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On Family Rigidity Theorems for Spin^c Manifolds

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ABSTRACT. In [**LM**], we proved a family version of the famous Witten rigidity theorems and several family vanishing theorems for elliptic genera. In this paper, we gerenalize our theorems [**LM**] in two directions. First we establish a family rigidity theorem for the Dirac operator on loop space twisted by general positive energy loop group representations. Second we prove a family rigidity theorem for *spin*^{c}-manifolds. Several vanishing theorems on both cases are also obtained.

0. Introduction

In [**W**], Witten considered the indices of elliptic operators on the free loop space $\mathcal{L}M$ of a manifold M. In particular the index of the formal signature operator on loop space is exactly the elliptic genus of Landweber-Stong. Witten made the conjecture about the rigidity of these elliptic operators which says that their S^1-equivariant indices on M are independent of $g \in S^1$. We refer the reader to [**T**], [**BT**], [**H**], [**K**], [**L**] and [**O**] for the history of the subject.

In [**Liu2**], the first author observed that these rigidity theorems are consequence of their modular invariance. This allowed him to give a simple and unified proof of the above conjectures of Witten. In [**Liu4**], it was proved the rigidity of the Dirac operator on loop space twisted by positive energy loop group representations of any level, while the Witten rigidity theorems are the special cases of level 1. An $\widehat{\mathfrak{A}}$-vanishing theorem for loop spaces with spin structure, which is an analogue of the famous $\widehat{\mathfrak{A}}$-vanishing theorem of Atiyah and Hirzebruch [**AH**], was also proved in [**Liu4**]. Recently, by using Liu's idea, Dessai [**D1**] proved a version of rigidity theorem for *spin*^{c}-manifolds.

The purpose of our paper is to generalize these results to family case.

Let M, B be two compact smooth manifolds, and $\pi : M \to B$ be a submersion with compact fibre X. Let a compact Lie group G act fiberwise on M, that is the action preserves each fiber of π. Let P be a family of elliptic operators along the fiber X, commuting with the action G. Then the family index of P is

$$\mathrm{Ind}(P) = \mathrm{Ker}P - \mathrm{Coker}P \in K_G(B). \tag{0.1}$$

Note that $\mathrm{Ind}(P)$ is a virtual G-representation. Let $\mathrm{ch}_g(\mathrm{Ind}(P))$ with $g \in G$ be the equivariant Chern character of $\mathrm{Ind}(P)$ evaluated at g.

To consider rigidity, we only need to restrict to the case when $G = S^1$. From now on we let $G = S^1$. A family elliptic operator P is called *rigid on equivariant Chern character level* with respect to this S^1-action, if $\mathrm{ch}_g(\mathrm{Ind}(P)) \in H^*(B)$ is independent of $g \in S^1$.

In [**LM**], several family rigidity and vanishing results for elliptic genera were obtained. As pointed out in [**LM**], by taking expansions in $H^*(B)$, from the family rigidity and vanishing theorems we get many higher level rigidity and vanishing results for characteristic numbers of the family. These characteristic numbers may not be the indices of any elliptic operators.

This paper is the continuation of [**LM**], and is naturally divided into two parts. In Section 1, we prove a family rigidity theorem of the Dirac operator on loop space twisted by positive energy loop group representations, and we also derive some vanishing theorems. In Section 2, we prove the family rigidity and vanishing theorems for *spin*^{c}-manifolds which generalize the results of Dessai [**D1**].

1. Loop groups and family rigidity theorems

This Section is organized as follows: In Section 1.1, we recall the modular invariance of the characters of the representations of affine Lie algebra. In Section 1.2, we state the family rigidity theorems of the Dirac operator on loop space twisted by general positive energy loop group representations for spin manifold. In Section 1.3, we prove the main theorem, Theorem 1.2. In Section 1.4, we derive some vanishing theorems.

1.1. Characters of affine Lie algebras. Let G be a simple, simply connected compact Lie group and LG be its loop group. There is a central extension $\widetilde{LG}$ of G

$$1 \to S^1 \to \widetilde{LG} \to LG \to 1. \tag{1.1}$$

The circle group S^1 acts on LG by the rotation R_θ, $R_\theta \nu(\theta') = \nu(\theta' - \theta)$. The action of S^1 on LG lifts (essentially uniquely) to an action on $\widetilde{LG}$. We say the representation U of LG is symmetric if $R_\theta U_\nu R_\theta^{-1} = U_{R_\theta \nu}$. We say a representation E of $\widetilde{LG}$ is positive energy [**PS**, Chap. 9] if

(a) E is a direct sum of irreducible representations;
(b) E is symmetric and $E^0 = \oplus_{j\in \mathbf{N}} E_j$ is dense in E, where $E_j = \{v \in E : R_\theta v = e^{-ijv}\}$ and E_j is a finite dimensional complex representation of G;
(c) The action of $\widetilde{LG} \rtimes S^1$ on E naturally extends to a smooth action of $\widetilde{LG} \rtimes \mathrm{Diff}^+(S^1)$, where $\mathrm{Diff}^+(S^1)$ is the group of orientation preserving diffeomorphisms of S^1.

Let $\mathfrak{g}$ be the Lie algebra of G. Let η be the Cartan subalgebra, W be the Weyl group of $\mathfrak{g}$. Denote by $Q = \Sigma_{i=1}^l \mathbf{Z}\alpha_i$, where $\{\alpha_i\}$ is the root basis, the root lattice of $\mathfrak{g}$. Then the affine Lie algebra associated to $\mathfrak{g}$ is

$$\widehat{L\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}K \oplus \mathbf{C}d, \tag{1.2}$$

where K (resp. d) is the infinitesimal generator of the central element (resp. the rotation of S^1) of $\widetilde{LG}$. $\widehat{L\mathfrak{g}}$ has the triangle decomposition

$$\widehat{L\mathfrak{g}} = \widehat{\eta}_- \oplus \widehat{\eta} \oplus \widehat{\eta}_+, \tag{1.3}$$

where $\widehat{\eta}_{\pm}$ are the nilpotent subalgebras and $\widehat{\eta} = \eta \otimes_{\mathbf{R}} \mathbf{C} \oplus \mathbf{C}K \oplus \mathbf{C}d$ is the Cartan subalgebra. Let $(,)$ be the normalized symmetric invariant bilinear form on $\widehat{L}\mathfrak{g}$ which extends the standard symmetric bilinear form on $\mathfrak{g}$, such that

(1.4) $(\mathbf{C}K \oplus \mathbf{C}d, \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}[t, t^{-1}]) = 0; \quad (K, K) = 0; \quad (d, d) = 0; \quad (K, d) = 1.$

Let $\widehat{\eta}^*$ be the dual of $\widehat{\eta}$ with respect to $(,)$. Let $\langle,\rangle$ denote the pairing between $\widehat{\eta}^*$ and $\widehat{\eta}$. Then the level of $\lambda \in \widehat{\eta}^*$ is defined to be $\langle \lambda, K\rangle$. Let $\Lambda_0, \delta \in \widehat{\eta}^*$ be the elements such that $\delta|_{\eta \oplus \mathbf{C}K} = 0, \langle \delta, d\rangle = 1$; $\Lambda_0|_{\eta\oplus\mathbf{C}d} = 0, \langle \Lambda_0, K\rangle = 1$.

It is known that $\widehat{L}\mathfrak{g}$ falls into class $X_N^{(1)}$ in the classification of Kac-Moody algebras [**Kac**, Chap 7]. An $\widehat{L}\mathfrak{g}$-module V is called a highest weight module with the highest weight $\Lambda \in \widehat{\eta}^*$ if there exists a non-zero vector $v \in V$ such that

(1.5) $\quad \widehat{\eta}_+(v) = 0; \quad h(v) = \Lambda(h)v, \quad \text{for} \quad h \in \widehat{\eta}; \quad \text{and} \quad U(\widehat{L}\mathfrak{g})(v) = V.$

where $U(\widehat{L}\mathfrak{g})$ is the universal envelopping algebra of $\widehat{L}\mathfrak{g}$. An irreducible representation $L(\Lambda)$ of $\widehat{L}\mathfrak{g}$ with the highest weight Λ is called of level $k = \langle \Lambda, K\rangle$. $L(\Lambda)$ is said to be integrable if $\Lambda \in P_+ = \{\lambda \in \widehat{\eta}^* : \langle \lambda, \alpha_i\rangle \in \mathbf{N} \quad \text{for all} \quad i\}$, the set of dominant integral weights. An integrable highest weight representation $L(\Lambda)$ of $\widehat{L}\mathfrak{g}$ can always be lifted to a representation of $\widetilde{L}G$ which turns out to be irreducible and of positive energy.

Since each $\widetilde{L}G$-module V has a weight space decomposition $V = \oplus_{\lambda\in\widehat{\eta}^*} V_\lambda$, we can define formal Kac-Weyl character of U as $\mathrm{ch}_V = \Sigma_{\lambda\in\widehat{\eta}^*}(\dim(V_\lambda))e^\lambda$.

The normalized character of $L(\Lambda)$ is $\chi_\Lambda = q^{m_\Lambda}\mathrm{ch}_{L(\Lambda)}$, where [**Kac**, (12.8.12)]

$$m_\Lambda = \frac{(\Lambda + 2\rho, \Lambda)}{2(m + h^\vee)} - \frac{m \dim \mathfrak{g}}{24(m + h^\vee)}, \tag{1.6}$$

where $h^\vee = \langle \rho, K\rangle$, the dual Coxeter number, $\rho = \overline{\rho} + h^\vee\Lambda_0$ [**Kac**, (6.2.8)], and $\overline{\rho}$ is half the sum of the positive roots of $\mathfrak{g}$. We call q^{m_Λ} the anomaly factor.

Let $M = \mathbf{Z}(W \cdot \theta)$ be a lattice in η^*, where θ is the long root in η, and W is the Weyl group of $\mathfrak{g}$. For any integer m, let $P_+^m = \{\lambda \in P_+ | \langle \lambda, K\rangle = m\}$ be the level m subset of the dominant integral weights.

If we choose an orthonormal basis $\{v_j\}_{j=1}^l$ of $\eta^* \otimes_{\mathbf{R}} \mathbf{C}$, such that for $v \in \widehat{\eta}^*$, then we have

$$v = 2\pi i(\Sigma_{j=1}^l z_j v_j - \tau\Lambda_0 + u\delta).$$

we denote $z = \Sigma_{s=1}^l z_s v_s \in \eta^* \otimes_{\mathbf{R}} \mathbf{C}$. Recall the classical theta functions associated to the lattice M is defined by

$$\Theta_\lambda(z, \tau) = e^{2\pi i m u} \sum_{\gamma \in M + m^{-1}\overline{\lambda}} e^{\pi i m\tau(\gamma,\gamma) + 2\pi i m(\gamma, z)} \tag{1.7}$$

Here $\overline{\lambda}$ means the orthogonal projection of λ from $\widehat{\eta}^*$ to $\eta^* \otimes_{\mathbf{R}} \mathbf{C}$ with respect to the bilinear form $(\cdot,\cdot)$, and $\gamma = \Sigma_{i=1}^l \gamma_i v_i$ with $(\gamma, z) = \Sigma_{i=1}^l \gamma_i z_i$. Then we can express χ_Λ as a finite sum

$$\chi_\Lambda(z, \tau) = \sum_{\lambda \in P^m \mathrm{mod}(mM + \mathbf{C}\delta)} c_\lambda^\Lambda(\tau)\Theta_\lambda(z, \tau), \tag{1.8}$$

Where P^m is the level m element in the integral weight lattice, and $\{c_\lambda^\Lambda(\tau)\}$ are some modular forms of weight $-\frac{1}{2}l$, which are called string functions in [**Kac**, §12.7, §13.10].

One of the important facts about the formal character is that χ_Λ is holomorphic in $\{v \in \widehat{\eta} : \mathrm{Re}(\delta, v) > 0\} = \{(z, \tau, u) \in \mathbf{C}^{l+2}, \mathrm{Im}(\tau) > 0\}$.

Now, we state the following important Kac-Peterson theorem on the modular transformation property of χ_Λ under $SL_2(\mathbf{Z})$ [**Kac**, Theorem 13.8].

THEOREM 1.1. *Let $\Lambda \in P_+^m$. Then*

$$\chi_\Lambda\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = e^{\pi i m (z,z)/\tau} \sum_{\Lambda' \in P_+^m \mathrm{mod} \mathbf{C}\delta} S_{\Lambda,\Lambda'} \chi_{\Lambda'}(z, \tau), \tag{1.9}$$

for some complex numbers $S_{\Lambda,\Lambda'}$, and

$$\chi_\Lambda(z, \tau + 1) = e^{2\pi i m_\Lambda} \chi_\Lambda(z, \tau). \tag{1.10}$$

By (1.8), for $\alpha \in M$, we also have

$$\begin{aligned} &\chi_\Lambda(z + \alpha, \tau) = \chi_\Lambda(z, \tau); \\ &\chi_\Lambda(z + \alpha\tau, \tau) = e^{2\pi i m (z,\alpha) + \pi i m (\alpha,\alpha)} \chi_\Lambda(z, \tau). \end{aligned} \tag{1.11}$$

This, together with its transformation formulas (1.9), (1.10), means that χ_Λ is an l-variable Jacobi form of index $m/2$ and weight 0.

1.2. Family rigidity theorem of general elliptic genera. Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X, and $\dim X = 2k$. We assume that the S^1 acts fiberwise on M, and TX has an S^1-equivariant spin structure. Let $\Delta(TX) = \Delta^+(TX) \oplus \Delta^-(TX)$ be the spinor bundle of TX. Let D^X be the Dirac operator on $\Delta(TX)$ which is defined fiberwise on the fiber X.

For a vector bundle F on M, we let

$$\begin{aligned} S_t(F) &= 1 + tF + t^2 S^2 F + \cdots, \\ \Lambda_t(F) &= 1 + tF + t^2 \Lambda^2 F + \cdots, \end{aligned} \tag{1.12}$$

be the symmetric and respectively the exterior power operations in $K(M)[[t]]$.

Assume E is an irreducible positive energy representation of $\widetilde{L}Spin(2l)$ and V is an S^1 equivariant vector bundle with structure group $Spin(2l)$ over M. Let Λ be the highest weight of E and m be the level of E. By the discussion in Section 1.1, we have the decomposition $E = \oplus_{n \geq 0} E_n$ under the action of R_θ. Here E_n is a finite dimensional representation of $Spin(2l)$. Let P be the frame bundle of V, which is a $Spin(2l)$ principal bundle. We define

$$\psi(E, V) = \Sigma_{n \geq 0} (P \times_{Spin(2l)} E_n) q^n \in K(M)[[q]]. \tag{1.13}$$

Let $p_1(\cdot)_{S^1}$ denote the first S^1-equivariant Pontrjagin class.

THEOREM 1.2. *For E an irreducible positive energy representation of $\widetilde{L}$Spin$(2l)$ of highest weight of level m, if $p_1(TX)_{S^1} = m p_1(V)_{S^1}$, then the elliptic operator*

$$D^X \otimes_{m=1}^\infty S_{q^m}(TX) \otimes \psi(E, V)$$

is rigid on equivariant Chern character level.

Theorem 1.2 actually holds for any semi-simple and simply connected Lie group, instead of $Spin(2l)$.

1.3. Proof of Theorem 1.2. For $\tau \in \mathbf{H} = \{\tau \in \mathbf{C}; \mathrm{Im}\tau > 0\}$, $q = e^{2\pi i\tau}$, $v \in \mathbf{C}$, let

$$(1.14)\quad \begin{aligned} \theta_3(v,\tau) &= c(q)\Pi_{n=1}^{\infty}(1+q^{n-1/2}e^{2\pi iv})\Pi_{n=1}^{\infty}(1+q^{n-1/2}e^{-2\pi iv}),\\ \theta_2(v,\tau) &= c(q)\Pi_{n=1}^{\infty}(1-q^{n-1/2}e^{2\pi iv})\Pi_{n=1}^{\infty}(1-q^{n-1/2}e^{-2\pi iv}),\\ \theta_1(v,\tau) &= c(q)q^{1/8}2\cos(\pi v)\Pi_{n=1}^{\infty}(1+q^{n}e^{2\pi iv})\Pi_{n=1}^{\infty}(1+q^{n}e^{-2\pi iv}),\\ \theta(v,\tau) &= c(q)q^{1/8}2\sin(\pi v)\Pi_{n=1}^{\infty}(1-q^{n}e^{2\pi iv})\Pi_{n=1}^{\infty}(1-q^{n}e^{-2\pi iv}). \end{aligned}$$

be the classical Jacobi theta functions [**Ch**], where $c(q) = \Pi_{n=1}^{\infty}(1-q^n)$.

Recall that we have the following transformation formulas of theta-functions [**Ch**]:

$$(1.15)\quad \begin{aligned} \theta(t+1,\tau) &= -\theta(t,\tau), & \theta(t+\tau,\tau) &= -q^{-1/2}e^{-2\pi it}\theta(t,\tau),\\ \theta_1(t+1,\tau) &= -\theta_1(t,\tau), & \theta_1(t+\tau,\tau) &= q^{-1/2}e^{-2\pi it}\theta_1(t,\tau). \end{aligned}$$

and

$$(1.16)\quad \begin{aligned} \theta\left(\frac{t}{\tau},-\frac{1}{\tau}\right) &= \frac{1}{i}\sqrt{\frac{\tau}{i}}e^{\frac{\pi it^2}{\tau}}\theta(t,\tau), & \theta(t,\tau+1) &= e^{\frac{\pi i}{4}}\theta(t,\tau),\\ \theta_1\left(\frac{t}{\tau},-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}e^{\frac{\pi it^2}{\tau}}\theta_2(t,\tau), & \theta_1(t,\tau+1) &= e^{\frac{\pi i}{4}}\theta_1(t,\tau),\\ \theta_2\left(\frac{t}{\tau},-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}e^{\frac{\pi it^2}{\tau}}\theta_1(t,\tau), & \theta_2(t,\tau+1) &= \theta_3(t,\tau),\\ \theta_3\left(\frac{t}{\tau},-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}e^{\frac{\pi it^2}{\tau}}\theta_3(t,\tau), & \theta_3(t,\tau+1) &= \theta_2(t,\tau). \end{aligned}$$

Let $g = e^{2\pi it} \in S^1$ be a topological generator of S^1. Let $\{M_\alpha\}$ be the fixed submanifolds of the circle action. Then $\pi : M_\alpha \to B$ be a submersion with fibre X_α. We have the following S^1-equivariant decomposition of TX

$$(1.17)\quad TX_{|M_\alpha} = N_1 \oplus \cdots \oplus N_h \oplus TX_\alpha,$$

Here N_γ is a complex vector bundle such that g acts on it by $e^{2\pi im_\gamma t}$. We denote the Chern roots of N_γ by $2\pi ix_\gamma^j$, and the Chern roots of $TX_\alpha \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm 2\pi iy_j'\}$. Let $\dim_{\mathbf{C}} N_\gamma = d(m_\gamma)$, and $\dim X_\alpha = 2k_\alpha$.

Let

$$(1.18)\quad V_{|M_\alpha} = V_1 \oplus \cdots \oplus V_{l_0},$$

be the equivariant decomposition of V restricted to M_α. Assume that g acts on V_v by $e^{2\pi in_vt}$, where some n_v may be zero. We denote the Chern roots of V_v by $2\pi iu_v^j$. Let us write $\dim_{\mathbf{R}} V_v = 2d(n_v)$. By [**Liu4**, §3.5], the equivariant Chern character of $\psi(E,V)$ can be obtained as $q^{-m_\Lambda}C_{E,V}(u+t,\tau)$, where

$$(1.19)\quad C_{E,V}(u+t,\tau) = \chi_E(U+T,\tau),$$

with $U+T = (u_1^j + n_1^jt, \ldots, u_{l_0}^j + n_{l_0}^jt)$.

For $g = e^{2\pi it}, t \in \mathbf{R}$, and $\tau \in \mathbf{H}$, $q = e^{2\pi i\tau}$, we let

$$(1.20)\quad F_{E,V}(t,\tau) = q^{m_\Lambda}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \bigotimes_{m=1}^{\infty} S_{q^m}(TX - \dim X) \otimes \psi(E,V))\Big).$$

For $f(x)$ a holomorphic function, we denote by $f(y')(TX^g) = \Pi_j f(y_j')$, the symmetric polynomial which gives characteristic class of TX^g, and similarly for

N_γ. Using the family Atiyah-Bott-Segal-Singer Lefschetz fixed point formula [**LM**, Theorem 1.1], (1.14), we find for $t \in [0,1] \setminus \mathbf{Q}$

(1.21)

$$F_{E,V}(t,\tau) = (2\pi i)^{-k} \sum_{\alpha} \pi_* \left[\theta'(0,\tau)^k \left(\frac{2\pi i y'}{\theta(y',\tau)}\right)(TX^g)\frac{C_{E,V}(u+t,\tau)}{\Pi_\gamma \theta(x_\gamma + m_\gamma t,\tau)(N_\gamma)}\right].$$

Considered as functions of (t,τ), we can obviously extend $F_{E,V}(t,\tau)$ to meromorphic functions on $\mathbf{C}\times\mathbf{H}$ with values in $H^*(B)$, and holomorphic in τ. Theorem 1.2 is equivalent to the statement that $F_{E,V}(t,\tau)$ is independent of t. To prove it, we will proceed as in [**Liu4**], [**LM**].

Lemma 1.1. *If $p_1(TX)_{S^1} = mp_1(V)_{S^1}$, then for $a, b \in 2\mathbf{Z}$,*

$$F_{E,V}(t + a\tau + b, \tau) = F_{E,V}(t,\tau). \tag{1.22}$$

Proof. By (1.15), for $a, b \in 2\mathbf{Z}$, $l \in \mathbf{Z}$, we have

$$\theta(x + l(t + a\tau + b),\tau) = e^{-\pi i(2lax + 2l^2at + l^2a^2\tau)}\theta(x + lt,\tau). \tag{1.23}$$

Since $mp_1(V)_{S^1} = p_1(TX)_{S^1}$, we have

$$m\Sigma_{v,j}(u_v^j + n_v t)^2 = \Sigma_j (y_j')^2 + \Sigma_{\gamma,j}(x_\gamma^j + m_\gamma t)^2. \tag{1.24}$$

This implies the equalities:

$$\begin{aligned} &m\Sigma_{v,j}(u_v^j)^2 = \Sigma_j(y_j')^2 + \Sigma_{\gamma,j}(x_\gamma^j)^2,\\ &m\Sigma_{v,j}n_v u_v^j = \Sigma_{\gamma,j} m_\gamma x_\gamma^j, \qquad \Sigma_\gamma m_\gamma^2 d(m_\gamma) = m\Sigma_v n_v^2 d(n_v). \end{aligned} \tag{1.25}$$

By using (1.11), (1.21), (1.23), and (1.25), we get (1.22). □

Now we will prove that $F_{E,V}(t,\tau)$ is holomorphic in t. Then, by Lemma 1.1, we get the rigidity theorem.

To prove $F_{E,V}(t,\tau)$ is holomorphic in t, we will examine the modular transformation property of $F_{E,V}(t,\tau)$ under the group $SL_2(\mathbf{Z})$.

Recall that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we define its modular transformation on $\mathbf{C}\times\mathbf{H}$ by

$$g(t,\tau) = \left(\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right). \tag{1.26}$$

Obviously, the two generators of $SL_2(\mathbf{Z})$ are $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. They act on $\mathbf{C}\times\mathbf{H}$ in the following way:

$$S(t,\tau) = \left(\frac{t}{\tau}, -\frac{1}{\tau}\right),\ T(t,\tau) = (t,\tau+1). \tag{1.27}$$

Let Ψ_τ be the scaling homomorphism from $\Lambda(T^*B)$ into itself : $\beta \to \tau^{\frac{1}{2}\deg\beta}\beta$. If α is a differential form on B, we denote by $\{\alpha\}^{(p)}$ the component of degree p of α.

Lemma 1.2. *For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we have*

$$F_{E,V}(g(t,\tau)) = (c\tau + d)^k \Psi_{c\tau+d} F_{gE,V}(t,\tau), \tag{1.28}$$

where $gE = \Sigma_\mu a_\mu E_\mu$ *is a finite complex linear combination of positive energy representations of* $\widetilde{L}Spin(2l)$ *of highest weight of level* m, *and we denote by*

$$F_{gE,V}(t,\tau) = (2\pi i)^{-k}\theta'(0,\tau)^k \sum_\mu \sum_\alpha a_\mu \pi_* \Big[\Big(\frac{2\pi i y'}{\theta(y',\tau)}\Big)(TX^g) \cdot \frac{C_{E_\mu,V}(u+t,\tau)}{\Pi_\gamma \theta(x_\gamma + m_\gamma t,\tau)(N_\gamma)}\Big]. \tag{1.29}$$

the complex linear combination of the equivariant Chern characters of the corresponding index bundles.

PROOF. Set

$$F(t,\tau) = \frac{\theta'(0,\tau)}{\theta(t,\tau)}. \tag{1.30}$$

By (1.16), we get

$$F(g(t,\tau)) = (c\tau+d)e^{-c\pi i t^2/(c\tau+d)}F((c\tau+d)t,\tau). \tag{1.31}$$

By Theorem 1.1, (1.19), it is easy to see that on M_α,

$$C_{E,V}(g(u+t,\tau)) = e^{cm\pi i\Sigma_{v,j}(u_v^j+n_v t)^2/(c\tau+d)}C_{gE,V}(u+t,\tau). \tag{1.32}$$

with

$$C_{gE,V}(u+t,\tau) = \Sigma_\mu a_\mu C_{E_\mu,V}(u+t,\tau). \tag{1.33}$$

By using (1.21), (1.31), (1.32), we get

$$\begin{aligned} F_{E,V}&\left(\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) \\ &= (2\pi i)^{-k}\sum_\alpha \pi_*\Big[\Big(2\pi i y' F\Big(y',\frac{a\tau+b}{c\tau+d}\Big)\Big)(TX^g) \\ &\quad \cdot \Pi_\gamma\Big(F\Big(x_\gamma+\frac{m_\gamma t}{c\tau+d},\frac{a\tau+b}{c\tau+d}\Big)(N_\gamma)\Big)C_{E,V}\Big(u+\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d}\Big)\Big] \\ &= (c\tau+d)^k(2\pi i)^{-k}\sum_\alpha \pi_*\Big[\Big(2\pi i y' F((c\tau+d)y',\tau)\Big)(TX^g) \\ &\quad \cdot \Pi_\gamma\Big(F((c\tau+d)x_\gamma+m_\gamma t,\tau)(N_\gamma)\Big)C_{gE,V}((c\tau+d)u+t,\tau)\Big] \end{aligned} \tag{1.34}$$

By (1.34), to prove (1.28), we only need prove the following equation for $p \in \mathbf{N}$,

$$\begin{aligned} &\Big\{\pi_*\Big[\Big(2\pi i y' F((c\tau+d)y',\tau)\Big)(TX^g) \\ &\quad \cdot \Pi_\gamma\Big(F((c\tau+d)x_\gamma+m_\gamma t,\tau)(N_\gamma)\Big)C_{gE,V}((c\tau+d)u+t,\tau)\Big]\Big\}^{(2p)} \\ &= (c\tau+d)^p\Big\{\pi_*\Big[\Big(2\pi i y' F(y',\tau)\Big)(TX^g) \\ &\quad \cdot \Pi_\gamma\Big(F(x_\gamma+m_\gamma t,\tau)(N_\gamma)\Big)C_{gE,V}(u+t,\tau)\Big]\Big\}^{(2p)}. \end{aligned} \tag{1.35}$$

By looking at the degree $2(p+k_\alpha)$ part, that is the $(p+k_\alpha)$-th homogeneous terms of the polynomials in x's, y''s and u's, on both sides, we get (1.35). The proof of Lemma 1.2 is complete. □

The following lemma is a generalization of [**Liu4**, Lemma 2.3], LEMMA 1.3. *For any $g \in SL_2(\mathbf{Z})$, the function $F_{gE,V}(t,\tau)$ is holomorphic in (t,τ) for $(t,\tau) \in \mathbf{R} \times \mathbf{H}$.*

PROOF. Let $z = e^{2\pi i t}$, and let $N = \max_{\alpha,\gamma}|m_\gamma|$. Denote by $D_N \subset \mathbf{C}^2$ the domain

$$|q|^{1/N} < |z| < |q|^{-1/N}, 0 < |q| < 1. \tag{1.36}$$

By (1.14), (1.19), (1.21) and (1.29), we know that in D_N, $F_{gE,V}(t,\tau)$ has a convergent Laurent series expansion of the form

$$\sum_\mu a_\mu q^{m_{\Lambda_\mu}} \sum_{j=0}^{\infty} b^g_{j\mu}(z) q^j \tag{1.37}$$

Here $\{b^g_{j\mu}(z)\}$ are rational functions of z with possible poles on the unit circle.

Now considered as a formal power series of q,

$$\bigotimes_{n=1}^{\infty} S_{q^n}(TX - \dim X) \otimes \Big(\sum_\mu a_\mu q^{m_{\Lambda_\mu}} \psi(E_\mu, V)\Big) = \sum_\mu a_\mu q^{m_{\Lambda_\mu}} \bigotimes_{j=0}^{\infty} V^g_{j,\mu} q^j$$

with $V^g_{j,\mu} \in K_{S^1}(M)$. Note that the terms in the above two sums correspond to each other. Now, we apply the family Lefschetz fixed point formula [**LM**, Theorem 1.1] to each $V^g_{j,\mu}$, for $t \in \mathbf{R} \setminus \mathbf{Q}$, we get

$$b^g_{j\mu}(z) = \mathrm{ch}_z(\mathrm{Ind}(D \otimes V^g_{j,\mu})). \tag{1.38}$$

But by [**S**, Proposition 2.2], we know that

$$K_{S^1}(B) \simeq K(B) \otimes R(S^1) \tag{1.39}$$

This implies that for $t \in \mathbf{R} \setminus \mathbf{Q}$, $z = e^{2\pi i t}$,

$$b^g_{j\mu}(z) = \Sigma_{l=-N(j)}^{N(j)} a^{g,\mu}_{l,j} z^l. \tag{1.40}$$

for $N(j)$ some positive integer depending on j and $a^{g,\mu}_{l,j} \in H^*(B)$. Since both sides are analytic functions of z, this equality holds for any $z \in \mathbf{C}$.

On the other hand, by multiplying $F_{gE,V}(t,\tau)$ by $f(z) = \Pi_{\alpha,\gamma}(1 - z^{m_\gamma})^{l' d(m_\gamma)}$ ($l' = \dim M$), we get holomorphic functions which have a convergent power series expansion of the form $\Sigma_\mu a_\mu q^{m_{\Lambda_\mu}} \Sigma_{j=0}^{\infty} c^g_{j\mu}(z) q^j$, with $\{c^g_{j\mu}(z)\}$ polynomial functions in D_N. Comparing the above two expansions, one gets

$$c^g_{j\mu}(z) = f(z) b^g_{j\mu}(z) \tag{1.41}$$

for each j. So by the Weierstrass preparation theorem, we get $F_{gE,V}(t,\tau)$ is holomorphic in D_N. □

Proof of Theorem 1.1. We will prove that $F_{E,V}$ is holomorphic on $\mathbf{C} \times \mathbf{H}$, which implies the rigidity theorem we want to prove.

From their expressions, we know the possible polar divisors of $F_{E,V}$ in $\mathbf{C} \times \mathbf{H}$ are of the form $t = \frac{n}{l}(c\tau + d)$ with n, c, d, l intergers and $(c,d) = 1$ or $c = 1$ and $d = 0$.

We can always find intergers a, b such that $ad - bc = 1$, and consider the matrix $g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$. By (1.28),

$$\Psi_{(-c\tau+a)} F_{gE,V}(t,\tau) = (-c\tau + a)^{-k} F_{E,V}\Big(\frac{t}{-c\tau + a}, \frac{d\tau - b}{-c\tau + a}\Big) \tag{1.42}$$

Now, if $t = \frac{n}{l}(c\tau + d)$ is a polar divisor of $F_{E,V}(t,\tau)$, then one polar divisor of $F_{gE,V}(t,\tau)$ is given by

$$\frac{t}{-c\tau + a} = \frac{n}{l}\Big(c\frac{d\tau - b}{-c\tau + a} + d\Big), \tag{1.43}$$

which exactly gives $t = n/l$. This contradicts Lemma 1.3, and completes the proof of Theorem 1.1. □

1.4. Family vanishing theorems. Recall that a (meromorphic) Jacobi form of index n and weight l over $L \rtimes \Gamma$, where L is an integral lattice in the complex plane $\mathbf{C}$ preserved by the modular subgroup $\Gamma \subset SL_2(\mathbf{Z})$, is a (meromorphic) function $F(t,\tau)$ on $\mathbf{C} \times \mathbf{H}$ such that

$$\begin{aligned} &F\left(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^l e^{2\pi i n(ct^2/(c\tau+d))} F(t,\tau), \\ &F(t + \lambda\tau + \mu, \tau) = e^{-2\pi i n(\lambda^2\tau + 2\lambda t)} F(t,\tau), \end{aligned} \tag{1.44}$$

where $(\lambda, \mu) \in L$, and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If F is holomorphic on $\mathbf{C} \times \mathbf{H}$, we say that F is a holomorphic Jacobi form.

For $N \in \mathbf{N}^*$, set

$$\Gamma(N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod} N) \right\}. \tag{1.45}$$

Recall that the equivariant cohomology group $H^*_{S^1}(M, \mathbf{Z})$ of M is defined by

$$H^*_{S^1}(M, \mathbf{Z}) = H^*(M \times_{S^1} ES^1, \mathbf{Z}). \tag{1.46}$$

where ES^1 is the S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(M, \mathbf{Z})$ is a module over $H^*(BS^1, \mathbf{Z})$ induced by the projection $\overline{\pi} : M \times_{S^1} ES^1 \to BS^1$. Let $p_1(V)_{S^1}, p_1(TX)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ be the equivariant first Pontrjagin classes of V and TX respectively. Also recall that

$$H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[u] \tag{1.47}$$

with u a generator of degree 2.

In this part, we suppose that there exists $n \in \mathbf{Z}$ such that

$$m p_1(V)_{S^1} - p_1(TX)_{S^1} = n \cdot \overline{\pi}^* u^2 \quad \text{in} \quad H^*_{S^1}(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}. \tag{1.48}$$

As in [**Liu4**], we call n the anomaly to rigidity.

THEOREM 1.3. *Let M, B, V and E be as in Theorem 1.2. Then for $p \in \mathbf{N}$, $\{F_{E,V}\}^{(2p)}$ is a holomorphic Jacobi form of index $n/2$ and weight $k+p$ over $(2\mathbf{Z})^2 \rtimes \Gamma(N(m))$.*

Here $N(m)$ is an integer depending on the level m and was given in [**Kac**], and m_Λ defined in (1.6).

PROOF. Now, by (1.48), we get

$$m\Sigma_{v,j}(u_v^j + n_v t)^2 - \Big(\Sigma_j (y_j')^2 + \Sigma_{\gamma,j}(x_\gamma^j + m_\gamma t)^2\Big) = n \cdot t^2. \tag{1.49}$$

This means

$$\begin{aligned} &m\Sigma_v n_v^2 d(n_v) - \Sigma_\gamma m_\gamma^2 d(m_\gamma) = n, \quad m\Sigma_{v,j} n_v u_v^j = \Sigma_{\gamma,j} m_\gamma x_\gamma^j, \\ &m\Sigma_{v,j}(u_v^j)^2 = \Sigma_j (y_j')^2 + \Sigma_{\gamma,j}(x_\gamma^j)^2. \end{aligned} \tag{1.50}$$

First by using (1.50), as in Lemma 1.1, for $(a,b) \in (2\mathbf{Z})^2$, we get

$$F_{E,V}(t + a\tau + b, \tau) = e^{-\pi i n(a^2\tau + 2at)} F_{E,V}(t,\tau). \tag{1.51}$$

Second by a theorem of Kac, Peterson and Wakimoto [**Kac**, Chapter 13], there exists an integer $N(m)$ such that for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N(m))$, we have

$$C_{E,V}(g(u+t,\tau)) = e^{mc\pi i \Sigma_{v,j}(u_v^j + n_v t)^2/(c\tau+d)} C_{E,V}(u+t,\tau). \tag{1.52}$$

Now, by using (1.21), (1.50) and (1.52), as Lemma 1.2, we get

$$F_{E,V}(g(t,\tau)) = (c\tau+d)^k e^{\pi i n c t^2/(c\tau+d)} \Psi_{c\tau+d} F_{E,V}(t,\tau). \tag{1.53}$$

As the same argument in the proof of Theorem 1.2 (see also [**Liu4**, Theorem 3.4]), we know $F_{E,V}$ is holomorphic on $\mathbf{C} \times \mathbf{H}$. By (1.51), (1.53), we get Theorem 1.3. □

The following lemma was established in [**EZ**, Theorem 1.2]:

LEMMA 1.4. *Let F be a holomorphic Jacobi form of index m and weight k. Then for fixed τ, $F(t,\tau)$, if not identically zero, has exactly $2m$ zeros in any fundamental domain for the action of the lattice on* $\mathbf{C}$.

This tells us that there are no holomorphic Jacobi forms of negative index. Therefore, if $m < 0$, F must be identically zero. If $m = 0$, it is easy to see that F must be independent of t. So we immediately get the following:

COROLLARY 1.1. *Let M, B, V, E and n be as in Theorem 1.3. If $n = 0$, the equivariant Chern character of the index bundle of*

$$\mathrm{Ind}\Big(D^X \bigotimes_{m=1}^{\infty} S_{q^m}(TX - \dim X) \otimes \psi(E,V)\Big)$$

is independent of $g \in S^1$. If $n < 0$, this equivariant Chern character is identically zero, in particular, the Chern character of this index bundle is zero.

2. Family rigidity theorems for *spin*^{c}-manifolds

The purpose of this Section is to prove a family version of the rigidity theorem for *spin*^{c}-manifolds.

This Section is organized as follows: In Section 2.1, we explain the equivariant family index theorem for *spin*^{c}-manifolds. In Section 2.2, we state our main result, Theorem 2.2. In Section 2.3, we prove Theorem 2.2. In Section 2.4, we prove a family version of the rigidity and vanishing theorem for *spin*^{c}-manifolds of [**D1**]. A family vanishing theorem of Witten genus for *spin*^{c}-manifolds is also obtained.

2.1. Equivariant family index theorem for *spin*^{c}-manifolds. By [**LM**, Theorem 1.1], we have the equivariant family index theorem for a family of equivariant elliptic operators. In fact, by the proof of [**LM**], we know we have a local version of [**LM**, Theorem 1.1] for the Dirac operator associated to the Clifford module in the sense of [**BeGeV**, §3.3, §10.3].

Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X with $\dim X = 2k$. We assume that the S^1 acts fiberwise on M, and TX has an S^1-equivariant *spin*^{c} structure. Let $\Delta(TX)$ be the complex spinor bundle for TX [**LaM**, Definition

D.9]. We denote D^c the corresponding *spin*^{c}-Dirac operator on the fibre X [**LaM**, Appendix D].

Let W be an S^1-equivariant complex vector bundle on M. Let $D^c \otimes W$ be the twisted *spin*^{c}-Dirac operator on $\Delta(TX) \otimes W$. Then $\mathrm{Ind}(D^c \otimes W) \in K_{S^1}(B)$.

Let $g = e^{2\pi i t} \in S^1$ be a generator of the action group. Let $\{M_\alpha\}$ be the fixed submanifolds of the circle action. Then $\pi : M_\alpha \to B$ be a submersion with fibre X_α. We have the following equivariant decomposition of TX

$$TX_{|M_\alpha} = N_1 \oplus \cdots \oplus N_h \oplus TX_\alpha, \tag{2.1}$$

Here N_γ is a complex vector bundle such that g acts on it by $e^{2\pi i m_\gamma t}$. So TX_α is naturally oriented. We denote the Chern roots of N_γ by $2\pi i x_\gamma^j$, and the Chern roots of $TX_\alpha \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm 2\pi i y_j'\}$. Let $\dim_{\mathbf{C}} N_\gamma = d(m_\gamma)$, $\dim X_\alpha = 2k_\alpha$.

We recall that the *spin*^{c}-structure on TX induces an S^1-equivariant complex line bundle L over M. Its equivariant Chern class $c_1(L)_{S^1}$ will also be denoted by $c_1(TX)_{S^1}$. We denote the Chern class $c_1(L)$ of L by $2\pi i c_1$. Let $i_\alpha : M_\alpha \to M$ be the inclusion, and let i_α^* denote the induced homomorphism in equivariant cohomology. If g acts on L on M_α by $e^{2\pi i l_c t}$, we have

$$i_\alpha^* c_1(TX)_{S^1} = 2\pi i(c_1 + l_c t). \tag{2.2}$$

We denote $\pi_* : H^*(M^g) \to H^*(B)$ the intergration along the fibre X^g. Now, we can reformulate the family Atiyah-Bott-Segal-Singer Lefschetz fixed point formula, [**LM**, Theorem 1.1] in this case,

THEOREM 2.1. *We have the following identity in* $H^*(B)$

$$\mathrm{ch}_g(\mathrm{Ind}(D^c \otimes W)) = \pi_* \left\{ \frac{\widehat{A}(TX^g)\mathrm{ch}_g(W) e^{\pi i(c_1 + l_c t)}}{\Pi_\gamma \left(e^{\pi i(x_\gamma + m_\gamma t)} - e^{-\pi i(x_\gamma + m_\gamma t)} \right)(N_\gamma)} \right\} \tag{2.3}$$

2.2. Family rigidity for *spin*^{c}-manifolds. In this part, we use the assumption of Section 2.1, we also use the notation of Sections 1 and 2.1.

For a vector bundle E on M, we denote by $\widetilde{E}$ the reduced vector bundle $E - \dim(E)$.

Let W be an S^1-equivariant complex vector bundle of rank r over M. Let $L_W = \det(W)$ the determinant line bundle of W on M. Let V be a dimension $2l$ real vector bundle on M with S^1-equivariant *spin*$(2l)$ structure. Let $\Delta(V) = \Delta^+(V) \oplus \Delta^-(V)$ be the spinor bundle of V.

Let $y = e^{2\pi i \beta}$ be a complex number, and we define the following elements in $K(M)[[q^{1/2}]]$:

(2.4)

$$\Theta_q(TX|W)_v = \bigotimes_{m=1}^{\infty} S_{q^m}(\widetilde{TX}) \otimes \Lambda_{-1}(W^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(\widetilde{W \otimes_{\mathbf{R}} \mathbf{C}}),$$

$$\Theta_q^\beta(TX|W)_v = \bigotimes_{m=1}^{\infty} S_{q^m}(\widetilde{TX}) \otimes \Lambda_{-y^{-1}}(\widetilde{W}^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-yq^n}(\widetilde{W}) \otimes \Lambda_{-y^{-1}q^n}(\widetilde{W}^*).$$

Let

$$\begin{aligned} R_1(V)_v &= \Delta(V) \otimes \textstyle\bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{V}), \\ R_2(V)_v &= \textstyle\bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(\widetilde{V}), \\ R_3(V)_v &= \textstyle\bigotimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(\widetilde{V}). \end{aligned} \tag{2.5}$$

For $g = e^{2\pi i t}, t \in \mathbf{R}$, $q = e^{2\pi i\tau}, \tau \in \mathbf{H}$, set

$$\begin{aligned} F_1(t,\tau) &= 2^{-l}\mathrm{ch}_g\Big(\mathrm{Ind}(D^c \otimes \Theta_q(TX|W)_v \otimes R_1(V)_v)\Big), \\ F_1^\beta(t,\tau) &= 2^{-l}\mathrm{ch}_g\Big(\mathrm{Ind}(D^c \otimes \Theta_q^\beta(TX|W)_v \otimes R_1(V)_v)\Big). \end{aligned} \tag{2.6}$$

We consider $F_1(t,\tau), F_1^\beta(t,\tau)$ as functions on $(t,\tau) \in \mathbf{C}\times\mathbf{H}$ with values in $H^*(B)$.
Recall that for $n \in \mathbf{N}^*$,

$$\Gamma_1(n) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\mathrm{mod} n) \right\}. \tag{2.7}$$

THEOREM 2.2. *If $p_1(V+W-TX)_{S^1} = n\cdot\overline{\pi}^* u^2$ ($n \in \mathbf{Z}$), $c_1(W)_{S^1} = c_1(TX)_{S^1}$ in $H^*_{S^1}(M,\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Q}$. Then*

i). *If $c_1(W) \equiv 0 \bmod(N)$ ($N \in \mathbf{N}, N > 1$), then for $y = e^{2\pi i\beta}$ an Nth root of unity, and $p \in \mathbf{N}$, $\{F_1^\beta(t,\tau)\}^{(2p)}$ is a holomorphic Jacobi form of index $n/2$ and weight $(k+p)$ over $(2N\mathbf{Z})^2 \rtimes \Gamma_1(2N)$.*

ii). *For $p \in \mathbf{N}$, $\{F_1(t,\tau)\}^{(2p)}$ is a holomorphic Jacobi form of index $n/2$ and weight $k-r+p$ over $(2\mathbf{Z})^2 \rtimes \Gamma_1(2)$. If $V = 0$, the same holds for $\Gamma_1(2)$ replaced by $SL_2(\mathbf{Z})$.*

Remark. If we replace the condition $c_1(W)_{S^1} = c_1(TX)_{S^1}$ by $\omega_2(TX) = \omega_2(W)$. Let $F_1(t,\tau)$, and let $F_1^\beta(t,\tau)$ be the equivariant Chern character of index bundles of $D^c \otimes (L_W \otimes L^{-1})^{1/2} \otimes \Theta_q(TX|W)_v \otimes R_1(V)_v$, $D^c \otimes (L_W \otimes L^{-1})^{1/2} \otimes \Theta_q^\beta(TX|W)_v \otimes R_1(V)_v$, we still have Theorem 2.2. In fact, we only need to take tensor product of $(L_W \otimes L^{-1})^{1/2}$ with the corresponding operators in the proof of Theorem 2.2. If $V = 0$, this result also generalizes [**Liu2**, Theorem B] to the family case.

Remark. If we replace the condition $c_1(W)_{S^1} = c_1(TX)_{S^1}$ by $c_1(W) = c_1(TX)$ in $H^*(M,\mathbf{Q})$, as Dessai remarked in [**D**, Lemma 3.4], then there exists $m \in \mathbf{Z}$ such that $c_1(W)_{S^1} - c_1(TX)_{S^1} = m\overline{\pi}^* u$. Now, for the functions $e^{\pi imt}F_1(t,\tau)$, $e^{\pi imt}F_1^\beta(t,\tau)$, we still have the result of Theorem 2.2. In fact this only multiplies all the functions in the proof of Theorem 2.2 by $e^{\pi imt}$.

2.3. Proof of Theorem 2.2. Let

$$V_{|M_\alpha} = V_1 \oplus \cdots \oplus V_{l_0}, \tag{2.8}$$

be the equivariant decomposition of V restricted to M_α. Assume that g acts on V_v by $e^{2\pi i n_v t}$, where some n_v may be zero. We denote the Chern roots of V_v by $2\pi i u_v^j$. Write $\dim_{\mathbf{R}} V_v = 2d(n_v)$. Similarly, let

$$W_{|M_\alpha} = W_1 \oplus \cdots \oplus W_{r_0}, \tag{2.9}$$

be the equivariant decomposition of W restricted to M_α. Assume that g acts on W_μ by $e^{2\pi i r_\mu t}$, where some r_μ may be zero. We denote the Chern roots of W_μ by $2\pi i\omega_\mu^j$. Write $\dim_{\mathbf{C}} W_\mu = d(r_\mu)$.

First note that the condition $c_1(W)_{S^1} = c_1(TX)_{S^1}$ means that

$$\Sigma_{\mu,j}(\omega_\mu^j + r_\mu t) = c_1 + l_c t. \tag{2.10}$$

We take $\beta = 1/N$. By applying the family Atiyah-Bott-Segal-Singer Lefschetz fixed point formula, Theorem 2.1, and using (2.10), for $g = e^{2\pi i t}, t \in \mathbf{R} \setminus \mathbf{Q}$, we get

$$
\begin{aligned}
F_1(t,\tau) &= (2\pi i)^{r-k} \frac{\theta'(0,\tau)^{k-r}}{\theta_1(0,\tau)^l} \sum_\alpha \pi_* \Big[\Big(\frac{2\pi i y'}{\theta(y',\tau)} \Big) (TX^g) \\
&\quad \frac{\Pi_v \theta_1(u_v + n_v t, \tau)(V_v)}{\Pi_\gamma \theta(x_\gamma + m_\gamma t, \tau)(N_\gamma)} \Pi_\mu \theta(\omega_\mu + r_\mu t, \tau)(W_\mu) \Big], \\
F_1^\beta(t,\tau) &= (2\pi i)^{-k} \frac{\theta'(0,\tau)^{k}}{\theta_1(0,\tau)^l \theta(\beta,\tau)^r} \sum_\alpha \pi_* \Big[\Big(\frac{2\pi i y'}{\theta(y',\tau)} \Big) (TX^g) \\
&\quad \frac{\Pi_v \theta_1(u_v + n_v t, \tau)(V_v)}{\Pi_\gamma \theta(x_\gamma + m_\gamma t, \tau)(N_\gamma)} \Pi_\mu \theta(\omega_\mu + r_\mu t + \beta, \tau)(W_\mu) \Big].
\end{aligned} \tag{2.11}
$$

In the following, we will consider $F_1(t,\tau)$, $F_1^\beta(t,\tau)$ as meromorphic functions on $(t,\tau) \in \mathbf{C} \times \mathbf{H}$ with values in $H^*(B)$.

LEMMA 2.1. *If $p_1(V + W - TX)_{S^1} = n \cdot \overline{\pi}^*(u^2)$ in $H^*_{S^1}(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$ for some integer n,*

i) *For $a, b \in 2\mathbf{Z}$,*

$$F_1(t + a\tau + b, \tau) = e^{-\pi i n (a^2\tau + 2at)} F_1(t,\tau). \tag{2.12}$$

ii) *For $a, b \in 2N\mathbf{Z}$,*

$$F_1^\beta(t + a\tau + b, \tau) = e^{-\pi i n (a^2\tau + 2at)} F_1^\beta(t,\tau). \tag{2.13}$$

PROOF. Since $p_1(V + W - TX)_{S^1} = n \cdot \overline{\pi}^* u^2$, we have

$$\Sigma_{v,j}(u_v^j + n_v t)^2 + \Sigma_{\mu,j}(\omega_\mu^j + r_\mu t)^2 - (\Sigma_j (y_j')^2 + \Sigma_{\gamma,j}(x_\gamma^j + m_\gamma t)^2) = nt^2. \tag{2.14}$$

This implies the equalities:

$$
\begin{aligned}
&\Sigma_{v,j}(u_v^j)^2 + \Sigma_{\mu,j}(\omega_\mu^j)^2 = \Sigma_j (y_j')^2 + \Sigma_{\gamma,j}(x_\gamma^j)^2, \\
&\Sigma_{v,j} n_v u_v^j + \Sigma_{\mu,j} \omega_\mu^j r_\mu = \Sigma_{\gamma,j} m_\gamma x_\gamma^j, \\
&\Sigma_v n_v^2 d(n_v) + \Sigma_\mu r_\mu^2 d(r_\mu) - \Sigma_\gamma m_\gamma^2 d(m_\gamma) = n.
\end{aligned} \tag{2.15}
$$

By (1.15), for $\theta_v = \theta, \theta_1$; $a, b \in 2\mathbf{Z}$, $l \in \mathbf{Z}$, we have

$$\theta_v(x + l(t + a\tau + b), \tau) = e^{-\pi i (2lax + 2l^2 at + l^2 a^2 \tau)} \theta_v(x + lt, \tau). \tag{2.16}$$

Let $F_{1,\alpha}$, $F_{1,\alpha}^\beta$ be the contribution of M_α to $F_1(t,\tau)$, $F_1^\beta(t,\tau)$. By using (2.11), (2.15), (2.16), we get for $a, b \in 2\mathbf{Z}$,

$$
\begin{aligned}
F_{1,\alpha}(t + a\tau + b, \tau) &= e^{-\pi i n (a^2\tau + 2at)} F_{1,\alpha}(t,\tau), \\
F_{1,\alpha}^\beta(t + a\tau + b, \tau) &= y^{-\Sigma_\mu r_\mu a} e^{-\pi i n (a^2\tau + 2at)} F_{1,\alpha}^\beta(t,\tau).
\end{aligned} \tag{2.17}
$$

Since by the assumption, $y^N = 1$, we get Lemma 2.1. □

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we define

$$
\varepsilon_A = \begin{cases} 1, & \text{if } (c,d) \equiv (0,1) \pmod 2, \\ 2, & \text{if } (c,d) \equiv (1,0) \pmod 2, \\ 3, & \text{if } (c,d) \equiv (1,1) \pmod 2 \end{cases} \tag{2.18}
$$

For $g = e^{2\pi i t}, t \in \mathbf{R}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, $j = 1, 2, 3$, we let

$$
\begin{aligned}
(2.19)\qquad F_j(t,\tau) &= \varepsilon(j)\mathrm{ch}_g\Big(\mathrm{Ind}(D^c \otimes \Theta_q(TX|W)_v \otimes R_j(V)_v)\Big),\\
F_j^{\beta}(t,\tau)^A &= \varepsilon(j)\mathrm{ch}_g\Big(\mathrm{Ind}(D^c \otimes L_W^{c\beta} \otimes \Theta_q^{(c\tau+d)\beta}(TX|W)_v \otimes R_j(V)_v)\Big).
\end{aligned}
$$

with $\varepsilon(j) = 2^{-l}$ for $j = 1$; 1 for $j = 2, 3$.

By applying the family Atiyah-Bott-Segal-Singer Lefschetz fixed point formula, Theorem 2.1, and using (2.10), for $g = e^{2\pi i t}, t \in \mathbf{R} \setminus \mathbf{Q}$, $j = 1, 2, 3$, we get

$$
\begin{aligned}
(2.20)\qquad F_j(t,\tau) &= (2\pi i)^{r-k}\frac{\theta'(0,\tau)^{k-r}}{\theta_j(0,\tau)^l}\sum_\alpha \pi_*\Big[\Big(\frac{2\pi i y'}{\theta(y',\tau)}\Big)(TX^g)\\
&\quad \cdot\frac{\Pi_v\theta_j(u_v+n_vt,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma+m_\gamma t,\tau)(N_\gamma)}\Pi_\mu\theta(\omega_\mu+r_\mu t,\tau)(W_\mu)\Big],\\
F_j^{\beta}(t,\tau)^A &= (2\pi i)^{-k}\frac{\theta'(0,\tau)^k}{\theta_j(0,\tau)^l\theta((c\tau+d)\beta,\tau)^r}\sum_\alpha \pi_*\Big[\Big(\frac{2\pi i y'}{\theta(y',\tau)}\Big)(TX^g)\\
&\quad \cdot\frac{\Pi_v\theta_j(u_v+n_vt,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma+m_\gamma t,\tau)(N_\gamma)}\\
&\quad \cdot\Pi_\mu\Big(e^{2\pi i c\beta(\omega_\mu+r_\mu t)}\theta(\omega_\mu+r_\mu t+(c\tau+d)\beta,\tau)\Big)(W_\mu)\Big].
\end{aligned}
$$

Remark. In fact, to define a S^1-action on $L_W^{c\beta}$, we must replace the S^1-action by its N-fold action. Here by abusing notation, we still say an S^1-action without causing any confusion.

LEMMA 2.2. *If* $p_1(V+W-TX)_{S^1} = n\cdot\overline{\pi}^*(u^2)$, *under the action* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, *we have*

$$
(2.21)\qquad \begin{aligned}
F_1(A(t,\tau)) &= (c\tau+d)^{k-r}e^{\pi i n c t^2/(c\tau+d)}\Psi_{c\tau+d}F_{\varepsilon_A}(t,\tau),\\
F_1^{\beta}(A(t,\tau)) &= (c\tau+d)^{k}e^{\pi i n c t^2/(c\tau+d)}\Psi_{c\tau+d}F_{\varepsilon_A}^{\beta}(t,\tau)^A,
\end{aligned}
$$

PROOF. As the equation for F_1 in (2.21) is very easy, we leave it to the interested reader. Here, we only prove (2.21) for F_1^{β}.

By (1.15), (1.16), $\frac{\theta_1(t,\tau)}{\theta_1(0,\tau)}$ is a Jacobi form of index 1/2 and weight 0 over $(2\mathbf{Z})^2 \rtimes \Gamma_1(2)$. This explains the index ε_A in the following equation. By (1.16), we get

$$
(2.22)\qquad \begin{aligned}
\frac{\theta'(0,\frac{a\tau+b}{c\tau+d})}{\theta(\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d})} &= (c\tau+d)e^{-\pi i\frac{ct^2}{c\tau+d}}\frac{\theta'(0,\tau)}{\theta(t,\tau)},\\
\frac{\theta(\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d})}{\theta(\beta,\frac{a\tau+b}{c\tau+d})} &= e^{\pi i c(\frac{t^2}{c\tau+d}-\beta^2(c\tau+d))}\frac{\theta(t,\tau)}{\theta(\beta(c\tau+d),\tau)},\\
\frac{\theta_1(\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d})}{\theta_1(0,\frac{a\tau+b}{c\tau+d})} &= e^{\pi i c\frac{t^2}{c\tau+d}}\frac{\theta_{\varepsilon_A}(t,\tau)}{\theta_{\varepsilon_A}(0,\tau)}.
\end{aligned}
$$

By (2.11), (2.22), we get

$$(2.23)\quad F_1^{\beta}\Big(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\Big)$$
$$= (2\pi i)^{-k} \sum_{\alpha} \pi_* \Big[\Big(2\pi i y' \frac{\theta'(0, \frac{a\tau+b}{c\tau+d})}{\theta(y', \frac{a\tau+b}{c\tau+d})}\Big)(TX^g)$$
$$\cdot \Pi_\gamma \Big(\frac{\theta'(0, \frac{a\tau+b}{c\tau+d})}{\theta(x_\gamma + m_\gamma \frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d})}\Big)(N_\gamma) \Pi_v \Big(\frac{\theta_1(u_v + n_v \frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d})}{\theta_1(0, \frac{a\tau+b}{c\tau+d})}\Big)(V_v)$$
$$\cdot \Pi_\mu \frac{\theta(\omega_\mu + r_\mu \frac{t}{c\tau+d} + \beta, \frac{a\tau+b}{c\tau+d})}{\theta(\beta, \frac{a\tau+b}{c\tau+d})}(W_\mu)\Big]$$
$$= (2\pi i)^{-k}(c\tau+d)^k e^{\pi i n c t^2/(c\tau+d)} \sum_{\alpha} \pi_* \Big[\Big(2\pi i y' \frac{\theta'(0,\tau)}{\theta((c\tau+d)y', \tau)}\Big)(TX^g)$$
$$\cdot \Pi_\gamma \Big(\frac{\theta'(0,\tau)}{\theta((c\tau+d)x_\gamma + m_\gamma t, \tau)}\Big)(N_\gamma) \Pi_v \Big(\frac{\theta_{\varepsilon_A}((c\tau+d)u_v + n_v t, \tau)}{\theta_{\varepsilon_A}(0,\tau)}\Big)(V_v)$$
$$\cdot \Pi_\mu \Big[e^{2\pi i c((c\tau+d)\omega_\mu + r_\mu t)\beta} \frac{\theta((c\tau+d)\omega_\mu + r_\mu t + (c\tau+d)\beta, \tau)}{\theta((c\tau+d)\beta, \tau)}\Big](W_\mu)\Big].$$

To prove (2.23), we will prove

(2.24)
$$\Big\{\pi_*\Big[\Big(\frac{2\pi i y'}{\theta((c\tau+d)y', \tau)}\Big)(TX^g) \frac{\Pi_v \theta_{\varepsilon_A}((c\tau+d)u_v + n_v t, \tau)(V_v)}{\Pi_\gamma \theta((c\tau+d)x_\gamma + m_\gamma t, \tau)(N_\gamma)}$$
$$\cdot \Pi_\mu \Big[e^{2\pi i c\beta((c\tau+d)\omega_\mu + r_\mu t)} \theta\Big((c\tau+d)\omega_\mu + r_\mu t + (c\tau+d)\beta, \tau\Big)\Big](W_\mu)\Big]\Big\}^{(2p)}$$
$$= (c\tau+d)^p \Big\{\pi_*\Big[\Big(\frac{2\pi i y'}{\theta(y', \tau)}\Big)(TX^g) \frac{\Pi_v \theta_{\varepsilon_A}(u_v + n_v t, \tau)(V_v)}{\Pi_\gamma \theta(x_\gamma + m_\gamma t, \tau)(N_\gamma)}$$
$$\cdot \Pi_\mu \Big[e^{2\pi i c\beta(\omega_\mu + r_\mu t)} \theta\Big(\omega_\mu + r_\mu t + (c\tau+d)\beta, \tau\Big)\Big](W_\mu)\Big]\Big\}^{(2p)}.$$

By looking at the degree $2(p+k_\alpha)$ part, that is the $(p+k_\alpha)$-th homogeneous terms of the polynomials in x's, y''s, u's and ω's on both sides, we immediately get (2.24).

The proof of Lemma 2.2 is complete. □

Since $F_j(t,\tau)$, $F_j^{\beta}(t,\tau)^A$ $(j = 1,2,3)$ are the equivariant Chern characters of the index bundles of some elliptic operators, the same proof as that of Lemma 1.3 gives the following

LEMMA 2.3.

i) *$F_j(t,\tau)$ $(j = 1,2,3)$ is holomorphic in $(t,\tau) \in \mathbf{R} \times \mathbf{H}$.*

ii) *If $c_1(W) \equiv 0 (\mathrm{mod} N)$, then for $A \in SL_2(\mathbf{Z})$, $j = 1,2,3$, $F_j^{\beta}(t,\tau)^A$ is holomorphic in $(t,\tau) \in \mathbf{R} \times \mathbf{H}$.*

This is the only essential place where we need the topological condition $c_1(W) \equiv 0 (\mathrm{mod} N)$ which insures the existence of $L_W^{c\beta}$, therefore the holomorphicity of $F_1^{\beta}(t,\tau)^A$ for $t \in \mathbf{R}$.

Proof of Theorem 2.2. Now, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2N)$, by (1.15), (2.20), we get

$$F^{\beta}_{\varepsilon_A}(t,\tau)^A = F^{\beta}_1(t,\tau). \tag{2.25}$$

By using the above three Lemmas, and proceeding as in the proof of Theorem 1.1, we know that $F^{\beta}_1(t,\tau)$ is holomorphic in $(t,\tau) \in \mathbf{C} \times \mathbf{H}$.

From Lemmas 2.1, 2.2, (2.25), we get Theorem 2.2. □

2.4. Family rigidity and vanishing theorems for ***spin^c*****-manifolds.** From Lemma 1.4 and Theorem 2.2, we get the following family rigidity and vanishing theorems for *spin*^{c}-manifolds.

THEOREM 2.3. *Let M, B, W, V as in Theorem 2.2. If $p_1(V + W - TX)_{S^1} = n \cdot \overline{\pi}^* u^2$ $(n \in \mathbf{Z})$ and $c_1(W)_{S^1} = c_1(TX)_{S^1}$ in $H^*_{S^1}(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$.*

i). *If $n = 0$, then $D^c \otimes \Theta_q(TX|W)_v \otimes R_1(V)_v$ is rigid. If, in addition, $c_1(W)$ is divisible by an integer $N \geq 2$, then $D^c \otimes \Theta^{\beta}_q(TX|W)_v \otimes R_1(V)_v$ is rigid for $y = e^{2\pi i\beta}$ an Nth root of unity.*

ii). *If $n < 0$, then the equivariant Chern character of the index bundle $D^c \otimes \Theta_q(TX|W)_v \otimes R_1(V)_v$ vanishes identically, in particular, the Chern character of this index bundle is zero. If, in addition, $c_1(W)$ is divisible by an integer $N \geq 2$, then the equivariant Chern character of the index bundle $D^c \otimes \Theta^{\beta}_q(TX|W)_v \otimes R_1(V)_v$ vanishes identically for $y = e^{2\pi i\beta}$ an Nth root of unity, in particular, the Chern character of this index bundle is zero.*

The following family vanishing Theorem generalizes [**LM**, Theorem 3.2] to family *spin*^{c}-manifolds.

THEOREM 2.4. *Let $\pi : M \to B$ be a fibration of compact connected manifolds with compact fibre X, and S^1 acts fiberwise and non-trivially on M. We suppose TX has a S^1-equivariant spin^c structure. If $c_1(TX) = 0$ in $H^*(M, \mathbf{Q})$, and if $p_1(TX)_{S^1} = -n \cdot \overline{\pi}^* u^2$ in $H^*_{S^1}(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$ for some integer n, then the equivariant Chern character of the index bundle, especially the Chern character of the index bundle of $D^c \otimes \otimes_{m=1}^{\infty} S_{q^m}(\widetilde{TX})$ is zero.*

Remark. Note that the condition $c_1(TX) = 0$ in $H^*(M, \mathbf{Q})$ does not mean the *Spin*^{c} structure is spin. This only insures that there exists $m \in \mathbf{Z}$, such that $c_1(TX)_{S^1} = m\overline{\pi}^* u$. So in fact, the difference between [**LM**, Theorem 3.2] and Theorem 2.4 are quite subtle.

As pointed out by Dessai [**D2**, §3], when the S^1-action is induced from an S^3 or nice Pin(2) action on M (In fact the S^3 and *Pin*(2) action need not act fiberwise on M), the condition $p_1(TX)_{S^1} = -n \cdot \overline{\pi}^* u^2$ in $H_{S^1}(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is also equivalent to $p_1(TX) = 0$ in $H^*(M, \mathbf{Q})$.

In [**HL**], some related result was proved for foliations.

Proof of Theorem 2.4. We only need to put $W = V = 0$ in Theorem 2.2. In fact, by (2.15), we know

$$\Sigma_j m_j^2 d(m_j) = -n. \tag{2.26}$$

So the case $n > 0$ can never happen. If $n = 0$, then all the exponents $\{m_j\}$ are zero, so the S^1-action can not have a fixed point. By (2.11), we get the result. For $n < 0$, by Remark in Section 2.2 and Theorem 2.3, we get the result. □

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