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From Local Index Theory to Bergman Kernel: A Heat Kernel Approach

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Dedicated to Gang Tian for his 60*th birthday.*

Abstract. The aim of this note is to explain a uniform approach of three different topics: Atiyah–Singer index theorem, holomorphic Morse inequalities and asymptotic expansion of Bergman kernel, by using heat kernels.

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0. Introduction

In this note, we explain how to give a uniform approach of three different topics: Atiyah–Singer index theorem, holomorphic Morse inequalities and asymptotic expansion of Bergman kernel, by using heat kernels.

Roughly the Atiyah–Singer index theorem announced in 1963, as one of the most important theorems in mathematics of 20 century, computes the index of an elliptic operator by using the characteristic classes, i.e., the topological way. Its heat kernel approach (as a solution of the McKean–Singer conjecture or as the local index theorem) was developed by Gilkey in his thesis in 1973 and also by Atiyah–Bott–Patodi, which needs to use Weyl's invariant theory and computes infinite examples to detect the final formula. In the 1980s, influenced by the supersymmetry in physics, Bismut and Getzler independently developed direct heat kernel proofs of the Atiyah–Singer index theorem. In modern index theory, the local index techniques plays a central role which allows us to study the more refined spectral invariants such as the analytic torsion and the eta invariant.

In complex geometry, the Atiyah–Singer index theorem reduces to the classical Riemann–Roch–Hirzebruch theorem, which computes the alternating sum of dimensions of the Dolbeault cohomology groups of a holomorphic vector bundle.

The holomorphic Morse inequalities give an asymptotic estimate of the dimension of each Dolbeault cohomology group of a pth tensor power of a line bundle when p goes to infinity. This was first established by Demailly in 1985 [7] answering a question of Siu after Siu's solution of Grauert–Riemenschneider conjecture, and Bismut [2] gave a heat kernel approach. If the line bundle is positive, then by the Kodaira vanishing theorem, for large p the associated Dolbeault cohomology group of positive degree is zero and the dimension of its zero degree part, i.e., the space of holomorphic sections of its pth tensor power, is given by the Riemann–Roch–Hirzebruch theorem. Its analytic refinement is the smooth kernel of the orthogonal projection from the space of smooth sections onto the space of holomorphic sections: the Bergman kernel. In his thesis [16] in 1990, Tian initiated the study of the asymptotic of Bergman kernels. Since then, it is a very active research direction.

In this note, we explain a uniform approach of the above three topics by using heat kernels, which is inspired a lot from the analytic localization techniques of Bismut–Lebeau in local index theory. The basic references of this note are [1, Chap. 4] on the local index theorem, and [6], [11, §1.6, §4.1], [10], where the readers can also find a complete list of references. In particular, based on our contributions with Dai, Liu and Marinescu, [11] gives a comprehensive study on holomorphic Morse inequalities and Bergman kernels and their applications. To keep this note in a reasonable size, we omit many technical details, and hope that this note can be served as an introduction of the subject and motivation to read the book [11] and recent developments.

This note is organized as follows: In Section 1, we explain the Atiyah–Singer index theorem and the basic ideas on its local version: the local index theorem. In Sections 2, 3, we show how to apply the ideas from the local index theory to give a heat kernel approach of the holomorphic Morse inequalities and Berman kernels.

This note is based on the three lectures I gave in January 2018 in the workshop 'International workshop on differential geometry' at Sydney in celebration of Professor Gang Tian's 60th birthday.

Notations: we denote by dim or $\dim_{\mathbb{C}}$ the complex dimension of a complex vector space. Denote also $\dim_{\mathbb{R}}$ the real dimension of a space. $\mathrm{supp}(f)$ means the support of a function f.

1. Local index theorem

In this section, we review briefly the Chern–Weil theory, the Atiyah–Singer index theorem for Dirac operators and the heat kernel proof of the local index theorem.

1.1. Chern–Weil Theory

Let X be a smooth manifold of dimension n. Let TX be its tangent bundle and T^*X its cotangent bundle. Let $\Omega^k(X) = \mathscr{C}^\infty(X, \Lambda^k(T^*X))$ be the space of smooth k-forms on X and $\Omega^\bullet(X) = \oplus_k \Omega^k(X)$, and $d : \Omega^k(X) \to \Omega^{k+1}(X)$ be the exterior differential operator.

Definition 1.1. Let E be a smooth manifold, and let $\pi : E \to X$ be a smooth map. Then E is called a complex vector bundle over X if there exist a covering $\{U_i\}_{i=1}^l$ of

X and a family of diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^m, \phi_i(v) = (\pi(v), \psi_i(v))$ such that if $U_i \cap U_j \neq \emptyset$, then for $x \in U_i \cap U_j$, $\psi_{ji}(x, \cdot) \in \mathrm{GL}(m, \mathbb{C})$, i.e., an invertible $\mathbb{C}$-linear map on $\mathbb{C}^m$, and is smooth on x, where $\psi_{ji}(x, \cdot)$ is given by

$$\begin{aligned} \phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^m \to (U_i \cap U_j) \times \mathbb{C}^m, \\ \phi_j \circ \phi_i^{-1}(x, w) = (x, \psi_{ji}(x, w)). \end{aligned} \tag{1.1}$$

That is, if we write $\psi_{ji}(x, w) = \psi_{ji}(x)w$, then $\psi_{ji}(x) \in \mathrm{GL}(m, \mathbb{C})$. For $x \in X$, $E_x := \pi^{-1}(x)$ is called the fiber of E at x. The integer m is called the rank of E and is denoted by $\mathrm{rk}(E)$. If $\mathrm{rk}(E) = 1$, then E is called a line bundle.

Denote by $\mathscr{C}^\infty(X, E)$ the space of smooth sections of E on X, i.e., the space of smooth maps from X to E such that its composition with π is the identity map on X. Denote by $\Omega^\bullet(X, E) := \mathscr{C}^\infty(X, \Lambda(T^*X) \otimes E)$ the space of smooth forms on X with values in E. We denote by $\mathscr{C}^\infty(X, \mathbb{C})$ the space of smooth $\mathbb{C}$-valued functions on X.

Definition 1.2. A map $\nabla^E : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, T^*X \otimes E)$ is called a connection if

1) ∇^E is $\mathbb{C}$-linear,
2) For any $s \in \mathscr{C}^\infty(X, E)$ and $\varphi \in \mathscr{C}^\infty(X, \mathbb{C})$,

$$\nabla^E(\varphi s) = d\varphi \otimes s + \varphi \nabla^E s. \tag{1.2}$$

A Hermitian metric h^E on E is a family of Hermitian products h^{E_x} on E_x which is smooth on $x \in X$. In this case, we call (E, h^E) a Hermitian vector bundle and as usual, we also denote h^E by $\langle\ \ \rangle$. A connection ∇^E is a Hermitian connection on (E, h^E) if for any $s_1, s_2 \in \mathscr{C}^\infty(X, E)$,

$$\left\langle \nabla^E s_1, s_2 \right\rangle + \left\langle s_1, \nabla^E s_2 \right\rangle = d \left\langle s_1, s_2 \right\rangle. \tag{1.3}$$

Let $\nabla^E : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, T^*X \otimes E)$ be a connection on E.

Definition 1.3. Let $\nabla^E : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ be the operator induced by ∇^E such that for any $\alpha \in \Omega^k(X)$ and $s \in \mathscr{C}^\infty(X, E)$,

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge \nabla^E s. \tag{1.4}$$

The operator $(\nabla^E)^2$ defines a homomorphism $R^E := (\nabla^E)^2 : E \to \Lambda^2(T^*X) \otimes E$. The $R^E \in \Omega^2(X, \mathrm{End}(E))$ is called the curvature operator of ∇^E.

Example. For $E = \mathbb{C}$, the exterior differential $d : \Omega^k(X, \mathbb{C}) \to \Omega^{k+1}(X, \mathbb{C})$ is a connection on the trivial line bundle $\mathbb{C}$ and $d^2 = 0$. The de Rham cohomology of X is defined by

$$H^k(X, \mathbb{C}) := \frac{\mathrm{Ker}\left(d|_{\Omega^k(X,\mathbb{C})}\right)}{\mathrm{Im}\left(d|_{\Omega^{k-1}(X,\mathbb{C})}\right)}, \quad H^\bullet(X, \mathbb{C}) = \bigoplus_{k=0}^n H^k(X, \mathbb{C}). \tag{1.5}$$

Theorem 1.4 (Chern–Weil). *For $f \in \mathbb{R}[z]$, i.e., f is a real polynomial on z, set*

$$F(R^E) = \mathrm{Tr}\left[f\left(\frac{i}{2\pi} R^E\right)\right] \in \Omega^{2\bullet}(X, \mathbb{C}). \tag{1.6}$$

Then $F(R^E)$ is closed. Moreover, its cohomology class $[F(R^E)] \in H^{2\bullet}(X,\mathbb{R})$ and it does not depend on the choice of the connection ∇^E.

Proof. By the definition of R^E, we have the Bianchi identity:

$$\big[\nabla^E, R^E\big] = \big[\nabla^E, (\nabla^E)^2\big] = 0. \tag{1.7}$$

Then

$$dF(R^E) = d\,\mathrm{Tr}\left[f\left(\frac{i}{2\pi}R^E\right)\right] = \mathrm{Tr}\left[\left[\nabla^E, f\left(\frac{i}{2\pi}R^E\right)\right]\right] = 0. \tag{1.8}$$

That is, $F(R^E)$ is closed.

Denote by $\pi : X\times\mathbb{R} \to X$ the natural projection. Let ∇_0^E, ∇_1^E be two connections on E. Then

$$\nabla^{\pi^* E} = (1-t)\nabla_0^E + t\nabla_1^E + dt\wedge\frac{\partial}{\partial t} \tag{1.9}$$

is a connection on π^*E, the pullback of E over $X\times\mathbb{R}$. Set $\nabla_t^E = (1-t)\nabla_0^E + t\nabla_1^E$. Its curvature $R_t^E = (\nabla_t^E)^2 \in \Omega^2\big(X, \mathrm{End}(E)\big)$ and $R^{\pi^*E} = (\nabla^{\pi^*E})^2 = R_t^E + dt\wedge\cdot$, thus there exists $Q_t \in \Omega^\bullet(X)$ such that

$$F(R^{\pi^*E}) = F(R_t^E) + dt\wedge Q_t. \tag{1.10}$$

Applying (1.8) for π^*E, we get $d^{X\times\mathbb{R}}F(R^{\pi^*E}) = 0$. By (1.10) and comparing the coefficient of dt in $d^{X\times\mathbb{R}}F(R^{\pi^*E}) = 0$, we obtain

$$\frac{\partial}{\partial t}F(R_t^E) = dQ_t. \tag{1.11}$$

Thus

$$F(R_1^E) - F(R_0^E) = d\int_0^1 Q_t dt, \tag{1.12}$$

which implies

$$\big[F(R_1^E)\big] = \big[F(R_0^E)\big] \in H^{2\bullet}(X,\mathbb{C}). \tag{1.13}$$

Finally, we can choose a Hermitian metric h^E on E and a Hermitian connection ∇^E on (E,h^E), then $\frac{i}{2\pi}R^E$ is self-adjoint with respect to h^E, thus $\mathrm{Tr}\Big[f\big(\frac{i}{2\pi}R^E\big)\Big]$ is a real differential form, which implies that $[F(R^E)] \in H^{2\bullet}(X,\mathbb{R})$. The proof of Theorem 1.4 is completed. □

Example. 1). For $f(z) = e^z$, the Chern character form of (E,∇^E) is

$$\mathrm{ch}(E,\nabla^E) = \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi}R^E\right)\right]. \tag{1.14}$$

The Chern character of E is

$$\mathrm{ch}(E) := \big[\,\mathrm{ch}(E,\nabla^E)\big] \in H^{2\bullet}(X,\mathbb{R}). \tag{1.15}$$

The first Chern form of (E,∇^E) is $c_1(E,\nabla^E) = \mathrm{Tr}\left[\frac{i}{2\pi}R^E\right]$. Its cohomology class is the first Chern class $c_1(E)$.

2). For $f(z) = \log(\frac{z}{1-e^{-z}})$, the Todd form of (E, ∇^E) is

$$\mathrm{Td}(E, \nabla^E) = \det\left[\frac{\frac{i}{2\pi}R^E}{1-e^{-\frac{i}{2\pi}R^E}}\right] = \exp\left\{\mathrm{Tr}\left[\log\left(\frac{\frac{i}{2\pi}R^E}{1-e^{-\frac{i}{2\pi}R^E}}\right)\right]\right\}. \tag{1.16}$$

The Todd class of E is

$$\mathrm{Td}(E) = \left[\,\mathrm{Td}(E, \nabla^E)\right] \in H^{2\bullet}(X, \mathbb{R}). \tag{1.17}$$

3). Let g^{TX} be a Riemannian metric on TX and ∇^{TX} be the Levi-Civita connection on (X, g^{TX}). The $\widehat{A}$-form of (TX, ∇^{TX}) is

$$\widehat{A}(TX, \nabla^{TX}) = \det{}^{1/2}\left[\frac{\frac{i}{4\pi}R^{TX}}{\sinh\left(\frac{i}{4\pi}R^{TX}\right)}\right]. \tag{1.18}$$

The $\widehat{A}$-genus of TX is

$$\widehat{A}(TX) = \left[\widehat{A}(TX, \nabla^{TX})\right] \in H^{4\bullet}(X, \mathbb{R}). \tag{1.19}$$

1.2. Atiyah–Singer index theorem

Let X be an n-dimensional compact spin manifold with n even (in particular, X is orientable) and g^{TX} be a Riemannian metric on X. Let $S(TX)$ be the spinor bundle of (TX, g^{TX}). Then $S(TX)$ is a $\mathbb{Z}_2$-graded vector bundle on X:

$$S(TX) = S^+(TX) \oplus S^-(TX). \tag{1.20}$$

For $U \in TX$, let $c(U) \in \mathrm{End}(S(TX))$ be the Clifford action of U on $S(TX)$. We will not explain in detail the Clifford action, but only recall that $c(U)$ exchange $S^+(TX)$ and $S^-(TX)$ and $c(U)^2 = -|U|^2_{g^{TX}}$.

The Levi-Civita connection ∇^{TX} on (X, g^{TX}) induces canonically the Clifford connection $\nabla^{S(TX)} = \nabla^{S^+(TX)} \oplus \nabla^{S^-(TX)}$ on $S(TX)$, i.e., the connection preserves the splitting (1.20) and compatible with the Clifford action:

$$\left[\nabla^{S(TX)}_V, c(U)\right] = c\left(\nabla^{TX}_V U\right) \quad \text{for } U, V \in \mathscr{C}^\infty(X, TX). \tag{1.21}$$

Let (E, h^E) be a Hermitian vector bundle on X. Let ∇^E be a Hermitian connection on (E, h^E). Denote by $\nabla^{S(TX)\otimes E}$ the connection on $S(TX)\otimes E$ induced by $\nabla^{S(TX)}$ and ∇^E.

Definition 1.5. The Dirac operator is defined by

$$D = \sum_{j=1}^n c(e_j)\nabla^{S(TX)\otimes E}_{e_j} : \mathscr{C}^\infty(X, S^\pm(TX)\otimes E) \to \mathscr{C}^\infty(X, S^\mp(TX)\otimes E), \tag{1.22}$$

where $\{e_j\}$ is an orthonormal frame of (TX, g^{TX}).

Let dv_X be the Riemannian volume form on (X, g^{TX}). For

$$s_1, s_2 \in \mathscr{C}^\infty(X, S(TX) \otimes E),$$

their Hermitian product is defined as

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle (x) dv_X(x).$$

The Dirac operator D is a first-order self-adjoint elliptic differential operator. As X is compact, D is a Fredholm operator, in particular, its kernel $\mathrm{Ker}(D)$ is a finite-dimensional complex vector space.

Set

$$D_\pm = D|_{\mathscr{C}^\infty(X, S^\pm(TX) \otimes E)}.$$

Then under the decomposition (1.20),

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}, \tag{1.23}$$

and D^2 preserves $\mathscr{C}^\infty(X, S^\pm(TX) \otimes E)$. As D is self-adjoint, $\mathrm{CoKer}(D_+)$, the cokernel of D_+, is $\mathrm{Ker}(D_-)$. Thus the index of D_+ is defined by

$$\mathrm{Ind}(D_+) = \dim \mathrm{Ker}(D_+) - \dim \mathrm{Ker}(D_-) \in \mathbb{Z}. \tag{1.24}$$

Theorem 1.6 (Atiyah–Singer index theorem (1963)).

$$\mathrm{Ind}(D_+) = \int_X \widehat{A}(TX)\mathrm{ch}(E). \tag{1.25}$$

1.3. Heat kernel and McKean–Singer formula

The heat kernel $e^{-tD^2}(x, y)$ is the smooth kernel of the heat operator e^{-tD^2} with respect to the Riemannian volume form $dv_X(y)$. The following result is well known.

Theorem 1.7. *For any $t > 0$ and $x, y \in X$,*

$$e^{-tD^2}(x, y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \psi_j(x) \otimes \psi_j(y)^*, \tag{1.26}$$

where ψ_j is a unit eigenfunction of D^2 corresponding to the eigenvalue λ_j with $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, $\lambda_j \to +\infty$ such that $\{\psi_j\}_j$ form a complete orthonormal basis of the space of L^2-integrable sections, $L^2(X, S(TX)\otimes E)$, and $\psi_j(y)^* \in (S(TX)\otimes E)^*_y$ is the metric dual of $\psi_j(y)$, i.e.,

$$\psi_j(y)^*(v) = \langle v, \psi_j(y) \rangle \quad \text{for } v \in (S(TX) \otimes E)_y. \tag{1.27}$$

Theorem 1.8 (McKean–Singer (1967)). *For any $t > 0$, we have*

$$\mathrm{Ind}(D_+) = \mathrm{Tr}_s \left[e^{-tD^2} \right] = \int_X \mathrm{Tr}_s \left[e^{-tD^2}(x, x) \right] dv_X(x), \tag{1.28}$$

where the supertrace Tr_s is given by

$$\mathrm{Tr}_s = \mathrm{Tr}\left|_{\mathscr{C}^\infty(X, S^+(TX) \otimes E)} - \mathrm{Tr}\right|_{\mathscr{C}^\infty(X, S^-(TX) \otimes E)}. \tag{1.29}$$

Proof. By (1.26),

$$\lim_{t\to+\infty} e^{-tD^2}(x,x) = \sum_{\lambda_j=0} \psi_j(x) \otimes \psi_j(x)^*,$$

which implies

$$\lim_{t\to+\infty} \operatorname{Tr}_s\left[e^{-tD^2}\right] = \operatorname{Ind}(D_+). \tag{1.30}$$

Then it suffices to prove that $\operatorname{Tr}_s\left[e^{-tD^2}\right]$ is independent of $t > 0$. In fact,

$$\begin{aligned}\frac{\partial}{\partial t}\operatorname{Tr}_s\left[e^{-tD^2}\right] &= -\operatorname{Tr}_s\left[D^2 e^{-tD^2}\right]\\ &= -\frac{1}{2}\operatorname{Tr}_s\left[\left[De^{-tD^2/2}, De^{-tD^2/2}\right]\right] = 0.\end{aligned} \tag{1.31}$$

Here for a $\mathbb{Z}_2$-graded vector space $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, the $\mathbb{Z}_2$-grading on $\operatorname{End}(\mathcal{E})$ is given by

$$\begin{aligned}\operatorname{End}(\mathcal{E})^+ &= \operatorname{Hom}(\mathcal{E}^+,\mathcal{E}^+) \oplus \operatorname{Hom}(\mathcal{E}^-,\mathcal{E}^-),\\ \operatorname{End}(\mathcal{E})^- &= \operatorname{Hom}(\mathcal{E}^+,\mathcal{E}^-) \oplus \operatorname{Hom}(\mathcal{E}^-,\mathcal{E}^+),\end{aligned} \tag{1.32}$$

and $[\cdot,\cdot]$ is the supercommutator of $\operatorname{End}(\mathcal{E})$, i.e.,

$$[A,B] = \begin{cases} AB - BA & \text{if } A \text{ or } B \in \operatorname{End}(\mathcal{E})^+,\\ AB + BA & \text{if } A, B \in \operatorname{End}(\mathcal{E})^-.\end{cases} \tag{1.33}$$

Then we verify easily as for matrices that $\operatorname{Tr}_s\big[[A,B]\big] = 0$. This completes the proof of Theorem 1.8. □

When $t \to 0$, classically the following asymptotic expansion of the heat kernel holds for any $k \in \mathbb{N}$:

$$e^{-tD^2}(x,x) = \sum_{j=-l}^{k} a_j(x) t^j + \mathscr{O}(t^{k+1}) \quad \text{uniformly on } X, \tag{1.34}$$

where $l = n/2$ and the coefficients $a_j(x)$ depend only on the restriction of D^2 on $B^X(x,\varepsilon)$, the ball in X of center x and radius ε for any $\varepsilon > 0$. Then the McKean–Singer formula implies that

$$\int_X \operatorname{Tr}_s\left[a_j(x)\right] dv_X(x) = \begin{cases} 0 & \text{for } j < 0,\\ \operatorname{Ind}(D_+) & \text{for } j = 0.\end{cases} \tag{1.35}$$

McKean–Singer conjectured that in fact a pointwise version of (1.35) holds, which they called the "miraculous cancellation". The solution of this conjecture is called the local index theorem stated as follows. For $\alpha \in \Omega(X)$, we denote $\alpha^{\max}$ the degree n component of the differential form α.

Theorem 1.9 (Local index theorem).

$$\operatorname{Tr}_s\left[a_j(x)\right]dv_X(x) = \begin{cases} 0 & \textit{for } j<0, \\ \left\{\widehat{A}(TX,\nabla^{TX})\operatorname{ch}(E,\nabla^E)\right\}_x^{\max} & \textit{for } j=0. \end{cases} \tag{1.36}$$

Equivalently,

$$\lim_{t\to 0}\operatorname{Tr}_s\left[e^{-tD^2}(x,x)\right]dv_X(x) = \left\{\widehat{A}(TX,\nabla^{TX})\operatorname{ch}(E,\nabla^E)\right\}_x^{\max}. \tag{1.37}$$

By Theorems 1.8 and 1.9, we get the Atiyah–Singer index theorem, Theorem 1.6.

Remark 1.10. Using the Bott periodicity theorem in K-theory, we obtain the index theorem for any elliptic operator P on X:

$$\operatorname{Ind}(P) = \int_{T^*X}\widehat{A}(TX)^2\operatorname{ch}\big(\sigma(P)\big). \tag{1.38}$$

Here $\sigma(P)$ is the principal symbol of P, which can be understood as an element in $K(T^*X)$, the K-group of T^*X.

1.4. Proof of the local index theorem

The proof presented here consists of Bismut–Lebeau's analytic localization techniques [3, §11] and Getzler rescaling trick. We need to compute the limit as $t\to 0$,

$$\lim_{t\to 0}\operatorname{Tr}_s\left[e^{-tD^2}(x,x)\right]dv_X(x).$$

Step 1. The asymptotic of $e^{-tD^2}(x,x)$ is local, i.e., only depends on the restriction of D^2 on any neighborhood of x.

Recall

$$e^{-a^2/2} = \int_{-\infty}^{+\infty}\cos(va)e^{-v^2/2}\frac{dv}{\sqrt{2\pi}}, \quad \text{for any } a\in\mathbb{R}. \tag{1.39}$$

Thus for the heat operator e^{-tD^2}, we have

$$e^{-tD^2} = \int_{-\infty}^{+\infty}\cos\left(v\sqrt{2t}D\right)e^{-v^2/2}\frac{dv}{\sqrt{2\pi}}. \tag{1.40}$$

We can formally verify (1.40) from (1.39) as D^2 is an infinite-dimensional diagonal matrix. Rigorously, the wave operator $\cos\big(v\sqrt{2t}D\big)$ is given by

$$\cos\big(vD\big)(x,y) = \sum\nolimits_j\cos(v\sqrt{\lambda_j})\psi_j(x)\otimes\psi_j(y)^*. \tag{1.41}$$

In fact, $w_t(x)=\cos\big(tD\big)(x,\cdot)$ is the fundamental solution of the equation

$$\left(\frac{\partial^2}{\partial t^2}+D^2\right)w_t(x)=0 \tag{1.42}$$

with the initial conditions

$$\lim_{t\to 0}w_t\phi=\phi, \quad \text{for any } \phi\in L^2(X,S(TX)\otimes E). \tag{1.43}$$

Using the energy estimates, we obtain the property of the finite propagation speed for the wave operator $\cos\big(tD\big)$:

$$\operatorname{supp}\cos\big(tD\big)(x,\cdot) \subset B^X(x,t) \tag{1.44}$$

and $\cos\big(tD\big)(x,\cdot)$ depends only on $D^2|_{B^X(x,t)}$.

Let $f:\mathbb{R}\to[0,1]$ be a smooth even function such that $f(v)=1$ for $|v|\leqslant \varepsilon/2$ and that $f(v)=0$ for $|v|\geqslant\varepsilon$. For $u>0$, set $F_u(a), G_u(a)$ even functions on $\mathbb{R}$ defined by

$$\begin{aligned} F_u(a) &= \int_{-\infty}^{+\infty} \cos(va) e^{-v^2/2} f(\sqrt{u}v)dv\big/\sqrt{2\pi}, \\ G_u(a) &= \int_{-\infty}^{+\infty} \cos(va) e^{-v^2/2}\Big(1-f(\sqrt{u}v)\Big)dv\big/\sqrt{2\pi}. \end{aligned} \tag{1.45}$$

Clearly, from (1.39) and (1.45),

$$F_u(a)+G_u(a) = e^{-a^2/2}. \tag{1.46}$$

From (1.44),

$$\operatorname{supp} F_u\big(\sqrt{u}D\big)(x,\cdot) \subset B^X(x,\varepsilon). \tag{1.47}$$

Clearly, from (1.45),

$$G_u\big(\sqrt{u}a\big) = \int_{|v|\geqslant\varepsilon/2} e^{iva}\exp\left(-\frac{v^2}{2u}\right)\big(1-f(v)\big)\frac{dv}{\sqrt{2\pi u}}. \tag{1.48}$$

Then

$$\begin{aligned} a^m G_u\big(\sqrt{u}a\big) &= i^m \int_{|v|\geqslant\varepsilon/2} e^{iva}\frac{\partial^m}{\partial v^m}\left[\exp\left(-\frac{v^2}{2u}\right)\big(1-f(v)\big)\right]\frac{dv}{\sqrt{2\pi u}} \\ &= \int_{|v|\geqslant\varepsilon/2} e^{iva}\exp\left(-\frac{v^2}{2u}\right)\sum_{j=0}^{m}\frac{\partial^j}{\partial v^j}\big(1-f(v)\big)P_j\left(\frac{1}{u},v\right)\frac{dv}{\sqrt{2\pi u}}, \end{aligned} \tag{1.49}$$

where $P_j(\frac{1}{u},v)$ are polynomials on $\frac{1}{u}$ and v. Thus, there exists $C>0$ such that for $u\in(0,1]$,

$$\Big|a^m G_u\big(\sqrt{u}a\big)\Big| \leqslant C\, e^{-\frac{\varepsilon^2}{16u}},\ \text{for any } a\in\mathbb{R}. \tag{1.50}$$

Again from (1.50) in view of D^2 as a diagonal matrix, we get the estimate of operator norm $\|\cdot\|^{0,0}$ from L^2 to L^2 as

$$\Big\|D^m G_u\big(\sqrt{u}D\big)\Big\|^{0,0} \leqslant C\, e^{-\frac{\varepsilon^2}{16u}}. \tag{1.51}$$

Using Sobolev inequalities and (1.51), we obtain uniformly on $x,y\in X$,

$$\Big|G_u\big(\sqrt{u}D\big)(x,y)\Big| \leqslant c_1 e^{-c_2/u}. \tag{1.52}$$

This concludes that the asymptotic expansion of $e^{-tD^2}(x,x)$ is local! Note that the heat operator e^{-tD^2} is defined globally by means of eigenvalues and eigenfunctions of D^2.

Step 2. Replace X by $\mathbb{R}^n$, we work on $\mathbb{R}^n$. Fix $x_0 \in X$. We identify $B^{T_{x_0}X}(0,4\varepsilon)$ to $B^X(x_0,4\varepsilon)$ by the exponential map: $v \to \exp_{x_0}(v)$. For $Z \in B^{T_{x_0}X}(0,4\varepsilon) \subset T_{x_0}X$, we identify $S(TX)_Z$, E_Z to $S(TX)_{x_0}$, E_{x_0} by parallel transport with respect to $\nabla^{S(TX)}$, ∇^E along the path $\gamma : [0,1] \to X, \gamma(s) = sZ$. Then we extend $D^2|_{B^{T_{x_0}X}(0,2\varepsilon)}$ to an operator on $\mathbb{R}^n$ which is the canonical (positive) Laplacian outside $B^{T_{x_0}X}(0,4\varepsilon)$.

Step 3. Rescaling. Set $\mathbf{E}_{x_0} = (S(TX)\otimes E)_{x_0}$. For $s \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbf{E}_{x_0})$, $Z \in \mathbb{R}^n$, set

$$\big(S_t s\big)(Z) = s\left(\frac{Z}{\sqrt{t}}\right), \quad L_2^t = S_t^{-1} t D^2 S_t. \tag{1.53}$$

Let $\{e_j\}_{j=1}^n$ be an oriented orthonormal basis of $T_{x_0}X$. For $1 \leqslant j \leqslant n$, $t \in (0,1]$, set

$$c_t(e_j) = \frac{1}{\sqrt{t}} e^j \wedge -\sqrt{t}\; i_{e_j} \in \operatorname{End}\big(\Lambda(T^*_{x_0}X)\big). \tag{1.54}$$

Let L_3^t be the operator obtained from L_2^t by replacing $c(e_j)$ by $c_t(e_j)$ in the explicit formula of the operator L_2^t. Then L_3^t acts on $\mathscr{C}^\infty(\mathbb{R}^n, (\Lambda(T^*X)\otimes E)_{x_0})$. We claim that as $t \to 0$,

$$\operatorname{Tr}_s\left[e^{-tD^2}(x_0,x_0)\right] = (-2i)^{n/2} \operatorname{Tr}\Big|_E \left[e^{-L_3^t}(0,0)\right]^{\max} + O(e^{-c/t}), \tag{1.55}$$

which follows from the simple linear algebra identity: for $1 \leqslant i_1 < \cdots < i_j \leqslant n$,

$$\operatorname{Tr}_s\Big|_{S(TX)}\big[c(e_{i_1})\cdots c(e_{i_j})\big] = \begin{cases} 0 & \text{if } j < n = 2l, \\ (-2i)^{n/2} & \text{if } j = n, \end{cases} \tag{1.56}$$

and thus

$$\operatorname{Tr}_s\left[e^{-L_2^t}(0,0)\right] = (-2i)^{n/2} t^{n/2} \operatorname{Tr}\Big|_E \left[e^{-L_3^t}(0,0)\right]^{\max}. \tag{1.57}$$

Theorem 1.11. *As $t \to 0$,*

$$L_3^t \to L_3^0 = -\sum_{j=1}^n \left[\frac{\partial}{\partial Z_j} + \frac{1}{4}\left\langle R^{TX}_{x_0} Z, \frac{\partial}{\partial Z_j}\right\rangle\right]^2 + R^E_{x_0}. \tag{1.58}$$

The following Lichnerowicz formula allows us to obtain (1.58):

$$D^2 = \Delta + \frac{1}{4} r^X + {}^c R^E, \tag{1.59}$$

where Δ is the (positive) Bochner Laplacian on $S(TX)\otimes E$ associated with the connection $\nabla^{S(TX)\otimes E}$, and r^X is the scale curvature of (X, g^{TX}) and for $\{e_j\}_{j=1}^n$

an orthonormal frame of (X, g^{TX}),

$$ {}^c R^E = \frac{1}{2} \sum_{i,j=1}^{n} R^E(e_i, e_j) c(e_i) c(e_j). \tag{1.60}$$

By using weighted Sobolev norm adapted from the structure of the operator L_3^t, we can obtain as $t \to 0$,

$$ e^{-L_3^t}(0,0) \to e^{-L_3^0}(0,0). \tag{1.61}$$

By Mehler's formula, we get

$$\begin{aligned} e^{-tL_3^0}(Z, Z') = (4\pi)^{-n/2} \exp(-tR^E_{x_0}) \det{}^{1/2} \left[\frac{R^{TX}_{x_0}}{e^{tR^{TX}_{x_0}/2} - e^{-tR^{TX}_{x_0}/2}} \right] \\ \times \exp \left\{ \left\langle - \frac{R^{TX}_{x_0}/4}{2 \tanh(tR^{TX}_{x_0}/2)} Z, Z \right\rangle - \left\langle \frac{R^{TX}_{x_0}/4}{2 \tanh(tR^{TX}_{x_0}/2)} Z', Z' \right\rangle \right. \\ \left. + \left\langle \frac{e^{tR^{TX}_{x_0}/4} R^{TX}_{x_0}/4}{2 \sinh(tR^{TX}_{x_0}/2)} Z, Z' \right\rangle \right\}. \end{aligned} \tag{1.62}$$

In particular,

$$ e^{-L_3^0}(0,0) = (4\pi)^{-n/2} \det{}^{1/2} \left[\frac{R^{TX}_{x_0}}{e^{R^{TX}_{x_0}/2} - e^{-R^{TX}_{x_0}/2}} \right] \exp(-R^E_{x_0}). \tag{1.63}$$

Combining (1.55), (1.61) and (1.63), we obtain

$$\begin{aligned} &\lim_{t \to 0} \operatorname{Tr}_s \left[e^{-tD^2}(x_0, x_0) \right] dv_X(x_0) \\ &\quad = (-2i)^{n/2} \operatorname{Tr}\Big|_E \left[e^{-L_3^0}(0,0) \right]^{\max} \\ &\quad = (-2i)^{n/2} (4\pi)^{-n/2} \left\{ \det{}^{1/2} \left[\frac{R^{TX}_{x_0}}{e^{R^{TX}_{x_0}/2} - e^{-R^{TX}_{x_0}/2}} \right] \operatorname{Tr} \left[e^{-R^E_{x_0}} \right] \right\}^{\max} \\ &\quad = \left\{ \det{}^{1/2} \left[\frac{R^{TX}_{x_0}/(2\pi i)}{e^{R^{TX}_{x_0}/(4\pi i)} - e^{-R^{TX}_{x_0}/(4\pi i)}} \right] \operatorname{Tr} \left[\exp\left(- \frac{R^E_{x_0}}{2\pi i} \right) \right] \right\}^{\max} \\ &\quad = \left\{ \widehat{A}(TX, \nabla^{TX}) \operatorname{ch}(E, \nabla^E) \right\}^{\max}. \end{aligned} \tag{1.64}$$

This completes the proof of (1.37). Then we finish the proof of the Atiyah–Singer index theorem. □

2. Holomorphic Morse inequalities

Let (X, J) be a compact complex manifold with complex structure J and $\dim_{\mathbb{C}} X = n$. Then we can identify the holomorphic tangent bundle $T^{(1,0)}X$ (resp. anti-holomorphic tangent bundle $T^{(0,1)}X$) as the eigenspace of J with eigenvalue i

(resp. $-i$) on $TX \otimes_{\mathbb{R}} \mathbb{C}$. Let $T^{*(0,1)}X$ be the anti-holomorphic cotangent bundle of X. Then formally,

$$\Lambda(T^{*(0,1)}X) = S(TX) \otimes (\det T^{(1,0)}X)^{1/2}, \tag{2.1}$$

here $\det F = \Lambda^{\mathrm{rk}(F)}F$ as the determinant line bundle of a vector bundle F.

If E is a holomorphic vector bundle on X. Let

$$\Omega^{0,\bullet}(X,E) = \mathscr{C}^\infty(X, \Lambda^\bullet(T^{*(0,1)}X) \otimes E)$$

be the space of anti-holomorphic differential forms on X with values in E. Then as in (1.4), we can define the Dolbeault operator

$$\overline{\partial}^E : \Omega^{0,k}(X,E) \to \Omega^{0,k+1}(X,E)$$

by using $\overline{\partial}^E$ on $\mathscr{C}^\infty(X,E)$ induced by the holomorphic structure on E. Moreover, $\big(\overline{\partial}^E\big)^2 = 0$. Denote by $H^\bullet(X,E)$ the Dolbeault cohomology of X with values in E, i.e.,

$$H^q(X,E) = \frac{\mathrm{Ker}(\overline{\partial}^E|_{\Omega^{0,q}(X,E)})}{\mathrm{Im}(\overline{\partial}^E|_{\Omega^{0,q-1}(X,E)})}. \tag{2.2}$$

Let g^{TX} be a J-invariant metric on TX and h^E be a Hermitian metric on E. Then they induce naturally an L^2-Hermitian product on $\Omega^{0,\bullet}(X,E)$ via

$$\langle s_1, s_2\rangle = \int_X \langle s_1, s_2\rangle(x) dv_X(x). \tag{2.3}$$

Let $\overline{\partial}^{E,*}$ be the formal adjoint of $\overline{\partial}^E$, and

$$D = \sqrt{2}\big(\overline{\partial}^E + \overline{\partial}^{E,*}\big). \tag{2.4}$$

Then

$$D^2 = 2\big(\overline{\partial}^E\overline{\partial}^{E,*} + \overline{\partial}^{E,*}\overline{\partial}^E\big). \tag{2.5}$$

Thus D^2 preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(X,E)$. By Hodge theory, we have

$$\mathrm{Ker}\left(D^2|_{\Omega^{0,q}(X,E)}\right) \simeq H^q(X,E) \quad \text{for any } q. \tag{2.6}$$

Remark 2.1. If (X, g^{TX}) is Kähler and (E, h^E) is a holomorphic Hermitian vector bundle on X with Chern connection ∇^E, i.e., the $(0,1)$-part of ∇^E is $\overline{\partial}^E$ and ∇^E is Hermitian, then D in (2.4) is the Dirac operator in (1.22) acting on $\Lambda^\bullet(T^{*(0,1)}X) \otimes E$.

Theorem 2.2 (Riemann–Roch–Hirzebruch Theorem).

$$\sum_{j=0}^{n}(-1)^j \dim H^j(X,E) = \int_X \mathrm{Td}(T^{(1,0)}X)\ \mathrm{ch}(E). \tag{2.7}$$

If X is projective, then Theorem 2.2 is the original Riemann–Roch–Hirzebruch theorem. If X is only a compact complex manifold, then (2.7) is a consequence of the Atiyah–Singer index theorem for the Spin^c Dirac operator and (2.6).

Question: How to estimate $\dim H^q(X, E)$ in geometric way? If it is not possible, then at least asymptotically?

The following Theorem 2.3 gives a positive answer to the above question. It is an analogue of the classical Morse inequalities: For a Morse function f on a compact manifold M, let $C_j(f)$ be the number of critical points of f with index j, then

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M,\mathbb{C}) \leqslant \sum_{j=0}^{q}(-1)^{q-j}C_j(f) \quad \text{for any } 0\leq q\leq \dim_{\mathbb{R}} M. \tag{2.8}$$

Let L be a holomorphic Hermitian line bundle on X. Set $L^p = L^{\otimes p}$, the pth tensor power of L. For $0 \leqslant j \leqslant n$, set

$$B_j^p = \dim H^j(X, L^p \otimes E). \tag{2.9}$$

Let h^L be a Hermitian metric on L and ∇^L be the Chern connection on (L, h^L) with curvature $R^L = (\nabla^L)^2$. We define $\dot{R}^L_x \in \mathrm{End}(T_x^{(1,0)}X)$ by

$$\langle \dot{R}^L u, \overline{v}\rangle = R^L(u, \overline{v}). \tag{2.10}$$

Set

$$\begin{aligned} &X(q) = \Big\{x\in X : iR^L_x \text{ non-degenerate}, \dot{R}^L_x \text{ has exactly } q \text{ negative eigenvalues}\Big\},\\ &X(\leqslant q) = \cup_{k\leqslant q} X(k). \end{aligned}\tag{2.11}$$

Theorem 2.3 (Demailly). *As $p \to +\infty$, the following strong Morse inequalities hold for every $q = 0, 1, \dots, n$:*

$$\sum_{j=0}^{q}(-1)^{q-j}B_j^p \leqslant \mathrm{rk}(E)\frac{p^n}{n!}\int_{X(\leqslant q)}(-1)^q\left(\frac{i}{2\pi}R^L\right)^n + o(p^n)\,, \tag{2.12}$$

with equality for $q = n$. In particular, we get the weak Morse inequalities

$$B_j^p \leqslant \mathrm{rk}(E)\frac{p^n}{n!}\int_{X(q)}(-1)^q\left(\frac{i}{2\pi}R^L\right)^n + o(p^n). \tag{2.13}$$

In 1987, Bismut gave a heat kernel proof of Demailly's holomorphic Morse inequalities by using probability theory. Here we gave a heat kernel proof by using Bismut–Lebeau's analytic localization techniques in local index theory [3, §11]. The starting point is the following analogue of the McKean–Singer formula in current context obtained first by Bismut [2]. As in (2.4), set

$$D_p = \sqrt{2}\big(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\big). \tag{2.14}$$

Theorem 2.4. *For any $u > 0$, $0 \leqslant q \leqslant n$, we have*

$$\sum_{j=0}^{q}(-1)^{q-j}B_j^p \leqslant \sum_{j=0}^{q}(-1)^{q-j}\operatorname{Tr}_j\left[e^{-\frac{u}{p}D_p^2}\right], \tag{2.15}$$

with equality for $q = n$. Again $\operatorname{Tr}_j\left[e^{-\frac{u}{p}D_p^2}\right]$ is the trace of $e^{-\frac{u}{p}D_p^2}$ on $\Omega^j(X, L^p \otimes E)$ which is given by

$$\operatorname{Tr}_j[e^{-\frac{u}{p}D_p^2}] = \int_X \operatorname{Tr}|_{\Lambda^j(T^{*(0,1)}X)\otimes L^p\otimes E}\left[e^{-\frac{u}{p}D_p^2}(x,x)\right]dv_X(x). \tag{2.16}$$

Note that

$$e^{-\frac{u}{p}D_p^2}(x,y) \in \bigoplus_{j=0}^{n} E_{p,x}^j \otimes E_{p,y}^{j,*}, \text{ with } E_p^j = \Lambda^j(T^{*(0,1)}X)\otimes L^p \otimes E. \tag{2.17}$$

As $\operatorname{End}(L) = \mathbb{C}$, thus

$$e^{-\frac{u}{p}D_p^2}(x,x) \in \bigoplus_{j=0}^{n}\operatorname{End}\left(\Lambda^j(T^{*(0,1)}X)\otimes E\right)_x.$$

Theorem 2.5 (Bismut). *For $u > 0$ fixed, as $p \to +\infty$, we have*

$$\begin{aligned}\exp\left(-\frac{u}{p}D_p^2\right)(x,x) &= (2\pi)^{-n}\frac{\det(\dot{R}^L)\exp(2u\omega_d)}{\det(1-\exp(-2u\dot{R}^L))}\otimes \operatorname{Id}_E\, p^n + o(p^n)\\ &= \prod_{j=1}^{n}\frac{a_j(x)\left(1+(e^{-2ua_j(x)}-1)\overline{w}^j\wedge i_{\overline{w}_j}\right)}{2\pi(1-e^{-2ua_j(x)})}\otimes \operatorname{Id}_E\, p^n + o(p^n)\,,\end{aligned} \tag{2.18}$$

where we choose an orthonormal basis w_j of $T^{(1,0)}X$ such that

$$\dot{R}^L(x) = \operatorname{diag}(a_1(x),\dots,a_n(x)) \in \operatorname{End}(T_x^{(1,0)}X), \tag{2.19}$$

and

$$\omega_d = -\sum_{j=1}^{n} a_j(x)\overline{w}^j\wedge i_{\overline{w}_j}. \tag{2.20}$$

If $\omega(\cdot, J\cdot) = g^{TX}(\cdot,\cdot)$, then $a_j(x) = 2\pi$.

Proof. Bismut used probability theory to prove the result. Our proof is based on the analytic localization techniques of Bismut–Lebeau.

Step 1. The problem is local! Recall that from (1.50) there exists $C > 0$ such that

$$\left|a^k G_u(\sqrt{u}a)\right| \leqslant Ce^{-\frac{\varepsilon^2}{16u}}, \quad \text{for any } u\in(0,1],\ a\in\mathbb{R}. \tag{2.21}$$

Thus for $u > 0$ fixed, there exists $C_k > 0$ such that for $p \in \mathbb{N}$,

$$\left\|D_p^k G_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}}D_p\right)\right\|^{0,0} \leqslant C_k e^{-\frac{\varepsilon^2 p}{32u}}. \tag{2.22}$$

Once we study carefully the Sobolev embedding theorem with parameter p, from (2.22) we know that there exist $c_1 > 0, c_2 > 0$ such that for any $x, y \in X$,

$$\left| G_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}} D_p\right)(x,y) \right| \leqslant c_1 e^{-c_2 p}. \tag{2.23}$$

But supp $F_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}} D_p\right)(x,\cdot) \subset B(x,\varepsilon)$ and $F_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}} D_p\right)(x,\cdot)$ only depends on the restriction of D_p to $B(x,\varepsilon)$.

Step 2. Replace X by $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, we work on $\mathbb{C}^n$.

Fix $x_0 \in X$. We identify $B^{T_{x_0}X}(0,4\varepsilon)$ to $B^X(x_0,4\varepsilon)$ by the exponential map: $v \to \exp_{x_0}(v)$. For $Z \in B^{T_{x_0}X}(0,4\varepsilon) \subset T_{x_0}X$, we identify $\Lambda^\bullet(T_Z^{*(0,1)}X)$, L_Z and E_Z to $\Lambda^\bullet(T_{x_0}^{*(0,1)}X)$, L_{x_0} and E_{x_0} by parallel transport with respect to $\nabla^{B,\Lambda^{0,*}}$, ∇^L and ∇^E along the path $\gamma : [0,1] \to X, \gamma(s) = sZ$, where $\nabla^{B,\Lambda^{0,*}}$ is the connection on $\Lambda^\bullet(T^{*(0,1)}X)$ induced by the Bismut connection ∇^B on $T^{(1,0)}X$, in particular, it preserves the $\mathbb{Z}$-grading on $\Lambda^\bullet(T^{*(0,1)}X)$.

Step 3. Rescaling. Once we trivialized L we can consider that D_p^2 acts on $\mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$ with $\mathbf{E}_{x_0} = (\Lambda^\bullet(T^{*(0,1)}X) \otimes E)_{x_0}$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $t = \frac{1}{\sqrt{p}}$, set

$$\big(S_t s\big)(Z) = s\left(\frac{Z}{t}\right), \qquad L_2^t = S_t^{-1} \frac{1}{p} D_p^2 S_t. \tag{2.24}$$

Then as $t \to 0$, with $\tau_{x_0} = \sum_{j=1}^n a_j(x_0)$,

$$L_2^t \to L_2^0 = -\sum_{j=1}^{2n} \left[\frac{\partial}{\partial Z_j} + \frac{1}{2} R_{x_0}^L\left(Z, \frac{\partial}{\partial Z_j}\right)\right]^2 - 2\omega_{d,x_0} - \tau_{x_0}. \tag{2.25}$$

Again (2.25) is obtained from the Lichnerowicz formula for D_p^2 obtained by Bismut:

$$D_p^2 = \Delta^{B,\Lambda^{0,*}} + p\ {}^c R^L + \text{0-order term independent of } p, \tag{2.26}$$

and $\Delta^{B,\Lambda^{0,*}}$ is the Bochner Laplacian acting on $\mathscr{C}^\infty(X, E_p)$ associated with $\nabla^{B,\Lambda^{0,*}}$, ∇^L and ∇^E.

From the finite propagation speed for the wave operator, for $u > 0$ fixed, we obtain as $p \to +\infty$,

$$e^{-\frac{u}{p} D_p^2}(x_0, x_0) = p^n e^{-uL_2^t}(0,0) + \mathscr{O}(e^{-cp}). \tag{2.27}$$

By using weighted Sobolev norms adapted from the structure of the operator L_2^t, we get

Theorem 2.6. *As $t \to 0$,*

$$e^{-uL_2^t}(0,0) \to e^{-uL_2^0}(0,0). \tag{2.28}$$

Finally, from (1.62) and (2.25), we get

$$e^{-uL_2^0}(0,0) = (2\pi)^{-n}\frac{\det(\dot{R}^L_{x_0})e^{2u\omega_{d,x_0}}}{\det(1-e^{-2u\dot{R}^L_{x_0}})}. \tag{2.29}$$

From Theorem 2.4, (2.27)–(2.29), as $p\to+\infty$,

$$\begin{aligned}\sum_{j=0}^{q}&(-1)^{q-j}B_j^p\\ \leqslant \sum_{j=0}^{q}&(-1)^{q-j}\int_X \operatorname{Tr}|_{\Lambda^j(T^{*(0,1)}X)\otimes E}\left[(2\pi)^{-n}\frac{\det(\dot{R}^L_{x_0})e^{2u\omega_{d,x_0}}}{\det(1-e^{-2u\dot{R}^L_{x_0}})}\otimes \mathrm{Id}_E\right]dv_X(x)\cdot p^n\\ &+o(p^n).\end{aligned} \tag{2.30}$$

One can verify directly that

$$\begin{aligned}\lim_{u\to+\infty}\int_X \operatorname{Tr}|_{\Lambda^j(T^{*(0,1)}X)}&\left[(2\pi)^{-n}\frac{\det(\dot{R}^L_{x_0})e^{2u\omega_{d,x_0}}}{\det(1-e^{-2u\dot{R}^L_{x_0}})}\right]dv_X(x)\\ &=\int_{X(j)}(-1)^j\frac{1}{n!}\left(\frac{iR^L}{2\pi}\right)^n.\end{aligned} \tag{2.31}$$

Combining (2.30) and (2.31) yields (2.12). □

3. Bergman kernels

3.1. Asymptotic expansion of Bergman kernels

Let (X,J) be a compact complex manifold with complex structure J and $\dim_{\mathbb{C}} X = n$. Let $(L,h^L),(E,h^E)$ be holomorphic Hermitian vector bundles on X and $\mathrm{rk}(L)=1$. Let ∇^L be the Chern connection on (L,h^L) with curvature

$$R^L = (\nabla^L)^2 \in \Omega^{1,1}(X,\mathrm{End}(L)) = \Omega^{1,1}(X,\mathbb{C}). \tag{3.1}$$

Assumption: $\omega = \frac{i}{2\pi}R^L$ is positive (equivalently, $w(\cdot,J\cdot)$ defines a metric on TX).

By the Kodaira vanishing theorem, we have for any $q>0$,

$$H^q(X,L^p\otimes E)=0 \qquad \text{for } p\gg 1. \tag{3.2}$$

Let g^{TX} be any J-invariant Riemannian metric on TX. Let P_p be the orthogonal projection from $\mathscr{C}^\infty(X,L^p\otimes E)$ onto $H^0(X,L^p\otimes E)$. Its smooth kernel is

$$P_p(x,y)=\sum_{i=1}^{d_p}S_i^p(x)\otimes(S_i^p(y))^*\in(L^p\otimes E)_x\otimes(L^p\otimes E)^*_y, \tag{3.3}$$

where $\{S_i^p\}_{i=1}^{d_p}$ ($d_p := \dim H^0(X,L^p\otimes E)$) is an orthonormal basis of $H^0(X,L^p\otimes E)$. In particular,

$$P_p(x,x)\in \mathrm{End}(L^p\otimes E)_x = \mathrm{End}(E)_x. \tag{3.4}$$

If $E = \mathbb{C}$, then

$$P_p(x,x) = \sum_{i=1}^{d_p} |S_i^p(x)|^2 : \ X \to [0, +\infty). \tag{3.5}$$

By the Riemann–Roch–Hirzebruch theorem and (3.2), we have for p large enough,

$$\begin{aligned}\int_X \operatorname{Tr}|_E[P_p(x,x)]dv_X(x) &= \dim H^0(X, L^p \otimes E) = \int_X \operatorname{Td}(T^{(1,0)}X)\operatorname{ch}(L^p\otimes E)\\ = \int_X \operatorname{Td}(T^{(1,0)}X)\operatorname{ch}(E)e^{p\omega} &= \operatorname{rk}(E)\int_X \frac{c_1(L)^n}{n!}p^n \qquad (3.6)\\ &+ \int_X \left(c_1(E) + \frac{\operatorname{rk}(E)}{2}c_1(T^{(1,0)}X)\right)\frac{c_1(L)^{n-1}}{(n-1)!}p^{n-1} + \mathscr{O}(p^{n-2}).\end{aligned}$$

Question: Whether as $p \to +\infty$,

$$\operatorname{Tr}|_E[P_p(x,x)]dv_X(x) = \operatorname{Td}(T^{(1,0)}X, \nabla^{T^{(1,0)}X})\operatorname{ch}(E, \nabla^E)_x e^{p\omega_x} + \mathscr{O}(p^{-\infty}), \tag{3.7}$$

where $\nabla^{T^{(1,0)}X}$ is the Chern connection on $(T^{(1,0)}X, g^{TX})$.

The following is a local version of the expansion.

Theorem 3.1 (Tian, Ruan, Catlin, Zelditch, Boutet de Monvel–Sjöstrand, Dai–Liu–Ma, Ma–Marinescu, . . .). *There exist $\boldsymbol{b}_j \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ such that for any k, as $p \to +\infty$, we have uniformly on X,*

$$p^{-n}P_p(x,x) = \sum_{j=0}^{k} \boldsymbol{b}_j(x)p^{-j} + \mathscr{O}(p^{-k-1}), \tag{3.8}$$

with

$$\boldsymbol{b}_0 = \det(\dot{R}^L/(2\pi))\operatorname{Id}_E. \tag{3.9}$$

The Kodaira embedding theorem shows that for $p \gg 1$, L^p give rise to holomorphic embeddings $\Phi_p : X \to \mathbb{P}(H^0(X, L^p)^*)$. Moreover, $L^p = \Phi_p^*\mathcal{O}(1)$ and $h^{L^p}(x) = P_p(x,x)h^{\Phi_p^*\mathcal{O}(1)}(x)$ (cf. [11, Theorem 5.1.3]). Here $\mathcal{O}(1)$ is the hyperplane line bundle on $\mathbb{P}(H^0(X, L^p)^*)$ with the metric $h^{\mathcal{O}(1)}$ induced naturally from the Hermitian product on $H^0(X, L^p)$. Thus

$$\frac{1}{p}\Phi_p^*\omega_{FS} - \omega = -\frac{i}{2\pi p}\overline{\partial}\partial \log P_p(x,x). \tag{3.10}$$

Where ω_{FS} is the Fubini–Study form on the projective space $\mathbb{P}(H^0(X, L^p)^*)$.

From (3.8) and (3.10), we know that the induced Fubini–Study forms via Kodaira embedding maps Φ_p is dense in the space of Kähler form in the Kähler class $c_1(L)$. More precisely,

Corollary 3.2 (Tian, Ruan). *For $k > 0$, there exists $C > 0$ such that*

$$\left|\frac{1}{p}\Phi_p^*(\omega_{\text{FS}}) - \omega\right|_{\mathscr{C}^k} \leqslant \frac{C}{p}\,. \tag{3.11}$$

Ruan improved Tian's asserting convergence in $\mathscr{C}^2$ topology with speed rate $p^{-1/2}$. The optimal convergence speed of the induced Fubini–Study forms in the symplectic case was obtained in [9].

3.2. Proof of the asymptotic expansion of Bergman kernels

In this subsection, we obtain the asymptotic behavior of $P_p(x,y)$ as $p \to +\infty$ via the analytic localization techniques of Bismut–Lebeau [3, §11]. The method also works in the symplectic case by Dai–Liu–Ma [6], also Ma–Marinescu [12]. The starting point of the approach is the following spectral gap result.

Theorem 3.3 (Bismut–Vasserot (1989); Ma–Marinescu, symplectic version (2002)). *There exists $C > 0$ such that for any $p \in \mathbb{N}^*$,*

$$\operatorname{Spec}(D_p^2) \subset \{0\} \cup [2p\mu_0 - C, +\infty), \tag{3.12}$$

where

$$\mu_0 = \inf_{\substack{x \in X,\\ 0 \neq u \in T_x^{(1,0)}X}} \frac{R^L(u,\overline{u})}{|u|^2}. \tag{3.13}$$

If $w(\cdot, J\cdot) = g^{TX}$, then $\mu_0 = 2\pi$.

Proof of Theorem 3.1. We divide the proof into three steps.

Step 1. The problem is local, i.e., module $\mathscr{O}(p^{-\infty})$, $P_p(x_0,\cdot)$ depends only on $D_p|_{B^X(x_0,\varepsilon)}$. Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that $f(v) = 1$ for $|v| \leqslant \varepsilon/2$ and that $f(v) = 0$ for $|v| \geqslant \varepsilon$. Take

$$F(a) = \left(\int_{-\infty}^{+\infty} f(v)dv\right)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)dv. \tag{3.14}$$

Then $F(0) = 1$ and for $p > C/\mu_0$,

$$P_p = F(D_p) - 1_{[\sqrt{p\mu_0},+\infty)}(|D_p|)F(D_p). \tag{3.15}$$

On one hand, by the finite propagation speed of solutions of wave equations, we have $\operatorname{supp} F(D_p)(x_0,\cdot) \subset B^X(x_0,\varepsilon)$ and $F(D_p)(x_0,\cdot)$ depends only on $D_p|_{B^X(x_0,\varepsilon)}$. On the other hand, as

$$\sup_{a\in\mathbb{R}} |a|^m |F(a)| \leqslant C_m, \tag{3.16}$$

which implies that the smooth kernel of the operator $1_{[\sqrt{p\mu_0},+\infty)}(|D_p|)F(D_p)$ has the following property: as $p \to +\infty$,

$$1_{[\sqrt{p\mu_0},+\infty)}(|D_p|)F(D_p)(x,y) = \mathscr{O}(p^{-\infty}). \tag{3.17}$$

As $F(D_p)(x,y) = 0$ if $d(x,y) > \varepsilon$, where $d(\ ,\ )$ is the Riemannian distance on (X, g^{TX}). Thus we know that if $d(x,y) > \varepsilon$, then

$$P_p(x,y) = \mathscr{O}(p^{-\infty}). \tag{3.18}$$

Step 2. We replace X by $\mathbb{R}^{2n} =: X_0$. We identify $B^{T_{x_0}X}(0,4\varepsilon)$ in $T_{x_0}X$ to $B^X(x_0,4\varepsilon)$ by the exponential map: $v \to \exp_{x_0}(v)$. For $Z \in B^{T_{x_0}X}(0,4\varepsilon) \subset T_{x_0}X$, we identify $\Lambda(T_Z^{*(0,1)}X)$, L_Z and E_Z to $\Lambda(T_{x_0}^{*(0,1)}X)$, L_{x_0} and E_{x_0} by parallel transport with respect to $\nabla^{B,\Lambda^{0,*}}$, ∇^L and ∇^E along the path $\gamma : [0,1] \to X, \gamma(s) = sZ$.

Step 3. Rescaling. Let $\rho : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = 1 \quad \text{if} \quad |v| < 2; \qquad \rho(v) = 0 \quad \text{if} \ |v| > 4. \tag{3.19}$$

Set $\varphi_\varepsilon(Z) = \rho(|Z|/\varepsilon)Z$. For the trivial vector bundle $L_0 := (L_{x_0}, h^{L_{x_0}})$, we defined a Hermitian connection on $X_0 := T_{x_0}X$ by

$$\nabla^{L_0}|_Z = \varphi_\varepsilon^* \nabla^L + \frac{1}{2}(1 - \rho^2(|Z|/\varepsilon))R^L_{x_0}(Z, \cdot). \tag{3.20}$$

The important observation is that the curvature $(\nabla^{L_0})^2$ of ∇^{L_0} is uniformly positive on $\mathbb{R}^{2n}$ and its small eigenvalues in the sense of (3.13) is bigger than $\frac{4}{5}\mu_0$ for ε small enough. We obtain a modified Dirac operator $D_{0,p}$ on X_0 with

$$\operatorname{Spec}(D^2_{0,p}) \subset \{0\} \cup \left[\frac{8}{5}p\mu_0 - C, +\infty\right). \tag{3.21}$$

Denote by $P_{0,p}$ the orthogonal projection from $L^2(X_0, E_{0,p})$ onto $\operatorname{Ker}(D^2_{0,p})$. Then

$$P_p = P_{0,p} + \mathscr{O}(p^{-\infty}). \tag{3.22}$$

For large p, we have

$$\begin{aligned} P_{0,p} &= e^{-\frac{u}{p}D^2_{0,p}} - e^{-\frac{u}{p}D^2_{0,p}} 1_{(p\mu_0, +\infty)}(D^2_{0,p}) \\ &= e^{-\frac{u}{p}D^2_{0,p}} - \int_u^\infty \frac{1}{p} D^2_{0,p} e^{-\frac{v}{p}D^2_{0,p}} dv. \end{aligned} \tag{3.23}$$

Then for u fixed we have the asymptotic expansion of $e^{-\frac{u}{p}D^2_{0,p}}$ and

$$\frac{1}{p}D^2_{0,p}e^{-\frac{u}{p}D^2_{0,p}} = \mathscr{O}(e^{-cu}).$$

This indicates that we can approximate the Bergman kernel by using heat kernels. The detail of this approach was first realized by Dai–Liu–Ma in [6] by using the analytic localization techniques of Bismut–Lebeau. In fact they obtain the full asymptotics of $P_p(x, y)$ as $p \to +\infty$. This approach works for the symplectic case, also the singular case with orbifold singularities. Ma–Marinescu [8, 11, 13, 14] use this kind of expansion to establish the Berezin–Toeplitz geometric quantization theory in symplectic case. The Berezin–Toeplitz theory has played an important role in the recent works on the asymptotics of analytic torsions [4], [15]. □

3.3. Coefficients of the asymptotic expansion of Bergman kernels

In the last part, we explain how to compute the coefficient in the expansion.

For $t = \frac{1}{\sqrt{p}}$, set

$$\big(S_t s\big)(Z) = s\left(\frac{Z}{t}\right), \ \mathscr{L}_t = S_t^{-1}\frac{1}{p}D^2_{0,p}S_t. \tag{3.24}$$

Then the Taylor expansion of $\mathscr{L}_t$ gives

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{k} t^r \mathcal{O}_r + \mathscr{O}(t^{k+1}), \tag{3.25}$$

where under the notation of (2.20),

$$\mathscr{L}_0 = \sum_j \left(-2\tfrac{\partial}{\partial z_j} + \frac{1}{2} a_j \overline{z}_j\right) \left(2\tfrac{\partial}{\partial \overline{z}_j} + \frac{1}{2} a_j z_j\right) + 2a_j \overline{w}^j \wedge i_{\overline{w}_j}. \tag{3.26}$$

Set

$$b_j = -2\tfrac{\partial}{\partial z_j} + \frac{1}{2} a_j \overline{z}_j\,, \quad b_j^+ = 2\tfrac{\partial}{\partial \overline{z}_j} + \frac{1}{2} a_j z_j\,, \quad \mathscr{L} = \sum_j b_j b_j^+. \tag{3.27}$$

Then

$$\mathscr{L}_0 = \mathscr{L} - 2\omega_{d,x_0} = \mathscr{L} + 2\sum_j a_j \overline{w}^j \wedge i_{\overline{w}_j}. \tag{3.28}$$

One verifies directly that for the spectrum of $\mathscr{L}$,

$$\mathrm{Spec}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})}) = \Big\{2\sum\nolimits_{j=1}^n \alpha_j a_j \,:\, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\Big\}, \tag{3.29}$$

and that an orthogonal basis of the eigenspace of $2\sum_{j=1}^n \alpha_j a_j$ is given by

$$b^\alpha \left(z^\beta \exp\left(-\frac{1}{4}\sum\nolimits_j a_j |z_j|^2\right)\right), \quad \text{with } \beta \in \mathbb{N}^n\,. \tag{3.30}$$

Thus

$$\mathrm{Ker}(\mathscr{L}) = \left\{z^\beta \exp\left(-\frac{1}{4}\sum\nolimits_j a_j |z_j|^2\right), \quad \text{with } \beta \in \mathbb{N}^n\right\}. \tag{3.31}$$

The orthogonal projection from $L^2(\mathbb{R}^{2n}, \mathbb{C})$ onto $\mathrm{Ker}(\mathscr{L})$ is the classical Bergman kernel on $\mathbb{C}^n$ associated with the trivial line bundle with metric

$$|1|_{h^L}(Z) = \exp\left(-\frac{1}{4}\sum\nolimits_j a_j |z_j|^2\right). \tag{3.32}$$

The classical Bergman kernel is given by

$$\mathcal{P}(Z, Z') = \prod_{j=1}^n \frac{a_j}{2\pi} \exp\left(-\frac{1}{4}\sum\nolimits_j a_j \left(|z_j|^2 + |z'_j|^2 - 2z_j \overline{z}'_j\right)\right). \tag{3.33}$$

As all our operators preserve $\mathbb{Z}$-grading of $\mathbf{E}_{x_0}$ and the degree $\geqslant 1$ part is zero. We can restrict all the following computation on 0-degree part, i.e., on $\mathscr{C}^\infty(X_0, E_{x_0})$.

Let $P_{0,t}$ be the spectral projection

$$P_{0,t} : L^2(X_0, E_{x_0}) \to \mathrm{Ker}(\mathscr{L}_t)$$

and $P_{0,t}(Z,Z)$ the smooth kernel of $P_{0,t}$. From (3.25), by the formal expansion of the resolvent for $|\lambda| = \mu_0/4$

$$(\lambda - \mathscr{L}_t)^{-1} = \sum_{r=0}^{\infty} t^r f_r(\lambda), \tag{3.34}$$

we obtain as $t \to 0$,

$$P_{0,t} = \frac{1}{2\pi i}\int_{|\lambda|=\mu_0/4} (\lambda - \mathscr{L}_t)^{-1} d\lambda = P^N + \frac{1}{2\pi i}\sum_{r=1}^{k} t^r \int_{|\lambda|=\mu_0/4} f_r(\lambda) d\lambda + \mathscr{O}(t^{k+1}), \tag{3.35}$$

with $P^N = \mathcal{P}\ \mathrm{Id}_E$. Then from (3.24)

$$P_{0,p}(Z,Z') = t^{-2n} P_{0,t}\left(\frac{Z}{t}, \frac{Z'}{t}\right). \tag{3.36}$$

From (3.22) and (3.36), the kernel of the coefficient of t^r in (3.35) gives the coefficient of $p^{-r/2}$ in the off-diagonal expansion of $p^{-n}P_p(Z,Z')$ by Dai–Liu–Ma, Ma–Marinescu. In particular, $\boldsymbol{b}_j$ in (3.8) is given by the evaluation of the kernel of the coefficient of t^{2j} in (3.35) at $(0,0)$.

Remark 3.4. In the Kähler case, i.e., $\omega(\cdot, J\cdot) = g^{TX}(\cdot,\cdot)$, then all $a_j = 2\pi$,

$$\mathcal{O}_1 = 0,$$

and

$$\boldsymbol{b}_1(x) = \big(-\mathscr{L}^{-1}\mathcal{O}_2 P^N - P^N \mathcal{O}_2 \mathscr{L}^{-1}\big)(0,0) = \frac{1}{8\pi}\big[r^X + 4R^E(w_j, \overline{w}_j)\big], \tag{3.37}$$

here $\{w_j\}$ is an orthonormal basis of $T^{(1,0)}_{x_0}X$ and r^X is the scalar curvature of (X, g^{TX}). Note that in the Kähler case, $\boldsymbol{b}_1$ was obtained first by Lu and Wang by using the pick section trick in complex analysis as in [16].

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