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On orbifold elliptic genus

Chongying Dong, Kefeng Liu, and Xiaonan Ma

ABSTRACT. An elliptic genus is defined and studied for a general orbifold. The main result is the rigidity property of the genus.

1. Introduction

The elliptic genus was derived as a partition function in quantum field theory [**27**]. Mathematically it is a beautiful combination of topology of manifolds, index theory and modular forms (cf. [**15**], [**10**]). The elliptic genus for smooth manifolds has been well-studied. Recently, Borisov and Libgober ([**3**], [**4**]) proposed some definitions of elliptic genus for certain singular spaces, especially for a complex orbifold which is a global quotient M/G, where the finite group G acts holomorphically on complex variety M. Similar definitions were introduced by string theorists in the 80s, in the study of orbifold string theory. One of their guiding principles is modular invariance. More recently orbifold string theory has attracted the attention of geometers and topologists. For example Chen and Ruan (cf. [**7**], [**23**]) have defined orbifold cohomology and orbifold quantum cohomology groups.

One of most important properties of the elliptic genus is its rigidity property under compact connected Lie group actions. For smooth manifolds, the rigidity and its generalizations have been well studied. Since the orbifold elliptic genus is the partition function of an orbifold string theory, it is natural to expect the rigidity property for the orbifold elliptic genus. Although the global quotients form a very important class of orbifolds, many interesting orbifolds are not global quotients. For example, most of the Calabi-Yau hypersurfaces of weighted projective spaces are not global quotients. In this paper we define an elliptic genus for a general orbifold which generalizes the definition of Borisov and Libgober, and prove their rigidity property. We actually introduce a more general elliptic genus involving twisted bundles and proved its rigidity. The idea of considering the weights in the definition of orbifold elliptic genus comes from [**3**] and our proof of the K-theory version of Witten's rigidity theorems [**21**, §4], [**22**, §4]. The proof of rigidity is again

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a combination of modular invariance and index theory, but now more complicated combinatorics are involved in the definition and proof.

This paper is organized as follows: In Sections 1 and 2 we review the equivariant index theorem on orbifolds. We define an orbifold elliptic genus and prove its rigidity for almost complex orbifolds in Section 3. Finally in Section 4 we introduce an orbifold elliptic genus for spin orbifolds; we will study its rigidity property on a later occasion.

The authors would like to than Jian Zhou for many interesting discussions regarding orbifold elliptic genus.

2. Equivariant index theorem for spin orbifolds

In this section and the next we recall notations for orbifolds, and explain the equivariant index theorem for orbifolds (cf. [**8**, Chap. 14], [**26**]).

We first recall the definition of orbifolds [1], which are called V-manifolds in [**11**], [**24**].

We consider the pair (G, V), where V is a connected smooth manifold, G is a finite group acting smoothly and effectively on V. A morphism $\Phi : (G, V) \to (G', V')$ is a family of open embeddings $\varphi : V \to V'$ satisfying:

i) For each $\varphi \in \Phi$, there is an injective group homomorphism $\lambda_\varphi : G \to G'$ such that φ is λ_φ-equivariant.

ii) For $g \in G', \varphi \in \Phi$, we define $g\varphi : V \to V'$ by $(g\varphi)(x) = g\varphi(x)$ for $x \in V$. If $(g\varphi)(V) \cap \varphi(V) \neq \phi$, then $g \in \lambda_\varphi(G)$.

iii) For $\varphi \in \Phi$, we have $\Phi = \{g\varphi, g \in G'\}$. This means G' acts transitively on Φ.

The morphism Φ induces a unique open embedding $i_\Phi : V/G \to V'/G'$ of orbit spaces.

DEFINITION 2.1. An orbifold $(X, \mathcal{U})$ is a paracompact Hausdorff space X together with a covering $\mathcal{U}$ of X consisting of connected open subsets such that

i) For $U \in \mathcal{U}$, $\mathcal{V}(U) = ((G_U, \widetilde{U}) \xrightarrow{\tau} U)$ is a ramified covering $\widetilde{U} \to U$ giving an identification $U \simeq \widetilde{U}/G_U$.

ii) For $U, V \in \mathcal{U}, U \subset V$, there is a morphism $\varphi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$ that covers the inclusion $U \subset V$.

iii) For $U, V, W \in \mathcal{U}, U \subset V \subset W$, we have $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$.

In the above definition, we can replace (G, V) by a category of manifolds with an additional structure such as orientation, Riemannian metric or complex structure. We understand that the morphisms (and the groups) preserve the specified structure. So we can define oriented, Riemannian or complex orbifolds.

REMARK 2.2. ([**13**, p143-144]) Let G be a compact Lie group (need not be connected) and M a smooth manifold with a smooth G-action. We assume that the action of G is effective and infinitesimally free. Then the quotient space M/G is an orbifold. Reciprocally, any orbifold X can be presented this way. For example,

[1] The definition of orbifold in this paper is the reduced orbifold in the sense of Chen-Ruan [**7**]. Let X be an orbifold in the sense of Chen-Ruan [**7**], let X_{red} be the corrresponding reduced orbifold. Then the elliptic operators and the charateristic classes on X will be reduced to the corresponding ones on X_{red} (cf. Definition 2.7). This means to work on index theory, we only need work on X_{red}.

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let $O(X)$ be the total space of the associated tangential orthonormal frame bundle. We know that $O(X)$ is a smooth manifold and the action of the orthogonal group $O(n)$ $(n = \dim X)$ is infinitesimally free on $O(X)$. The X is identified canonically with the orbifold $O(X)/O(n)$.

Let X be an oriented orbifold, with singular set ΣX. For $x \in X$, there exists a small neighbourhood $(G_x, \widetilde{U}_x) \xrightarrow{\tau_x} U_x$ such that $\widetilde{x} = \tau_x^{-1}(x) \in \widetilde{U}_x$ is a fixed point of G_x. Such G_x is unique up to isomorphisms for each $x \in X$ [**24**, p468]. Let (1), $(h_x^1), \cdots, (h_x^{\rho_x})$ be all the conjugacy classes in G_x. Let $Z_{G_x}(h_x^j)$ be the centralizer of h_x^j in G_x. One also denotes by $\widetilde{U}_x^{h_x^j}$ the fixed points of h_x^j in $\widetilde{U}_x$. There is a natural bijection

$$\begin{aligned} \{(y, (h_y^j)) | y \in U_x, j = 1, \cdots, \rho_y\} \\ \simeq \amalg_{j=1}^{\rho_x} \widetilde{U}_x^{h_x^j} / Z_{G_x}(h_x^j). \end{aligned} \tag{2.1}$$

So we can define globally [**11**, p77],

$$\widetilde{\Sigma X} = \{(x, (h_x^j)) | x \in X, G_x \neq 1, j = 1, \cdots, \rho_x\}. \tag{2.2}$$

Then $\widetilde{\Sigma X}$ has a natural orbifold structure defined by

$$\left\{ (Z_{G_x}(h_x^j)/K_x^j, \widetilde{U}_x^{h_x^j}) \to \widetilde{U}_x^{h_x^j} / Z_{G_x}(h_x^j) \right\}_{(x, U_x, j)}. \tag{2.3}$$

Here K_x^j is the kernel of the representation $Z_{G_x}(h_x^j) \to \text{Diffeo}\,(\widetilde{U}_x^{h_x^j})$. The number $m = |K_x^j|$ is called the multiplicity of $\widetilde{\Sigma X}$ in X at (x, h_x^j). Since the multiplicity is locally constant on $\widetilde{\Sigma X}$, we may assign the multiplicity m_i to each connected component X_i of $\widetilde{\Sigma X}$. In a sense $\widetilde{\Sigma X}$ is a resolution of singularities of X.[2]

DEFINITION 2.3. A mapping π from an orbifold X to an orbifold X' is called smooth if for $x \in X, y = \pi(x)$, there exist orbifold charts $(G_x, \widetilde{U}_x), (G'_y, \widetilde{U'}_y)$ together with a smooth mapping $\phi : \widetilde{U}_x \to \widetilde{U'}_y$ and a homomorphism $\rho : G_x \to G'_y$ such that ϕ is ρ-equivariant and $\tau'_y \circ \phi = \pi \circ \tau_x$. Thus we have the following commutative diagram :

$$\begin{array}{ccc} \widetilde{U}_x & \xrightarrow{\psi} & \widetilde{U'}_y \\ \tau' \downarrow & & \tau' \downarrow \\ U_x & \xrightarrow{\pi} & U'_y \end{array}$$

DEFINITION 2.4. An orbifold vector bundle ξ over an orbifold $(X, \mathcal{U})$ is defined as follows: ξ is an orbifold and for $U \in \mathcal{U}$, $(G_U^\xi, \widetilde{p}_U : \widetilde{\xi}_U \to \widetilde{U})$ is a G_U^ξ-equivariant vector bundle and $(G_U^\xi, \widetilde{\xi}_U)$ (resp. $(G_U^\xi/K_U, \widetilde{U}), K_U = \text{Ker}(G_U^\xi \to \text{Diffeo}(\widetilde{U}))$) is the orbifold structure of ξ (resp. X). In general, G_U^ξ does not act effectively on $\widetilde{U}$, i.e. $K_U \neq \{1\}$. If G_U^ξ acts effectively on $\widetilde{U}$ for $U \in \mathcal{U}$, we say ξ is a proper orbifold vector bundle.

REMARK 2.5. ([**13**, p144] Let G be a compact Lie group acting effectively and infinitesimally freely on M. Then each G-equivariant bundle $E \to M$ defines a proper orbifold vector bundle $E/G \to M/G$, and vice versa.

[2] It has nothing to do with of singularities by birational morphisms studied in algebraic geometry.

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In the following, we will always denote by $(G_x, \widetilde{U}_x)$ $(x \in X)$ the orbifold chart as above. For $h \in G_x$, we have the following h-equivariant decomposition of $T\widetilde{U}_x \otimes_{\mathbf{R}} \mathbf{C}$ as a real vector bundle on $\widetilde{U}_x^h$,

$$T\widetilde{U}_x \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{\lambda \in \mathbf{Q} \cap]0,1[} N_{\lambda(h)} \bigoplus T\widetilde{U}_x^h \otimes_{\mathbf{R}} \mathbf{C}. \tag{2.4}$$

Here $N_{\lambda(h)}$ is the complex vector bundle over $\widetilde{U}_x^h$ with h acting by $e^{2\pi i\lambda}$ on it. The complex conjugation provides a $\mathbf{C}$ anti-linear isomorphism between $N_{\lambda(h)}$ and $\overline{N_{(1-\lambda)(h)}}$. If the order of h is even, this produces a real structure on $N_{\frac{1}{2}(h)}$, so this bundle is the complexification of a real vector bundle $N^{\mathbf{R}}_{\frac{1}{2}(h)}$ on $\widetilde{U}_x^h$. Thus, $T\widetilde{U}_x$ is isomorphic, as a real vector bundle, to

$$T\widetilde{U}_x \simeq \bigoplus_{\lambda \in \mathbf{Q} \cap]0,\frac{1}{2}[} N_{\lambda(h)} \bigoplus N^{\mathbf{R}}_{\frac{1}{2}(h)} \bigoplus T\widetilde{U}_x^h. \tag{2.5}$$

Note that $N_{\lambda(h)}$ (resp. $N^{\mathbf{R}}_{\frac{1}{2}(h)}$) extends to a complex (resp. real) vector bundle on $\widetilde{\Sigma X}$. We will still denote them by $N_{\lambda(h)}$, $N^{\mathbf{R}}_{\frac{1}{2}(h)}$.

Assume that a compact Lie group H acts differentiably on X. If $\gamma \in H$, let $X^\gamma = \{x \in X, \gamma x = x\}$. In the index theorem, we will use the following orbifold as fixed point set of γ which is a resolution of singularities of X^γ [**8**, p180]. For $x \in X^\gamma$, then on local chart $(G_x, \widetilde{U}_x)$, $\gamma_{\widetilde{U}}$ acts on $\widetilde{U}_x$ as a linear map. The compatibility condition for $\gamma_{\widetilde{U}}$ means that there exists an automorphism α of G_x such that for each $g \in G_x$, $\gamma_{\widetilde{U}} \circ g \circ \gamma_{\widetilde{U}}^{-1} = \alpha(g)$. For $h \in G_x$, let $(h)_\gamma = \{gh\alpha(g)^{-1}; g \in G_x\}$ be the γ conjugacy class in G_x. Let

$$\widehat{U}_x^{(h)_\gamma} = \{(y, h_1) \in \widetilde{U}_x \times G_x | (h_1 \circ \gamma_{\widetilde{U}})(y) = y, h_1 \in (h)_\gamma\}. \tag{2.6}$$

Let $\widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}}$ be the fixed point set of $h \circ \gamma_{\widetilde{U}}$ in $\widetilde{U}_x$, then $\widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}}$ is connected, and $x \in \widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}}$.

For $g \in G_x$, g acts on $\widehat{U}_x^{(h)_\gamma}$ by the transformation

$$(y, h) \to (g(y), g \circ h \circ \alpha(g)^{-1}).$$

Indeed, if $(h \circ \gamma_{\widetilde{U}})(y) = y$, as $\alpha(g)^{-1} \circ \gamma_{\widetilde{U}} = \gamma_{\widetilde{U}} \circ g^{-1} \circ \gamma_{\widetilde{U}}^{-1} \circ \gamma_{\widetilde{U}} = \gamma_{\widetilde{U}} \circ g^{-1}$, we know

$$(gh \circ \alpha(g)^{-1})\gamma_{\widetilde{U}} \circ g(y) = gh \circ \gamma_{\widetilde{U}}(y) = g(y). \tag{2.7}$$

Let $Z^\gamma_{h,G_x} = \{g \in G_x, gh \circ \alpha(g)^{-1} = h\}$, $K^\gamma_{h,G_x} = \mathrm{Ker}\{Z^\gamma_{h,G_x} \to \mathrm{Diffeo}(\widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}})\}$. Then

$$(Z^\gamma_{h,G_x}/K^\gamma_{h,G_x}, \widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}}) \to \widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}}/Z^\gamma_{h,G_x} = \widehat{U}_x^{(h)_\gamma}/G_x \tag{2.8}$$

defines an orbifold. We denote it by $\widetilde{X}^\gamma$. Clearly, $m(\widetilde{X}^\gamma) = |K^\gamma_{h,G_x}|$ is local constant on $\widetilde{X}^\gamma$.

DEFINITION 2.6. The oriented orbifold X is spin if there exists 2-sheeted covering of $SO(X)$ ($SO(X)$ is the oriented orthonormal frame bundle of TX), such that for $U \in \mathcal{U}$, there exists a principal Spin(n) bundle $\mathrm{Spin}(\widetilde{U})$ on $\widetilde{U}$, such that $\mathrm{Spin}(X)_{|U} \to SO(X)_{|U}$ is induced by $\mathrm{Spin}(\widetilde{U}) \to SO(\widetilde{U})$, and $\mathrm{Spin}(\widetilde{U})$ also verifies the corresponding compatible condition.

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Then $\mathrm{spin}(X)$ is clearly a smooth manifold.

Assume that orbifold X is spin. Let h^{TX} be a metric on TX and $S(TX) = S^+(TX) \oplus S^-(TX)$ the corresponding orbifold spinor bundle on X. Let $c(\cdot)$ be the Clifford action of TX on $S(TX)$. Let $\nabla^{S(TX)}$ be the connection on $S(TX)$ induced by the Levi-Civita connection ∇^{TX} on TX. Let W be a complex orbifold vector bundle on X. Let ∇^W be a connection on W. Then $\nabla^{S(TX)\otimes W} = \nabla^{S(TX)} \otimes 1 + 1 \otimes \nabla^W$ is a connection on $S(TX) \otimes W$. Let $\Gamma(S^{\pm}(TX) \otimes W)$ be the set of C^∞ sections of $S^{\pm}(TX) \otimes W$ on X. Let $D^X \otimes W$ be the Dirac operator on $\Gamma(S^+(TX) \otimes W)$ to $\Gamma(S^-(TX) \otimes W)$, defined by

$$D^X \otimes W = \sum_{i=1}^{\dim X} c(e_i) \nabla_{e_i}^{S^+(TX)\otimes W}. \tag{2.9}$$

Here $\{e_i\}$ is an orthonormal basis of TX.

Let H be a compact Lie group. If $\gamma \in H$ acts on X and lifts to $\mathrm{Spin}(X)$ and W, then $\nabla^{S(TX)}$ is γ invariant and we can always find a γ invariant connection ∇^W on W. Note that $D^X \otimes W$ is a γ invariant elliptic operator on X. For $x \in X$, let $K_x^W = \mathrm{Ker}(G_x^W \xrightarrow{\tau} G_x)$. On $\widehat{U}^{(h)_\gamma}$, let N be the normal bundle of $\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}$ in $\widetilde{U}_x$. Let W^0 be the subbundle of W on $\widehat{U}_x$ which is K_x^W-invariant. Then W^0 extends to a proper orbifold vector bundle on X. We have the following decompositions:

$$N = \bigoplus_{0<\theta<\pi} N_\theta \oplus N_\pi, \qquad W^0 = \bigoplus_{0\le\theta<2\pi} W_\theta, \tag{2.10}$$

where N_θ, W_θ (resp. N_π) are complex (resp. real) vector bundles on which $h \circ \gamma_{\widetilde{U}}$ acts as multiplication by $e^{i\theta}$. Then ∇^{TX} induces connection ∇^{N_θ} on N_θ, and $\nabla^{TX} = \oplus \nabla^{N_\theta} \oplus \nabla^{TX^\gamma}$. Let $R^W, R^{W^0}, R^{N_\theta}, R^{TX^\gamma}$ be the curvatures of ∇^W, ∇^{W^0}, ∇^{N_θ}, ∇^{TX^γ} (∇^{W^0} is the connection on W^0 induced by ∇^W).

DEFINITION 2.7. For $h \in G_x, g = h \circ \gamma_{\widetilde{U}}$, $0 < \theta \le \pi$, set

$$\begin{aligned}
&\mathrm{ch}_g(W, \nabla^W) = \\
&\qquad \frac{1}{|K_x^W|} \sum_{h_1 \in G_x^W, \tau(h_1)=h} \mathrm{Tr}\Big[(h_1 \circ \gamma_{\widetilde{U}}) \exp(\frac{-R^W}{2\pi i})\Big] = \mathrm{Tr}\Big[g \exp(\frac{-R^{W^0}}{2\pi i})\Big], \\
&\qquad \widehat{A}(T\widetilde{U}^g, \nabla^{T\widetilde{U}^g}) = \det{}^{1/2}\Big(\frac{\frac{i}{4\pi} R^{T\widetilde{U}^g}}{\sinh(\frac{i}{4\pi} R^{T\widetilde{U}^g})}\Big), \\
&\qquad \widehat{A}_\theta(N_\theta, \nabla^{N_\theta}) = \Big[i^{\frac{1}{2}\dim N_\theta} \det{}^{1/2}\Big(1 - g \exp(\frac{i}{2\pi} R^{N_\theta})\Big)\Big]^{-1}, \\
&\qquad \widehat{A}_g(N, \nabla^N) = \prod_{0<\theta\le\pi} \widehat{A}_\theta(N_\theta, \nabla^{N_\theta}).
\end{aligned} \tag{2.11}$$

If we denote by $\{x_j, -x_j\}$ $(j = 1, \cdots, l)$ the Chern roots of N_θ, $T\widetilde{U}^g$ (where we consider N_θ as a real vector bundle) such that Πx_j defines the orientation of N_θ

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and $T\widetilde{U}^g$, then

$$\widehat{A}(T\widetilde{U}^g, \nabla^{T\widetilde{U}}) = \prod_j \frac{x_j}{2} / \sinh(\frac{x_j}{2}), \tag{2.12}$$

$$\widehat{A}_\theta(N_\theta, \nabla^{N_\theta}) = 2^{-l} \prod_{j=1}^{l} \frac{1}{\sinh \frac{1}{2}(x_j + i\theta)} = \prod_{j=1}^{l} \frac{e^{\frac{1}{2}(x_j+i\theta)}}{e^{x_j+i\theta} - 1}.$$

Recall that for $\gamma \in H$, the Lefschetz number $\mathrm{Ind}_\gamma(D^X \otimes W)$, which is the index of $D^X \otimes W$ if $\gamma = 1$, is defined by

$$\mathrm{Ind}_\gamma(D^X \otimes W) = \mathrm{Tr}\gamma|\mathrm{Ker}D^X \otimes W - \mathrm{Tr}\gamma|\mathrm{Coker}D^X \otimes W\,. \tag{2.13}$$

By using the heat kernel, as in [**8**, Th. 14.1], we get

THEOREM 2.8. *For $\gamma \in H$, we have the following equality:*

$$\mathrm{Ind}_\gamma(D^X \otimes W) = \sum_{F \in \widetilde{X}^\gamma} \frac{1}{m(F)} \int_F \alpha_F, \tag{2.14}$$

where α_F is the characteristic class

$$\widehat{A}(T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}, \nabla^{T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}})\Pi_{0<\theta\le\pi}\widehat{A}_\theta(N_\theta, \nabla^{N_\theta})\mathrm{ch}_{h\circ\gamma}(W, \nabla^W)$$

on $\widetilde{U}_x^{h\circ\gamma_{\widetilde{U}}}$.

Let S^1 act differentiably on X. Let $X^{S^1} = \{x \in X, \gamma(x) = x, \text{for all}\, \gamma \in S^1\}$. Let V be the canonical basis of $\mathrm{Lie}(S^1) = \mathbf{R}$. For $x \in X$, let V_X be the smooth vector field on $(G_x, \widetilde{U}_x)$ corresponding to V. Then V_X is G_x-invariant [**8**, p181]. We still denote by V_X the corresponding smooth vector field on X. We have $X^{S^1} = \{x \in X, V_X(x) = 0\}$.

For $x \in X$, let $(1), \cdots (h_x^j), \cdots$ be the conjugacy classes of G_x. Let $\widetilde{X}^{S^1} = \{(x, (h_x^j)) | x \in X^{S^1}, h_x^j \in G_x\}$. Then $\widetilde{X}^{S^1}$ has a natural orbifold structure defined by

$$\{Z_{G_x}(h_x^j)/K_x^{j,V}, \widetilde{U}_V^{h_x^j}\} \to (\widetilde{U}_V^{h_x^j}/Z_{G_x}(h_x^j), (h_x^j)) \tag{2.15}$$

where $\widetilde{U}_V^{h_x^j} = \widetilde{U}_x^{h_x^j} \cap \{y \in \widetilde{U}_x | V_X(y) = 0\}$ and $K_x^{j,V}$ is the kernel of the natural map $Z_{G_x}(h_x^j) \to \mathrm{Diffeo}\{\widetilde{U}_V^{h_x^j}\}$.

We have the following decomposition of smooth vector bundles on $\widetilde{U}_V^h$:

$$\begin{aligned} N_{\lambda(h)} &= \oplus_j N_{\lambda,j}, \\ N^{\mathbf{R}}_{\frac{1}{2}(h)} &= \oplus_{j>0} N_{\frac{1}{2},j} \oplus N^{\mathbf{R}}_{\frac{1}{2},0}, \\ T\widetilde{U}^h &= \oplus_{j>0} N_{0,j} \oplus T\widetilde{U}_V^{h_x^j}, \\ W^0 &= \oplus_{\lambda,j} W^0_{\lambda,j}. \end{aligned} \tag{2.16}$$

Note that $N_{\lambda,j}, N_{\frac{1}{2},j}, N_{0,j}$ and $W^0_{\lambda,j}$ extend to complex vector bundles on $\widetilde{X}^{S^1}$, and $\gamma = e^{2\pi i t} \in S^1$ acts on them as multiplication by $e^{2\pi i j t}$. Also, $N^{\mathbf{R}}_{\frac{1}{2},0}$ and $T\widetilde{U}_V^{h_x^j}$ extend to real vector bundles on $\widetilde{X}^{S^1}$, and S^1 acts trivially on them. In fact, $T\widetilde{U}^h = T\widetilde{U}_V^{h_x^j} \oplus_{v\neq 0} N_{0,v,\mathbf{R}}$, where $N_{0,v,\mathbf{R}}$ denotes the underling real bundle of the complex vector bundle N_v on which $g \in S^1$ acts by multiplying by g^v. Since we can

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choose either N_v or $\overline{N}_v$ as the complex vector bundle for $N_{v,\mathbf{R}}$, in what follows, we always assume $N_{\frac{1}{2},j}, N_{0,j}$ are zero if $j<0$.

By (2.16), for given $a \in \mathbf{C}$, the eigenspace of $h \circ \gamma_{\widetilde{U}}$ with eigenvalue a is equal to the sum of the above elements $N_{\lambda,j}$ such that

$$e^{2\pi i(\lambda+tj)} = a. \tag{2.17}$$

Let $A \subset \mathbf{R}$ consist of $a \in \mathbf{R}$ such that there exists $x \in X^{S^1}$, such that more than one non-zero $N_{\lambda,j}$ on $\widetilde{U}_V^{h_x^j}$ is in the eigenspace of $h \circ \gamma_{\widetilde{U}}$ with eigenvalue $e^{2\pi i a}$. As X is compact, A is a discrete set of $\mathbf{R}$.

If $\gamma = e^{2\pi i t}, t \in \mathbf{R} \setminus A$, then $\widetilde{X}^{\gamma} = \widetilde{X}^{S^1}$ by the construction. An immediate consequence of Theorem 2.8 is the following.

THEOREM 2.9. *Under the condition of Theorem 2.8, for $t \in \mathbf{R} \setminus A$, $\gamma = e^{2\pi i t}$, we have*

$$\mathrm{Ind}_{\gamma}(D^X \otimes W) = \sum_{F \in \widetilde{X}^{S^1}} \frac{1}{m(F)} \int_F \alpha_F, \tag{2.18}$$

where α_F is the characteristic class

$$\frac{\widehat{A}(T\widetilde{U}_V^h, \nabla^{T\widetilde{U}_V^h}) \sum_{\lambda,j} e^{2\pi i(\lambda+tj)} \mathrm{ch}(W^0_{\lambda,j}, \nabla^{W^0})}{\Pi_{\lambda,j} i^{\frac{1}{2}\dim N_{\lambda,j}} \det^{1/2}\Big(1 - e^{2\pi i(\lambda+tj)} \exp(\frac{i}{2\pi} R^{N_{\lambda,j}})\Big)}$$

on $\widetilde{U}_V^h$.

3. Equivariant index theorem for almost complex orbifolds

If X is an almost complex orbifold, then on the orbifold chart $(G_x, \widetilde{U}_x)$ for $x \in X$, we have the following h-equivariant decomposition of $T\widetilde{U}_x$ as a complex vector bundle on $\widetilde{U}_x^h$

$$T\widetilde{U}_x \simeq \bigoplus_{\lambda \in \mathbf{Q} \cap [0,1[} N_{\lambda(h)}. \tag{3.1}$$

Here $N_{\lambda(h)}$ are complex vector bundles over $\widetilde{U}_x^h$ with h acting by $e^{2\pi i \lambda}$ on it, and $N_{0(h)}$ is $T\widetilde{U}_x^h$. Again $N_{\lambda(h)}$ extends to a complex vector bundle on $\widetilde{\Sigma X}$. We will still denote it by $N_{\lambda(h)}$.

Let W be an orbifold complex vector bundle on X. Let $D^X \otimes W$ be the Spin^c Dirac operator on $\Lambda(T^{*(0,1)}X) \otimes W$ [**16**, Appendix D].

Let H be a compact Lie group acting on X. We assume that the action H on X lifts on W, and preserves the complex structures of TX and W. Now for $\gamma \in H$, the decomposition (2.10) on $\widehat{U}_x^{(h)_\gamma}$ also preserves the complex structure of the normal bundle N. We denote by R^N the curvature of ∇^N as complex vector bundle. Then

$$N = \bigoplus_{0<\theta<2\pi} N_{\theta}, \tag{3.2}$$
$$W^0 = \bigoplus_{0\le\theta<2\pi} W_{\theta}.$$

Here N_θ, W_θ are complex vector bundles on which $h \circ \gamma_{\widetilde{U}}$ acts as multiplication by $e^{i\theta}$. The following theorem is proved in [**8**, Th. 14.1].

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THEOREM 3.1. *Let*

$$\mathrm{Td}(T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}, \nabla^{T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}}) = \det\left(\frac{-R^{T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}}/2i\pi}{1-\exp(-R^{T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}}/2i\pi)}\right)$$

be the Chern-Weil Todd form of $T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}$. *Then we have*

$$\mathrm{Ind}_\gamma(D^X\otimes W) = \sum_{F\in\widetilde{X}^\gamma}\frac{1}{m(F)}\int_F\alpha_F. \tag{3.3}$$

Here on $\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}$, α_F *is the characteristic class*

$$\frac{\mathrm{Td}(T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}, \nabla^{T\widetilde{U}^{h\circ\gamma_{\widetilde{U}}}})\mathrm{ch}_{h\circ\gamma}(W,\nabla^W)}{\det(1-(h\circ\gamma)\exp(\frac{i}{2\pi}R^N))}.$$

If $H=S^1$, on $\widetilde{U}_V^h$ as in (2.15), we have the following decomposition of complex vector bundles,

$$N_{\lambda(h)} = \oplus_j N_{\lambda(h),j}$$

$$T\widetilde{U}^h = \oplus N_{0,j}\oplus T\widetilde{U}_V^{h_x^j}.$$

Here $N_{\lambda(h),j}, N_{0,j}$ extend to complex vector bundles on $\widetilde{X}^{S^1}$, and $\gamma = e^{2\pi it}\in S^1$ acts on them as multiplication by $e^{2\pi ijt}$. By Theorem 3.1, we get

THEOREM 3.2. *Under the condition of Theorem 3.1, for* $t\in\mathbf{R}\setminus A$, $\gamma = e^{2\pi it}$, *we have*

$$\mathrm{Ind}_\gamma(D^X\otimes W) = \sum_{F\in\widetilde{X}^{S^1}}\frac{1}{m(F)}\int_F\alpha_F.$$

Here on $\widetilde{U}_V^h$, α_F *is the characteristic class*

$$\frac{\mathrm{Td}(T\widetilde{U}_V^h, \nabla^{T\widetilde{U}_V^h})\sum_{\lambda,j}e^{2\pi i(\lambda+tj)}\mathrm{ch}(W^0_{\lambda,j},\nabla^{W^0})}{\Pi_{\lambda,j}\det\left(1-e^{2\pi i(\lambda+tj)}\exp(\frac{i}{2\pi}R^{N_{\lambda,j}})\right)}.$$

Note that we can also get Theorems 2.9 and 3.2 from [**26**, Theorem 1].

4. Elliptic genus for almost complex orbifolds

In this section, we define an elliptic genus for a general almost complex orbifold and prove its rigidity property. We are using the setting of Section 2.

For $\tau\in\mathbf{H}=\{\tau\in\mathbf{C};\mathrm{Im}\tau>0\}$, $q=e^{2\pi i\tau}$, $t\in\mathbf{C}$, let

$$\theta(t,\tau) = c(q)q^{1/8}2\sin(\pi t)\prod_{k=1}^\infty(1-q^ke^{2\pi it})\prod_{k=1}^\infty(1-q^ke^{-2\pi it}). \tag{4.1}$$

be the classical Jacobi theta function [**6**], where $c(q)=\prod_{k=1}^\infty(1-q^k)$. Set

$$\theta'(0,\tau) = \frac{\partial\theta(\cdot,\tau)}{\partial t}|_{t=0}. \tag{4.2}$$

Recall the following transformation formulas for the theta-functions [**6**]:

$$\theta(t+1,\tau) = -\theta(t,\tau),\qquad \theta(t+\tau,\tau) = -q^{-1/2}e^{-2\pi it}\theta(t,\tau), \tag{4.3}$$
$$\theta(\frac{t}{\tau},-\frac{1}{\tau}) = \frac{1}{i}\sqrt{\frac{\tau}{i}}e^{\frac{\pi it^2}{\tau}}\theta(t,\tau),\quad \theta(t,\tau+1) = e^{\frac{\pi i}{4}}\theta(t,\tau).$$

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For a complex or real vector bundle F on a manifold X, let

$$\text{(4.4)} \qquad \begin{aligned} \mathrm{Sym}_q(F) &= 1 + qF + q^2\mathrm{Sym}^2 F + \cdots, \\ \Lambda_q(F) &= 1 + qF + q^2\Lambda^2 F + \cdots, \end{aligned}$$

be the symmetric and the exterior power operations on F, respectively.

Let X be an almost complex orbifold, and $\dim_{\mathbf{C}} X = l$. In this Section, all vector bundles are complex.

Under the notation of (3.1), let $F(x,h) = \sum_\lambda \lambda \dim_{\mathbf{C}} N_{\lambda(h)}$ be the fermionic shift, then $F : X \cup \widetilde{\Sigma X} \to \mathbf{Q}$ is locally constant. For a connected component $X_i \subset X \cup \widetilde{\Sigma X}$, we define $F(X_i)$ to be the values of F on X_i.

Let W be a proper orbifold complex vector bundle on X with $\dim_{\mathbf{C}} W = m$; then W^0 in (3.2) is W. Now for the vector bundle W, the fermionic shift $F(X_i, W) = \sum_\lambda \lambda \dim W_\lambda$ is well defined on each connected component $X_i \subset X \cup \widetilde{\Sigma X}$. For $x \in X$, $y = e^{2\pi i z}$, we use the orbifold chart $(G_x, \widetilde{U}_x)$. For $h \in G_x$, by (3.1), we define on $\widetilde{U}_x^h$,

(4.5)

$$\begin{aligned} \Theta^z_{q,x,(h)}(TX) = &\bigotimes_{\lambda \in \mathbf{Q}\cap[0,1[} \left(\bigotimes_{k=1}^\infty \left(\Lambda_{-y^{-1}q^{k-1+\lambda(h)}} N^*_{\lambda(h)} \otimes \Lambda_{-yq^{k-\lambda(h)}} N_{\lambda(h)} \right)\right) \\ &\bigotimes_{\lambda \in \mathbf{Q}\cap]0,1[} \left(\bigotimes_{k=1}^\infty \left(\mathrm{Sym}_{q^{k-1+\lambda(h)}} N^*_{\lambda(h)} \otimes \mathrm{Sym}_{q^{k-\lambda(h)}} N_{\lambda(h)} \right)\right) \\ &\bigotimes_{k=1}^\infty \left(\mathrm{Sym}_{q^k} N_0^* \otimes \mathrm{Sym}_{q^k} N_0 \right), \end{aligned}$$

$$\begin{aligned} \Theta^z_{q,x,(h)}(TX|W) = &\bigotimes_{\lambda \in \mathbf{Q}\cap[0,1[} \left(\bigotimes_{k=1}^\infty \left(\Lambda_{-y^{-1}q^{k-1+\lambda(h)}} W^*_{\lambda(h)} \otimes \Lambda_{-yq^{k-\lambda(h)}} W_{\lambda(h)} \right)\right) \\ &\bigotimes_{\lambda \in \mathbf{Q}\cap]0,1[} \left(\bigotimes_{k=1}^\infty \left(\mathrm{Sym}_{q^{k-1+\lambda(h)}} N^*_{\lambda(h)} \otimes \mathrm{Sym}_{q^{k-\lambda(h)}} N_{\lambda(h)} \right)\right) \\ &\bigotimes_{k=1}^\infty \left(\mathrm{Sym}_{q^k} N_0^* \otimes \mathrm{Sym}_{q^k} N_0 \right). \end{aligned}$$

One verifies that each coefficient of q^a ($a \in \mathbf{Q}$) in $\Theta^z_{q,x,(h)}(TX)$ and $\Theta^z_{q,x,(h)}(TX|W)$ defines an orbifold vector bundle on $\widetilde{\Sigma X}$. We denote the restrictions of $\Theta^z_{q,x,(h)}(TX)$ and $\Theta^z_{q,x,(h)}(TX|W)$ to the connected component X_i of $X \cup \widetilde{\Sigma X}$ by $\Theta^z_{q,X_i}(TX)$ and $\Theta^z_{q,X_i}(TX|W)$ respectively. Note that $\Theta^z_{q,X_i}(TX)$ is the usual Witten element on X_i (see **[10]** and **[27]**). Also if X is a manifold the elliptic genus defined here is the Witten element on X :

$$\text{(4.6)} \quad \Theta^z_q(TX) = \bigotimes_{k=1}^\infty \left(\Lambda_{-y^{-1}q^{k-1}} T^*X \otimes \Lambda_{-yq^k} TX \right) \bigotimes_{k=1}^\infty \left(\mathrm{Sym}_{q^k} T^*X \otimes \mathrm{Sym}_{q^k} TX \right).$$

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DEFINITION 4.1. The orbifold elliptic genus of X is defined to be

$$F(y,q) = y^{\frac{l}{2}} \sum_{X_i \subset X \cup \widetilde{\Sigma X}} y^{-F(X_i)} \mathrm{Ind}(D^{X_i} \otimes \Theta^{z}_{q,X_i}(TX)). \tag{4.7}$$

More generally, we define the orbifold elliptic genus associated to W as

$$F(y,q,W) = y^{\frac{m}{2}} \sum_{X_i \subset X \cup \widetilde{\Sigma X}} y^{-F(X_i,W)} \mathrm{Ind}(D^{X_i} \otimes \Theta^{z}_{q,X_i}(TX|W)). \tag{4.8}$$

If X is a global quotient M/G where the action of a finite group G on an almost complex manifold M preserves its complex structure, the equation (4.7) coincides with [**3**, Definition 4.1].

We next prove that the orbifold elliptic genus is rigid for an S^1 action on X.

Let S^1 act on X, preserving the complex structure on TX, and lifting to W. We define the Lefschetz number for $\gamma \in S^1$,

$$F_\gamma(y,q,W) = y^{\frac{m}{2}} \sum_{X_i \subset X \cup \widetilde{\Sigma X}} y^{-F(X_i,W)} \mathrm{Ind}_\gamma(D^{X_i} \otimes \Theta^{z}_{q,X_i}(TX|W)) \,. \tag{4.9}$$

Let P be a compact manifold with an infinitesimally free action by a compact Lie group G (need not be connected), with $X = P/G$ the corresponding orbifold. We still denote by W the corresponding vector bundle on P for W. Then $K_X = \det(T^{(1,0)}X)$ and $K_W = \det W$ are naturally induced by complex line bundles on P, which we will still denote by K_X, K_W. We may also consider K_X, K_W as orbifold line bundles on X.

We will assume that S^1 acts on P and commutes with the G-action such that it induces the S^1 action on X (For example, S^1 acts naturally on $SO(X)$, the oriented orthonormal frame bundle of TX, and induces the S^1 action on X.).

Recall that the equivariant cohomology group $H^*_{S^1}(P,\mathbf{Z})$ of P is defined by

$$H^*_{S^1}(P,\mathbf{Z}) = H^*(P \times_{S^1} ES^1, \mathbf{Z}). \tag{4.10}$$

where ES^1 is the universal S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(P,\mathbf{Z})$ is a module over $H^*(BS^1,\mathbf{Z})$ induced by the projection $\overline{\pi}$: $P \times_{S^1} ES^1 \to BS^1$. Let $p_1(W)_{S^1}, p_1(TX)_{S^1} \in H^*_{S^1}(P,\mathbf{Z})$ be the equivariant first Pontrjagin classes of W and TX respectively. Also recall that

$$H^*(BS^1,\mathbf{Z}) = \mathbf{Z}[u] \tag{4.11}$$

with u a generator of degree 2.

Recall from [**17**, Theorem B] that for smooth manifold X one needs the conditions

$$p_1(W-TX)_{S^1} = 0, \quad c_1(W-TX)_{S^1} = 0. \tag{4.12}$$

for the rigidity theorem.

Note that if the connected component X_i of $X \cup \widetilde{\Sigma X}$ is defined by $(\widetilde{U}^h_x, Z_{G_x}(h))$, for $g \in Z_{G_x}(h)$, set $U^{h,g}_V = \{b \in \widetilde{U}_x | hb = gb = b, V_X(b) = 0\}$. Then the connected component X_{ik} of $\widetilde{X^{S^1}_i}$ is defined by $(U^{h,g}_V, Z_{Z_{G_x}(h)}(g))$. We have the following

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decomposition of complex vector bundles on $U_V^{h,g}$,

$$
(4.13) \qquad T\widetilde{U}_x = \sum_{\lambda(h),\lambda(g)\in\mathbf{Q}\cap[0,1[,v\in\mathbf{Z}} N_{\lambda(h),\lambda(g),v},
$$
$$
W = \sum_{\lambda(h),\lambda(g)\in\mathbf{Q}\cap[0,1[,v\in\mathbf{Z}} W_{\lambda(h),\lambda(g),v}.
$$

where $g,h\in G_x$ (resp. $\gamma=e^{2\pi it}\in S^1$) act on $N_{\lambda(h),\lambda(g),v}, W_{\lambda(h),\lambda(g),v}$ as multiplication by $e^{2\pi i\lambda(g)}$, $e^{2\pi i\lambda(h)}$ (resp. $e^{2\pi ivt}$). Let $2\pi i x^j_{\lambda(h),\lambda(g),v}$, $2\pi i w^j_{\lambda(h),\lambda(g),v}$ be the formal Chern roots of $N_{\lambda(h),\lambda(g),v}$, $W_{\lambda(h),\lambda(g),v}$ respectively. To simplify the notation, we will omit the superscript j.

Then $N_{\lambda(h),\lambda(g),v}$, $W_{\lambda(h),\lambda(g),v}$ extend to orbifold vector bundles on X_{ik}. Now the natural generalization of (4.12) for orbifolds is the following: there exists $n\in\mathbf{N}$, such that on each connected component X_{ik} of $\widetilde{X_i^{S^1}}$,

$$
(4.14) \qquad \sum_{\lambda(h),\lambda(g),v,j} \Big[(w^j_{\lambda(h),\lambda(g),v}+\lambda(g)-\tau\lambda(h)+vu)^2 - (x^j_{\lambda(h),\lambda(g),v}+\lambda(g)-\tau\lambda(h)+vu)^2\Big] = n\overline{\pi}^*u^2 \in H^*(X_{ik},\mathbf{Q})[\tau,u],
$$

$$
(4.15) \qquad \sum_{\lambda(h),\lambda(g),v,j} (w^j_{\lambda(h),\lambda(g),v}+\lambda(g)-\tau\lambda(h)+vu) - \sum_{\lambda(h),\lambda(g),v,j} (x^j_{\lambda(h),\lambda(g),v}+\lambda(g)-\tau\lambda(h)+vu) = 0 \in H^*(X_{ik},\mathbf{Q})[\tau,u].
$$

THEOREM 4.2. *Assume that S^1 acts on P inducing the S^1-action on X, and lifts to W, and $c_1(W)=0$ in $H^*(P,\mathbf{Z})$. Also assume that there exists $n\in\mathbf{Z}$ such that equations (4.14) and (4.15) hold. Then we have*

i) If $n=0$, then $F_\gamma(y,q,W)$ is constant on $\gamma\in S^1$.

ii) If $n<0$, then $F_\gamma(y,q,W)=0$.

Note that, in case $W=TX$, the conditions (4.14), (4.15) are automatic, and as a consequence we get the rigidity and vanishing theorems for the usual orbifold elliptic genus $F(y,q)$. In particular we know that for a Calabi-Yau almost complex orbifold manifold X, $F(y,q)$ is rigid for any y.

Proof: To prove Theorem 3.1, we only need prove for any $N\in\mathbf{N}$, $N>1$, i) and ii) holds for any N-th root of unity $y=e^{2\pi iz}$. From now on, we assume that y is an N-th root of unity. Using Theorem 3.2, for $\gamma=e^{2\pi it}, t\in\mathbf{R}\setminus A$, $y=e^{2\pi iz}, q=e^{2\pi i\tau}$, we get

$$
(4.16) \qquad F_\gamma(y,q,W) = y^{\frac{m}{2}} \sum_{X_i\subset X\cup\widetilde{\Sigma X}} y^{-F(X_i,W)} \sum_{F\subset\widetilde{X}_i^{S^1}} \frac{1}{m(F)}\int_F \alpha_F,
$$

Recall that V_X is the smooth vector field generated by S^1-action on X. For $x\in X$, take the orbifold chart $(G_x,\widetilde{U}_x)$. If $X_i\subset X\cup\widetilde{\Sigma X}$ is represented by $\widetilde{U}_x^h/Z_{G_x}(h)$ on $\widetilde{U}_x$ as in (2.3), the normal bundle $N_{X_i,g,V}=N_{U_V^{h,g}/\widetilde{U}_x^h}$ of $U_V^{h,g}$ in $\widetilde{U}_x^h$ extends to an orbifold vector bundle on $\widetilde{X}_i^{S^1}$. By Theorem 3.2, the contribution of the chart

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$(G_x, \widetilde{U}_x)$ for $\mathrm{Ind}_\gamma(D^{X_i} \otimes \Theta^z_{q,X_i}(TX|W))$ is

$$\frac{1}{|Z_{G_x}(h)|} \sum_{gh=hg, g\in G_x} \int_{U_V^{h,g}} \frac{\mathrm{Td}(TU_V^{h,g})\mathrm{ch}_{g\circ\gamma}(\Theta^z_{q,x,(h)}(TX|W))}{\det(1-(g\circ\gamma)e^{-\frac{1}{2\pi i}R^{N_{X_i,g,V}}})}. \tag{4.17}$$

Let

$$\begin{aligned} N_v &= \sum_{\lambda(h),\lambda(g)\in \mathbf{Q}\cap[0,1[} N_{\lambda(h),\lambda(g),v}, \\ W_v &= \sum_{\lambda(h),\lambda(g)\in \mathbf{Q}\cap[0,1[} W_{\lambda(h),\lambda(g),v}. \end{aligned} \tag{4.18}$$

As the vector field V_X commutes with the action of G_x, N_v and W_v extend to vector bundles on $\widetilde{X}_i^{S^1}$.

For a holomorphic function $f(x)$ we denote by

$$f(x)(N_{\lambda(h),\lambda(g),v}) = \prod_j f(x^j_{\lambda(h),\lambda(g),v})$$

the symmetric polynomial which gives characteristic class of $N_{\lambda(h),\lambda(g),v}$, etc. Let $F_\gamma(y,q,W)_{|\widetilde{U}_x}$ be the contribution of the chart $(G_x, \widetilde{U}_x)$ for $F_\gamma(y,q,W)$. Then by (4.17) we have

(4.19)

$$\begin{aligned} &F_\gamma(y,q,W)_{|\widetilde{U}_x} = \\ &\frac{1}{|G_x|} \sum_{gh=hg; g,h\in G_x} y^{\frac{m}{2}-F(X_i,W)} \int_{U_V^{h,g}} \frac{\mathrm{Td}(TU_V^{h,g})\mathrm{ch}_{g\circ\gamma}\Theta^z_{q,x,(h)}(TX|W)}{\det(1-(g\circ\gamma)e^{-\frac{1}{2\pi i}R^{N_{X_i,g,V}}})} \\ &= \frac{1}{|G_x|} \sum_{gh=hg; g,h\in G_x} y^{\frac{m}{2}-F(X_i,W)} \int_{U_V^{h,g}} \frac{(2\pi i x)(N_{0(h),0(g),0})}{\prod_{\lambda(g),v}(1-e^{2\pi i(x+\lambda(g)+tv)})(N_{0(h),\lambda(g),v})} \\ &\times \prod_{k=1}^{+\infty}\prod_{\lambda(g),v} \left\{ \left[\prod_{\lambda(h)>0} \left(\frac{(1-y^{-1}q^{k-1+\lambda(h)}e^{2\pi i(-w-\lambda(g)-tv)})}{(1-q^{k-1+\lambda(h)}e^{2\pi i(-x-\lambda(g)-tv)})} \right.\right.\right. \\ &\left.\left. \times \frac{(1-yq^{k-\lambda(h)}e^{2\pi i(w+\lambda(g)+tv)})(W_{\lambda(h),\lambda(g),v})}{(1-q^{k-\lambda(h)}e^{2\pi i(x+\lambda(g)+tv)})(N_{\lambda(h),\lambda(g),v})} \right) \right] \\ &\left. \times \frac{(1-y^{-1}q^{k-1}e^{2\pi i(-w-\lambda(g)-tv)})(1-yq^k e^{2\pi i(w+\lambda(g)+tv)})(W_{0(h),\lambda(g),v})}{(1-q^k e^{2\pi i(-x-\lambda(g)-tv)})(1-q^k e^{2\pi i(x+\lambda(g)+tv)})(N_{0(h),\lambda(g),v})} \right\} \\ &= (i^{-1}c(q)q^{1/8})^{l-m}\frac{1}{|G_x|} \sum_{gh=hg; g,h\in G_x} \int_{U_V^{h,g}} (2\pi i x)(N_{0(h),0(g),0}) \\ &\times \prod_{\lambda(h),\lambda(g),v} \frac{(\theta(w+\lambda(g)-\tau\lambda(h)+z+tv,\tau)e^{-2\pi i z\lambda(h)})(W_{\lambda(h),\lambda(g),v})}{\theta(x+\lambda(g)-\tau\lambda(h)+tv,\tau)(N_{\lambda(h),\lambda(g),v})}. \end{aligned}$$

Here we have used (4.1), (4.15) to get the last equality of (4.19).

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If we consider $F_\gamma(y,q,W)$ as a function of (t,z,τ), we can extend it to a meromorphic function on $\mathbf{C}\times\mathbf{H}\times\mathbf{C}$. From now on, we denote

$$(i^{-1}c(q)q^{1/8})^{m-l}\frac{\theta'(0,\tau)^l}{\theta(z,\tau)^m}F_\gamma(y,q,W)$$

by $F(t,\tau,z)$. Set

$$F(t,\tau,z)_{|\widetilde{U}_x} = (i^{-1}c(q)q^{1/8})^{m-l}\frac{\theta'(0,\tau)^l}{\theta(z,\tau)^m}F_\gamma(y,q,W)_{|\widetilde{U}_x}. \tag{4.20}$$

Now, the equation (4.14) implies the equalities

$$\begin{aligned}
&\sum_{\lambda(h),\lambda(g),v} w^2_{\lambda(h),\lambda(g),v} - \sum_{\lambda(h),\lambda(g),v} x^2_{\lambda(h),\lambda(g),v} = 0,\\
&\sum_{\lambda(h),\lambda(g)} v\ w_{\lambda(h),\lambda(g),v} - \sum_{\lambda(h),\lambda(g)} v\ x_{\lambda(h),\lambda(g),v} = 0,\\
&\sum_{\lambda(h),\lambda(g)} \lambda(h)\lambda(g)\left(\dim W_{\lambda(h),\lambda(g)} - \dim N_{\lambda(h),\lambda(g)}\right) = 0,\\
&\sum_{\lambda(h),\lambda(g),v} \lambda(h)v\ \left(\dim W_{\lambda(h),\lambda(g),v} - \dim N_{\lambda(h),\lambda(g),v}\right) = 0,\\
&\sum_{\lambda(h),\lambda(g),v} \lambda(g)v\ \left(\dim W_{\lambda(h),\lambda(g),v} - \dim N_{\lambda(h),\lambda(g),v}\right) = 0,\\
&\sum_{\lambda(g)} \lambda(g)^2(\dim W_{\lambda(g)} - \dim N_{\lambda(g)}) = 0,\\
&\sum_{v} v^2(\dim W_v - \dim N_v) = n.
\end{aligned} \tag{4.21}$$

By (4.1), for $a,b\in 2\mathbf{Z}$, $k\in\mathbf{N}$,

$$\theta(x+k(t+a\tau+b),\tau) = e^{-\pi i(2kax+2k^2at+k^2a^2\tau)}\theta(x+kt,\tau). \tag{4.22}$$

As $c_1(K_W)=0$ in $H^*(P,\mathbf{Z})$, by the same argument as [**10**, §8] or [**22**, Lemma 2.1, Remark 2.6], $\sum_u v\dim W_v$ is constant on each connected component of X.

By (4.19), (4.21), (4.22), we know for $a,b\in 2\mathbf{Z}$,

$$F(t+a\tau+b,\tau,z) = e^{-2\pi iza\sum_v v\dim W_v}e^{-\pi in(a^2\tau+2at)}F(t,\tau,z). \tag{4.23}$$

Especially, for $a,b\in 2N\mathbf{Z}$,

$$F(t+a\tau+b,\tau,z) = e^{-\pi in(a^2\tau+2at)}F(t,\tau,z).$$

For $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, we define its modular transformation on $\mathbf{C}\times\mathbf{H}$ by

$$A(t,\tau) = \left(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right). \tag{4.24}$$

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By (4.19), under the action $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we have

$$\begin{aligned} \frac{\theta(A(t_1,\tau))}{\theta(A(t_2,\tau))} &= e^{\pi ic(t_1^2-t_2^2)/(c\tau+d)}\frac{\theta(t_1,\tau)}{\theta(t_2,\tau)}, \\ \frac{\theta'(A(0,\tau))}{\theta(A(t,\tau))} &= (c\tau+d)e^{-\pi ict^2/(c\tau+d)}\frac{\theta'(0,\tau)}{\theta(t,\tau)}. \end{aligned} \tag{4.25}$$

For $g,h\in G_x$, by looking at the degree $2\dim_{\mathbf{C}} U_V^{h,g}$ part, that is the $\dim_{\mathbf{C}} U_V^{h,g}$-th homogeneous terms of the polynomials in x, w's, on both sides of the following equation, we get

$$\begin{aligned} &\int_{U_V^{h,g}} (x)(N_{0(h),0(g),0}) \prod_{\lambda(h),\lambda(g),v} \Bigg[\frac{e^{2\pi iczw(c\tau+d)}\theta(w(c\tau+d)+\lambda(g)(c\tau+d)}{\theta(x(c\tau+d)+(c\tau+d)\lambda(g)} \\ &\qquad \frac{-(a\tau+b)\lambda(h)+z(c\tau+d)+tv,\tau)(W_{\lambda(h),\lambda(g),v})}{-(a\tau+b)\lambda(h)+tv,\tau)(N_{\lambda(h),\lambda(g),v})}\Bigg] \\ &= \int_{U_V^{h,g}} (x)(N_{0(h),0(g),0}) \prod_{\lambda(h),\lambda(g),v} \Bigg[\frac{e^{2\pi iczw}\theta(w+\lambda(g)(c\tau+d)}{\theta(x+(c\tau+d)\lambda(g)} \\ &\qquad \frac{-(a\tau+b)\lambda(h)+z(c\tau+d)+tv,\tau)(W_{\lambda(h),\lambda(g),v})}{-(a\tau+b)\lambda(h)+tv,\tau)(N_{\lambda(h),\lambda(g),v})}\Bigg]. \end{aligned} \tag{4.26}$$

By (4.3), (4.19), (4.21), (4.25) and (4.26), we easily derive the following identity:

$$\begin{aligned} F(A(t,\tau),z)_{|\widetilde{U}_x} &= \frac{1}{|G_x|}(c\tau+d)^l e^{\pi icnt^2/(c\tau+d)}\frac{\theta'(0,\tau)^l}{\theta(z(c\tau+d),\tau)^m} \\ &\qquad \sum_{gh=hg;g,h\in G_x}\int_{U_V^{g,h}}(2\pi ix)(N_{0(h),0(g),0}) \\ &\times \prod_{\lambda(h),\lambda(g),v}\Bigg\{\left(e^{2\pi icz((w+(\lambda(g)-\frac{a\tau+b}{c\tau+d}\lambda(h)))(c\tau+d)+tv)}e^{-2\pi iz\lambda(h)}\right)(W_{\lambda(h),\lambda(g),v}) \\ &\quad \times\frac{\theta(w+(c\tau+d)\lambda(g)-(a\tau+b)\lambda(h)+z(c\tau+d)+tv,\tau)(W_{\lambda(h),\lambda(g),v})}{\theta(x+(c\tau+d)\lambda(g)-(a\tau+b)\lambda(h)+tv,\tau)(N_{\lambda(h),\lambda(g),v})}\Bigg\}. \end{aligned} \tag{4.27}$$

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By (4.3), (4.15), (4.21), (4.22), (4.27), we have

$$(4.28)\quad (c\tau+d)^{-l}e^{-\pi icnt^2/(c\tau+d)}\frac{\theta(z(c\tau+d),\tau)^m}{\theta'(0,\tau)^l}F(A(t,\tau),z)_{|\widetilde{U}_x}$$
$$=\frac{1}{|G_x|}\sum_{gh=hg;g,h\in G_x}\int_{U_V^{h,g}}(2\pi ix)(N_{0(h),0(g),0})$$
$$\prod_{\lambda(h),\lambda(g),v}\left[\left(e^{2\pi icz(d\lambda(g)-b\lambda(h)+\tau(c\lambda(g)-a\lambda(h)))}e^{2\pi icz(w+tv)}\right.\right.$$
$$\left.\times e^{-2\pi iz(c\lambda(g)-a\lambda(h)+\lambda(g^{-c}h^a))(c\tau+d)}e^{-2\pi iz\lambda(h)}\right)(W_{\lambda(h),\lambda(g),v})$$
$$\left.\times\frac{\theta(w+\lambda(g^dh^{-b})-\tau\lambda(g^{-c}h^a)+z(c\tau+d)+tv,\tau)(W_{\lambda(h),\lambda(g),v})}{\theta(x+\lambda(g^dh^{-b})-\tau\lambda(g^{-c}h^a)+tv,\tau)(N_{\lambda(h),\lambda(g),v})}\right]$$
$$=\frac{1}{|G_x|}\sum_{gh=hg;g,h\in G_x}\int_{U_V^{h,g}}(2\pi ix)(N_{0(h),0(g),0})$$
$$\prod_{\lambda(h),\lambda(g),v}\left[\left(e^{2\pi icz(w+tv)}e^{-2\pi iz(c\tau+d)\lambda(g^{-c}h^a)}\right)(W_{\lambda(h),\lambda(g),v})\right.$$
$$\left.\frac{\theta(w+\lambda(g^dh^{-b})-\tau\lambda(g^{-c}h^a)+z(c\tau+d)+tv,\tau)(W_{\lambda(h),\lambda(g),v})}{\theta(x+\lambda(g^dh^{-b})-\tau\lambda(g^{-c}h^a)+tv,\tau)(N_{\lambda(h),\lambda(g),v})}\right].$$

Recall that $c_1(K_W)=0$; this implies that $\sum_v v\dim W_v$ is constant on each connected component of X.

Now, observe that if $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, then as g,h run through all pairs of commuting elements in G_x, the elements $g^{-c}h^a, g^dh^{-b}$ run through all pairs of commuting elements in G_x as well. Then by (4.28),

$$(4.29)\quad F(A(t,\tau),z)=e^{2\pi icz\sum_v v\dim W_v}(c\tau+d)^le^{\pi icnt^2/(c\tau+d)}F(t,\tau,(c\tau+d)z).$$

The following lemma implies that the index theory comes in to cancel part of the poles of the functions F.

LEMMA 4.3. *For $z\in\mathbf{C}$, the function $F(t,\tau,z)$ is holomorphic in (t,τ) for $(t,\tau)\in\mathbf{R}\times\mathbf{H}$.*

The proof of the above Lemma is the same as the proof of [**17**, Lemma 1.3] or [**19**, Lemma 2.3]. To be complete, we include a proof here.

Proof of Lemma 4.3 : Let $x=e^{2\pi it}$, and $M_1=\max_{N_v\neq 0}|v|$, set $M_2=\inf_{\lambda\in]0,1[,N_{\lambda(h)}\neq 0}\{\lambda,1-\lambda\}$ under the notation of (4.13), and max, inf should also consider for all the component of the fixe set of S^1. Set $M=M_1/M_2$. Denote by $D_M\subset\mathbf{C}^2$ the domain

$$(4.30)\quad |q|^{1/M}<|x|<|q|^{-1/M}, 0<|q|<1.$$

By (4.4), (4.19), we know that in D_M, $F(t,\tau,z)$ has a convergent Laurent series expansion of the form

$$(4.31)\quad \sum_i\sum_{j=0}^{\infty}b_{ij}(x)q^{j/L}$$

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where $L \in \mathbf{N}$, the subscript i means the sum on X_i, each component of $X \cup \widetilde{\Sigma X}$, $L \in \mathbf{N}$, and $\{b_{ij}(x)\}$ are rational functions of x with possible poles on the unit circle.

Now considered as a formal power series of q,

$$y^{\frac{m}{2}-F(X_i,W)}\Theta_{q,X_i}(TX|W) = \sum_{j=0}^{\infty} a_{ij}(y)V_{ij}q^{j/L}$$

with V_{ij} are S^1 equivariant vector bundle on X_i, and $a_{ij}(y)$ are polynomial on $y^{1/L}$. Note that the terms in the above two sums correspond to each other. Now, we apply the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula to each $V_{j,\mu}^A$, for $t \in \mathbf{R} \setminus A$, we get

$$b_{ij}(x) = a_{ij}(y)\mathrm{Ind}_x(D^{X_i} \otimes V_{ij}). \tag{4.32}$$

This implies that for $t \in \mathbf{R} \setminus A, x = e^{2\pi i t}$,

$$b_{ij}(x) = \sum_{l=-N(j)}^{N(j)} a_{ij}^k x^k. \tag{4.33}$$

for $N(ij)$ some positive integer depending on i, j and $a_{ij}^k \in \mathbf{C}$. Since both sides are analytic functions of x, this equality holds for any $x \in \mathbf{C}$.

On the other hand, multiplying $F(t,\tau,z)$ by $f(x) = \prod_{\alpha,N_v\neq 0}(1 - x^v e^{2\pi i\lambda(g)})^{lv}$ ($l = \dim X$) wih α running over all the connected component of S^1 fixed point set of $X \cup \widetilde{\Sigma X}$, we get holomorphic functions which have a convergent power series expansion of the form $\sum_{j=0}^{\infty} c_j(x)q^{j/L}$, with $\{c_j(x)\}$ polynomial functions in D_M. Comparing the above two expansions, one gets

$$c_j(x) = f(x)\sum_i b_{ij}(x) \tag{4.34}$$

for each j. So by the Weierstrass preparation theorem, we get $F(t,\tau,z)$ is holomorphic in D_M. □

Now, we return to the proof of Theorem 4.2. Note that the possible polar divisors of F in $\mathbf{C} \times \mathbf{H}$ are of the form $t = \frac{k}{j}(c\tau + d)$ with k, c, d, j integers and $(c,d) = 1$. Then there are integers a, b such that $ad-bc = 1$. Set $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$. We have

$$F(t,\tau,(-c\tau+a)z) = e^{-2\pi i c z\sum_v v \dim W_v}(-c\tau+a)^{-l}e^{\pi i c n t^2/(-c\tau+a)}F\Big(A(t,\tau),z\Big).$$

Now, if $t = \frac{k}{j}(c\tau + d)$ is a polar divisor of $F(t,\tau,z)$, then one polar divisor of $F(t,\tau,(-c\tau+a)z)$ is given by

$$\frac{t}{-c\tau+a} = \frac{k}{j}\Big(c\frac{d\tau-b}{-c\tau+a} + d\Big), \tag{4.35}$$

which exactly gives $t = k/j$. This contradicts Lemma 4.3.

This means $F(t,\tau,z)$ is holomorphic function on $\mathbf{C} \times \mathbf{H}$.

For fixed $\tau \in \mathbf{H}$, if $F(\cdot,\tau,z)$ isn't identically zero, we let δ be the contour $z_0+2s, z_0+2+2s\tau$, $z_0+2+2(1-s)\tau$, $z_0+2(1-s)\tau$ ($s \in [0,1]$), such that $F(\cdot,\tau,z)$ does not have any zero on δ. Then by (4.23),

$$\frac{1}{2\pi i}\int_\delta \frac{1}{F(t,\tau,z)}\frac{\partial}{\partial t}F(t,\tau,z)dt = 4n. \tag{4.36}$$

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This means that $F(t,\tau,z)$ has exactly $4n$ zeros inside δ. Therefore, if $n<0$, $F(t,\tau,z)$ must be identically zero. If $n=0$, $F(t,\tau,z)$ is a double periodic holomorphic function, it must be independent of t. Thus we completes the proof of Theorem 4.2. □

5. Elliptic genus for spin orbifolds

We are following the setting of Section 1.

For $\tau \in \mathbf{H} = \{\tau \in \mathbf{C}; \mathrm{Im}\tau > 0\}$, $q = e^{2\pi i \tau}$, let

$$\theta_3(v,\tau) = c(q)\prod_{k=1}^{\infty}(1+q^{k-1/2}e^{2\pi i v})\prod_{k=1}^{\infty}(1+q^{k-1/2}e^{-2\pi i v}). \tag{5.1}$$

be the other three classical Jacobi theta-functions [6], where $c(q)=\Pi_{k=1}^{\infty}(1-q^k)$.

Let X be a compact orbifold, $\dim_{\mathbf{R}} X = 2n$. We assume that X and $\widetilde{\Sigma X}$ are spin in the sense of Definition 2.6. For $x \in X$, take the orbifold chart $(G_x, \widetilde{U}_x)$. By (2.5), for $h \in G_x$, we define on $\widetilde{U}_x^h$

$$\begin{aligned}(5.2)\quad \Theta'_{q,x,(h)}(TX) = & \bigotimes_{\lambda\in\mathbf{Q}\cap]0,\frac{1}{2}[}\Big(\bigotimes_{k=1}^{\infty}\Big(\Lambda_{q^{k-1+\lambda(h)}}N^*_{\lambda(h)}\otimes\Lambda_{q^{k-\lambda(h)}}N_{\lambda(h)}\Big)\\ & \bigotimes_{k=1}^{\infty}\Big(\mathrm{Sym}_{q^{k-1+\lambda(h)}}N^*_{\lambda(h)}\otimes\mathrm{Sym}_{q^{k-\lambda(h)}}N_{\lambda(h)}\Big)\Big)\\ & \bigotimes_{k=1}^{\infty}\Big(\Lambda_{q^{k-\frac{1}{2}}}N^{\mathbf{R}}_{\frac{1}{2}(h)}\otimes\mathrm{Sym}_{q^{k-\frac{1}{2}}}N^{\mathbf{R}}_{\frac{1}{2}(h)}\Big)\otimes\bigotimes_{k=1}^{\infty}(\Lambda_{q^k}(T\widetilde{U}_x^h)\otimes\mathrm{Sym}_{q^k}(T\widetilde{U}_x^h)).\end{aligned}$$

It is easy to verify that each coefficient of q^a $(a\in\mathbf{Q})$ in $\Theta'_{q,x,(h)}(TX)$ defines an orbifold vector bundle on $X\cup\widetilde{\Sigma X}$. We denote it by $\Theta'_{q,X_i}(TX)$ on the connected component X_i of $X\cup\widetilde{\Sigma X}$. Especially, $\Theta'_{q,X}(TX)$ is the usual Witten element

$$\Theta'_q(TX) = \otimes_{k=1}^{\infty}\Big(\Lambda_{q^k}(TX)\otimes\mathrm{Sym}_{q^k}(TX)\Big). \tag{5.3}$$

on X. We propose the following definition for the elliptic genus of a spin orbifold:

Definition 5.1. The orbifold elliptic genus of X is

$$F(q) = \sum_{X_i\subset X\cup\widetilde{\Sigma X}}\mathrm{Ind}\Big(D^{X_i}\otimes(S^+(TX_i)\oplus S^-(TX_i))\otimes\Theta'_{q,X_i}(TX)\Big). \tag{5.4}$$

Let S^1 act on X and preserve the spin structure of $X\cup\widetilde{\Sigma X}$. We define the Lefschetz number for $\gamma\in S^1$

$$F_{d_s,\gamma}(q) = \sum_{X_i\subset X\cup\widetilde{\Sigma X}}\mathrm{Ind}_\gamma\Big(D^{X_i}\otimes(S^+(TX_i)\oplus S^-(TX_i))\otimes\Theta_{q,X_i}(TX)\Big) \tag{5.5}$$

On local chart $(G_x,\widetilde{U}_x)$, for $h,g\in G_x$, $gh=hg$, so as in (2.5), on $\widetilde{U}_x^h$, we have

$$T\widetilde{U}_x = N_0\oplus\bigoplus_{\lambda(h)\in]0,\frac{1}{2}[}N_{\lambda(h)}\oplus N^{\mathbf{R}}_{\frac{1}{2}(h)}. \tag{5.6}$$

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here h acts on the real vector bundles $N_0 = T\widetilde{U}_x^h$, $N^{\mathbf{R}}_{\frac{1}{2}(h)}$ as multiplication by $1, e^{\pi i}$, and h acts on complex vector bundles $N_{\lambda(h)}$ as multiplication by $e^{2\pi i\lambda(h)}$. Now on $\widetilde{U}_x^{h,g}$, the fixed point set of g on $\widetilde{U}_x^h$, we have the following decomposition

$$(5.7)\quad N_{\lambda(h)} = \bigoplus_{\lambda(g)\in[0,\frac{1}{2}]} N_{\lambda(h),\lambda(g)} \oplus \bigoplus_{\lambda(g)\in]0,\frac{1}{2}[} N_{\lambda(h),1-\lambda(g)} \quad \text{for} \quad 0<\lambda(h)<\frac{1}{2},$$

$$N_0 = \bigoplus_{\lambda(g)\in[0,\frac{1}{2}]} N_{0,\lambda(g)},$$

$$N^{\mathbf{R}}_{\frac{1}{2}(h)} = \bigoplus_{\lambda(g)\in[0,\frac{1}{2}]} N_{\frac{1}{2}(h),\lambda(g)}.$$

Here $N_{\lambda(h),\lambda(g)}$ $(\lambda(h),\lambda(g)\in\{0,\frac{1}{2}\})$ are real vector bundles on $\widetilde{U}_x^{h,g}$, and $N_{\lambda(h),\lambda(g)}$ (for ($\lambda(h)$ or $\lambda(g)$ not in $\{0,\frac{1}{2}\}$)) are complex vector bundles on $\widetilde{U}_x^{h,g}$. The elements h, g act on $N_{\lambda(h),\lambda(g)}$ as multiplication by $e^{2\pi i\lambda(h)}$, $e^{2\pi i\lambda(g)}$ respectively. Again $N_{\lambda(h),\lambda(g)}$ extends to a vector bundle on $\widetilde{X}_i^{S^1}$.

For a holomorphic function $p(x)$ we denote by

$$p(x)(N_{\lambda(h),\lambda(g)}) = \prod_j p(x^j_{\lambda(h),\lambda(g)})$$

the symmetric polynomial which gives characteristic class of $N_{\lambda(h),\lambda(g)}$. Then the contribution of the chart $(G_x, \widetilde{U}_x)$ for $F_{d_s,\gamma}(q)$ is

$$(5.8)\quad F_{d_s}(t,\tau)_{|\widetilde{U}_x} = \frac{i^{-n}}{|G_x|} \sum_{gh=hg;g,h\in G_x} \int_{U_V^{h,g}} \left(2\pi i x \frac{\theta_1(x_{0(h),0(g)},\tau)}{\theta(x,\tau)}\right)(N_{0(h),0(g)})$$
$$\prod_{\substack{\lambda(h)\in\{0,\frac{1}{2}\},0\le\lambda(g)\le\frac{1}{2}\\ (\lambda(h),\lambda(g))\neq(0,0)}} \frac{\theta_1(x+\lambda(g)-\tau\lambda(h),\tau)}{\theta(x+\lambda(g)-\tau\lambda(h),\tau)}(N_{\lambda(h),\lambda(g)})$$
$$\prod_{0<\lambda(h)<\frac{1}{2}} \Bigg[\prod_{0\le\lambda(g)\le\frac{1}{2}} \frac{\theta_1(x+\lambda(g)-\tau\lambda(h),\tau)}{\theta(x+\lambda(g)-\tau\lambda(h),\tau)}(N_{\lambda(h),\lambda(g)})$$
$$\prod_{0<\lambda(g)<\frac{1}{2}} \frac{\theta_1(x-\lambda(g)-\tau\lambda(h),\tau)}{\theta(x-\lambda(g)-\tau\lambda(h),\tau)}(N_{\lambda(h),1-\lambda(g)})\Bigg].$$

We plan to return to the study of their rigidity and vanishing properties on a later occasion.

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Department of Mathematics, University of California, Santa Cruz, CA 95064
E-mail address: dong@math.ucsc.edu

Department of Mathematics, University of California, Los Angles, CA, 90095-1555
E-mail address: liu@math.ucla.edu

Humboldt-Universität zu Berlin, Institut für Mathematik, Rudower Chaussee 25, 12489 Berlin, Germany
E-mail address: ma@math.polytechnique.fr
Current address: Centre de Mathématiques, Ecole Polytechnique, 91128 Palaiseau Cedex, France

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