in G$, set*

$$\log\big(\|\ \|^2_{\lambda_G(\xi)}\big)(g) = \log\big(|\ |^2_{\lambda_G(\xi)}\big)(g) - \frac{\partial\theta^X(g)}{\partial s}(0). \tag{1.12}$$

The formal symbol $\|\ \|_{\lambda_G(\xi)}$ will be called a Quillen metric on the equivariant determinant $\lambda_G(\xi)$.

b) Some characteristic classes.

Let X be a complex manifold. Let L be a holomorphic vector bundle over X. Let h^L be a Hermitian metric on L. Let ∇^L be the holomorphic Hermitian connection on (L, h^L). Let R^L be its curvature.

Let g be a holomorphic section of $\mathrm{End}(L)$. We assume that g is an isometry of L. Then g is parallel with respect to ∇^L.

Let $1, \mathrm{e}^{i\theta_1}, \ldots, \mathrm{e}^{i\theta_q}$ $(0 < \theta_j < 2\pi)$ be the locally constant distinct eigenvalues of g acting on L on X. Let $L^{\theta_0}, L^{\theta_1}, \ldots, L^{\theta_q}$ $(\theta_0 = 0)$ be the corresponding eigenbundles. Then L splits holomorphically as an orthogonal sum

$$L = L^{\theta_0} \oplus \cdots \oplus L^{\theta_q}. \tag{1.13}$$

Let $h^{L^{\theta_0}}, \ldots, h^{L^{\theta_q}}$ be the Hermitian metrics on $L^{\theta_0}, \ldots, L^{\theta_q}$ induced by h^L. Then ∇^L induces the holomorphic Hermitian connections $\nabla^{L^{\theta_0}}, \ldots, \nabla^{L^{\theta_q}}$ on $(L^{\theta_0}, h^{L^{\theta_0}}), \ldots, (L^{\theta_q}, h^{L^{\theta_q}})$. Let $R^{L^{\theta_0}}, \ldots, R^{L^{\theta_q}}$ be their curvatures.

If A is a (q,q) matrix, set

$$\mathrm{Td}(A) = \det\Big(\frac{A}{1-\mathrm{e}^{-A}}\Big), \quad e(A) = \det(A), \quad \mathrm{ch}(A) = \mathrm{Tr}[\exp(A)]. \tag{1.14}$$

The genera associated to Td and e are called the Todd genus and the Euler genus.

Definition 1.3. — *Set*

$$\begin{cases} \mathrm{Td}_g(L, h^L) = \mathrm{Td}\Big(\dfrac{-R^{L^{\theta_0}}}{2i\pi}\Big) \displaystyle\prod_{j=1}^q \frac{\mathrm{Td}}{e}\Big(\frac{-R^{L^{\theta_j}}}{2i\pi} + i\theta_j\Big), \\ \mathrm{Td}'_g(L, h^L) = \dfrac{\partial}{\partial b}\Big[\mathrm{Td}\Big(\dfrac{-R^{L^{\theta_0}}}{2i\pi} + b\Big) \\ \qquad\qquad \times \displaystyle\prod_{j=1}^q \frac{\mathrm{Td}}{e}\Big(\frac{-R^{L^{\theta_j}}}{2i\pi} + i\theta_j + b\Big)\Big]_{b=0}, \\ (\mathrm{Td}_g^{-1})'(L, h^L) = \dfrac{\partial}{\partial b}\Big[\mathrm{Td}^{-1}\Big(\dfrac{-R^{L^{\theta_0}}}{2i\pi} + b\Big) \\ \qquad\qquad \times \displaystyle\prod_{j=1}^q \Big(\frac{\mathrm{Td}}{e}\Big)^{-1}\Big(\frac{-R^{L^{\theta_j}}}{2i\pi} + i\theta_j + b\Big)\Big]_{b=0}, \\ \mathrm{ch}_g(L, h^L) = \mathrm{Tr}\Big[g \exp\Big(\dfrac{-R^L}{2i\pi}\Big)\Big]. \end{cases} \tag{1.15}$$

Then the forms in (1.15) are closed forms on X, and their cohomology class does not depend on the g-invariant metric h^L. We denote these cohomology classes by $\mathrm{Td}_g(L)$, $\mathrm{Td}'_g(L), \ldots, \mathrm{ch}_g(L)$.

2. Equivariant analytic torsion forms and anomaly formulas.

This section is organized as follows. In a), we describe the Kähler fibrations. In b), we construct the Levi-Civita superconnection in the sense of [B1]. In c), we indicate results concerning the equivariant superconnection forms. In d), we construct the equivariant analytic torsion forms. In e), we prove the anomaly formulas, along the lines of [B5], [BKö].

a) Kähler fibrations.

Let $\pi : M \to B$ be a holomorphic submersion with compact fibre X. Let TM, TB be the holomorphic tangent bundles to M, B. Let TX be the holomorphic relative tangent bundle TM/B. Let J^{TX} be the complex structure on the real tangent bundle $T_{\mathbb{R}}X$. Let h^{TX} be a Hermitian metric on TX.

Let $T^H M$ be a vector subbundle of TM, such that

$$TM = T^H M \oplus TX. \tag{2.1}$$

We now define the Kähler fibration as in [BGS2, Def. 1.4].

Definition 2.1. — *The triple $(\pi, h^{TX}, T^H M)$ is said to define a Kähler fibration if there exists a smooth real 2-form ω of complex type $(1,1)$, which has the following properties:*

(a) *ω is closed;*

(b) *$T^H_{\mathbb{R}} M$ and $T_{\mathbb{R}}X$ are orthogonal with respect to ω;*

(c) *if $X, Y \in T_{\mathbb{R}}X$, then $\omega(X,Y) = \langle X, J^{TX} Y\rangle_{h^{TX}}$.*

Now we recall a simple result of [BGS2, Thms. 1.5 and 1.7].

Theorem 2.2. — *Let ω be a real smooth 2-form on M of complex type $(1,1)$, which has the following two properties:*

(a) *ω is closed;*

(b) *the bilinear map* $X, Y \in T_{\mathbb{R}}X \to \omega(J^{TX}X, Y)$ *defines a Hermitian product* h^{TX} *on* TX.

For $x \in M$, *set*

$$T_x^H M = \left\{ Y \in T_x M\,;\ \text{ for any } X \in T_x X,\ \omega(X, \overline{Y}) = 0 \right\}. \tag{2.2}$$

Then $T^H M$ *is a subbundle of* TM *such that* $TM = T^H M \oplus TX$. *Also* $(\pi, h^{TX}, T^H M)$ *is a Kähler fibration, and* ω *is an associated* $(1,1)$*-form.*

A smooth real $(1,1)$*-form* ω' *on* M *is associated to the Kähler fibration* $(\pi, h^{TX}, T^H M)$ *if and only if there is a real smooth closed* $(1,1)$*-form* η *on* B *such that*

$$\omega' - \omega = \pi^* \eta. \tag{2.3}$$

b) The Bismut superconnection of a Kähler fibration.

Let ω^M be a real (1,1)-form on M taken as in Theorem 2.2.

Let ξ be a complex vector bundle on M. Let h^ξ be a Hermitian metric on ξ. Let ∇^{TX}, ∇^ξ be the holomorphic Hermitian connections on $(TX, h^{TX}), (\xi, h^\xi)$. Let R^{TX}, L^ξ be the curvatures of ∇^{TX}, ∇^ξ. Let $\nabla^{\Lambda(T^{*(0,1)}X)}$ be the connection induced by ∇^{TX} on $\Lambda(T^{*(0,1)}X)$. Let $\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the connection on $\Lambda(T^{*(0,1)}X) \otimes \xi$,

$$\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi} = \nabla^{\Lambda(T^{*(0,1)}X)} \otimes 1 + 1 \otimes \nabla^\xi. \tag{2.4}$$

Definition 2.3. — *For* $0 \le p \le \dim X$, $b \in B$, *let* E_b^p *be the vector space of* $\mathcal{C}^\infty$ *sections of* $(\Lambda^p(T^{*(0,1)}X) \otimes \xi)_{|X_b}$ *over* X_b. *Set*

$$E_b = \bigoplus_{p=0}^{\dim X} E_b^p, \quad E_b^+ = \bigoplus_{p\,\text{even}} E_b^p, \quad E_b^- = \bigoplus_{p\,\text{odd}} E_b^p. \tag{2.5}$$

As in [B1, §1f)], [BGS2, §1d)], we can regard the E_b's as the fibres of a smooth $\mathbb{Z}$-graded infinite dimensional vector bundle E over the base B. Smooth sections of E over B will be identified with smooth sections of $\Lambda(T^{*(0,1)}X) \otimes \xi$ over M.

Let $\langle\ \rangle$ be the Hermitian product on E associated to h^{TX}, h^ξ defined in (1.2).

If $U \in T_{\mathbb{R}}B$, let U^H be the lift of U in $T_{\mathbb{R}}^H M$, so that $\pi_* U^H = U$.

Definition 2.4. — *If* $U \in T_{\mathbb{R}}B$, *if* s *is a smooth section of* E *over* B, *set*

$$\nabla^E_U s = \nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{U^H} s. \tag{2.6}$$

By [B1, §1f)], ∇^E is a connection on the infinite dimensional vector bundle E. Let $\nabla^{E'}$ and $\nabla^{E''}$ be the holomorphic and anti-holomorphic parts of ∇^E.

For $b \in B$, let $\bar\partial^{X_b}$ be the Dolbeault operator acting on E_b, and let $\bar\partial^{X_b *}$ be its formal adjoint with respect to the Hermitian product (1.2). Set

$$D^X = \bar\partial^{X_b} + \bar\partial^{X_b *}. \tag{2.7}$$

Let $c(T_{\mathbb{R}}X)$ be the Clifford algebra of $(T_{\mathbb{R}}X, h^{TX})$. The bundle $\Lambda(T^{*(0,1)}X)\otimes\xi$ is a $c(T_{\mathbb{R}}X)$-Clifford module. In fact, if $U \in TX$, let $U' \in T^{*(0,1)}X$ correspond to U by the metric h^{TX}. If $U, V \in TX$, set

$$c(U) = \sqrt{2}\, U'\wedge, \quad c(\overline{V}) = -\sqrt{2}\, i_{\overline{V}}. \tag{2.8}$$

Let P^{TX} be the projection $TM \simeq T^H M \oplus TX \to TX$.

If U, V are smooth vector fields on B, set

$$T(U^H, V^H) = -P^{TX}[U^H, V^H]. \tag{2.9}$$

Then T is a tensor. By [BGS2, Thm. 1.7], we know that as a 2-form, T is of complex type (1,1).

Let $f_1, \ldots, f_{2m}$ be a base of $T_{\mathbb{R}}B$, and let $f^1, \ldots, f^{2m}$ be the dual base of $T^*_{\mathbb{R}}B$.

Definition 2.5.

$$c(T) = \frac{1}{2}\sum_{1\le\alpha,\beta\le 2m} f^\alpha f^\beta c\big(T(f^H_\alpha, f^H_\beta)\big). \tag{2.10}$$

Then $c(T)$ *is a section of* $\big(\Lambda(T^*_{\mathbb{R}}B)\widehat{\otimes}\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes\xi)\big)^{\mathrm{odd}}$. *Similarly, if* $T^{(1,0)}$, $T^{(0,1)}$ *denote the components of* T *in* $T^{(1,0)}X, T^{(0,1)}X$, *we also define* $c(T^{(1,0)}), c(T^{(0,1)})$ *as in* (2.10), *so that*

$$c(T) = c\big(T^{(1,0)}\big) + c(T^{(0,1)}). \tag{2.11}$$

DEFINITION 2.6. — *For $u>0$, let B_u be the Bismut superconnection constructed in* [B1, §3], [BGS2, §2a)],

$$\left\{\begin{aligned} &B''_u = \nabla^{E''} + \sqrt{u}\,\bar\partial^X - \frac{c(T^{(1,0)})}{2\sqrt{2u}}, \\ &B'_u = \nabla^{E'} + \sqrt{u}\,\bar\partial^{X*} - \frac{c(T^{(0,1)})}{2\sqrt{2u}}, \\ &B_u = B'_u + B''_u. \end{aligned}\right. \tag{2.12}$$

Let N_V be the number operator defining the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X)\otimes\xi$ and on E. N_V acts by multiplication by p on $\Lambda^p(T^{*(0,1)}X)\otimes\xi$. If $U,V\in T_{\mathbb{R}}B$, set

$$\omega^{H\overline{H}}(U,V) = \omega^M(U^H,V^H). \tag{2.13}$$

DEFINITION 2.7. — *For $u>0$, set*

$$N_u = N_V + \frac{i\,\omega^{H\overline{H}}}{u}\cdot \tag{2.14}$$

In the rest of this subsection, we recall the definition of the tensor S [B1, Def. 1.8] which will be used in the proof of Theorem 2.13.

Let $h^{T_{\mathbb{R}}B}$ be a Riemannian metric on $T_{\mathbb{R}}B$. Let $\nabla^{T_{\mathbb{R}}B}$ be the Levi-Civita connection on $(T_{\mathbb{R}}B, h^{T_{\mathbb{R}}B})$. The metric $h^{T_{\mathbb{R}}B}$ and the connection $\nabla^{T_{\mathbb{R}}B}$ lift to a metric $h^{T^H_{\mathbb{R}}M}$ and a connection $\nabla^{T^H_{\mathbb{R}}M}$ on $T^H_{\mathbb{R}}M$. Let $h^{T_{\mathbb{R}}M} = h^{T^H_{\mathbb{R}}M}\oplus h^{T_{\mathbb{R}}X}$ be the metric on $T_{\mathbb{R}}M = T^H_{\mathbb{R}}M\oplus T_{\mathbb{R}}X$ which is the orthogonal sum of the metrics $h^{T^H_{\mathbb{R}}M}$ and $h^{T_{\mathbb{R}}X}$. Let $\langle\ ,\ \rangle_{h^{T_{\mathbb{R}}M}}$ denote the correponding scalar product on $T_{\mathbb{R}}M$.

Let $\nabla^{T_{\mathbb{R}}X}$ be the connection on $T_{\mathbb{R}}X$ induced by ∇^{TX}. Let $\nabla^{T_{\mathbb{R}}M,L}$ be the Levi-Civita connection on $(T_{\mathbb{R}}M, h^{T_{\mathbb{R}}M})$. Let $\nabla^{T_{\mathbb{R}}M} = \nabla^{T^H_{\mathbb{R}}M}\oplus\nabla^{T_{\mathbb{R}}X}$ be the connection on $T_{\mathbb{R}}M = T^H_{\mathbb{R}}M\oplus T_{\mathbb{R}}X$. Set

$$S = \nabla^{T_{\mathbb{R}}M,L} - \nabla^{T_{\mathbb{R}}M}. \tag{2.15}$$

Then S is a 1-form on M taking values in antisymmetric elements of $\mathrm{End}(T_{\mathbb{R}}M)$. By [B1, Thm. 1.9], the $(3,0)$ tensor $\langle S(\cdot)\cdot,\cdot\rangle_{h^{T_{\mathbb{R}}M}}$ does not depend on $h^{T_{\mathbb{R}}B}$. By (2.15), for $U,V\in T_{\mathbb{R}}X$,

$$S(U)V = S(V)U. \tag{2.16}$$

c) Equivariant superconnection forms and double transgression formulas.

At first, we assume that the direct image $R^\bullet\pi_*\xi$ of ξ by π is locally free. For $b \in B$, let $H(X_b, \xi_{|X_b})$ be the cohomology of the sheaf of holomorphic sections of $\xi_{|X_b}$. Then the $H(X_b, \xi_{|X_b})$'s are the fibres of a $\mathbb{Z}$-graded holomorphic vector bundle $H(X, \xi_{|X})$ on B, and $R^\bullet\pi_*\xi = H(X, \xi_{|X})$. So we will write indifferently $R^\bullet\pi_*\xi$ or $H(X, \xi_{|X})$.

By (1.4), the $K(X_b, \xi_{|X_b})$ are the fibres of a smooth vector bundle $K(X, \xi_{|X})$ over B. By [BGS3, Thm. 3.5], the isomorphism of the fibre (1.4) induces a smooth isomorphism of $\mathbb{Z}$-graded vector bundles on B

$$H(X, \xi_{|X}) \simeq K(X, \xi_{|X}). \tag{2.17}$$

Then $K(X, \xi_{|X})$ inherits a Hermitian product from $(E, \langle\ \rangle)$. Let $h^{H(X,\xi|X)}$ be the corresponding smooth metric on $H(X, \xi_{|X})$. Let $P^{H(X,\xi|X)}$ be the orthogonal projection operator from E on $H(X, \xi_{|X}) \simeq K(X, \xi_{|X})$. Let $\nabla^{H(X,\xi|X)}$ be the holomorphic Hermitian connection on $(H(X, \xi_{|X}), h^{H(X,\xi|X)})$.

Let G be a compact Lie group. We assume that G acts holomorphically on M, B, ξ, and that ξ, M are G-equivariant (vector) bundles over M, B. We also assume ω^M, h^ξ are G-invariant. Then $R^\bullet\pi_*\xi$ is also a G–equivariant vector bundle over B, and $h^{H(X,\xi|X)}$ is also G-invariant.

For $g \in G$, set

$$M^g = \big\{x \in M \,;\, gx = x\big\}, \quad B^g = \big\{x \in B \,;\, gx = x\big\}. \tag{2.18}$$

Then we have a holomorphic submersion $\pi^g : M^g \to B^g$ with compact fibre X^g.

Definition 2.8. — *Let P^B be the vector space of smooth forms on B, which are sums of forms of type (p,p). Let $P^{B,0}$ be the vector space of the forms $\alpha \in P^B$ such that there exist smooth forms β, γ on B for which $\alpha = \partial\beta + \bar\partial\gamma$.*

We define $P^{M^g}, P^{M^g,0}$, $P^{B^g}, P^{B^g,0}$ in the same way.

Let Φ be the homomorphism $\alpha \mapsto (2i\pi)^{-\deg\alpha/2}\alpha$ of $\Lambda^{\mathrm{even}}(T^*_{\mathbb{R}}B)$ into itself.

In the rest of the section, we will construct an equivariant analytic torsion form $T_g(\omega^W, h^\xi) \in P^{B^g}$ corresponding to the fibration $\pi : \pi^{-1}(B^g) \to B^g$. Without loss generality, we may and we will assume that $B^g = B$.

Theorem 2.9. — *For $u > 0$, the forms $\Phi \mathrm{Tr}_s[g\exp(-B_u^2)]$ and $\Phi \mathrm{Tr}_s[gN_u\exp(-B_u^2)]$ lie in P^{B^g}. The forms $\Phi \mathrm{Tr}_s[g\exp(-B_u^2)]$ are closed and that their cohomology class is constant. Moreover,*

$$\frac{\partial}{\partial u}\Phi \mathrm{Tr}_s\big[g\exp(-B_u^2)\big] = -\frac{1}{u}\frac{\bar{\partial}\partial}{2i\pi}\Phi \mathrm{Tr}_s\big[gN_u\exp(-B_u^2)\big]. \tag{2.19}$$

Proof. — Since g commutes with N_u, B_u, *etc.*, by proceeding as in [BGS2, Thm. 2.9], we have Theorem 2.9. □

Put

$$\begin{cases} C_{-1,g} = \displaystyle\int_{X^g} \frac{\omega^M}{2\pi}\,\mathrm{Td}_g(TX,h^{TX})\,\mathrm{ch}_g(\xi,h^\xi), \\ C_{0,g} = \displaystyle\int_{X^g} \big(-\mathrm{Td}'_g(TX,h^{TX}) \\ \qquad\qquad + \dim X\cdot \mathrm{Td}_g(TX,h^{TX})\big)\,\mathrm{ch}_g(\xi,h^\xi). \end{cases} \tag{2.20}$$

Set

$$\begin{cases} \mathrm{ch}_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big) \\ \qquad\qquad = \displaystyle\sum_{k=0}^{\dim X}(-1)^k\,\mathrm{ch}_g\big(H^k(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big), \\ \mathrm{ch}'_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big) \\ \qquad\qquad = \displaystyle\sum_{k=0}^{\dim X}(-1)^k k\,\mathrm{ch}_g\big(H^k(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big). \end{cases} \tag{2.21}$$

Theorem 2.10. — *As $u \to 0$*

$$\Phi \mathrm{Tr}_s\big[g\exp(-B_u^2)\big] = \int_{X^g}\mathrm{Td}_g(TX,h^{TX})\,\mathrm{ch}_g(\xi,h^\xi) + O(u). \tag{2.22}$$

There are forms $C'_{j,g} \in P^{B^g}$ $(j \geq -1)$ such that for $k \in \mathbb{N}$, as $u \to 0$

$$\Phi \mathrm{Tr}_s\big[gN_u\exp(-B_u^2)\big] = \sum_{j=-1}^{k} C'_{j,g}u^j + O(u^{k+1}). \tag{2.23}$$

Also

$$C'_{-1,g} = C_{-1,g}, \quad C'_{0,g} = C_{0,g} \text{ in } P^{B^g}/P^{B^g,0}. \tag{2.24}$$

As $u \to +\infty$

$$\left\{\begin{aligned} &\Phi\mathrm{Tr}_s\big[g\exp(-B_u^2)\big] = \mathrm{ch}_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big) + O\Big(\frac{1}{\sqrt{u}}\Big),\\ &\Phi\mathrm{Tr}_s[gN_u\exp(-B_u^2)] = \mathrm{ch}'_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big) + O\Big(\frac{1}{\sqrt{u}}\Big).\end{aligned}\right. \tag{2.25}$$

Proof. — By combining the technique of [BGS2, Thms. 2.2, 2.16] and [B7, Thms. 4.9–4.11], we have the equations (2.22), (2.23), (2.24).

Equation (2.25) was stated in [BKö, Thm. 3.4] if $g = 1$. By proceeding as in [BeGeV, Thm. 9.23], we also have (2.25). □

d) Higher analytic torsion forms.

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, set

$$\zeta_1(s) = -\frac{1}{\Gamma(s)}\int_0^1 u^{s-1}\big(\Phi\mathrm{Tr}_s\big[gN_u\exp(-B_u^2)\big] - \mathrm{ch}'_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big)\big)\,\mathrm{d}u.$$

Using (2.23), we see that $\zeta_1(s)$ extends to a holomorphic function of $s \in \mathbb{C}$ near $s = 0$.

For $s \in \mathbb{C}$, $\mathrm{Re}(s) < \frac{1}{2}$, set

$$\zeta_2(s) = -\frac{1}{\Gamma(s)}\int_1^{+\infty} u^{s-1}\big(\Phi\mathrm{Tr}_s[gN_u\exp(-B_u^2)] - \mathrm{ch}'_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big)\big)\,\mathrm{d}u.$$

Then by (2.25), $\zeta_2(s)$ is a holomorphic function of s.

Definition 2.11. — *Set*

$$T_g(\omega^M,h^\xi) = \frac{\partial}{\partial s}(\zeta_1+\zeta_2)(0). \tag{2.26}$$

Then $T_g(\omega^M,h^\xi)$ is a smooth form on B^g. Using (2.23), (2.25), we get

$$\begin{aligned} T_g(\omega^M,h^\xi) = &-\int_0^1 \Big(\Phi\mathrm{Tr}_s\big[gN_u\exp(-B_u^2)\big] - \frac{C'_{-1,g}}{u} - C'_{0,g}\Big)\frac{\mathrm{d}u}{u}\\ &-\int_1^{+\infty}\Big(\Phi\mathrm{Tr}_s\big[gN_u\exp(-B_u^2)\big] - \mathrm{ch}'_g\big(H(X,\xi_{|X}),h^{H(X,\xi_{|X})}\big)\Big)\frac{\mathrm{d}u}{u}\\ &+ C'_{-1,g} + \Gamma'(1)\big(C'_{0,g} - \mathrm{ch}'_g(H(X,\xi_{|X}),h^{H(X,\xi_{|X})})\big). \end{aligned} \tag{2.27}$$

THEOREM 2.12. — *The form $T_g(\omega^M, h^\xi)$ lies in P^{B^g}. Moreover,*

$$\frac{\bar\partial\partial}{2i\pi} T_g(\omega^M, h^\xi) = \mathrm{ch}_g\big(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}\big) - \int_{X^g} \mathrm{Td}_g(TX, h^{TX})\,\mathrm{ch}_g(\xi, h^\xi). \tag{2.28}$$

Proof. — As we saw before, the forms $\Phi\mathrm{Tr}_s[gN_u \exp(-B_u^2)]$ lie in P^{B^g}. So the form $T_g(\omega^M, h^\xi) \in P^{B^g}$. Using Theorem 2.10 and equation (2.19), the proof of our Theorem 2.12 proceeds as the proof of [BGS2, Thm. 2.20]. □

e) Anomaly formulas for the analytic torsion forms.

Now let (ω'^M, h'^ξ) be another couple of objects similar to (ω^M, h^ξ). We denote by a "$\,'\,$" the objects associated to (ω'^M, h'^ξ).

By [BGS1, §1(f)], there are uniquely defined Bott-Chern classes

$$\widetilde{\mathrm{Td}}_g(TX, g^{TX}, g'^{TX}), \widetilde{\mathrm{ch}}_g(\xi, h^\xi, h'^\xi) \in P^{M^g}/P^{M^g,0},$$
$$\widetilde{\mathrm{ch}}_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}, h'^{H(X,\xi_{|X})}) \in P^{B^g}/P^{B^g,0}$$

such that

$$\begin{cases} \dfrac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{Td}}_g(TX, g^{TX}, g'^{TX}) = \mathrm{Td}_g(TX, g'^{TX}) - \mathrm{Td}_g(TX, g^{TX}), \\ \dfrac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{ch}}_g(\xi, h^\xi, h'^\xi) = \mathrm{ch}_g(\xi, h'^\xi) - \mathrm{ch}_g(\xi, h^\xi), \\ \dfrac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{ch}}_g\big(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}, h'^{H(X,\xi_{|X})}\big) \\ \qquad = \mathrm{ch}_g\big(H(X,\xi_{|X}), h'^{H(X,\xi_{|X})}\big) - \mathrm{ch}_g\big(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}\big). \end{cases} \tag{2.29}$$

Let C be a smooth section of $T^*_{\mathbb{R}}X \widehat{\otimes} \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes \xi)$. Let $e_1, \ldots, e_{2\ell}$ be an orthonormal base of $T^*_{\mathbb{R}}X$. We use the notation

$$\big(\nabla_{e_i}^{\Lambda(T^{*(0,1)}X)\otimes\xi} + C(e_i)\big)^2 = \sum_{i=1}^{2\ell}\big(\nabla_{e_i}^{\Lambda(T^{*(0,1)}X)\otimes\xi} + C(e_i)\big)^2 - \nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{\Sigma_{i=1}^{2\ell}\nabla^{TX}_{e_i}e_i} - C\Big(\sum_{i=1}^{2\ell}\nabla^{TX}_{e_i}e_i\Big).$$

THEOREM 2.13. — *The following identity holds in $P^{B^g}/P^{B^g,0}$:*

$$T_g(\omega'^M, h'^\xi) - T_g(\omega^M, h^\xi) = \widetilde{\mathrm{ch}}_g\big(H(X,\xi_{|X}), h^{H(X,\xi|X)}, h'^{H(X,\xi|X)}\big) - \int_{X^g}\Big[\widetilde{\mathrm{Td}}_g(TX, h^{TX}, h'^{TX})\,\mathrm{ch}_g(\xi, h^\xi) + \mathrm{Td}_g(TX, h'^{TX})\,\widetilde{\mathrm{ch}}_g(\xi, h^\xi, h'^\xi)\Big]. \tag{2.30}$$

In particular, the class of $T_g(\omega, h^\xi)$ in $P^{B^g}/P^{B^g,0}$ only depends on (h^{TX}, h^ξ).

Proof. — Assume first that $h^\xi = h'^\xi$. Let $c \in [0,1] \to \omega^M_c$ be a smooth family of G-invariant (1,1)-forms on M verifying the assumptions of Theorem 2.2 such that $\omega^M_0 = \omega^M$, $\omega^M_1 = \omega'^M$. Then all the objects considered in Section 2 a)–d) now depend on the parameter c. Most of the time, we will omit the subscript c. The upper-dot "$\cdot$" is often used instead of $\partial/\partial c$.

Recall that we assume that $B^g = B$. Set

$$\begin{cases} Q = -*^{-1}\dot{*}, \\ Q^{H(X,\xi|X)} = P^{H(X,\xi|X)} Q P^{H(X,\xi|X)}. \end{cases} \tag{2.31}$$

Let $e_1, \dots, e_{2\ell}$ be an orthonormal base of $T_{\mathbb{R}}X$ with respect to h^{TX}_c. Let $f_1, \dots, f_{2m}$ be a base of $T_{\mathbb{R}}B$, and that $f^1, \dots, f^{2m}$ is the corresponding dual base of $T^*_{\mathbb{R}}B$. Set

$$M_u = -\frac{i}{4}\dot{\omega}(e_j, e_k)c(e_j)c(e_k) - \frac{i}{\sqrt{2u}}\dot{\omega}(f^H_\alpha, e_j)f^\alpha c(e_j) - \frac{i\,\dot{\omega}^{H\overline{H}}}{2u}(f_\alpha, f_\beta)f^\alpha f^\beta - \frac{1}{4}\dot{\omega}(e_j, J^{TX}e_j). \tag{2.32}$$

By the arguments of [BGS2, Thm. 2.11], we know there is $p \in \mathbb{N}$, $\mu_j \in P^{B^g}$, $(j \geq -p)$ such that as $u \to 0$, we have the asymptotic expansion

$$\Phi\mathrm{Tr}_s\big[gM_u \exp(-B_u^2)\big] = \sum_{j=-p}^{k} \mu_j u^j + O(u^{k+1}). \tag{2.33}$$

By proceeding as in [BKö, §§2–3], we easily find an analogue of [BKö, Thm. 3.16],

$$\dot{T}_g(\omega^M, h^\xi) = \mu_0 + \Phi\mathrm{Tr}_s\big[gQ^{H(X,\xi|X)}\exp(-(\nabla^{H(X,\xi|X)})^2)\big] + \frac{\overline{\partial}}{\sqrt{2i\pi}}\theta^1(0) + \frac{\partial}{\sqrt{2i\pi}}\theta^2(0) + \frac{\overline{\partial}\partial}{2i\pi}\theta^3(0). \tag{2.34}$$

In (2.34), the $\theta^i(0)$ $(i = 1,2,3)$ have universal expressions in terms of g, ω^M_c, h^ξ as in [BKö].

Let $da, d\bar{a}$ be two odd Grassmann variables which anticommute with the other odd elements in $\Lambda(T^*_{\mathbb{R}}B)$ or $c(T_{\mathbb{R}}X)$. Set

$$L_u = -B_u^2 - da\, u\frac{\partial B_u}{\partial u} - d\bar{a}[B_u, -M_u] + da\, d\bar{a}\Big(-\frac{\partial}{\partial u}(uM_u)\Big). \tag{2.35}$$

If $\alpha \in \mathbb{C}(da, d\bar{a})$, let $[\alpha]^{da\, d\bar{a}} \in \mathbb{C}$ be the coefficient of $da\, d\bar{a}$ in the expansion of α. By a formula analogous of [BKö, Thm. 3.17], we know that the class of $-\mu_0$ in $P^{B^g}/P^{B^g,0}$ coincides with the class of the constant term in the asymptotic expansion of $\Phi\mathrm{Tr}_s[g\exp(L_u)]^{da\, d\bar{a}}$ when $u \to 0$.

Recall that the $(3,0)$ tensor $\langle S(\cdot)\cdot,\cdot\rangle$ was defined in (2.15). Let ∇'_u be the connection on $\Lambda(da\oplus d\bar{a})\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}B)\widehat{\otimes}\Lambda(T^{*(0,1)}X)\widehat{\otimes}\xi$ along the fibres X,

$$\begin{aligned}\nabla'_u = \nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi} &+ \frac{1}{u}\langle S(\cdot)e_j, f^H_\alpha\rangle\sqrt{\frac{u}{2}}c(e_j)f^\alpha + \frac{1}{2u}\langle S(\cdot)f^H_\alpha, f^H_\beta\rangle f^\alpha f^\beta \\ &- \frac{da}{2u}\sqrt{\frac{u}{2}}c(\cdot) - \frac{i\,\dot{\omega}}{u}(e_k, .)d\bar{a}\sqrt{\frac{u}{2}}c(e_k) - i\,\dot{\omega}(f^H_\alpha, .)\frac{d\bar{a}f^\alpha}{u}\cdot\end{aligned} \tag{2.36}$$

Let K^X be the scalar curvature of the fiber (X, h^{TX}). Set

$$L'^\xi = L^\xi + \frac{1}{2}\mathrm{Tr}[R^{TX}]. \tag{2.37}$$

By [BKö, Thm. 3.18], we get

$$\begin{aligned}L_u = \frac{u}{2}(\nabla'_{u,e_i})^2 &- \nabla_{e_i}\big(\dot{\omega}(e_j, J^{TX}e_j)\big)\frac{d\bar{a}\sqrt{u}\,c(e_i)}{4\sqrt{2}} \\ &- \nabla_{f^H_\alpha}\big(\dot{\omega}(e_j, J^{TX}e_j)\big)\frac{d\bar{a}\,f^\alpha}{4} + \frac{da\,d\bar{a}}{4}\dot{\omega}(e_j, J^{TX}e_j) \\ &- \frac{uK^X}{8} - \frac{u}{4}c(e_i)c(e_j)L'^\xi(e_i, e_j) - \sqrt{\frac{u}{2}}c(e_i)f^\alpha L'^\xi(e_i, f^H_\alpha) \\ &- \frac{f^\alpha f^\beta}{2}L'^\xi(f^H_\alpha, f^H_\beta).\end{aligned} \tag{2.38}$$

Let $P_u(x, x', b)$ ($b \in B$, $x, x' \in X_b$) be the smooth kernel associated to $\exp(L_u)$ with respect to $\mathrm{d}v_X(x')/(2\pi)^{\dim X}$. Then

$$\Phi\mathrm{Tr}_s\big[g\exp(L_u)\big] = \int_X \Phi\mathrm{Tr}_s\big[gP_u(g^{-1}x, x, b)\big]\frac{\mathrm{d}v_X(x)}{(2\pi)^{\dim X}}\cdot \tag{2.39}$$

Let $N_{X^g/X} = TX/TX^g$ be the normal bundle to X^g in X. We identify $N_{X^g/X}$ with the orthogonal bundle to TX^g in TX. By standard estimates on heat kernels, for $b \in B$, the problem of calculating the limit of (2.39) when $u \to 0$ can be localized to an open neighbourhood $\mathcal{U}_\varepsilon$ of X_b^g on X_b. Using normal geodesic coordinates to X_b^g in X_b, we will identify $\mathcal{U}_\varepsilon$ to an ε-neighbourhood of X^g in $N_{X^g/X,\mathbb{R}}$.

Since we have used normal geodesic coordinates to X^g in X, if $(x,z) \in N_{X^g/X}$,

$$g^{-1}(x,z) = (x, g^{-1}z). \tag{2.40}$$

Let $\mathrm{d}v_{X^g}$, $\mathrm{d}v_{N_{X^g/X}}$ be the Riemannian volume forms on TX^g, $N_{X^g/X}$ induced by h^{TX}. Let $k(x,z)$ ($x \in X^g$, $z \in N_{X^g/X,\mathbb{R}}$, $|z| < \varepsilon$) be defined by

$$\mathrm{d}v_X = k(x,z)\,\mathrm{d}v_{X^g}(x)\,\mathrm{d}v_{N_{X^g/X}}(z). \tag{2.41}$$

Then

$$k(x,0) = 1.$$

Take $x_0 \in X_b^g$. By using the finite propagation speed as in [B5, § 11b)], we may replace X_b by $(TX)_{x_0} \simeq \mathbb{C}^\ell$ with $0 \in (TX)_{x_0}$ representing x_0 and we may assume the extended fibration over $\mathbb{C}^\ell$ coincides with the given fibration over $B(0,\varepsilon)$.

Take $y \in \mathbb{C}^\ell$, set $Y = y + \bar{y}$. We trivialize

$$\Lambda(da \oplus d\bar{a})\widehat{\otimes}\Lambda(T_{\mathbb{R}}^* B)\widehat{\otimes}\Lambda(T^{*(0,1)}X)\widehat{\otimes}\xi$$

by parallel transport along the curve $t \mapsto tY$ with respect to ∇'_u.

Let $\rho(Y)$ be a $\mathcal{C}^\infty$ function over $\mathbb{C}^\ell$ which is equal to 1 if $|Y| \le \frac{1}{4}\varepsilon$, and equal to 0 if $|Y| \ge \frac{1}{2}\varepsilon$. Let H_{x_0} be the vector space of smooth sections of $(\Lambda(da \oplus d\bar{a})\widehat{\otimes}\Lambda(T_{\mathbb{R}}^* B)\ \widehat{\otimes}\Lambda(T^{*(0,1)}X)\widehat{\otimes}\xi)_{x_0}$ over $(T_{\mathbb{R}}X)_{x_0}$. Let Δ^{TX} be the standard Laplacian on $(T_{\mathbb{R}}X)_{x_0}$ with respect to the metric $h^{TX_{x_0}}$. For $u > 0$, let L_u^1 be the operator

$$L_u^1 = \big(1 - \rho^2(Y)\big)\Big(-\frac{1}{2}u\Delta^{TX}\Big) - \rho^2(Y)L_u. \tag{2.42}$$

For $u > 0$, $s \in H_{x_0}$, set

$$R_u s(Y) = s\Big(\frac{Y}{\sqrt{u}}\Big), \quad L_u^2 = R_u^{-1} L_u^1 R_u. \tag{2.43}$$

Let $e_1, \ldots, e_{2\ell'}$ be an orthonormal base of $(T_{\mathbb{R}}X^g)_{x_0}$, and let $e_{2\ell'+1}, \ldots, e_{2\ell}$ be an orthonormal base of $N_{X^g/X,\mathbb{R},x_0}$.

Let L_u^3 be the operator obtained from L_u^2 by replacing the Clifford variables $c(e_j)$ $(1 \leq j \leq 2\ell')$ by the operators $\sqrt{2/u}\, e^j - \sqrt{u/2}\, i_{e_j}$.

Let $P_u^i(z,z')$ $(z,z' \in (T_{\mathbb{R}}X)_{x_0},\ i = 1,2,3)$ be the smooth kernel associated to $\exp(-L_u^i)$ with respect to $\mathrm{d}v_{TX_{x_0}}(z')/(2\pi)^{\dim X}$. By using the finite propagation speed and (2.42), there exist $c, C > 0$ such that for $z \in N_{X^g/X,\mathbb{R},x_0}$, $|z| \leq \frac{1}{8}\varepsilon$, $u \in\,]0,1]$, we have

$$\left|P_u(g^{-1}(x_0,z),(x_0,z))k(x_0,z) - P_u^1(g^{-1}z,z)\right| \leq c\exp\Big(-\frac{C}{u^2}\Big). \tag{2.44}$$

By the discussion after (2.39), (2.41), we get

$$\begin{aligned}
&\lim_{u\to 0}\Phi\mathrm{Tr}_s[g\exp(L_u)] \\
&\qquad = \lim_{u\to 0}\int_{\mathcal{U}_{\varepsilon/8}}\Phi\mathrm{Tr}_s\big[gP_u(g^{-1}x,x)\big]\frac{\mathrm{d}v_X(x)}{(2\pi)^{\dim X}} \\
&\qquad = \lim_{u\to 0}\int_{x\in X^g}\int_{\substack{|z|\leq\varepsilon/8\\ z\in N_{X^g/X,\mathbb{R}}}}\Phi\mathrm{Tr}_s\big[gP_u(g^{-1}(x,z),(x,z))\big] \\
&\qquad\qquad\qquad\qquad k(x,z)\frac{\mathrm{d}v_{X^g}(x)\,\mathrm{d}v_{N_{X^g/X}}(z)}{(2\pi)^{\dim X}}.
\end{aligned}\tag{2.45}$$

If $\alpha \in \mathbb{C}(e^j, i_{e_j})_{(1\leq j\leq 2\ell')}$, let $[\alpha]^{\max} \in \mathbb{C}$ be the coefficient of $e^1 \wedge \ldots \wedge e^{2\ell'}$ in the expansion of α. Then by proceeding as in [B5, Prop. 11.12], if $z \in N_{X^g/X,\mathbb{R}}$, we get

$$\begin{aligned}
&\mathrm{Tr}_s\big[gP_u^1(g^{-1}z,z)\big] \\
&\qquad = (-i)^{\dim X^g}u^{-\dim N_{X^g/X}}\Big[\mathrm{Tr}_s\Big[gP_u^3\Big(\frac{g^{-1}z}{\sqrt{u}},\frac{z}{\sqrt{u}}\Big)\Big]^{\max}\Big]^{da\,d\bar a}.
\end{aligned}\tag{2.46}$$

For $q, r \in \mathbb{N}$, $O_q(|Y|^r)$ will denote an expression in

$$\big(\Lambda(da\oplus d\bar a)\widehat\otimes\Lambda(T_{\mathbb{R}}^*B)\widehat\otimes c(T_{\mathbb{R}}X)\widehat\otimes\,\mathrm{End}(\xi)\big)_{x_0}$$

which has the following two properties:

- For $k \in \mathbb{N}$, $k \leq r$, its derivatives of order k are $O(|Y|^{r-k})$ as $|Y| \to 0$.
- It is of total length $\leq q$ with respect to the obvious $\mathbb{Z}$-grading of $(\Lambda(da\oplus d\bar a)\widehat\otimes\Lambda(T_{\mathbb{R}}^*B)\ \widehat\otimes c(T_{\mathbb{R}}X)\widehat\otimes\,\mathrm{End}(\xi))_{x_0}$.

Let Γ' be the connection form for ∇_1' in the trivialization of $(\Lambda(da\oplus d\bar a)\widehat\otimes\Lambda(T_{\mathbb{R}}^*B)\widehat\otimes\Lambda(T^{*(0,1)}X)\widehat\otimes\xi)$ with respect to ∇_1'. By using [ABoP, Prop. 3.7], we see that for $Y \in T_{\mathbb{R}}X$,

$$\Gamma_Y' = \frac{1}{2}(\nabla_1'^{\,2})_{x_0}(Y,\cdot) + O_2\big(|Y|^2\big). \tag{2.47}$$

LEMMA 2.1. — *The following identity holds:*

$$
\begin{aligned}
(2.48)\quad \nabla'^{,2}_1 = {} & \frac{1}{4}\langle \nabla^{TX,2} e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\mathrm{Tr}[\nabla^{TX,2}] \\
& + \frac{1}{2}\big\langle (SP^{TX}S + \nabla^{TX}S) f^H_\alpha, f^H_\beta \big\rangle f^\alpha \wedge f^\beta \\
& + \frac{1}{2}\big\langle (\nabla^{TX}S) e_i, f^H_\alpha \big\rangle \sqrt{2}\, c(e_i) f^\alpha \\
& - i\big(\dot{\omega}(e_k,\cdot)\langle S(\cdot)e_k, f^H_\alpha\rangle - (\nabla_{\!\cdot}\, \dot{\omega})(f^H_\alpha,\cdot)\big) f^\alpha d\bar{a} \\
& - \frac{i}{\sqrt{2}}(\nabla_{\!\cdot}\, \dot{\omega})(e_k,\cdot) d\bar{a}\, c(e_k) + da\, d\bar{a}(i\dot{\omega}).
\end{aligned}
$$

Proof of Lemma 2.1. — If $da = d\bar{a} = 0$, (2.48) is exactly [B6, Prop. 11.8]. In general, by using (2.16), (2.36), we obtain straightforwardly the extra-contributions of $da, d\bar{a}$ to $\nabla'^{,2}_1$. $\square$

By [B1, Thm. 4.14] (*cf.* [B6, (11.61)]), for $X, Y \in T_{\mathbb{R}}X$, $Z, W \in T_{\mathbb{R}}M$

$$
\begin{aligned}
(2.49)\quad & \big\langle \nabla^{TX,2}(X,Y)P^{TX}Z, P^{TX}W \big\rangle + \big\langle (SP^{TX}S)(X,Y)Z, W\big\rangle \\
& + \big\langle (\nabla^{TX}S)(X,Y)Z, W\big\rangle = \big\langle \nabla^{TX,2}(Z,W)X, Y\big\rangle.
\end{aligned}
$$

Let $R^{TX}{}_{|M^g}, L^\xi{}_{|M^g}, \ldots$ be the restrictions of $R^{TX}, L^\xi, \ldots$ over M^g. Let ∇_{e_i} be the ordinary differentiation operator on $(T_{\mathbb{R}}X)_{x_0}$ in the direction e_i. By (2.38), (2.47), (2.48) and (2.49), as $u \to 0$,

$$
\begin{aligned}
(2.50)\quad L^3_u \to L^3_0 = {} & -\frac{1}{2}\Big(\nabla_{e_j} + \frac{1}{2}\langle R^{TX}{}_{|M^g} Y, e_j\rangle \\
& - d\bar{a}\, a_1(Y, e_j) + da\, d\bar{a}\Big(\frac{i}{2}\dot{\omega}(Y, e_j)\Big)\Big)^2 \\
& - d\bar{a}\, a_2 - \frac{da\, d\bar{a}}{4}\dot{\omega}(e_j, J^{TX}e_j) + L'^\xi{}_{|M^g},
\end{aligned}
$$

and $a_1 \in \Lambda^2(T^*_{\mathbb{R}}X)_{x_0} \otimes (T^*_{\mathbb{R}}X \oplus T^*_{\mathbb{R}}B)_{x_0}$, $a_2 \in (T^*_{\mathbb{R}}X \oplus T^*_{\mathbb{R}}B)_{x_0}$. Let

$$
\begin{aligned}
(2.51)\quad L^{3'}_0 = {} & -\frac{1}{2}\Big(\nabla_{e_j} + \frac{1}{2}\langle R^{TX}{}_{|M^g} Y, e_j\rangle + da\, d\bar{a}\Big(\frac{i}{2}\dot{\omega}(Y, e_j)\Big)\Big)^2 \\
& - \frac{da\, d\bar{a}}{4}\dot{\omega}(e_j, J^{TX}e_j) + L'^\xi{}_{|M^g}.
\end{aligned}
$$

Let $P^{3'}_0(z, z')$ $(z, z' \in (T_{\mathbb{R}}X)_{x_0})$ be the heat kernel of $\exp(-L^{3'}_0)$ over $(T_{\mathbb{R}}X)_{x_0}$ with respect to $dv_{TX_{x_0}}(z')/(2\pi)^{\dim X}$.

By proceeding as in [B5, §§11g)–11i)], we have: There exist $\gamma > 0$, $c > 0$, $C > 0$, $r \in \mathbb{N}$ such that for $u \in]0,1]$, $z, z' \in (T_{\mathbb{R}}X)_{x_0}$, we have

$$\begin{cases} \left|P_u^3(z,z')\right| \leq c\big(1+|z|+|z'|\big)^r \exp\big(-C|z-z'|^2\big), \\ \left|(P_u^3 - P_0^3)(z,z')\right| \leq cu^\gamma\big(1+|z|+|z'|\big)^r \exp\big(-C|z-z'|^2\big). \end{cases} \tag{2.52}$$

From (2.46), (2.50)–(2.52), we get

$$\begin{aligned} &\lim_{u\to 0} \int_{\substack{|z|\leq \varepsilon/8 \\ z\in N_{X^g/X,\mathbb{R}}}} \Phi \mathrm{Tr}_s\big[gP_u^1(g^{-1}z,z)\big]\, dv_{N_{X^g/X}}(z) \qquad (2.53)\\ &= \lim_{u\to 0} \int_{\substack{|z|\leq \varepsilon/8\sqrt{u} \\ z\in N_{X^g/X,\mathbb{R}}}} (-i)^{\dim X^g} \big\{\Phi \mathrm{Tr}_s[gP_u^3(g^{-1}z,z)]^{\max}\big\}^{da\,d\bar a}\, dv_{N_{X^g/X}}(z) \\ &= \int_{N_{X^g/X,\mathbb{R}}} (-i)^{\dim X^g} \big\{\Phi \mathrm{Tr}_s[gP_0^3(g^{-1}z,z)]^{\max}\big\}^{da\,d\bar a}\, dv_{N_{X^g/X}}(z) \\ &= \int_{N_{X^g/X,\mathbb{R}}} (-i)^{\dim X^g} \big\{\Phi \mathrm{Tr}_s[gP_0^{3'}(g^{-1}z,z)]^{\max}\big\}^{da\,d\bar a}\, dv_{N_{X^g/X}}(z). \end{aligned}$$

Clearly for $U, V \in T_{\mathbb{R}}X$,

$$\dot\omega(U,V) = \Big\langle U, J^{TX}(h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}V\Big\rangle. \tag{2.54}$$

So

$$\begin{aligned} L_0^{3'} = -\frac{1}{2}\Big(\nabla_{e_i} + \frac{1}{2}\Big\langle\Big(R^{TX}{}_{|M^g} - i\,da\,d\bar a J^{TX}(h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\Big)Y, e_i\Big\rangle\Big)^2 \\ + L^{\xi}{}_{|M^g} - \frac{1}{2}\Big(\mathrm{Tr}\big[R^{TX}{}_{|M^g}\big] + da\,d\bar a \mathrm{Tr}\Big[(h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\Big]\Big). \end{aligned} \tag{2.55}$$

Let $1, e^{i\theta_1}, \ldots, e^{i\theta_q}$ ($0 < \theta_j < 2\pi$, $1 \leq j \leq q$) be the locally constant distinct eigenvalues of g acting on TX over M^g. Let $N_{X^g/X}^{\theta_j}$ be the corresponding eigenbundles. Let $h^{TX^g}, h^{N_{X^g/X}^{\theta_j}}$ be the Hermitian metrics on TX^g, $N_{X^g/X}^{\theta_j}$ induced by h^{TX}. Let $R^{TX^g}, R^{N_{X^g/X}^{\theta_j}}$ be their curvatures as in Section 1b). By proceeding as in [B4, (3.16)–(3.21)],

$$\begin{aligned} &(-i)^{\dim X^g} \int_{N_{X^g/X,\mathbb{R}}} \big\{\Phi \mathrm{Tr}_s[gP_0^{3'}(g^{-1}z,z)]^{\max}\big\}^{da\,d\bar a} \frac{dv_{N_{X^g/X}}(z)}{(2\pi)^{\dim X}} \qquad (2.56)\\ &= \Big\{\frac{\partial}{\partial b}\Big[\mathrm{Td}\Big(\frac{-R^{TX^g}}{2i\pi} - b(h^{TX^g})^{-1}\frac{\partial h^{TX^g}}{\partial c}\Big) \\ &\times \prod_{j=1}^q \frac{\mathrm{Td}}{e}\Big(\frac{-R^{N_{X^g/X}^{\theta_j}}}{2i\pi} - b\big(h^{N_{X^g/X}^{\theta_j}}\big)^{-1}\frac{\partial h^{N_{X^g/X}^{\theta_j}}}{\partial c} + i\,\theta_j\Big)\Big]_{b=0} \mathrm{ch}_g(\xi, h^{\xi})\Big\}^{\max}. \end{aligned}$$

By (2.44), (2.53) and (2.56), we know the limit of (2.45) when $u \to 0$. By using [BGS1, Rem. 1.28 and Cor. 1.30] and proceeding as in [BKö, §3h)], we obtain Theorem 2.13 in the case where $h^{\xi} = h'^{\xi}$.

To prove (2.30) in the full generality, one only needs to consider the case where $\omega^M = \omega'^M$. Then by using Theorem 2.12 and by proceeding as in [BGS1, §1f)], *i.e.* by replacing B by $B \times \mathbb{P}^1$, one easily obtains (2.30) in this special case. □

3. The equivariant Quillen norm of the canonical section $\boldsymbol{\sigma}$.

This section is organized as follows. In a), we describe the canonical section σ. In b), we announce a formula for the equivariant Quillen norm of σ.

In this section, we use the same notation as in Section 1.

a) The canonical section $\boldsymbol{\sigma}$.

Let M, B be compact complex manifolds of complex dimension n and m. Let $\pi : M \to B$ be a holomorphic submersion with fibre X. Let ξ be a holomorphic vector bundle on M. Let G be a compact Lie group. We assume that ξ, M are G-equivariant holomorphic bundles over M, B.

We assume that the sheaves $R^k\pi_*\xi$ $(0 \le k \le \dim X)$ are locally free.

If given $W \in \widehat{G}$, λ_W, μ_W are complex lines, if $\lambda = \bigoplus_{W\in\widehat{G}} \lambda_W$, $\mu = \bigoplus_{W\in\widehat{G}} \mu_W$, set

$$\lambda^{-1} = \bigoplus_{W\in\widehat{G}} \lambda_W^{-1}, \quad \lambda \otimes \mu = \bigoplus_{W\in\widehat{G}} \lambda_W \otimes \mu_W. \tag{3.1}$$

Now we use the notation of Section 1. Set

$$\begin{cases} \lambda_G(\xi) = \det\big(H(M,\xi), G\big)^{-1} = \bigoplus\limits_{W\in\widehat{G}} \lambda_W(\xi), \\ \lambda_G(R^k\pi_*\xi) = \det\big(H(B, R^k\pi_*\xi), G\big)^{-1}, \\ \lambda_G(R^\bullet\pi_*\xi) = \bigotimes\limits_{k=0}^{\dim X} \big(\lambda_G(R^k\pi_*\xi)\big)^{(-1)^k} = \bigoplus\limits_{W\in\widehat{G}} \lambda_W(R^\bullet\pi_*\xi). \end{cases} \tag{3.2}$$

By proceeding as in [BerB, §1b)] and [B5, §3b)], for $W \in \widehat{G}$, the line $\lambda_W(\xi) \otimes \lambda_W^{-1}(R^\bullet\pi_*\xi)$ has a canonical nonzero section σ_W. Set

$$\sigma = \bigoplus_{W\in\widehat{G}} \sigma_W \in \lambda_G(\xi) \otimes \lambda_G^{-1}(R^\bullet\pi_*\xi). \tag{3.3}$$

b) A formula for the Quillen norm of the canonical section $\boldsymbol{\sigma}$.

Let h^{TM}, h^{TB} be G-invariant Kähler metrics on TM and TB. Let h^{TX} be the metric induced by h^{TM} on TX. Let h^ξ be a G-invariant Hermitian metric on ξ. Let $h^{H(X,\xi|X)}$ be the L^2-metric on $H(X,\xi_{|X})$ with respect to h^{TX}, h^ξ as in Section 2 c).

We have the exact sequence of G-equivariant holomorphic Hermitian vector bundles on M,

$$0 \to TX \longrightarrow TM \longrightarrow \pi^*TB \to 0. \tag{3.4}$$

By a construction of [BGS1,§1f)], there is a uniquely defined class of forms $\widetilde{\mathrm{Td}}_g(TM,TB, h^{TM}, h^{TB}) \in P^{M^g}/P^{M^g,0}$, such that

$$\begin{aligned}\frac{\bar\partial\partial}{2i\pi}\widetilde{\mathrm{Td}}_g(TM,TB,h^{TM},h^{TB}) &= \mathrm{Td}_g(TM,h^{TM}) \\ &\quad - \pi^*\big(\mathrm{Td}_g(TB,h^{TB})\big)\,\mathrm{Td}_g(TX,h^{TX}).\end{aligned} \tag{3.5}$$

Let ω^M be the Kähler form of h^{TM}. Let $T_g(\omega^M, h^\xi) \in P^{B^g}$ be the analytic torsion form constructed in Section 2 c). Let $\| \quad \|_{\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)}$ be the G-equivariant Quillen metric on the line $\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)$ attached to the metrics $h^{TM}, h^\xi, h^{TB}, h^{H(X,\xi|X)}$ on $TM, \xi, TB, R^\bullet\pi_*\xi$.

Now we state the main result of this paper, which extends [BerB, Thm. 3.1].

THEOREM 3.1. — *For $g \in G$, the following identity holds:*

$$\begin{aligned}\log\big(\|\sigma\|^2_{\lambda_G(\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_*\xi)}\big)(g) &= -\int_{B^g} \mathrm{Td}_g(TB,h^{TB})T_g(\omega^M,h^\xi) \\ &\quad + \int_{M^g} \widetilde{\mathrm{Td}}_g(TM,TB,h^{TM},h^{TB})\,\mathrm{ch}_g(\xi,h^\xi).\end{aligned} \tag{3.6}$$

Proof. — The proof of Theorem 3.1 will be given in Sections 4-9. □

Remark 3.2. — By Theorem 2.13, to prove Theorem 3.1 for any Kähler metrics h^{TM}, h^{TB}, we only need to establish (3.6) for one given metrics h^{TM}, h^{TB}. So by replacing h^{TM} by $h^{TM} + \pi^* h^{TB}$, we may and we will assume that $\widetilde{h}^{TM}$ is a Kähler metric on TM and

$$h^{TM} = \widetilde{h}^{TM} + \pi^* h^{TB}. \tag{3.7}$$

4. A proof of Theorem 3.1.

This section is organized as follows. In a), we introduce a 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ as in [BerB, §3a)]. In b), we state eight intermediate results which we need for the proof of Theorem 3.1 whose proofs are delayed to Sections 5–9. In c), we prove Theorem 3.1.

In this section, we make the same assumption as in Section 3. Also, we assume that h^{TM} is given by formula (3.7). In the sequel, $g \in G$ is fixed once and for all.

a) A fundamental closed 1-form.

Recall that N_V denotes the number operator of $\Lambda(T^{*(0,1)}X)$. Let N_H be the number operator of $\Lambda(T^{*(0,1)}B)$. By (2.2), we have the identification of smooth vector bundles over M

$$TM \simeq TX \oplus T^H M, \quad T^H M \simeq \pi^* TB. \tag{4.1}$$

This identification determines an identification of $\mathbb{Z}$-graded bundles of algebra on M

$$\Lambda(T^{*(0,1)}M) = \Lambda(T^{*(0,1)}B) \widehat{\otimes} \Lambda(T^{*(0,1)}X). \tag{4.2}$$

So the operators N_V and N_H act naturally on $\Lambda(T^{*(0,1)}M)$. Of course, $N = N_V + N_H$ defines the total grading of $\Lambda(T^{*(0,1)}M) \otimes \xi$ and $\Omega(M, \xi)$.

Definition 4.1. — *For $T > 0$, let h_T^{TM} be the Kähler metric on TM*

$$h_T^{TM} = \frac{1}{T^2} \widetilde{h}^{TM} + \pi^* h^{TB}. \tag{4.3}$$

Let $\langle\ \rangle_T$ be the Hermitian product (1.2) on $\Omega(M, \xi)$ attached to the metrics h_T^{TM}, h^ξ. Let D_T^M be the corresponding operator constructed in (1.3) acting on $\Omega(M, \xi)$. Let $*_T$ be the Hodge operator associated to the metric h_T^{TM}. Then $*_T$ acts on $\Lambda(T_{\mathbb{R}}^* M) \otimes \xi$.

THEOREM 4.2. — *Let $\alpha_{u,T}$ be the* 1*-form on* $\mathbb{R}_+^* \times \mathbb{R}_+^*$

$$\alpha_{u,T} = \frac{2du}{u} \operatorname{Tr}_s\left[gN \exp(-u^2 D_T^{M,2})\right] + dT \operatorname{Tr}_s\Big[g *_T^{-1} \frac{\partial *_T}{\partial T} \exp(-u^2 D_T^{M,2})\Big]. \tag{4.4}$$

Then $\alpha_{u,T}$ is closed.

Proof. — Clearly g is an even operator which commutes with the operators $\bar{\partial}^M, \bar{\partial}_T^{M*}, *_T, N_V, N_H$. By using [BerB, (4.27), (4.28), (4.30)], the proof of Theorem 4.2 is identical to the proof of [BerB, Thm. 4.3]. $\square$

Take ϵ, A, T_0, $0 < \epsilon \leq 1 \leq A < +\infty$, $1 \leq T_0 < +\infty$. Let $\Gamma = \Gamma_{\epsilon,A,T_0}$ be the oriented contour in $\mathbb{R}_+^* \times \mathbb{R}_+^*$

The contour Γ is made of four oriented pieces $\Gamma_1, \dots, \Gamma_4$ indicated above. For $1 \leq k \leq 4$, set

$$I_k^0 = \int_{\Gamma_k} \alpha. \tag{4.5}$$

THEOREM 4.3. — *The following identity holds:*

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{4.6}$$

Proof. — This follows from Theorem 4.2. $\square$

b) Eight intermediate results.

Let $\bar{\partial}^{B*}$ be the formal adjoint of the operator $\bar{\partial}^B$ acting on $\Omega(B, R^\bullet \pi_* \xi)$, with respect to the metrics $h^{TB}, h^{H(X,\xi|X)}$. Set

$$D^B = \bar{\partial}^B + \bar{\partial}^{B*}, \quad F = \operatorname{Ker} D^B. \tag{4.7}$$

By Hodge theory,

$$H^\bullet(B, R^\bullet\pi_*\xi) \simeq F. \tag{4.8}$$

Let Q be the orthogonal projection from $\Omega(B, R^\bullet\pi_*\xi)$ on F with respect to the Hermitian product (1.2) attached to the metrics $h^{TB}, h^{H(X,\xi|X)}$. Set $Q^\perp = 1 - Q$.

Let $a \in \,]0,1]$ be such that the operator $D^{B,2}$ has no eigenvalues in $]0, 2a]$.

Definition 4.4. — *For $T > 0$, set*

$$E_T = \operatorname{Ker} D_T^{M,2}. \tag{4.9}$$

Let P_T be the orthogonal projection operator from $\Omega(M,\xi)$ on E_T with respect to $\langle\ \rangle_T$.

Let $E_T^{[0,a]}$ (resp. $E_T^{]0,a]}$) be the direct sum of the eigenspaces of $D_T^{M,2}$ associated to eigenvalues $\lambda \in [0,a]$ (resp. $\lambda \in \,]0,a]$). Let $D_T^{M,2,[0,a]}$ (resp. $D_T^{M,2,]0,a]}$) be the restriction of $D_T^{M,2}$ to $E_T^{[0,a]}$ (resp. $E_T^{]0,a]}$). Let $P_T^{[0,a]}$ (resp. $P_T^{]0,a]}$) be the orthogonal projection operator from $\Omega(M,\xi)$ on $E_T^{[0,a]}$ (resp. $E_T^{]0,a]}$) with respect to $\langle\ \rangle_T$. Set $P_T^{]a,+\infty[} = 1 - P_T^{[0,a]}$.

For $0 \le k \le n$, $g \in G$, set

$$\chi_g(\xi) = \operatorname{Tr}_s\big[g_{|H(M,\xi)}\big], \quad \chi_g(R^k\pi_*\xi) = \operatorname{Tr}_s\big[g_{|H(B,R^k\pi_*\xi)}\big]. \tag{4.10}$$

Then by the Lefchetz fixed point formula of Atiyah-Bott [ABo],

$$\begin{cases} \chi_g(\xi) = \displaystyle\int_{M^g} \operatorname{Td}_g(TM)\operatorname{ch}_g(\xi), \\ \chi_g(R^k\pi_*\xi) = \displaystyle\int_{B^g} \operatorname{Td}_g(TY)\operatorname{ch}_g(R^k\pi_*\xi). \end{cases} \tag{4.10}$$

We now state eight intermediate results contained in Theorems 4.5–4.12 which play an essential role in the proof of Theorem 3.1. The proof of Theorems 4.5–4.12 are deferred to Sections 5–9.

Theorem 4.5. — *For any $u > 0$,*

$$\lim_{T\to+\infty} \operatorname{Tr}_s\big[gN\exp(-u^2 D_T^{M,2})\big] = \operatorname{Tr}_s\big[gN\exp(-u^2 D^{B,2})\big]. \tag{4.12}$$

For any $u>0$, there exists $C>0$ such that for $T\geq 1$,

$$\left|\operatorname{Tr}_s\left[gN_V\exp(-u^2D_T^{M,2})\right]-\sum_{j=0}^{\dim X}(-1)^j j\chi_g(R^j\pi_*\xi)\right|\leq \frac{C}{T}\cdot \tag{4.13}$$

For any $\varepsilon>0$, there exists $C>0$ such that for $u\geq\varepsilon$, $T\geq 1$,

$$\left|\operatorname{Tr}\left[g\exp(-u^2D_T^{M,2})\right]\right|\leq C. \tag{4.14}$$

Theorem 4.6. — *For any $u>0$,*

$$\lim_{T\to+\infty}\operatorname{Tr}_s\left[gN\exp(-u^2D_T^{M,2})P_T^{]a,+\infty[}\right] = \operatorname{Tr}_s\left[gN\exp(-u^2D^{B,2})Q^{\perp}\right]. \tag{4.15}$$

There exist $c>0, C>0$ such that for $u\geq 1$, $T\geq 1$,

$$\left|\operatorname{Tr}\left[gN\exp(-uD_T^{M,2})P_T^{]a,+\infty[}\right]\right|\leq c\exp(-Cu). \tag{4.16}$$

Theorem 4.7. — *The following identity holds:*

$$\lim_{T\to+\infty}\operatorname{Tr}\left[gD_T^{M,2,[0,a]}\right]=0. \tag{4.17}$$

For $T\geq 1$ large enough, for $0\leq i\leq\dim M$,

$$\operatorname{Tr}\left[g_{|E_T^{[0,a],i}}\right]=\sum_{j=0}^{i}\operatorname{Tr}\left[g_{|H^j(B,R^{i-j}\pi_*\xi)}\right]. \tag{4.18}$$

Let (E_r,d_r) $(r\geq 2)$ be the Leray spectral sequence associated to π,ξ. By [Ma1, Thm. II.2.1], the Dolbeault complex $(\Omega(M,\xi),\bar\partial^M)$ filtered as in [BerB, §1a)] calculates the Leray spectral sequence. Then as in [BerB, §4], for $r\geq 2$, E_r is equipped with a metric h^{E_r} associated to h^{TM},h^{TB},h^ξ. For $r\geq 2$, let ${}_r|\ |_{\lambda_G(\xi)}$ be the corresponding metric on $\lambda_G(\xi)\simeq\det(E_r,G)^{-1}$ defined as in (1.8).

For $r\geq 1$, let $N_{|E_r},N_{H|E_r},N_{V|E_r}$ be the restrictions of N,N_H,N_V to E_r.

Theorem 4.8. — *The following identity holds:*

$$\lim_{T\to+\infty}\Big\{\operatorname{Tr}_s\left[gN\log(D_T^{M,2,[0,a]})\right] + 2\sum_{r\geq 2}(r-1)\big(\operatorname{Tr}_s[gN_{|E_r}]-\operatorname{Tr}_s[gN_{|E_{r+1}}]\big)\log(T)\Big\} = \log\left(\frac{{}_\infty|\ |_{\lambda_G(\xi)}}{{}_2|\ |_{\lambda_G(\xi)}}\right)^2(g). \tag{4.18}$$

For $T\geq 1$, let $|\ |_{\lambda_G(\xi),T}$ be the L_2 metric on the line $\lambda_G(\xi)$ associated to the metrics h_T^{TM},h^ξ on TM,ξ defined in (1.8).

THEOREM 4.9. — *The following identity holds:*

$$\lim_{T\to+\infty}\Big\{\log\Big(\frac{|\ |_{\lambda_G(\xi),T}}{|\ |_{\lambda_G(\xi)}}\Big)^2(g) + 2\big(-\dim X\chi_g(\xi)+\mathrm{Tr}_s[gN_{V|E_\infty}]\big)\log(T)\Big\} = \log\Big(\frac{{}^\infty|\ |_{\lambda_G(\xi)}}{|\ |_{\lambda_G(\xi)}}\Big)^2(g). \tag{4.20}$$

For $u>0$, let B_u be the Bismut superconnection on $\Omega(X,\xi_{|X})$ constructed in Definition 2.6 which is attached to h^{TM}, h^ξ on TM,ξ. Let $\widetilde{N}_u$ be the operator defined in (2.14) associated to the metric $\widetilde{h}^{TM}$.

THEOREM 4.10. — *For any $T\ge 1$,*

$$\lim_{\varepsilon\to 0}\mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})\exp(-\varepsilon^2 D^{M,2}_{T/\varepsilon})\Big] = \frac{2}{T}\int_{B^g}\mathrm{Td}_g(TB,h^{TB})\Phi\mathrm{Tr}_s\big[g\widetilde{N}_{T^2}\exp(-B^2_{T^2})\big]-\frac{2}{T}\dim X\chi_g(\xi). \tag{4.21}$$

Let $\omega^M,\widetilde{\omega}^M,\omega^B$ be the Kähler forms associated to $h^{TM},\widetilde{h}^{TM},h^{TB}$. Let ∇^{TM}_T be the holomorphic Hermitian connection on (TM,h^{TM}_T), and let R^{TM}_T be its curvature.

THEOREM 4.11. — *There exists $C>0$ such that for $\varepsilon\in\,]0,1]$, $\varepsilon\le T\le 1$,*

$$\Big|\mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})\exp(-\varepsilon^2 D^{M,2}_{T/\varepsilon})\Big]-\frac{2}{T^3}\int_{M^g}\frac{\widetilde{\omega}^M}{2\pi}\,\mathrm{Td}_g(TM)\,\mathrm{ch}_g(\xi) + \int_{M^g}\frac{\partial}{\partial b}\mathrm{Td}_g\Big(\frac{-R^{TM}_{T/\varepsilon}}{2i\pi}-b(h^{TM}_{T/\varepsilon})^{-1}\frac{\partial}{\partial T}(h^{TM}_{T/\varepsilon})\Big)_{b=0}\mathrm{ch}_g(\xi,h^\xi)\Big|\le C. \tag{4.22}$$

THEOREM 4.12. — *There exist $\delta\in\,]0,1]$, $C>0$ such that for $\varepsilon\in\,]0,1]$, $T\ge 1$,*

$$\Big|\mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})\exp(-\varepsilon^2 D^{M,2}_{T/\varepsilon})\Big] - \frac{2}{T}\Big(\sum_{j=0}^{\dim X}(-1)^j j\chi_g(R^j\pi_*\xi)-\dim X\chi_g(\xi)\Big)\Big|\le\frac{C}{T^{1+\delta}}\cdot \tag{4.22}$$

Theorems 4.5–4.9 can be obtained formally from [BerB, Thms. 4.8–4.12] by introducing in the right place the operator g. This will permit us to transfer formally the discussion in [BerB, Sect. 4] to our situation.

c) Proof of Theorem 3.1.

By Theorem 2.12,

$$\mathrm{ch}_g(R^\bullet \pi_* \xi) = \int_{X^g} \mathrm{Td}_g(TX)\,\mathrm{ch}_g(\xi). \tag{4.24}$$

We also have the obvious equality

$$\mathrm{Td}'_g(TM) = \pi^*\big(\mathrm{Td}'_g(TB)\big)\,\mathrm{Td}_g(TX) + \pi^*\big(\mathrm{Td}_g(TB)\big)\,\mathrm{Td}'_g(TX). \tag{4.25}$$

By Theorem 4.3, Theorems 4.5–4.12, and proceeding as in [BerB, §4c),d)], using (4.24), (4.25), we get (3.6). □

5. A proof of Theorems 4.5, 4.6 and 4.7.

The proof of Theorems 4.5, 4.6 and 4.7 is essentially the same as the proof of [BerB, Theorems 4.8, 4.9 and 4.10] given in [BerB, §5], where the corresponding results were established when G is trivial. Now we use the notation of [BerB, §5].

At first, for each $U \in TB$, $(gU)^H = gU^H$, so the operator C_T in [BerB, (5.7)] commutes with the action of G.

Let $\langle\ \rangle_\infty$ be the Hermitian product on E_0^0 associated to the metrics $\pi^* h^{TB} \oplus h^{TX}$, h^ξ on TM, ξ defined by (1.2).

Let $E_{1,T}, E_0^\mu, E_{1,T}^\mu$ $(\mu \geq 0)$ be the vector spaces defined in [BerB, Def. 5.12]. Then for any $T > 0$, the linear isometric embedding J_T of $E_{1,\infty}$ in $E_{1,T}$ defined in [BerB, Def. 5.16] is G-equivariant. Let $E_{1,T}^{0,\perp}$ be the orthogonal space to $E_{1,T}^0$ in E_0^0 with respect to $\langle\ \rangle_\infty$. It follows from the previous considerations that for any $T > 0$, the orthogonal splitting $E_0^0 = E_{1,T}^0 \oplus E_{1,T}^{0,\perp}$ of E_0^0 considered in [BerB, (5.29)] is G-invariant, *i.e.* G acts on $E_{1,T}^0$ and $E_{1,T}^{0,\perp}$.

Therefore the matrix of the unitary operator g with respect to the splitting $E_0^0 = E_{1,T}^0 \oplus E_{1,T}^{0,\perp}$ can be written in the form

$$g = \begin{bmatrix} g_{0,T} & 0 \\ 0 & g_{1,T} \end{bmatrix}, \tag{5.1}$$

and moreover

$$g_{0,T} J_T = J_T g. \tag{5.2}$$

The proof of Theorems 4.5 , 4.6 and 4.7 then proceeds as in [BerB, §5 c)–g)]. □

6. A proof of Theorems 4.8–4.9.

In this section, we give a proof of Theorems 4.8 and 4.9. These generalize [BerB, §6], where the corresponding results were proved in the case where G is trivial.

At first we can verify the formulas of [BerB,Theorems 6.1–6.5] are G-equivariant. By using [B5, Thm. 1.4], and by proceeding as in [BerB, §6(d)], we obtain (4.19).

By proceeding as in [BerB, §6(e)], we get (4.20).

This completes the proof of Theorems 4.8 and 4.9. □

7. A proof of Theorem 4.10.

This section is organized as follows. In a), we show that the proof of (4.21) can be localized near $\pi^{-1}(B^g)$. In b), given $b_0 \in B^g$, we replace M by $(T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$, and rescaling on certain Clifford variables. In c), we prove (4.21).

Recall that in this section, we will calculate the asymptotics as $\varepsilon \to 0$ of certain supertraces involving $\varepsilon D^M_{T/\varepsilon}$ for a fixed $T \geq 1$.

In this section, we use the same notation as in Section 4.

a) The proof is local on $\pi^{-1}(B^g)$.

Let $\mathrm{d}v_M$ (resp. $\mathrm{d}v_B$, resp. $\mathrm{d}v_X$) be the Riemannian volume form on M (resp. B, resp. on the fibre X) associated to the metric $\pi^* h^{TB} \oplus h^{TX}$ on $TM \simeq \pi^* TB \oplus TX$ (resp. h^{TB} on TB, resp. h^{TX} on TX).

Let d^B, d^M be the distance functions on B, M associated to h^{TB}, h^{TM}. Let α^B, α^M be the injective radius of B, M. In the sequel, we assume that given $0 < \alpha < \alpha_0 < \frac{1}{4} \inf\{\alpha^B, \alpha^M\}$ are chosen small enough so that if $y \in B$, $d^B(g^{-1}y, y) \leq \alpha$, then $d^B(y, B^g) \leq \frac{1}{4}\alpha_0$, and if $x \in M$, $d^M(g^{-1}x, x) \leq \alpha$, then $d^M(x, M^g) \leq \frac{1}{4}\alpha_0$. If $x \in B$, let $B^B(x, \alpha)$ be the open ball of center x and radius α in B.

Let f be a smooth even function defined on $\mathbb{R}$ with values in $[0, 1]$, such that

$$f(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{2}\alpha, \\ 0 & \text{for } |t| \geq \alpha. \end{cases} \tag{7.1}$$

Set

$$g(t) = 1 - f(t). \tag{7.2}$$

Definition 7.1. — *For* $u \in]0,1]$, $a \in \mathbb{C}$, *set*

$$\begin{cases} F_u(a) = \displaystyle\int_{-\infty}^{+\infty} \exp(i\,ta\sqrt{2}\,)\exp\Big(\frac{-t^2}{2}\Big)f(ut)\frac{\mathrm{d}t}{\sqrt{2\pi}}, \\ G_u(a) = \displaystyle\int_{-\infty}^{+\infty} \exp(i\,ta\sqrt{2}\,)\exp\Big(\frac{-t^2}{2}\Big)g(ut)\frac{\mathrm{d}t}{\sqrt{2\pi}}. \end{cases} \tag{7.3}$$

Clearly

$$F_u(a) + G_u(a) = \exp(-a^2). \tag{7.4}$$

The functions $F_u(a)$, $G_u(a)$ are even holomorphic functions. So there exist holomorphic functions $\widetilde{F}_u(a)$, $\widetilde{G}_u(a)$ such that

$$F_u(a) = \widetilde{F}_u(a^2), \quad G_u(a) = \widetilde{G}_u(a^2). \tag{7.5}$$

The restrictions of $F_u, G_u, \widetilde{F}_u, \widetilde{G}_u$ to $\mathbb{R}$ lie in the Schwartz space $S(\mathbb{R})$.

From (7.4), we deduce that

$$\exp(-\varepsilon^2 D^{M,2}_{T/\varepsilon}) = F_\varepsilon(\varepsilon D^M_{T/\varepsilon}) + G_\varepsilon(\varepsilon D^M_{T/\varepsilon}). \tag{7.6}$$

Proposition 7.2. — *For* $\delta > 0$ *fixed, there exist* $c > 0, C > 0$ *such that for* $0 < \varepsilon \le \delta$, $T \ge 1$,

$$\Big|\mathrm{Tr}_s\Big[g *_T^{-1} \frac{\partial}{\partial T}(*_T)G_{\frac{\varepsilon}{T}}(\frac{\varepsilon}{T}D^M_T)\Big]\Big| \le c\exp\Big(-\frac{CT^2}{\varepsilon^2}\Big). \tag{7.7}$$

Proof. — The proof of our theorem is as same as the proof of [BerB, Prop. 8.3]. □

For $T \ge 1$ fixed, we use (7.7) with $\varepsilon = T$ and T replace by T/ε, we find

$$\Big|\mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon})G_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big]\Big| \le c\exp\Big(-\frac{C}{\varepsilon^2}\Big). \tag{7.8}$$

Let $F_\varepsilon(\varepsilon D^M_{T/\varepsilon})(x,x')$ $(x,x' \in M)$ be the smooth kernel associated to $F_\varepsilon(\varepsilon D^M_{T/\varepsilon})$ with respect to the volume form $\mathrm{d}v_M(x')/(2\pi)^{\dim M}$. Using (7.3) and finite propagation speed [CP, §7.8], [T, §4.4], it is clear that for $\varepsilon \in]0,1]$, $T \geq 1$, $x, x' \in M$, if $d^B(\pi x, \pi x') \geq \alpha$, then

$$F_\varepsilon(\varepsilon D^M_{T/\varepsilon})(x,x') = 0, \tag{7.9}$$

and moreover, given $x \in M$, $F_\varepsilon(\varepsilon D^M_{T/\varepsilon})(x,\cdot)$ only depends on the restriction of $D^M_{T/\varepsilon}$ to $\pi^{-1}(B^B(\pi x, \alpha))$.

Let $N_{B^g/B}$ be the normal bundle to B^g in B. We identify $N_{B^g/B}$ to the orthogonal bundle to TB^g in TB. Let $h^{N_{B^g/B}}$ be the metric on $N_{B^g/B}$ induced by h^{TB}. Let $\mathrm{d}v_{N_{B^g/B}}$ be the Riemannian volume form on $(N_{B^g/B,\mathbb{R}}, h^{N_{B^g/B}})$. Let $c(N_{B^g/B,\mathbb{R}})$, $c(T_\mathbb{R}X)$ be the Clifford algebras of $(N_{B^g/B,\mathbb{R}}, h^{N_{B^g/B}}), (T_\mathbb{R}X, h^{TX})$. For $U \in T_\mathbb{R}B$, $V \in T_\mathbb{R}X$, let $c(U), c(V)$ denote the corresponding Clifford multiplication operators acting on $\pi^*\Lambda(T^{*(0,1)}B), \Lambda(T^{*(0,1)}X)$ associated to h^{TB}, h^{TX} defined as in (2.8). Set

$$A'_{\varepsilon,T} = \Big(\frac{T}{\varepsilon}\Big)^{N_V} \varepsilon D^M_{T/\varepsilon} \Big(\frac{T}{\varepsilon}\Big)^{-N_V}. \tag{7.10}$$

Then by (7.10), we get

$$\mathrm{Tr}_s\Big[g *^{-1}_{T/\varepsilon} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big] = \mathrm{Tr}_s\Big[g *^{-1}_{T/\varepsilon} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(A'_{\varepsilon,T})\Big]. \tag{7.11}$$

Let $F_\varepsilon(A'_{\varepsilon,T})(x,x')$ $(x, x' \in M)$ be the smooth kernel associated to the operator $F_\varepsilon(A'_{\varepsilon,T})$ with respect to $\mathrm{d}v_M(x')/(2\pi)^{\dim M}$.

Let $\mathcal{U}_{\alpha_0}(B^g)$ be the set of $b \in B$ such that $d^B(b, B^g) < \alpha_0$. We identify $\mathcal{U}_{\alpha_0}(B^g)$ to $\{(b,Y)\,;\ b \in B^g,\ Y \in N_{B^g/B,\mathbb{R}},\ |Y| \leq \alpha_0\}$ by using geodesic coordinates normal to B^g in B. By (7.9) and the choice of α, α_0, we get

$$\begin{aligned}\int_M \mathrm{Tr}_s\Big[g *^{-1}_{T/\varepsilon} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(A'_{\varepsilon,T})(g^{-1}x,x)\Big] \frac{\mathrm{d}v_M}{(2\pi)^{\dim M}} \\ = \int_{B^g} \int_{\substack{|Y|\leq \alpha_0/4 \\ Y \in N_{B^g/B,\mathbb{R}}}} \int_X \mathrm{Tr}_s\Big[g *^{-1}_{T/\varepsilon} \frac{\partial}{\partial T}(*_{T/\varepsilon}) \\ F_\varepsilon(A'_{\varepsilon,T})(g^{-1}(b,Y,x),(b,Y,x))\Big] \frac{\mathrm{d}v_M}{(2\pi)^{\dim M}}.\end{aligned} \tag{7.12}$$

By (7.8), (7.11), (7.12), we see that the proof of Theorem 4.10 is local near $\pi^{-1}(B^g)$.

b) Rescaling of the variable Y and of the Clifford variables.

Let $\nabla^{TB}, \nabla^{TX}, \nabla^{\xi}$ be the holomorphic Hermitian connections on (TB, h^{TB}), (TX, h^{TX}) and (ξ, h^{ξ}). Let R^{TB}, R^{TX}, L^{ξ} be the corresponding curvatures.

Taking $b_0 \in B^g$, we identify $B^B(b_0, \alpha_0)$ with $B(0, \alpha_0) \subset (TB)_{b_0} = \mathbb{C}^m$ by using normal coordinates.

Take $y \in \mathbb{C}^m$, $|y| \leq \alpha_0$, set $Y = y + \bar{y}$. We identify $TB_{|Y}$ to $TB_{\{0\}}$ by parallel transport along the curve $t \mapsto tY$ with respect to the connection ∇^{TB}. We lift horizontally the paths $t \in \mathbb{R}_+^* \mapsto tY$ into paths $t \in \mathbb{R}_+^* \mapsto x_t \in M$ with $x_t \in X_{tY}$, $\mathrm{d}x_t/\mathrm{d}t \in T_{\mathbb{R}}^H M$. If $x_0 \in X_{b_0}$, we identify TX_{x_t}, ξ_{x_t} to TX_{x_0}, ξ_{x_0} by parallel transport along the curve $t \mapsto x_t$ with respect to the connections $\nabla^{TX}, \nabla^{\xi}$. These trivializations induce corresponding trivializations of $\Lambda(T^{*(0,1)}B)$, $\Lambda(T^{*(0,1)}M) \otimes \xi$.

Let $\Omega_{b_0} = \Omega(X_{b_0}, \xi_{|X_{b_0}})$ be the vector space of smooth sections of $(\Lambda(T^{*(0,1)}X) \otimes \xi)_{|X_{b_0}}$ on X_{b_0}. Then Ω_{b_0} is naturally equipped with a Hermitian product $\langle\ \rangle$ attached to $h^{TX|X_{b_0}}, h^{\xi|X_{b_0}}$ defined in (1.2).

Recall that the operator D^X is defined in (2.7). Under our trivialization, $\operatorname{Ker} D^X{}_{|B^B(b_0,\alpha_0)}$ is a $\mathbb{Z}$-graded smooth vector subbundle of Ω_{b_0} on $B^B(b_0, \alpha_0)$.

By [BerB, §8b)], there is also a smooth $\mathbb{Z}$-graded vector bundle $K \subset \Omega_{b_0}$ over $(T_{\mathbb{R}}B)_{b_0} \simeq \mathbb{R}^{2m}$ which coincides with $\operatorname{Ker} D^X$ on $B(0, 2\alpha_0)$, with $\operatorname{Ker} D^X_{b_0}$ over $T_{\mathbb{R}}B \backslash B(0, 3\alpha_0)$ and such that if $K^\perp$ is the orthogonal bundle to K in Ω_{b_0},

$$K^\perp \cap \operatorname{Ker} D^X_{b_0} = \{0\}. \tag{7.13}$$

Let P_Y ($Y \in \mathbb{R}^{2m}$) be the orthogonal projection operator from Ω_{b_0} on K_Y. Set $P_Y^\perp = 1 - P_Y$.

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\varphi(t) = \begin{cases} 1 & \text{for } |t| \leq \alpha_0, \\ 0 & \text{for } |t| \geq 2\alpha_0. \end{cases} \tag{7.14}$$

Let Δ^{TB} be the standard Laplacian on $(T_{\mathbb{R}}B)_{b_0}$ with respect to the metric $h^{TB|b_0}$. Let H_{b_0} be the vector space of smooth sections of $\pi^*\Lambda(T^{*(0,1)}B)_{b_0} \otimes (\Lambda(T^{*(0,1)}X) \otimes \xi)_{|X_{b_0}}$ over $(T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$. Let $L^1_{\varepsilon,T}$ be the operator

$$L^1_{\varepsilon,T} = \varphi^2\big(|Y|\big) A'^2_{\varepsilon,T} + \big(1 - \varphi^2(|Y|)\big)\Big(\frac{-\varepsilon^2 \Delta^{TB}}{2} + T^2 P_Y^\perp D^{X,2}_{b_0} P_Y^\perp\Big). \tag{7.15}$$

For $(Y,x)\in (T_{\mathbb{R}}B)_{b_0}\times X_{b_0}$, $\varepsilon>0$, $s\in H_{b_0}$, set

$$S_\varepsilon s(Y,x)=s(Y/\varepsilon,x). \tag{7.16}$$

Put

$$L^2_{\varepsilon,T}=S_\varepsilon^{-1}L^1_{\varepsilon,T}S_\varepsilon. \tag{7.17}$$

Let $\mathcal{O}_p$ be the set of differential operators acting on smooth sections of $(\Lambda(T^{*(0,1)}X)\otimes\xi)_{X_{b_0}}$ over $\mathbb{R}^{2m}\times X_{b_0}$. Then we find that

$$L^2_{\varepsilon,T}\in c(T_{\mathbb{R}}B)\widehat{\otimes}\mathcal{O}_p.$$

Let $f_1,\dots,f_{2m'}$ be an orthonormal basis of $(T_{\mathbb{R}}B^g)_{b_0}$, let $f_{2m'+1},\dots,f_{2m}$ be an orthonormal basis of $N_{B^g/B,\mathbb{R},b_0}$.

Definition 7.3. — *For $\varepsilon>0$, set*

$$c_\varepsilon(f_j)=\frac{\sqrt2}{\varepsilon}f^j\wedge-\frac{\varepsilon}{\sqrt2}i_{f_j},\quad 1\le j\le 2m'. \tag{7.18}$$

Let $L^3_{\varepsilon,T},M^3_{\varepsilon,T}$ be obtained from $L^2_{\varepsilon,T}$, $*^{-1}_{T/\varepsilon}\partial/\partial T(*_{T/\varepsilon})$ by replacing the Clifford variables $c(f_j)$ $(1\le j\le 2m')$ by the operators $c_\varepsilon(f_j)$.

For $b_0\in B^g$, $Y\in N_{B^g/B,\mathbb{R},b_0}$, $|Y|\le\alpha_0$, let $k(b_0,Y)$ be defined by $dv_B(b_0,Y)=k(b_0,Y)dv_{B^g}(b_0)dv_{N_{B^g/B}}(Y)$. Let $dv_{(TB)_{b_0}}$ be the Riemannian volume form on $((TB)_{b_0},h^{TB}_{b_0})$.

Let $P^i_{\varepsilon,T}((Y,x),(Y',x'))$, $\widetilde{F}_\varepsilon(L^i_{\varepsilon,T})\big((Y,x),(Y',x')\big)$ $\big((Y,x),(Y',x')\in (T_{\mathbb{R}}B)_{b_0}\times X_{b_0}\big)$ $(i=1,2,3)$ be the smooth kernels associated to $\exp(-L^i_{\varepsilon,T})$, $\widetilde{F}_\varepsilon(L^i_{\varepsilon,T})$ calculated with respect to $\mathrm{d}v_{(TB)_{b_0}}(Y')\,\mathrm{d}v_{X_{b_0}}(x')/(2\pi)^{\dim M}$. Using finite propagation speed [CP, §7.8], [T, §4.4], we see that if $(Y,x)\in N_{B^g/B,\mathbb{R},b_0}\times X_{b_0}$, $|Y|<\frac14\alpha_0$, then

$$F_\varepsilon(A'_{\varepsilon,T})\big(g^{-1}(b_0,Y,x),(b_0,Y,x)\big)k(b_0,Y) = \widetilde{F}_\varepsilon(L^1_{\varepsilon,T})\big(g^{-1}(Y,x),(Y,x)\big). \tag{7.19}$$

We observe that for any $k\in\mathbb{N}$, $c>0$, there is $C>0,C'>0$ such that for $\varepsilon>0$,

$$\sup_{|\operatorname{Im}(a)|\le c}|a|^k\cdot\big|\widetilde{F}_\varepsilon(a^2)-\exp(-a^2)\big|\le C\exp\Big(\frac{-C'}{\varepsilon^2}\Big). \tag{7.20}$$

Using (7.20), and proceeding as in [BerB, Prop. 8.2], we find for $T\ge1$ fixed, there exist $c,C>0$ such that for $|Y|,|Y'|<\frac14\alpha_0$,

$$\big|\big(\widetilde{F}_\varepsilon(L^1_{\varepsilon,T})-\exp(-L^1_{\varepsilon,T})\big)((Y,x),(Y',x'))\big|\le c\exp\Big(\frac{-C}{\varepsilon^2}\Big). \tag{7.21}$$

By (7.19), (7.21), we can replace $F_\varepsilon(A'_{\varepsilon,T})$ by $\exp(-L^1_{\varepsilon,T})$ in (7.12).

We know that $P^3_{\varepsilon,T}((Y,x),(Y',x'))$ lies in

$$\big(\operatorname{End}\big(\Lambda(T^*_{\mathbb{R}}B^g)\big)\widehat{\otimes}c(N_{B^g/B,\mathbb{R}})\big)_{b_0}\widehat{\otimes}c(T_{\mathbb{R}}X_{b_0})\widehat{\otimes}\operatorname{End}(\xi).$$

Then $M^3_{\varepsilon,T}P^3_{\varepsilon,T}(g^{-1}(Y,x),(Y,x))$ can be expanded in the form

$$\begin{aligned}(7.22)\quad & M^3_{\varepsilon,T}P^3_{\varepsilon,T}\big(g^{-1}(Y,x),(Y,x)\big)\\ &= \sum_{\substack{1\le i_1<\dots<i_p\le 2m'\\ 1\le j_1<\dots<j_q\le 2m'}} f^{i_1}\wedge\ldots\wedge f^{i_p}\wedge i_{f_{j_1}}\ldots\wedge i_{f_{j_q}}\widehat{\otimes}R^{i_1\cdots i_p;j_1\cdots j_q},\end{aligned}$$

with $R^{i_1\cdots i_p;j_1\cdots j_q}(g^{-1}(Y,x),(Y,x)) \in c(N_{B^g/B,\mathbb{R}})_{b_0}\widehat{\otimes}c(T_{\mathbb{R}}X_{b_0})\widehat{\otimes}\operatorname{End}(\xi)$. Set

$$(7.23)\qquad \big[M^3_{\varepsilon,T}P^3_{\varepsilon,T}\big(g^{-1}(Y,x),(Y,x)\big)\big]^{\max} = R^{1,\dots,2m'}\big(g^{-1}(Y,x),(Y,x)\big).$$

Proposition 7.4. — *If $Y\in N_{B^g/B,\mathbb{R},b_0}$, $x\in X_{b_0}$, the following identity holds:*

$$\begin{aligned}(7.24)\quad & \operatorname{Tr}_s\Big[g\,*^{-1}_{T/\varepsilon}\frac{\partial}{\partial T}\big(*_{T/\varepsilon}\big)P^1_{\varepsilon,T}(g^{-1}(Y,x),(Y,x))\Big]\\ &= (-i)^{\dim B^g}\varepsilon^{-2\dim N_{B^g/B}}\operatorname{Tr}_s\big[g[M^3_{\varepsilon,T}P^3_{\varepsilon,T}(g^{-1}(\varepsilon^{-1}Y,x),(\varepsilon^{-1}Y,x))]^{\max}\big].\end{aligned}$$

Proof. — Since g acts like the identity on $\Lambda(T^{*(0,1)}B^g)$, $g\in c(N_{B^g/B,\mathbb{R}})_{b_0}\widehat{\otimes}\, c(T_{\mathbb{R}}X_{b_0})\,\widehat{\otimes}\operatorname{End}(\xi)$. Therefore the rescaling of the Clifford variable in (7.18) has no effect on g. Identity (7.24) is now a trivial consequence of [Ge]. □

c) Proof of Theorem 4.10.

Recall that for $u>0$, the Bismut superconnection B_u associated to h^{TM} and h^{ξ} was constructed in Section 2b). Also we observe that B_u is unchanged if h^{TM} is changed into $\widetilde{h}^{TM}$.

Recall that R^{TB} is the curvature of ∇^{TB}. Let $R^{TB}|_{B^g}$, $\widetilde{\omega}^{H\overline{H}}|_{B^g}$ be the restriction of R^{TB}, $\widetilde{\omega}^{H\overline{H}}$ on B^g. Also ∇_{f_α} denote the ordinary differentiation operator on $(T_{\mathbb{R}}B)_{b_0}$ in the direction f_α. Then by (7.18), as in [BerB, (7.30), (7.35)], we have as $\varepsilon\to 0$

$$(7.25)\qquad L^3_{\varepsilon,T}\longrightarrow L^3_{0,T},$$

and for $Y \in (T_{\mathbb{R}}B)_{b_0}$,

$$\text{(7.26)}\quad \mathrm{e}^{-\frac{i\widetilde{\omega}^{H\overline{H}}_{|B^g}}{2T^2}} L^3_{0,T}(Y)\,\mathrm{e}^{\frac{i\widetilde{\omega}^{H\overline{H}}_{|B^g}}{2T^2}} = -\frac{1}{2}\Big(\nabla_{f_\alpha} + \frac{1}{2}\langle R^{TB}{}_{|B^g}Y, f_\alpha\rangle_{h^{TB}}\Big)^2 + \frac{1}{2}\operatorname{Tr}(R^{TB}{}_{|B^g}) + B^2_{T^2|B^g}.$$

By [BerB, (7.36)], (7.18), as [BerB, (7.38)], we get , as $\varepsilon \to 0$

$$\text{(7.27)}\qquad M^3_{\varepsilon,T} \longrightarrow M^3_{0,T} = \frac{2}{T}(N_V - \dim X) + \frac{2i\,\widetilde{\omega}^{H\overline{H}}_{|B^g}}{T^3}\cdot$$

By [B4, (3.16)–(3.21)], [BerB, §7d)], we have

$$\text{(7.28)}\quad \int_{N_{B^g/B,\mathbb{R},b_0}}\int_{X_{b_0}} \operatorname{Tr}_s\big[g[M^3_{0,T}P^3_{0,T}(g^{-1}(Y,x),(Y,x))]^{\max}\big] \frac{\mathrm{d}v_{N_{B^g/B}}(Y)\,\mathrm{d}v_{X_{b_0}}(x)}{(2\pi)^{\dim M}}$$
$$= i^{\dim B^g}\,\frac{2}{T}\Big\{\operatorname{Td}_g(TB,h^{TB})\Phi\operatorname{Tr}_s\big[g(\widetilde{N}_{T^2} - \dim X)\exp(-B^2_{T^2})\big]\Big\}^{\max}.$$

Theorem 7.5. — *For $T \geq 1$ fixed, there exist $c > 0$, $C > 0$, $r \in \mathbb{N}$ such that for $\varepsilon \in]0,1]$, $(Y,x),(Y',x') \in (T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$,*

$$\text{(7.29)}\quad \big|(P^3_{\varepsilon,T} - P^3_{0,T})((Y,x),(Y',x'))\big| \leq c\big(1 + |Y| + |Y'|\big)^r \exp\big(-C|Y - Y'|^2\big).$$

To prove Theorem 7.5, we establish at first an uniform estimate on the kernel $P^3_{\varepsilon,T}$.

Theorem 7.6. — *For $T \geq 1$ fixed, there is $C > 0$ such that for $k \in \mathbb{N}$, there exist $c > 0, r \in \mathbb{N}$ such that for any $\varepsilon \in]0,1]$, $(Y,x),(Y',x') \in (T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$,*

$$\text{(7.30)}\quad \sup_{|\alpha|,|\alpha'|\leq k}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^\alpha \partial Y'^{\alpha'}} P^3_{\varepsilon,T}\big((Y,x),(Y',x')\big)\Big| \leq c\big(1 + |Y| + |Y'|\big)^r \exp\big(-C|Y - Y'|^2\big).$$

Proof of Theorem 7.6. — Set

$$\text{(7.31)}\qquad g_\varepsilon(Y) = 1 + \big(1 + |Y|^2\big)^{\frac{1}{2}}\varphi\Big(\frac{\varepsilon|Y|}{2}\Big).$$

Let $\mathbb{E}^0$ be the vector space of square integrable sections of $(\Lambda(T^*_{\mathbb{R}}B^g)\widehat{\otimes}\Lambda(\overline{N}^*_{B^g/B}))_{b_0}\widehat{\otimes}(\Lambda(T^{*(0,1)}X)\otimes\xi)_{|X_{b_0}}$ over $(T_{\mathbb{R}}B)_{b_0}\times X_{b_0}$. For $0 \le q \le 2m' = 2\dim B^g$, let $\mathbb{E}^0_q$ be the vector space of square integrable sections of $(\Lambda^q(T^*_{\mathbb{R}}B^g)\ \widehat{\otimes}\Lambda(\overline{N}^*_{B^g/B}))_{b_0}\widehat{\otimes}(\Lambda(T^{*(0,1)}X)\otimes\xi)_{|X_{b_0}}$. Then $\mathbb{E}^0 = \bigoplus_{q=0}^{2m'}\mathbb{E}^0_q$. Similarly, if $p\in\mathbb{R}$, $\mathbb{E}^p$ and $\mathbb{E}^p_q$ denote the corresponding p^{th} Sobolev spaces. If $s\in\mathbb{E}^0_q$, set

(7.32)
$$|s|^2_{\varepsilon,0} = \int_{(T_{\mathbb{R}}B)_{b_0}\times X_{b_0}} \big|s(Y,x)\big|^2 g_\varepsilon(Y)^{2(2m'-q)}\,\frac{\mathrm{d}v_{(TB)_{b_0}}(Y')\,\mathrm{d}v_{X_{b_0}}(x')}{(2\pi)^{\dim M}}.$$

Let $\langle\ \rangle_{\varepsilon,0}$ be the Hermitian product attached to $|\ |_{\varepsilon,0}$. If $\mathcal{L}\in\mathrm{End}(E^0)$, let $\|\mathcal{L}\|^{0,0}_{\varepsilon,0}$ be the corresponding norm of $\mathcal{L}$. If $s\in\mathbb{E}^1$, put

$$|s|^2_{\varepsilon,1} = |s|^2_{\varepsilon,0} + \sum_\alpha |\nabla_{f_\alpha}s|^2_{\varepsilon,0} + \sum_i |\nabla_{e_i}s|^2_{\varepsilon,0}. \tag{7.33}$$

Let $\Delta = -\Delta^{TB} + D^{X,2}_{b_0}$. Using the technique in [BerB, § 9d)], especially [BerB, (9.51)] (in our situation, T is fixed), where we replace the Sobolev norms [BerB, (9.49), (9.50)] by (7.32) and (7.33), we find for any $k,k'\in\mathbb{N}$, there exists $C'>0$ such that for $\varepsilon\in\,]0,1]$,

$$\big\|\Delta^k\exp(-L^3_{\varepsilon,T})\Delta^{k'}\big\|^{0,0}_{\varepsilon,0} \le C'. \tag{7.34}$$

Take $p\in\mathbb{N}$. Let J^0_{p,b_0} be the set of square integrable sections of $(\Lambda(T^*_{\mathbb{R}}B^g)\widehat{\otimes}\Lambda(\overline{N}^*_{B^g/B}))_{b_0}\widehat{\otimes}(\Lambda(T^{*(0,1)}X)\otimes\xi)_{|X_{b_0}}$ over

$$\big\{(Y,x)\in(T_{\mathbb{R}}B)_{b_0}\times X_{b_0}\ ;\ x\in X_{b_0}, |Y|\le p+\tfrac{1}{2}\big\}.$$

We equip J^0_{p,b_0} with the Hermitian product for $s\in J^0_{p,b_0}$,

$$|s|^2 = \int_{|Y|\le p+1/2}\int_{X_{b_0}} \big|s(Y,x)\big|^2\,\frac{\mathrm{d}v_{(TB)_{b_0}}(Y')\,\mathrm{d}v_{X_{b_0}}(x')}{(2\pi)^{\dim M}}. \tag{7.35}$$

If $\mathcal{L}\in\mathrm{End}(J^0_{p,b_0})$, let $\|\mathcal{L}\|_{p,\infty}$ be the corresponding norm of $\mathcal{L}$ with respect to $|\ |$.

Obviously, there is $C>0$ such that for any $p\in\mathbb{N}$, $s\in J^0_{p,b_0}$

$$|s|\le|s|_{\varepsilon,0}\le C(1+p)^{2m'}|s|. \tag{7.36}$$

By (7.34) and (7.36), we find for any $k, k' \in \mathbb{N}$, there exists $C' > 0$ such that for $\varepsilon \in]0,1]$, $p \in \mathbb{N}$,

$$\left\| \Delta^k \exp(-L^3_{\varepsilon,T}) \Delta^{k'} \right\|_{p,\infty} \leq C'(1+p)^{2m'}. \tag{7.37}$$

Using (7.37) and Sobolev's inequalities, we see that for $k, k' \in \mathbb{N}$, there exist $C > 0$, $r > 0$ such that for $p \in \mathbb{N}$, $\varepsilon \in]0,1]$,

$$\sup_{|Y|,|Y'| \leq p+1/4} \left| \Delta^k_{(Y,x)} \Delta^{k'}_{(Y',x')} P^3_{\varepsilon,T}((Y,x),(Y',x')) \right| \leq C(1+p)^r.$$

So we get the bounds in (7.30) with $C = 0$.

To get the required $C > 0$, we proceed as in the proof of [B5, Thm. 11.14].

Let $u \in \mathbb{R} \to k(u)$ be a smooth even function such that

$$k(u) = \begin{cases} 0 & \text{for } |u| \leq \frac{1}{2}, \\ 1 & \text{for } |u| \geq 1. \end{cases} \tag{7.38}$$

For $q \in \mathbb{R}^*_+$, $a \in \mathbb{C}$, set

$$K_q(a) = 2 \int_0^{+\infty} \cos\bigl(t\sqrt{2}\, a\bigr) \exp\Bigl(-\frac{t^2}{2}\Bigr) k\Bigl(\frac{t}{q}\Bigr) \frac{dt}{\sqrt{2\pi}} \cdot \tag{7.39}$$

Clearly, $K_q(a)$ is an even holomorphic function of a, therefore, there is a holomorphic function $a \in \mathbb{C} \to \widetilde{K}_q(a)$ such that

$$K_q(a) = \widetilde{K}_q(a^2). \tag{7.40}$$

Given $c > 0$, set

$$\begin{cases} V_c = \Bigl\{ \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) \geq \dfrac{(\mathrm{Im}\,\lambda)^2}{4c^2} - c^2 \Bigr\}, \\ \Gamma_c = \Bigl\{ \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) = \dfrac{(\mathrm{Im}\,\lambda)^2}{4c^2} - c^2 \Bigr\}. \end{cases} \tag{7.41}$$

Then by [B5, (11.53)], for any $c > 0$, there exists $C' > 0$ for which given $m, m' \in \mathbb{N}$, there exists $C > 0$, such that for $q \geq 1$,

$$\sup_{a \in V_c} |a|^m \cdot \left| \widetilde{K}^{(m')}_q(a) \right| \leq C \exp(-C' q^2). \tag{7.42}$$

Also

$$\widetilde{K}_q(L^3_{\varepsilon,T}) = \frac{1}{2\pi i}\int_{\Gamma_c} \frac{\widetilde{K}_q(\lambda)}{\lambda - L^3_{\varepsilon,T}}\,\mathrm{d}\lambda. \tag{7.43}$$

Let $\widetilde{K}_q(L^3_{\varepsilon,T})((Y,x),(Y',x'))$ be the smooth kernel associated to $\widetilde{K}_q(L^3_{\varepsilon,T})$ calculated with respect to $dv_{(TB)_{b_0}}(Y')dv_{X_{b_0}}(x')/(2\pi)^{\dim M}$. Using (7.42) and proceeding as in [BerB, §9d)], where we always replace the Sobolev norms [BerB, (9.49), (9.50)] by (7.32) and (7.33), we get the following estimation which is an analog of [B5, (11.59)] : there is $C_0>0$ such that for $k\in\mathbb{N}$, there exist $C>0$, $r\in\mathbb{N}$ for which given $q\in\mathbb{N}$, $(Y,x),(Y',x')\in(T_{\mathbb{R}}B)_{b_0}\times X_{b_0}$, $\varepsilon\in[0,1]$, then

$$\begin{aligned}\sup_{|\alpha|,|\alpha'|\le k}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^{\alpha}\partial Y'^{\alpha'}}\widetilde{K}_q(L^3_{\varepsilon,T})\big((Y,x),(Y',x')\big)\Big|&\\ \le C\big(1+|Y|+|Y'|\big)^r\exp(-C_0q^2).&\end{aligned} \tag{7.44}$$

If $t\ge q$, then $k(t/q)=1$. Using finite propagation speed for the solution of hyperbolic equations for $\cos(s\sqrt{L^3_{\varepsilon,T}})$ [CP, §7.8], [T, 4.4], we find there is a fixed constant $C'_0>0$ such that for $q\in\mathbb{N}^*$,

$$P^3_{\varepsilon,T}\big((Y,x),(Y',x')\big) = \widetilde{K}_q(L^3_{\varepsilon,T})\big((Y,x),(Y',x')\big) \tag{7.45}$$

if $|Y-Y'|\ge C'_0q$.

From (7.44), (7.45), we deduce that there exist $C_0, C'_0>0$ for which given $k\in\mathbb{N}$, there exist $C>0$, $r\in\mathbb{N}$ for which given $q\in\mathbb{N}^*$, $(Y,x),(Y',x')\in(T_{\mathbb{R}}B)_{b_0}\times X_{b_0}$, $\varepsilon\in[0,1]$, then

$$\begin{aligned}\sup_{|\alpha|,|\alpha'|\le k}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^{\alpha}\partial Y'^{\alpha'}}P^3_{\varepsilon,T}\big((Y,x),(Y',x')\big)\Big|&\\ \le C\big(1+|Y|+|Y'|\big)^r\exp(-C_0q^2)\quad\text{if } |Y-Y'|\ge C'_0q.&\end{aligned} \tag{7.46}$$

For $(Y,x),(Y',x')\in(T_{\mathbb{R}}B)_{b_0}\times X_{b_0}$, let $q\in\mathbb{N}$ such that

$$C'_0q\le|Y-Y'|\le C'_0(q+1).$$

By (7.30) with $C=0$ and (7.46), we get

$$\begin{aligned}\sup_{|\alpha|,|\alpha'|\le k}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^{\alpha}\partial Y'^{\alpha'}}P^3_{\varepsilon,T}\big((Y,x),(Y',x')\big)\Big|&\\ \le C\big(1+|Y|+|Y'|\big)^r\exp(-C_0q^2)&\\ \le C\big(1+|Y|+|Y'|\big)^r\exp\Big(-C_0\Big(\frac{|Y-Y'|}{C'_0}-1\Big)^2\Big).&\end{aligned} \tag{7.47}$$

The proof of Theorem 7.6 is completed. □

Proof of Theorem 7.5. — Using (7.25) and Theorem 7.6, and proceeding as in [B5, §11 i)], [BL, §11 q)], we have Theorem 7.5. □

Using Theorem 7.5, (7.19), (7.21), (7.24) and (7.28), we get over B^g

$$
\begin{aligned}
(7.48)\quad & \lim_{\varepsilon\to 0}\int_{\substack{|Y|\le\alpha_0/4\\ Y\in N_{B^g/B,\mathbb{R}}}}\int_X \mathrm{Tr}_s\Big[g*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})\\
& F_\varepsilon(A'_{\varepsilon,T})(g^{-1}(b,Y,x),(b,Y,x))\Big]k(b,Y)\,\frac{\mathrm{d}v_{N_{B^g/B}}(Y)\,\mathrm{d}v_{X_b}(x')}{(2\pi)^{\dim M}}\\
& = \frac{2}{T}\Big\{\,\mathrm{Td}_g(TB,h^{TB})\Phi\mathrm{Tr}_s\big[g(\widetilde{N}_{T^2}-\dim X)\exp(-B_{T^2}^2)\big]\Big\}^{\max}.
\end{aligned}
$$

By (7.7), (7.12) and (7.48), the proof of Theorem 4.10 is completed. □

8. A proof of Theorem 4.11.

This section is organized as follows. In a), we reformulate Theorem 4.11. In b), we indicate that the proof is localized near $\pi^{-1}(B^g)$ by Proposition 7.2. In c), we prove the estimate (8.1).

In this section, we make the same assumption and we use the same notation as in Sections 4 and 7.

a) A reformulation of Theorem 4.11.

THEOREM 8.1. — *There exists* $C>0$ *such that for* $0<u\le 1$, $T\ge 1$,

$$
\begin{aligned}
(8.1)\quad & \Big|\mathrm{Tr}_s\Big[g*_T^{-1}\frac{\partial}{\partial T}(*_T)\exp\Big(-\frac{u^2}{T^2}D_T^{M,2}\Big)\Big]\\
& \qquad -\frac{2}{u^2}\int_{M^g}\frac{\widetilde{\omega}^{TM}}{2\pi T}\,\mathrm{Td}_g(TM)\,\mathrm{ch}_g(\xi)\\
& + \int_{M^g}\frac{\partial}{\partial b}\,\mathrm{Td}_g\Big(\frac{-R_T^{TM}}{2i\pi}-b(h_T^{TM})^{-1}\frac{\partial}{\partial T}(h_T^{TM})\Big)_{b=0}\mathrm{ch}_g(\xi,h^\xi)\Big|\le\frac{Cu^2}{T}\cdot
\end{aligned}
$$

Remark 8.2. — Theorem 8.1 implies Theorem 4.11. In fact, for $0<\varepsilon\le 1$, $\varepsilon\le T\le 1$ we use (8.1), with $u=T$ and T replaced by T/ε, then we find that the right-hand side of (8.1) is dominated by

$$CT^2\,\frac{\varepsilon}{T}=C\varepsilon T\le C\varepsilon.$$

So we have proved (4.22).

b) Localization of the problem near $\pi^{-1}(B^g)$.

By Proposition 7.2 and the argument in Section 7b), the proof of (8.1) can be localized near B^g. Thus, we are entitled to choose $b_0 \in B^g$ as in Section 7b), to replace M by $\mathbb{C}^m \times X_{b_0}$ and to trivialize the vector bundles as indicated in Section 7b). Then we will prove (8.1) in this situation.

c) Proof of Theorem 8.1.

By (7.10),

$$A'_{1/T,1} = T^{N_V} \frac{1}{T} D_T^M T^{-N_V}. \tag{8.2}$$

Therefore

$$\begin{aligned}(8.3)\quad \operatorname{Tr}_s\Big[g *_T^{-1} \Big(\frac{\partial}{\partial T} *_T\Big) \exp\Big(-\frac{u^2}{T^2} D_T^{M,2}\Big)\Big] \\ = \operatorname{Tr}_s\Big[g *_T^{-1} \Big(\frac{\partial}{\partial T} *_T\Big) \exp\big(-u^2 A'^2_{1/T,1}\big)\Big].\end{aligned}$$

We will use the notation of Section 7 with ε replaced by $1/T$, and T by 1. By (7.25), we see that as $T \to +\infty$

$$L^3_{1/T,1} \longrightarrow L^3_{0,1}. \tag{8.4}$$

Let $P^i_{\varepsilon,T,u}((Y,x),(Y',x'))$ $((Y,x),(Y',x') \in (T_{\mathbb{R}}B)_{b_0} \times X_{b_0})$ $(i = 1,2,3)$ be the smooth kernel associated to the operator $\exp(-u^2 L^i_{\varepsilon,T})$ calculated with respect to $dv_{(TB)_{b_0}}(Y')\, dv_{X_{b_0}}(x')/(2\pi)^{\dim M}$. For Y in $N_{B^g/B,\mathbb{R},b_0}$, $x \in X_{b_0}$, set

$$Q_{\varepsilon,u}(Y,x) = \operatorname{Tr}_s\big[g\big[M^3_{\varepsilon,1} P^3_{\varepsilon,1,u}\big(g^{-1}(Y,x),(Y,x)\big)\big]^{\max}\big]. \tag{8.5}$$

By (7.24), for $Y \in N_{B^g/B,\mathbb{R},b_0}$, $x \in X_{b_0}$, we have

$$\begin{aligned}(8.6)\quad \operatorname{Tr}_s\Big[g *_T^{-1} \Big(\frac{\partial}{\partial T} *_T\Big) P^1_{1/T,1,u}\big(g^{-1}(Y,x),(Y,x)\big)\Big] \\ = (-i)^{\dim B^g} T^{2\dim N_{B^g/B}} \frac{1}{T} Q_{1/T,u}(TY,x).\end{aligned}$$

By (8.6) and the argument of Section 7b), to calculate the asymptotics of (8.3) as $u \to 0$ uniformly in $T \geq 1$, we have to find the asymptotics as $u \to 0$ of

$$\int_{Y \in N_{B^g/B,\mathbb{R}}} \int_X Q_{1/T,u}(Y,x) \frac{dv_{X_{b_0}}(x)\, dv_{N_{B^g/B}}(Y)}{(2\pi)^{\dim M}}. \tag{8.7}$$

Let $d^X(x,x')$ be the distance function on $(X_{b_0}, h^{TX_{b_0}})$. Then

$$d\big((Y,x),(Y',x')\big) = \big(|Y-Y'|^2 + d^X(x,x')^2\big)^{1/2}$$

is a distance function on $(T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$.

Proposition 8.3. — *There exist* $c, C > 0, p, r \in \mathbb{N}$ *such that for any* $(Y,x),(Y',x') \in (T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$, $\varepsilon \in [0,1]$, $u \in]0,1]$,

$$\begin{aligned}(8.8)\quad \big|u^p P^3_{\varepsilon,1,u}\big((Y,x),(Y',x')\big)\big| \leq c\big(1+|Y|+|Y'|\big)^r \\ \times \exp\Big(-C\,\frac{|Y-Y'|^2+d^X(x,x')^2}{u^2}\Big).\end{aligned}$$

Proof. — At first, using the technique in [BerB, §9d)], where we replace the Sobolev norms [BerB, (9.49), (9.50)] by (7.32) and (7.33), the bounds in (8.8) with $C=0$ are obtained. To get the required $C>0$, we proceed as in the proof of Theorem 7.6.

Using finite propagation speed for the solution of hyperbolic equations for $\cos(s\sqrt{L^3_{\varepsilon,1}})$ [CP, §7.8], [T, §4.4], we find there is a fixed constant $c'>0$ such that for $\varepsilon \in [0,1]$, $u \in]0,1]$, $q \geq 1$,

$$\begin{aligned}(8.9)\quad P^3_{\varepsilon,1,u}\big((Y,x),(Y',x')\big) = \widetilde{K}_{q/u}(u^2 L^3_{\varepsilon,1})\big((Y,x),(Y',x')\big) \\ \text{if}\quad d\big((Y,x),(Y',x')\big) \geq c'q.\end{aligned}$$

By using the proof of Theorem 7.6, and [B5, Thm. 11.14], there is $C>0$, $c>0$, $p,r \in \mathbb{N}$ such that for $q \in \mathbb{N}^*$, $(Y,x),(Y',x') \in (T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$, $\varepsilon \in [0,1]$, $u \in]0,1]$,

$$\begin{aligned}(8.10)\quad \big|u^p \widetilde{K}_{q/u}(u^2 L^3_{\varepsilon,1})\big((Y,x),(Y',x')\big)\big| \\ \leq c\big(1+|Y|+|Y'|\big)^r \exp\Big(-\frac{Cq^2}{u^2}\Big).\end{aligned}$$

By (8.8) with $C=0$, (8.9) and (8.10), as (7.47), we have (8.8). □

Let $N_{X^g/X}$ be the normal bundle to X^g in X. We identify $N_{X^g/X}$ to the orthogonal bundle to TX^g in TX. Let $h^{N_{X^g/X}}$ be the metric on $N_{X^g/X}$ induced by $h^{TX}|_{X_{b_0}}$. Let $dv_{N_{X^g/X}}$ be the Riemannian volume form on $(N_{X^g/X,\mathbb{R}}, h^{N_{X^g/X}})$.

By (8.8), to calculate the asymptotics of (8.7) as $u \to 0$, we can localize near $\{0\} \times X^g_{b_0}$. We identify $\mathcal{U}_{\alpha_0}(\{0\} \times X^g_{b_0})$ to

$$\big\{(Y,x,X)\,;\ Y \in N_{B^g/B,\mathbb{R},b_0},\ x \in X^g,\ X \in N_{X^g/X,\mathbb{R}},\ |Y|,|X| \leq \alpha_0\big\}$$

by geodesic coordinates normal to $\{0\} \times X^g_{b_0}$ in $(T_{\mathbb{R}}B)_{b_0} \times X$.

For $Y \in (T_{\mathbb{R}}B)_{b_0}, x \in X^g$, $X \in N_{X^g/X,\mathbb{R}}, |Y|, |X| \leq \frac{1}{4}\alpha_0$, let $k'(Y,x,X)$ be defined by

$$\mathrm{d}v_X(Y,x,X) = k'(Y,x,X)\,\mathrm{d}v_{N_{X^g/X}}(X)\,\mathrm{d}v_{X^g}(x). \tag{8.11}$$

By standard results on heat kernel (*cf.* [BeGeV, Thm. 2.30]), we find there exist smooth functions $a'_{T,-n}(x), \dots, a'_{T,0}(x)$ $(x \in M^g)$ such that as $u \to 0$, for $x \in X^g_{b_0}$

$$\int_{\substack{X \in N_{X^g/X,\mathbb{R}}, |X| \leq \alpha_0/4 \\ Y \in N_{B^g/B,\mathbb{R}}, |Y| \leq \alpha_0/4}} Q_{1/T,u}\big(Y,(x,X)\big) k'(Y,x,X) \frac{\mathrm{d}v_{N_{X^g/X}}(X)\,\mathrm{d}v_{N_{B^g/B}}(Y)}{(2\pi)^{\dim M}} = \sum_{j=-n}^{0} a'_{T,j}(x) u^{2j} + O(u^2). \tag{8.12}$$

Also the $a'_{T,j}(x)$ only depend on the operator $L^3_{1/T,1}$ and its higher derivatives on x. By (8.4), $a'_{T,j}(x)$ is continuous on $T \in [1,+\infty]$.

By (7.12), (7.27), (8.4)–(8.8), (8.12), we know that there exist $a_{T,j}$ depending continuously on $T \in [1,+\infty]$ such that for any $u \in]0,1]$, $T \in [1,+\infty]$

$$\left|\mathrm{Tr}_s\Big[g *_T^{-1} \frac{\partial}{\partial T}(*_T) \exp\Big(-\frac{u^2}{T^2} D_T^{M,2}\Big)\Big] - \sum_{j=-\dim M}^{0} a_{T,j} u^{2j}\right| \leq \frac{cu^2}{T} \cdot \tag{8.13}$$

Set

$$\begin{cases} b_{-1,g} = \displaystyle\int_{M^g} \frac{\widetilde{\omega}^M}{2\pi}\, \mathrm{Td}_g(TM)\,\mathrm{ch}_g(\xi), \\ b_{0,g} = \displaystyle\int_{M^g} \frac{\partial}{\partial b}\Big[\mathrm{Td}_g\Big(\frac{-R_T^{TM}}{2i\pi} - b(h_T^{TM})^{-1}\frac{\partial h_T^{TM}}{\partial T}\Big)\Big]_{b=0} \mathrm{ch}_g(\xi, h^\xi). \end{cases} \tag{8.14}$$

By [B5, (2.44), (2.63)] which extends [BGS3, Thm. 1.22], for $T \geq 1$ fixed, as $u \to 0$

$$\mathrm{Tr}_s\Big[g *_T^{-1} \frac{\partial}{\partial T}(*_T) \exp(-u^2 D_T^{M,2})\Big] = \frac{2}{u^2}\frac{b_{-1,g}}{T^3} - b_{0,g} + O(u^2). \tag{8.15}$$

By comparing (8.13) and (8.15), we get

$$a_{T,j} = 0 \quad \text{if} \quad j < -1, \quad a_{T,-1} = \frac{2}{T} b_{-1,g}, \quad a_{T,0} = -b_{0,g}. \tag{8.16}$$

By (8.13) and (8.16), we get (8.1). $\square$

9. A proof of Theorem 4.12.

This section is organized as follows. In a), as in [BerB, §9], we reduce the problem to a local problem near B^g. In b), we summarize very briefly the content of [BerB, §9 c)]. In c), we establish key estimates on the kernel of $\widetilde{F}_\varepsilon(L^3_{\varepsilon,T})$. In d), we prove Theorem 4.12.

We use the same notation as in Sections 4 and 7.

a) Finite propagation speed and localization.

Proposition 9.1. — *There exists $C>0$, such that for $0<\varepsilon\leq 1$, $T\geq 1$*

$$\begin{aligned}(9.1)\quad \Big|\mathrm{Tr}_s\Big[g *^{-1}_{T/\varepsilon}\frac{\partial}{\partial T}(*_{T/\varepsilon})G_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big]\\ -\frac{2}{T}\Big(\sum_{j=0}^{\dim X}(-1)^j j\chi_g(R^j\pi_*\xi)-\dim X\chi_g(\xi)\Big)G_\varepsilon(0)\Big|\leq\frac{C}{T^2}.\end{aligned}$$

Proof. — For $v>0$, set

$$H_v(a)=\int_{-\infty}^{+\infty}\exp(i\,t\sqrt{2}\,a)\exp\Big(-\frac{t^2}{2v^2}\Big)g(t)\frac{dt}{v\sqrt{2\pi}}.$$

Clearly

$$G_v(a)=H_v\Big(\frac{a}{v}\Big).$$

By an analogue of the McKean Singer formula [MKS], we find that

$$(9.2)\qquad \mathrm{Tr}_s\big[gN_VH_\varepsilon(D^B)\big]=\sum_{j=0}^{\dim X}(-1)^j j\chi_g(R^j\pi_*\xi)H_\varepsilon(0).$$

Using (9.2) and proceeding as in [BerB, Prop. 9.1], we have (9.1). □

By (7.6) and (9.1), to establish Theorem 4.12, we only need to establish the following result.

THEOREM 9.2. — *If $\alpha > 0$ is small enough, there exist $\delta > 0$, $C > 0$, such that for $0 < \varepsilon \leq 1$, $T \geq 1$*

$$\begin{aligned}(9.3)\quad \Big|\mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big] \\ - \frac{2}{T}\Big(\sum_{j=0}^{\dim X}(-1)^j j\chi_g(R^j\pi_*\xi) - \dim X\chi_g(\xi)\Big)F_\varepsilon(0)\Big| \leq \frac{C}{T^{1+\delta}}.\end{aligned}$$

Proof. — The remainder of the section is devoted to the proof of Theorem 9.2. □

By (7.11), we deduce that

$$(9.4)\quad \mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big] = \mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) \widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})\Big].$$

Let $\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})(x,x')(x,x' \in M)$ be the smooth kernel associated to $\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})$ with respect to $dv_M(x')/(2\pi)^{\dim M}$. Using finite propagation speed, as in (7.9), it is clear that if $x \in M$, $\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})(x,\cdot)$ only depends on the restriction of $A'_{\varepsilon,T}$ to $\pi^{-1}(B^B(\pi x,\alpha))$.

As in Section 7, the proof of (9.3) is local near $\pi^{-1}(B^g)$.

b) The matrix structure of the operator $L^3_{\epsilon,T}$ as $T \to +\infty$.

We use the same trivializations and notation as in Section 7.

Also by using (7.19), (7.24), for $Y \in (N_{B^g/B,\mathbb{R}})_{b_0}$, we get

$$\begin{aligned}(9.5)\quad &\mathrm{Tr}_s\Big[g *_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) \widetilde{F}_\varepsilon(L^1_{\varepsilon,T})(g^{-1}(Y,x),(Y,x))\Big] \\ &= (-i)^{\dim B^g}\varepsilon^{-2\dim N_{B^g/B}}\mathrm{Tr}_s\Big[gM^3_{\varepsilon,T}\widetilde{F}_\varepsilon(L^3_{\varepsilon,T})\big(g^{-1}(\varepsilon^{-1}Y,x),(\varepsilon^{-1}Y,x)\big)\Big]^{\max}.\end{aligned}$$

Recall that the vector bundle K and the operators P, S_ε were defined in (7.13) and (7.16). Let $\mathbb{F}^0_\varepsilon$ be the vector space of square integrable sections of $\Lambda(T^*_{\mathbb{R}}B^g) \mathbin{\widehat{\otimes}} \Lambda(\overline{N}^*_{B^g/B}) \mathbin{\widehat{\otimes}} S_\varepsilon^{-1*}K$ over $(T_{\mathbb{R}}B)_{b_0}$. Then $\mathbb{F}^0_\varepsilon$ is a Hilbert subspace of $\mathbb{E}^0$. Let $\mathbb{F}^{0,\perp}_\varepsilon$ be its orthogonal complement in $\mathbb{E}^0$. Let p_ε be the orthogonal projection operator from $\mathbb{E}^0$ on $\mathbb{F}^0_\varepsilon$, set $p^\perp_\varepsilon = 1 - p_\varepsilon$. Then if $s \in \mathbb{E}^0$,

$$(9.6)\qquad p_\varepsilon s(Y) = P_{\varepsilon Y}s(Y,\cdot) \quad \text{for} \quad Y \in T_{\mathbb{R}}B.$$

Put

$$
\left\{
\begin{aligned}
&E_{\varepsilon,T}=p_\varepsilon L^3_{\varepsilon,T}p_\varepsilon, \quad &&F_{\varepsilon,T}=p_\varepsilon L^3_{\varepsilon,T}p^\perp_\varepsilon,\\
&G_{\varepsilon,T}=p^\perp_\varepsilon L^3_{\varepsilon,T}p_\varepsilon, \quad &&H_{\varepsilon,T}=p^\perp_\varepsilon L^3_{\varepsilon,T}p^\perp_\varepsilon.
\end{aligned}
\right.
\tag{9.7}
$$

Then we write $L^3_{\varepsilon,T}$ in matrix form with respect to the splitting $\mathbb{E}^0=\mathbb{F}^0_\varepsilon\oplus\mathbb{F}^{0,\perp}_\varepsilon$,

$$
L^3_{\varepsilon,T}=\begin{bmatrix} E_{\varepsilon,T} & F_{\varepsilon,T}\\ G_{\varepsilon,T} & H_{\varepsilon,T}\end{bmatrix}.
\tag{9.8}
$$

Recall that L^ξ, R^{TX} are the curvatures of (ξ,∇^ξ), (TX,∇^{TX}), and that the $(3,0)$-tensor $\langle S(\cdot)\cdot,\cdot\rangle$ is defined in Section 2b). In the sequel, $[\ ,\ ]_+$ denotes an anticommutator.

Theorem 9.3. — *There exist operators $E_\varepsilon, F_\varepsilon, G_\varepsilon, H_\varepsilon$ such that as $T\to+\infty$,*

$$
\left\{
\begin{aligned}
&E_{\varepsilon,T}=E_\varepsilon+O\Big(\frac{1}{T}\Big), \quad &&F_{\varepsilon,T}=TF_\varepsilon+O(1),\\
&G_{\varepsilon,T}=TG_\varepsilon+O(1), \quad &&H_{\varepsilon,T}=T^2H_\varepsilon+O(T).
\end{aligned}
\right.
\tag{9.9}
$$

Set

$$
\begin{aligned}
Q_\varepsilon=\varphi^2(\varepsilon|Y|)\Big\{-\frac{1}{2}\Big[&\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{e_i},\\
&\sum_{\alpha=1}^{2m'}\big\langle S(e_i)f^H_\alpha,e_j\big\rangle_{h^{TX}}\Big(f^\alpha\wedge-\frac{\varepsilon^2}{2}i_{f_\alpha}\Big)\frac{c(e_j)}{\sqrt{2}}\\
&+\frac{\varepsilon}{2}\sum_{\alpha=2m'+1}^{2m}\big\langle S(e_i)f^H_\alpha,e_j\big\rangle_{h^{TX}}c(f_\alpha)c(e_j)\Big]_+\\
+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{2m'}&\Big(f^\alpha\wedge-\frac{\varepsilon^2}{2}i_{f_\alpha}\Big)c(e_j)\Big(L^\xi+\frac{1}{2}\mathrm{Tr}[R^{TX}]\Big)(f_\alpha,e_i)\\
+\sum_{\alpha=2m'+1}^{2m}&\frac{\varepsilon}{2}c(f^H_\alpha)c(e_j)\Big(L^\xi+\frac{1}{2}\mathrm{Tr}[R^{TX}]\Big)(f_\alpha,e_i)\Big\}.
\end{aligned}
\tag{9.10}
$$

Then $Q_\varepsilon(\mathbb{F}^0_\varepsilon)\subset\mathbb{F}^{0,\perp}_\varepsilon$, and

$$
\left\{
\begin{aligned}
&F_\varepsilon=p_\varepsilon Q_\varepsilon p^\perp_\varepsilon, \qquad G_\varepsilon=p^\perp_\varepsilon Q_\varepsilon p_\varepsilon,\\
&H_\varepsilon=p^\perp_\varepsilon\big(\varphi^2(\varepsilon|Y|)D^{X,2}_{\varepsilon Y}+\big(1-\varphi^2(\varepsilon|Y|)\big)D^{X,2}_{b_0}\big)p^\perp_\varepsilon.
\end{aligned}
\right.
\tag{9.11}
$$

Proof. — For a fixed $\varepsilon > 0$, the analysis of the matrix structure of $L^3_{\varepsilon,T}$ as $T \to +\infty$ is the same as in [BerB, §9c)]. Of course, the rescaling on the Clifford variables which depends on $\varepsilon > 0$, is different, but this does not introduce any extra difficulty.

So Theorem 9.3 holds for essentially the same reasons as in [BerB, Theorem 9.3]. Especially, by [BerB, (7.33), (9.37)], we get (9.10). □

c) Uniform bounds on the kernel of $\widetilde{F}_\epsilon(L^3_{\epsilon,T})$.

We now establish an extension of [BerB, Thm. 9.6].

THEOREM 9.4. — *There exists* $C > 0$, *for which if* $k \in \mathbb{N}$, *there exist* $C' > 0$, $r \in \mathbb{N}$ *such that if* $|\alpha|, |\alpha'| \leq k$, $\varepsilon \in]0,1]$, $T \geq 1$, $(Y,x),(Y',x') \in (T_{\mathbb{R}}B)_{b_0} \times X_{b_0}$,

$$\tag{9.12} \begin{aligned} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^\alpha \partial Y'^{\alpha'}} \widetilde{F}_\varepsilon(L^3_{\varepsilon,T})\big((Y,x),(Y',x')\big)\Big| \\ \leq C'\big(1+|Y|+|Y'|\big)^r \exp\big(-C|Y-Y'|^2\big). \end{aligned}$$

Proof. — Recall that $\langle\ \rangle_{\varepsilon,0}$ is the Hermitian product on $\mathbb{E}^0$ defined by (7.32). If $s \in \mathbb{E}^1$, put

$$\tag{9.13} \begin{aligned} |s|^2_{\varepsilon,T,1} = T^2|P^\perp_{\varepsilon Y}s|^2_{\varepsilon,0} + |P_{\varepsilon Y}s|^2_{\varepsilon,0} \\ + \sum_\alpha |\nabla_{f_\alpha}s|^2_{\varepsilon,0} + T^2\sum_i |\nabla_{e_i}P^\perp_{\varepsilon Y}s|^2_{\varepsilon,0}. \end{aligned}$$

The bounds in (9.12) with $C = 0$ are easily obtained by proceeding as in [BerB,Thm. 9.6], where we replace the Sobolev norms [BerB, (9.49), (9.50)] by (7.32) and (9.13). To get the required $C > 0$, we proceed as in the proof of Theorem 7.6 where we use the Sobolev norms (7.32) and (9.13). □

d) Proof of Theorem 9.2.

Let $\mathbb{F}_\varepsilon$ be the vector space of smooth sections of $\Lambda(T^*_{\mathbb{R}}B^g)\widehat{\otimes}\Lambda(\overline{N}^*_{B^g/B})\widehat{\otimes}S^{-1*}_\varepsilon K$ over $(T_{\mathbb{R}}B)_{b_0}$. Let Ξ_ε be the operator from $\mathbb{F}_\varepsilon$ to itself

$$\tag{9.14} \Xi_\varepsilon = E_\varepsilon - F_\varepsilon H^{-1}_\varepsilon G_\varepsilon.$$

One verifies easily that Ξ_ε is an elliptic second order differential operator acting on $\mathbb{F}_\varepsilon$.

The operator $(\varepsilon D^B)^2$ acts on smooth sections of $\Lambda(T^{*(0,1)}B)\widehat{\otimes}\operatorname{Ker} D^X$. Therefore by proceeding as before, *i.e.* by rescaling the coordinate Y and the Clifford variables $c(f_\beta)(1 \leq \beta \leq 2m')$, we construct from $(\varepsilon D^B)^2$ an operator Σ^3_ε, which acts on smooth sections of $\Lambda(T^*_{\mathbb{R}}B^g)\widehat{\otimes}\Lambda(\overline{N}^*_{B^g/B})$ $\widehat{\otimes}\, S^{-1*}_\varepsilon K$ over $B(0,2\alpha/\varepsilon)$. Then as [BerB, Prop. 9.9], we have

Proposition 9.5. — *Over $B(0,\alpha/\varepsilon)$, one has the identity*

$$\Xi_\varepsilon = \Sigma^3_\varepsilon. \tag{9.15}$$

Let $\widetilde{F}_\varepsilon(\Xi_\varepsilon)(Y,Y'), \widetilde{F}_\varepsilon(\Sigma^3_\varepsilon)(Y,Y')(Y,Y' \in (T_{\mathbb{R}}B)_{b_0})$ be the smooth kernels associated to the operator $\widetilde{F}_\varepsilon(\Xi_\varepsilon)$, $\widetilde{F}_\varepsilon(\Sigma^3_\varepsilon)$ with respect to $dv_{T_{\mathbb{R}}B}(Y')/(2\pi)^{\dim B}$. Using (9.15) and finite propagation speed, it is clear that for $|Y|,|Y'| \leq \alpha/4\varepsilon$,

$$\widetilde{F}_\varepsilon(\Xi_\varepsilon)(Y,Y') = \widetilde{F}_\varepsilon(\Sigma^3_\varepsilon)(Y,Y'). \tag{9.16}$$

Here, the minor difference with [BerB] is that here only the Clifford variables $c(f_\ell)$ $(1 \leq \ell \leq 2\dim B^g)$ are rescaled, while in [BerB], the Clifford variables $c(f_\ell)$ $(1 \leq \ell \leq 2\dim B)$ were rescaled. Because our Clifford rescaling introduces fewer diverging terms than in [BerB, §9], so we have the following analogue of [BerB, Thm. 9.8]: There exists $C > 0$ such that for $0 \leq \varepsilon \leq 1$, $T \geq 1$,

$$\left\|\widetilde{F}_\varepsilon(L^3_{\varepsilon,T}) - P_{\varepsilon Y}\widetilde{F}_\varepsilon(\Xi_\varepsilon)P_{\varepsilon Y}\right\|^{0,0}_{\varepsilon,0} \leq \frac{C}{\sqrt{T}}. \tag{9.17}$$

Now by using (7.27), (9.5), (9.12), (9.16), (9.17), and by proceeding as in [BerB, §9 g)] and [B5, §13 j)], we obtain Theorem 9.2. □

Acknowledgements. — I learned subjects from Professor Jean-Michel Bismut. I am very much indebted to him for very helpful discussions, suggestions and for the time he spent in teaching me. Without his help, this paper may never have appeared in this form. I would also like to thank K. Köhler for discussion on his papers. I would also like thank ICTP, Trieste, especially Professor M.S. Narasimhan for hospitality. Finally I thank the referee for helpful comments.

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Manuscrit reçu le 17 juin 1999,
révisé le 24 novembre 1999,
accepté le 14 février 2000.

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