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Rigidity and vanishing theorems in K-theory

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Abstract. In this Note we announce some new rigidity and vanishing results in the equivariant K-theory. These results generalize the famous Witten rigidity theorems. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Théorèmes de rigidité et d'annulation dans la K-théorie

Résumé. *Dans cette Note, nous annonçons des résultats de rigidité et d'annulation dans la K-théorie équivariante. Ces résultats étendent les théorèmes de rigidité de Witten dans le contexte de la K-théorie équivariante.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit X une variété compacte, orientée et de dimension paire. On suppose que X admet une action de S^1 et que X est munie d'une structure spinorielle S^1-invariante.

Soit $g^{\mathrm{T}X}$ une métrique S^1-invariante sur $\mathrm{T}X$. Soit $\mathrm{S}(\mathrm{T}X) = \mathrm{S}^+(\mathrm{T}X) \oplus \mathrm{S}^-(\mathrm{T}X)$ le fibré des spineurs $\mathbf{Z}_2$-gradués sur $(\mathrm{T}X, g^{\mathrm{T}X})$. Suivant Witten [9], on pose

$$\Theta'_q(\mathrm{T}X) = \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\mathrm{T}X \otimes_{\mathbf{R}} \mathbf{C}) \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}X \otimes_{\mathbf{R}} \mathbf{C}) = \sum_{n=0}^{+\infty} R_n q^n,$$

avec $R_n \in K(X)$.

Witten a conjecturé dans [9] que pour tout $n \in \mathbf{N}$, le nombre de Lefschetz $\mathrm{L}(g)_n$ de l'opérateur de Dirac twisté, qui envoit $\Gamma(\mathrm{S}^+(\mathrm{T}X) \otimes \mathrm{S}(\mathrm{T}X) \otimes R_n)$ dans $\Gamma(\mathrm{S}^-(\mathrm{T}X) \otimes \mathrm{S}(\mathrm{T}X) \otimes R_n)$, ne dépend pas $g \in \mathrm{S}^1$.

La conjecture de Witten a été démontrée par Taubes [8], Bott–Taubes [2] et Liu [5] etc.

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Dans [6], Liu et Ma ont étendu la conjecture de Witten à une situation en famille. Ils ont démontré des résultats de rigidité et d'annulation au niveau du caractère de Chern équivariant pour la famille d'opérateurs de Dirac twistés décrit ci-dessus.

Dans cette Note, nous annonçons des résultats de rigidité et d'annulation au niveau de la K-théorie équivariante, qui raffinent les résultats de Liu–Ma [6]. Les détails de la preuve et les extensions sont développés dans [7].

In this Note, we announce the proofs of the K-theory versions of the famous rigidity and vanishing theorems for elliptic genera. Details and further extensions will be developed in [7].

1. A family rigidity theorem for the Witten elements

For simplicity, we will focus on the discussion of the rigidity for one of the elliptic genera. For more general rigidity and vanishing results, we refer the reader to [7].

Let $\pi : M \to B$ be a smooth fibration of compact manifolds with fibre X and $\dim X = 2\ell$. Let $\mathrm{T}X$ be the vertical tangent bundle of the fibration $\pi : M \to B$. We make the assumption that S^1 acts fiberwise on M, and that $\mathrm{T}X$ admits an S^1-equivariant spin structure. Let $g^{\mathrm{T}X}$ be an S^1-invariant metric on $\mathrm{T}X$. Let $\mathrm{S}(\mathrm{T}X) = \mathrm{S}^+(\mathrm{T}X) \oplus \mathrm{S}^-(\mathrm{T}X)$ be the $\mathbf{Z}_2$-graded bundle of spinors of $(\mathrm{T}X, g^{\mathrm{T}X})$.

For a complex (resp. real) vector bundle E over M, let

$$\mathrm{Sym}_t(E) = 1 + tE + t^2\mathrm{Sym}^2 E + \cdots,$$
$$\Lambda_t(E) = 1 + tE + t^2\Lambda^2 E + \cdots$$

be the symmetric and exterior power operations of E (resp. $E \otimes_{\mathbf{R}} \mathbf{C}$) in $K(M)[[t]]$ respectively. Following Witten [9], set

$$\Theta'_q(\mathrm{T}X) = \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\mathrm{T}X) \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}X) = \sum_{n=0}^{+\infty} R_n q^n, \tag{1.1}$$

with each $R_n \in K(M)$.

For any $n \in \mathbf{N}$, $b \in B$, let $D_b^X \otimes R_n$ denote the twisted *signature* operator on $X_b = \pi^{-1}(b)$ mapping from $\Gamma\big((\mathrm{S}^+(\mathrm{T}X) \otimes \mathrm{S}(\mathrm{T}X) \otimes R_n)|_{X_b}\big)$ to $\Gamma\big((\mathrm{S}^-(\mathrm{T}X) \otimes \mathrm{S}(\mathrm{T}X) \otimes R_n)|_{X_b}\big)$. Then $\{D_b^X \otimes R_n\}_{b\in B}$ is a smooth family of twisted signature operators which we denote by $D^X \otimes R_n$. The family operator $D^X \otimes R_n$ is clearly S^1-equivariant. Thus, its index bundle $\mathrm{Ind}(D^X \otimes R_n)$, in the sense of Atiyah and Singer [1], lies in $K_{\mathrm{S}^1}(B)$. Let $\big(\mathrm{Ind}(D^X \otimes R_n)\big)^{\mathrm{S}^1} \in K(B)$ denote the S^1-invariant part of $\mathrm{Ind}(D^X \otimes R_n)$. We say that $D^X \otimes R_n$ is *rigid on the equivariant K-theory level* if $\mathrm{Ind}(D^X \otimes R_n) = \big(\mathrm{Ind}(D^X \otimes R_n)\big)^{\mathrm{S}^1}$.

We can now state the main result of this Note as follows:

THEOREM 1.1. – *For any $n \in \mathbf{N}$, the family operator $D^X \otimes R_n$ is rigid on the equivariant K-theory level.*

Remark 1.2. – When B is a point, Theorem 1.1 was conjectured by Witten [9] and was proved by Taubes [8], Bott–Taubes [2] and Liu [5], etc. When B is not a point, Theorem 1.1 refines a result of Liu–Ma [6] to the equivariant K-theory level.

In order to outline a proof of Theorem 1.1, we will first state in the next section a K-theory version of the equivariant family index theorem for the considered operators.

2. An equivariant family index theorem for circle actions

Let F be the fixed point set of the S^1-action on M. Then $\pi : F \to B$ is a fibration with compact fibre denoted by Y. One has the following splitting of $\mathrm{T}X$ over F,

$$\mathrm{T}X|_F = \mathrm{T}Y \bigoplus_{v \neq 0} N_{v,\mathbf{R}}, \tag{2.1}$$

where $N_{v,\mathbf{R}}$ denotes the underlying real bundle of the complex vector bundle N_v on which S^1 acts by sending g to g^v. Since we can choose either N_v or $\overline{N}_v$ as the complex vector bundle for $N_{v,\mathbf{R}}$, in what follows we may and we will assume that

$$\mathrm{T}X|_F = \mathrm{T}Y \bigoplus_{0<v} N_v, \tag{2.2}$$

where N_v is the complex vector bundle on which S^1 acts by sending g to g^v (here N_v can be zero).

Let $\mathrm{T}Y$ carry the orientation induced from those of $\mathrm{T}X$ and the N_v's via (2.2). Let D^Y be the family signature operator along the fibers Y. If E is an S^1-equivariant Hermitian vector bundle over F carrying with an S^1-invariant Hermitian connection, we denote by $D^Y \otimes E$ the associated family twisted signature operator. Then the index bundle of $D^Y \otimes E$ lies in $K_{\mathrm{S}^1}(B)$. For any $h \in \mathbf{Z}$, let $\mathrm{Ind}(D^Y \otimes E, h)$ denote the component of $\mathrm{Ind}(D^Y \otimes E)$ of weight h with respect to the induced S^1-representation. In what follows, if $R(q) = \sum_{m \in \mathbf{Z}} R_m q^m \in K_{\mathrm{S}^1}(M)[[q]]$, we will also denote $\mathrm{Ind}(D^X \otimes R_m, h)$ by $\mathrm{Ind}(D^X \otimes R(q), m, h)$.

The main result of this section can be stated as follows:

THEOREM 2.1. – *For $m,\ h \in \mathbf{Z}$, we have the following identity in $K(B)$,*

$$\begin{aligned} &\mathrm{Ind}\big(D^X \otimes \Theta'_q(\mathrm{T}X), m, h\big) \\ &\quad = \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}\Bigg(D^{Y_\alpha} \otimes \Theta'_q(\mathrm{T}X) \otimes \mathrm{Sym}\Bigg(\bigoplus_{0<v} N_v\Bigg) \otimes \Lambda\Bigg(\bigoplus_{0<v} N_v\Bigg), m, h\Bigg), \end{aligned} \tag{2.3}$$

where α runs over the connected components of F.

Proof. – Theorem 2.1 is proved in [7] by using the analytic arguments in [10] and [11]. □

3. Proof of Theorem 1.1

For $p \in \mathbf{N}$, we define the following elements in $K_{\mathrm{S}^1}(F)[[q]]$:

$$\begin{aligned} \mathcal{F}_p(X) &= \bigotimes_{0<v} \Bigg(\bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(N_v) \bigotimes_{n>pv} \mathrm{Sym}_{q^n}(\overline{N}_v)\Bigg) \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}Y), \\ \mathcal{F}'_p(X) &= \bigotimes_{\substack{0<v \\ 0 \leqslant n \leqslant pv}} \big(\mathrm{Sym}_{q^{-n}}(N_v) \otimes \det N_v\big), \end{aligned}$$

$$\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X) \otimes \Lambda\Bigg(\bigoplus_{0<v} N_v\Bigg) \otimes \Bigg(\det\Bigg(\bigoplus_{0<v} N_v\Bigg)\Bigg)^{-1} \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\mathrm{T}X). \tag{3.1}$$

Then

$$\mathcal{F}^0(X) = \Theta'_q(\mathrm{T}X) \otimes \mathrm{Sym}\Bigg(\bigoplus_{0<v} N_v\Bigg) \otimes \Lambda\Bigg(\bigoplus_{0<v} N_v\Bigg).$$

Set

$$e(N)=\sum_{0<v}v^2\dim N_v,\qquad d'(N)=\sum_{0<v}v\dim N_v. \tag{3.2}$$

We now state two intermediate results on the relations between the family indices on the fixed point set.

PROPOSITION 3.1. – *For $h,\ p,\ m\in\mathbf{Z}$, $p>0$, we have the following identity in $K(B)$,*

$$\sum_\alpha(-1)^{\Sigma_{0<v}\dim N_v}\mathrm{Ind}\left(D^{Y_\alpha}\otimes\Theta'_q(\mathrm{T}X)\otimes\mathrm{Sym}\left(\bigoplus_{0<v}N_v\right)\otimes\Lambda\left(\bigoplus_{0<v}N_v\right),m,h\right)$$
$$=\sum_\alpha(-1)^{\Sigma_{0<v}\dim N_v}\mathrm{Ind}\left(D^{Y_\alpha}\otimes\mathcal{F}^{-p}(X),m+\frac{1}{2}p^2e(N)+\frac{1}{2}pd'(N),h\right). \tag{3.3}$$

PROPOSITION 3.2. – *For $h,\ p\in\mathbf{Z}$, $p>0$, $m\in\mathbf{Z}$, on each connected component F_α of F, we have the following identity in $K(B)$,*

$$\mathrm{Ind}\left(D^{Y_\alpha}\otimes\mathcal{F}^{-p}(X),m+\frac{1}{2}p^2e(N)+\frac{1}{2}pd'(N),h\right)=\mathrm{Ind}(D^{Y_\alpha}\otimes\mathcal{F}^0(X),m+ph,h).$$

Propositions 3.1 and 3.2 are proved in [7], where, inspired by Taubes [8], we introduce certain shifting operations for vector bundles over F and study the behaviour of the involved family indices under the shifting operations. Moreover, in the proof of Proposition 3.1, we make use a key idea in [8] to reduce the problem to the fixed point set of the induced $\mathbf{Z}_n$-actions. For more details, *see* [7].

Proof of Theorem 1.1. – By (3.1), Theorem 2.1 and Propositions 3.1, 3.2, for $p\in\mathbf{Z}$, $p>0$, we get the following identity in $K(B)$,

$$\mathrm{Ind}\big(D^X\otimes\Theta'_q(\mathrm{T}X),m,h\big)=\mathrm{Ind}(D^X\otimes\Theta'_q(\mathrm{T}X),m',h), \tag{3.4}$$

with

$$m'=m+ph. \tag{3.5}$$

Note that by (1.1), if $m<0$, for $h\in\mathbf{Z}$, we have

$$\mathrm{Ind}\big(D^X\otimes\Theta'_q(\mathrm{T}X),m,h\big)=0\quad\text{in } K(B). \tag{3.6}$$

Let $m_0,\ h\in\mathbf{Z}$ with $h\neq 0$ be fixed:
(i) if $h>0$, we take $m'=m_0$, then when p is big enough, we get $m<0$;
(ii) if $h<0$, we take $m=m_0$, then as p is big enough we get $m'<0$.
From (3.4), (3.6) and the above discussion, we get Theorem 1.1. □

4. Vanishing results and further remarks

In some sense, our proof given in [7] may be considered as a K-theory version of the proof given by Bott–Taubes [2] of the Witten rigidity theorem, which was also inspired by the ideas in Taubes' proof [8]. While on the other hand, the proof in [7] is self-contained and the arguments in [7], even in the case where the base B is a point, are different from the ones in the papers of Bott–Taubes [2], Liu [5] and Taubes [8]. Moreover, our method in [7] is quite general and allows us to deal with systematically more general situations than what was described in this note. We refer to [7] for more results and discussions. Here, for the conclusion of this Note, we only state one of the vanishing results, which follows from our techniques together with an observation of Dessai [3].

THEOREM 4.1. – *Assume that M is connected and that $\frac{1}{2}p_1(\mathrm{T}X) = 0$, where $p_1(\mathrm{T}X)$ is the first Pontryagin class of $\mathrm{T}X$. If the S^1-action on M is non-trivial, and is induced from a fiberwise S^3-action on M which also preserves the spin structure on $\mathrm{T}X$, then the index bundle of the family twisted* Dirac *operator $\mathcal{D}^X \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}X)$ is identically zero in $K_{\mathrm{S}^1}(B)$.*

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