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GEOMETRIC HYPOELLIPTIC LAPLACIAN AND ORBITAL INTEGRALS [after Bismut, Lebeau and Shen]

by **Xiaonan MA**

INTRODUCTION

In 1956, Selberg expressed the trace of an invariant kernel acting on a locally symmetric space $Z = \Gamma\backslash G/K$ as a sum of certain integrals on the orbits of Γ in G, the so called "orbital integrals," and he gave a geometric expression for such orbital integrals for the heat kernel when $G = \mathrm{SL}_2(\mathbb{R})$, and the corresponding locally symmetric space is a compact Riemann surface of constant negative curvature. In this case, the orbital integrals are one to one correspondence with the closed geodesics in Z. In the general case, Harish-Chandra worked on the evaluation of orbital integrals from the 1950s until the 1970s. He could give an algorithm to reduce the computation of an orbital integral to lower dimensional Lie groups by the discrete series method. Given a reductive Lie group, in a finite number of steps, there is a formula for such orbital integrals. See Section 3.5 for a brief description of Harish-Chandra's Plancherel theory.

It is important to understand the different properties of orbital integrals even without knowing their explicit values. The orbital integrals appear naturally in Langlands program.

About 15 years ago, Bismut gave a natural construction of a Hodge theory whose corresponding Laplacian is a hypoelliptic operator acting on the total space of the cotangent bundle of a Riemannian manifold. This operator interpolates formally between the classical elliptic Laplacian on the base and the generator of the geodesic flow. We will describe recent developments in the theory of the hypoelliptic Laplacian, and we will explain two consequences of this program, the explicit formula obtained by Bismut for orbital integrals, and the recent solution by Shen of Fried's conjecture (dating back to 1986) for locally symmetric spaces. The conjecture predicts the equality of the analytic torsion and of the value at 0 of the Ruelle dynamical zeta function associated with the geodesic flow.

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We will describe in more detail these two last results.

Let G be a connected reductive Lie group, let $\mathfrak{g}$ be its Lie algebra, let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution of G. Let $K \subset G$ be the maximal compact subgroup of G given by the fixed-points of θ, and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the corresponding Cartan decomposition of $\mathfrak{g}$.

Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action of G on $\mathfrak{g}$ and also under θ. We assume B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Then $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ is a K-invariant scalar product on $\mathfrak{g}$ that is such that the Cartan decomposition is an orthogonal splitting.

Let $C^{\mathfrak{g}} \in U(\mathfrak{g})$ be the Casimir element of G. If $\{e_i\}_{i=1}^{m}$ is an orthonormal basis of $\mathfrak{p}$ and $\{e_i\}_{i=m+1}^{m+n}$ is an orthonormal basis of $\mathfrak{k}$, set

(0.1)
$$B^*(\mathfrak{g}) = -\frac{1}{2} \sum_{1 \leq i,j \leq m} \left| [e_i, e_j] \right|^2 - \frac{1}{6} \sum_{m+1 \leq i,j \leq m+n} \left| [e_i, e_j] \right|^2, \quad \mathscr{L} = \frac{1}{2} C^{\mathfrak{g}} + \frac{1}{8} B^*(\mathfrak{g}).$$

Let E be a finite dimensional Hermitian vector space, let $\rho^E : K \to \mathrm{U}(E)$ be a unitary representation of K. Let $F = G \times_K E$ be the corresponding vector bundle over the symmetric space $X = G/K$. Then $\mathscr{L}$ descends to a second order differential operator $\mathscr{L}^X$ acting on $C^\infty(X, F)$. For $t > 0$, let $e^{-t\mathscr{L}^X}(x, x')$ be the smooth kernel of the heat operator $e^{-t\mathscr{L}^X}$.

Assume $\gamma \in G$ is semisimple. Then up to conjugation, there exist $a \in \mathfrak{p}, k \in K$ such that $\gamma = e^a k^{-1}$ and $\mathrm{Ad}(k)a = a$. Let $\mathrm{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}^X}\right]$ denote the corresponding orbital integral of $e^{-t\mathscr{L}^X}$ (cf. (3.22), (3.46)). If $\gamma = 1$, then the orbital integral associated with $1 \in G$ is given by

$$\mathrm{Tr}^{[\gamma=1]}\left[e^{-t\mathscr{L}^X}\right] = \mathrm{Tr}^F\left[e^{-t\mathscr{L}^X}(x, x)\right] \tag{0.2}$$

which does not depend on $x \in X$.

Let $Z(\gamma) \subset G$ be the centralizer of γ, and let $\mathfrak{z}(\gamma)$ be its Lie algebra. Set $\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}$, $\mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}$. Then $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$.

Set $\mathfrak{z}_0 = \mathrm{Ker}(\mathrm{ad}(a)), \mathfrak{k}_0 = \mathfrak{z}_0 \cap \mathfrak{k}$. Let $\mathfrak{z}_0^\perp$ be the orthogonal space to $\mathfrak{z}_0$ in $\mathfrak{g}$. Let $\mathfrak{k}_0^\perp(\gamma)$ be the orthogonal space to $\mathfrak{k}(\gamma)$ in $\mathfrak{k}_0$, and $\mathfrak{z}_0^\perp(\gamma)$ be the orthogonal space to $\mathfrak{z}(\gamma)$ in $\mathfrak{z}_0$, so that $\mathfrak{z}_0^\perp(\gamma) = \mathfrak{p}_0^\perp(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma)$. For a self-adjoint matrix Θ, set $\widehat{A}(\Theta) = \det^{1/2}\left[\frac{\Theta/2}{\sinh(\Theta/2)}\right]$. For $Y \in \mathfrak{k}(\gamma)$, set

$$(0.3) \quad J_\gamma(Y) = \left|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{-1/2} \frac{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{k}(\gamma)})} \times \left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\left(1-e^{-i\,\mathrm{ad}(Y)}\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-e^{-i\,\mathrm{ad}(Y)}\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

If $\gamma = 1$, then the above equation reduces to $J_1(Y) = \frac{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{p}})}{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{k}})}$ for $Y \in \mathfrak{k} = \mathfrak{k}(1)$.

Theorem 0.1 (Bismut's orbital integral formula [**12**], Theorem 6.1.1)
Assume $\gamma \in G$ is semisimple. Then for any $t > 0$, we have

$$(0.4) \quad \mathrm{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}^X}\right] = (2\pi t)^{-\dim\mathfrak{p}(\gamma)/2} e^{-\frac{|a|^2}{2t}} \int_{\mathfrak{k}(\gamma)} J_\gamma(Y)\,\mathrm{Tr}^E\left[\rho^E(k^{-1})e^{-i\rho^E(Y)}\right] e^{-\frac{|Y|^2}{2t}} \frac{dY}{(2\pi t)^{\dim\mathfrak{k}(\gamma)/2}}.$$

There are some striking similarities of Equation (0.4) with the Atiyah-Singer index formula, where the $\widehat{A}$-genus of the tangent bundle appears. Here the $\widehat{A}$-function of both $\mathfrak{p}$ and $\mathfrak{k}$ parts (with different roles) appear naturally in the integral (0.4).

A more refined version of Theorem 0.1 for the orbital integral associated with the wave operator is given in [**12**, Theorem 6.3.2] (cf. Theorem 3.12).

Let $\Gamma \subset G$ be a discrete cocompact torsion free subgroup. The above objets constructed on X descend to the locally symmetric space $Z = \Gamma\backslash X$ and $\pi_1(Z) = \Gamma$. We denote by $\mathscr{L}^Z$ the corresponding differential operator on Z. Let $[\Gamma]$ be the set of conjugacy classes in Γ. The Selberg trace formula (cf. (3.28), (3.64)) for the heat kernel of the Casimir operator on Z says that

$$(0.5) \qquad \mathrm{Tr}[e^{-t\mathscr{L}^Z}] = \sum_{[\gamma]\in[\Gamma]} \mathrm{Vol}\left(\Gamma\cap Z(\gamma)\backslash Z(\gamma)\right)\mathrm{Tr}^{[\gamma]}[e^{-t\mathscr{L}^X}].$$

Each term $\mathrm{Tr}^{[\gamma]}[\cdot]$ in (0.5) is evaluated in (0.4).

Assume $m = \dim X$ is odd now. Let $\rho : \Gamma \to U(\mathbf{q})$ be a unitary representation. Then $F = X \times_\Gamma \mathbb{C}^{\mathbf{q}}$ is a flat Hermitian vector bundle on $Z = \Gamma\backslash X$. Let $T(F)$ be the analytic torsion associated with F on Z (cf. Definition 5.1), which is a regularized determinant of the Hodge Laplacian for the de Rham complex associated with F.

In 1986, Fried discovered a surprising relation of the analytic torsion to dynamical systems. In particular, for a compact orientable hyperbolic manifold, he identified the value at zero of the Ruelle dynamical zeta function associated with the closed geodesics in Z and with ρ, to the corresponding analytic torsion, and he conjectured that a similar result should hold for general compact locally homogenous manifolds. In 1991, Moscovici-Stanton **[54]** made an important progress in the proof of Fried's conjecture for locally symmetric spaces. The following recent result of Shen establishes Fried's conjecture for arbitrary locally symmetric spaces, and Theorem 0.1 is one important ingredient in Shen's proof.

Given $[\gamma] \in [\Gamma]\backslash\{1\}$, let $B_{[\gamma]}$ be the space of closed geodesics in Z which lie in the homotopy class $[\gamma]$, and let $l_{[\gamma]}$ be the length of the geodesic associated with γ in Z. The group $\mathbb{S}^1$ acts on $B_{[\gamma]}$ by rotations. This action is locally free. Denote by $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) \in \mathbb{Q}$ the orbifold Euler characteristic number for the quotient orbifold $\mathbb{S}^1\backslash B_{[\gamma]}$. Let

$$n_{[\gamma]} = \left|\mathrm{Ker}\left(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]})\right)\right| \tag{0.6}$$

be the generic multiplicity of $B_{[\gamma]}$.

THEOREM 0.2 (**[62]**). — *For any unitary representation $\rho : \Gamma \to U(\mathbf{q})$,*

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \mathrm{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{n_{[\gamma]}} e^{-\sigma l_{[\gamma]}}\right) \tag{0.7}$$

is a well-defined meromorphic function on $\mathbb{C}$. If $H^\bullet(Z,F) = 0$, then $R_\rho(\sigma)$ is holomorphic at $\sigma = 0$ and

$$R_\rho(0) = T(F)^2. \tag{0.8}$$

This article is organized as follows. In Section 1, we describe Bismut's program on the geometric hypoelliptic Laplacian in de Rham theory, and we give its applications. In Section 2, we introduce the heat kernel on smooth manifolds and the basic ideas in the heat equation proof of the Lefschetz fixed-point formulas, which will serve as a model for the proof of Theorem 0.1. In Section 3, we review orbital integrals, their relation to Selberg trace formula, and we state Theorem 0.1. In Section 4, we give the basic ideas in how to adapt the construction of the hypoelliptic Laplacian of Section 1 in the context of locally symmetric spaces in order to establish Theorem 0.1. In Section 5, we concentrate on Shen's solution of Fried's conjecture.

Notation. — If A is a $\mathbb{Z}_2$-graded algebra, if $a, b \in A$, the supercommutator $[a, b]$ is given by

$$[a, b] = ab - (-1)^{\deg a\cdot\deg b} ba. \tag{0.9}$$

If B is another $\mathbb{Z}_2$-graded algebra, we denote by $A\widehat{\otimes}B$ the $\mathbb{Z}_2$-graded tensor product, such that the $\mathbb{Z}_2$-degree of $a\widehat{\otimes}b$ is given by $\deg a + \deg b$, and where the product is given by

$$(a\widehat{\otimes}b) \cdot (c\widehat{\otimes}d) = (-1)^{\deg b\cdot\deg c} ac \widehat{\otimes} bd. \tag{0.10}$$

If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded vector space, and $\tau = \pm 1$ on $E^\pm$, for $u \in \mathrm{End}(E)$, the supertrace $\mathrm{Tr}_s[u]$ is given by

$$\mathrm{Tr}_s[u] = \mathrm{Tr}[\tau u]. \tag{0.11}$$

In what follows, we will often add a superscript to indicate where the trace or supertrace is taken.

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1. FROM HYPOELLIPTIC LAPLACIANS TO THE TRACE FORMULA

In this section, we describe some basic ideas taken from Bismut's program on the geometric hypoelliptic Laplacian and its applications to geometry and dynamical systems.

A differential operator P is hypoelliptic if for every distribution u defined on an open set U such that Pu is smooth, then u is smooth on U. Elliptic operators are hypoelliptic, but there are hypoelliptic differential operators which are not elliptic. Classical examples are Kolmogorov operator $\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ [**44**] and Hörmander's generalization $\sum_{j=1}^k X_j^2 + X_0$ on Euclidean spaces [**42**]. Along this line, see for example Helffer-Nier's [**38**] recent book and Lebeau's work [**46**] on the hypoelliptic estimates and Fokker-Planck operators.

In 1978, Malliavin [**50**] introduced the so-called 'Malliavin calculus' to reprove Hörmander's regularity result [**42**] from a probabilistic point of view. Malliavin calculus was further developed by Bismut [**4**] and Stroock [**63**].

About 15 years ago, Bismut initiated a program whose purpose is to study the applications of hypoelliptic second order differential operators to differential geometry.

In [**6**], Bismut constructed a (geometric) hypoelliptic Laplacian on the total space of the cotangent bundle T^*M of a compact Riemannian manifold M, that depends on a parameter $b > 0$. This hypoelliptic Laplacian is a deformation of the usual Laplacian on M. More precisely, when $b \to 0$, it converges to the Laplacian on M in a suitable sense, and when $b \to +\infty$, it converges to the generator of the geodesic flow. In this way, properties of the geodesic flow on M are potentially related to the spectral properties of the Laplacian on M.

We now explain briefly Bismut's hypoelliptic Laplacian in de Rham theory. Let (M, g^{TM}) be a compact Riemannian manifold of dimension m. Let $(\Omega^\bullet(M), d)$ be the de Rham complex of M, let d^* be the formal L_2 adjoint of d, and let $\square^M = (d+d^*)^2$ be the Hodge Laplacian acting on $\Omega^\bullet(M)$.

Let $\pi : \mathscr{M} \to M$ be the total space of the cotangent bundle T^*M. Let Δ^V be the Laplacian along the fibers T^*M, and let $\mathscr{H}$ be the function on $\mathscr{M}$ defined by

$$\mathscr{H}(x,p) = \frac{1}{2}|p|^2 \quad \text{for } p \in T^*_x M, x \in M. \tag{1.1}$$

Let $Y^{\mathscr{H}}$ be the Hamiltonian vector field on $\mathscr{M}$ associated with $\mathscr{H}$ and with the canonical symplectic form on $\mathscr{M}$. Then $Y^{\mathscr{H}}$ is the generator of the geodesic flow. Let $L_{Y^{\mathscr{H}}}$ denote the corresponding Lie derivative operator acting on $\Omega^\bullet(\mathscr{M})$. For $b > 0$, the Bismut hypoelliptic Laplacian on $\mathscr{M}$ is given by

$$\mathscr{L}_b = \frac{1}{b^2}\alpha + \frac{1}{b}\beta + \vartheta, \tag{1.2}$$

with

$$\alpha = \frac{1}{2}(-\Delta^V + |p|^2 - m + \cdots), \quad \beta = -L_{Y^{\mathscr{H}}} + \cdots, \tag{1.3}$$

where the dots and ϑ are geometric terms which we will not be made explicit. The operator $\mathscr{L}_b$ is essentially the weighted sum of the harmonic oscillator along the fiber, minus the generator of the geodesic flow $-L_{Y^{\mathscr{H}}}$ along the horizontal direction. (1)

The vector space $\mathrm{Ker}(\alpha)$ is spanned by the function $\exp(-|p|^2/2)$. We identify $\Omega^\bullet(M)$ to $\mathrm{Ker}(\alpha)$ by the map $s \to \pi^* s \exp(-|p|^2/2)/\pi^{m/4}$. Let P be the standard L_2-projector from $\Omega^\bullet(\mathscr{M})$ on $\mathrm{Ker}(\alpha)$. Then by [**6**, Theorem 3.14],

$$P(\vartheta - \beta\alpha^{-1}\beta)P = \frac{1}{2}\square^M. \tag{1.4}$$

In [**6**], Equation (1.4) is used to prove that as $b \to 0$, we have the formal convergence of resolvents

$$(\lambda - \mathscr{L}_b)^{-1} \to P\Big(\lambda - \frac{1}{2}\square^M\Big)^{-1} P. \tag{1.5}$$

Bismut-Lebeau [**20**] set up the proper analysis foundation for the study of the hypoelliptic Laplacian $\mathscr{L}_b$. They not only proved a corresponding version of the Hodge theorem, but they also studied the precise properties of its resolvent and of the corresponding heat kernel. Since $\mathscr{M}$ is noncompact, they needed to refine the hypoelliptic estimates of Hörmander in order to control hypoellipticity at infinity. They developed the adequate theory of semiclassical pseudodifferential operators with parameter $\hbar = b$ and obtained the proper version of the convergence of resolvents in (1.5). They developed also a hypoelliptic local index theory which is itself a deformation of classical elliptic local index theory.

In [**20**], Bismut-Lebeau defined a hypoelliptic version of the analytic torsion of Ray-Singer [**56**] associated with the elliptic Hodge Laplacian in (1.4). The main result in

(1) On Euclidean spaces, all geometric terms vanish and the operator $\mathscr{L}_b$ acting on functions reduces to the Fokker-Planck operator.

[**20**] is the proof of the equality of the hypoelliptic torsion with the Ray-Singer analytic torsion.

In his thesis [**61**], Shen studied the Witten deformation of the hypoelliptic Laplacian for a Morse function on the base manifold, and identified the hypoelliptic torsion to the combinatory torsion. Shen's work gives a new proof of Bismut-Lebeau's result on the equality of the hypoelliptic torsion and the Ray-Singer analytic torsion.

This article concentrates on applications of the hypoelliptic Laplacian to orbital integrals. We will briefly summarize other applications.

A version of Theorem 0.1 for compact Lie groups can be found in [**7**]. In [**7**, Theorem 4.3], as a test of his ideas, Bismut gave a new proof of the classical explicit formula for the scalar heat kernel in terms of the coroots lattice [**29**] for a simple simply connected compact Lie group, by using the hypoelliptic Laplacian on the total space of the cotangent bundle of the group. In [**8**], Bismut also constructed a hypoelliptic Dirac operator which is a hypoelliptic deformation of the usual Dirac operator.

In [**14**, Theorem 0.1], Bismut established a Grothendieck-Riemann-Roch theorem for a proper holomorphic submersion $\pi : M \to B$ of complex manifolds in Bott-Chern cohomology. For compact Kähler manifolds, Bott-Chern cohomology coincides with de Rham cohomology. In the general situation considered in [**14**], the elliptic methods of [**5**], [**18**] are known to fail, and hypoelliptic methods seem to be the only way to obtain this result.

As in the case of the Dirac operator, there does not exist a universal hypoelliptic Laplacian which works for all situations, there are several hypoelliptic Laplacians. To attack a specific (geometric) problem, we need to construct the corresponding hypoelliptic Laplacian. Still all the hypoelliptic Laplacians have naturally the same structure, but the geometric terms depend on the situation. Probability theory plays an important role, both formally and technically in its construction and in its use.

In this article, we will not touch the analytic and probabilistic aspects of the proofs. We will explain how to give a natural construction of the hypoelliptic Laplacian which is needed in order to establish Theorem 0.1. The method consists in giving a cohomological interpretation to orbital integrals, so as to reduce their evaluation to methods related to the proof of Lefschetz fixed-point formulas. Theorem 0.1 gives a direct link of the trace formula to index theory.

We hope this article can be used as an invitation to the original papers [**6, 7, 8, 12, 14, 17**] and to several surveys on this topic [**9, 10, 11, 13, 15, 16**] and [**47**].

2. HEAT KERNEL AND LEFSCHETZ FIXED-POINT FORMULA

This section is organized as follows. In Section 2.1, we explain some basic facts about heat kernels. In Section 2.2, we review the heat equation proof of the Lefschetz fixed-point formula. This proof will be used as a model for the proof of the main theorem of this article.

2.1. A brief introduction to the heat kernel

Let M be a compact manifold of dimension m. Let TM be the tangent bundle, T^*M be the cotangent bundle, and let g^{TM} be a Riemannian metric on M. Let h^F be a Hermitian metric on F. Let $C^\infty(M,F)$ be the space of smooth sections of F on M. Let $\langle\cdot,\cdot\rangle$ be the L_2-Hermitian product on $C^\infty(M,F)$ defined by the integral of the pointwise product with respect to the Riemannian volume form dx. We denote by $L_2(M,F)$ the vector space of L_2-integrable sections of F on M.

Let $\nabla^F : C^\infty(M,F)\to C^\infty(M,T^*M\otimes F)$ be a Hermitian connection on (F,h^F) and let $\nabla^{F,*}$ be its formal adjoint. Then the (negative) Bochner Laplacian Δ^F acting on $C^\infty(M,F)$, is defined by

$$-\Delta^F=\nabla^{F,*}\nabla^F. \tag{2.1}$$

The operator $-\Delta^F$ is an essentially self-adjoint second order elliptic operator. Let ∇^{TM} be the Levi-Civita connection on (TM,g^{TM}). We can rewrite it as

$$-\Delta^F=-\sum_{i=1}^m\Big((\nabla^F_{e_i})^2-\nabla^F_{\nabla^{TM}_{e_i}e_i}\Big), \tag{2.2}$$

where $\{e_i\}_{i=1}^m$ is a local smooth orthonormal frame of (TM,g^{TM}).

For a self-adjoint section $\Phi\in C^\infty(M,\mathrm{End}(F))$ (for any $x\in M$ that $\Phi_x\in\mathrm{End}(F_x)$ is self-adjoint), set

$$-\Delta^F_\Phi=-\Delta^F-\Phi. \tag{2.3}$$

Then the heat operator $e^{t\Delta^F_\Phi}:L_2(M,F)\to L_2(M,F)$ for $t>0$ of $-\Delta^F_\Phi$ is the unique solution of

$$\begin{cases}\big(\frac{\partial}{\partial t}-\Delta^F_\Phi\big)e^{t\Delta^F_\Phi}=0\\ \lim_{t\to 0}e^{t\Delta^F_\Phi}s=s\in L_2(M,F) \qquad \text{for any } s\in L_2(M,F).\end{cases} \tag{2.4}$$

For $x,x'\in M$, let $e^{t\Delta^F_\Phi}(x,x')\in F_x\otimes F^*_{x'}$ be the Schwartz kernel of the operator $e^{t\Delta^F_\Phi}$ with respect to the Riemannian volume element dx'. Classically, $e^{t\Delta^F_\Phi}$ is smooth in $x,x'\in M, t>0$.

Since M is compact, the operator $-\Delta^F_\Phi$ has discrete spectrum, consisting of eigenvalues $\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots$ counted with multiplicities, with $\lambda_k\to+\infty$ as $k\to+\infty$. Let $\{\varphi_j\}_{j=1}^{+\infty}$ be a system of orthonormal eigenfunctions such that

$-\Delta_{\Phi}^{F}\varphi_j = \lambda_j\varphi_j$. Then $\{\varphi_j\}_{j=1}^{+\infty}$ is an orthonormal basis of $L_2(M,F)$. The heat kernel can also be written as (cf. [**3**, Proposition 2.36], [**48**, Appendix D])

$$e^{t\Delta_{\Phi}^{F}}(x,x') = \sum_{j=1}^{+\infty} e^{-t\lambda_j}\varphi_j(x)\otimes\varphi_j(x')^* \tag{2.5}$$

where $\varphi_j(x')^* \in F_{x'}^*$ is the metric dual of $\varphi_j(x') \in F_{x'}$.

The trace of the heat operator is given by

$$\mathrm{Tr}[e^{t\Delta_{\Phi}^{F}}] = \sum_{j=1}^{+\infty} e^{-t\lambda_j}. \tag{2.6}$$

The (heat) trace $\mathrm{Tr}[e^{t\Delta_{\Phi}^{F}}]$ involves the full spectrum information of operator Δ_{Φ}^{F} and has many applications.

In general, it is difficult to evaluate explicitly $\mathrm{Tr}[e^{t\Delta_{\Phi}^{F}}]$ for $t>0$. However, we will explain the explicit formula obtained by Bismut for locally symmetric spaces and its connection with Selberg trace formula.

Remark 2.1. — Let $\pi : \widetilde{M} \to M$ be the universal cover of M with fiber $\pi_1(M)$, the fundamental group of M. Then geometric data on M lift to $\widetilde{M}$, and we will add a $\widetilde{\ }$ to denote the corresponding objets on $\widetilde{M}$. It's well-known (see for instance [**49**, (3.18)]) that if $\widetilde{x}, \widetilde{x}' \in \widetilde{M}$ are such that $\pi(\widetilde{x}) = x, \pi(\widetilde{x}') = x'$, we have

$$e^{t\Delta_{\Phi}^{F}}(x,x') = \sum_{\gamma\in\pi_1(M)} \gamma e^{t\widetilde{\Delta}_{\Phi}^{F}}(\gamma^{-1}\widetilde{x}, \widetilde{x}'), \tag{2.7}$$

where the right-hand side is uniformly convergent.

2.2. The Lefschetz fixed-point formulas

Let $\Omega^\bullet(M) = \bigoplus_j \Omega^j(M) = \bigoplus_j C^\infty(M, \Lambda^j(T^*M))$ be the vector space of smooth differential forms on M (with values in $\mathbb{R}$), which is $\mathbb{Z}$-graded by degree. Let $d : \Omega^j(M) \to \Omega^{j+1}(M)$ be the exterior differential operator. Then $d^2 = 0$ so that $(\Omega^\bullet(M), d)$ forms the de Rham complex. The de Rham cohomology groups of M are defined by

$$H^j(M,\mathbb{R}) = \frac{\mathrm{Ker}(d|_{\Omega^j(M)})}{\mathrm{Im}(d|_{\Omega^{j-1}(M)})}, \quad H^\bullet(M,\mathbb{R}) = \bigoplus_{j=0}^{m} H^j(M,\mathbb{R}). \tag{2.8}$$

They are canonically isomorphic to the singular cohomology of M.

Let $d^* : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ be the formal adjoint of d with respect to the scalar product $\langle\cdot,\cdot\rangle$ on $\Omega^\bullet(M)$, i.e., for all $s, s' \in \Omega^\bullet(M)$,

$$\langle d^*s, s'\rangle := \langle s, ds'\rangle. \tag{2.9}$$

Set

$$D = d + d^*. \tag{2.10}$$

Then D is a first order elliptic differential operator, and we have

$$D^2 = dd^* + d^*d. \tag{2.11}$$

The operator D^2 is called the Hodge Laplacian, it is an operator of the type (2.3) for $F = \Lambda^\bullet(T^*M)$, which preserves the $\mathbb{Z}$-grading on $\Omega^\bullet(M)$. By Hodge theory, we have the isomorphism,

$$\operatorname{Ker}(D|_{\Omega^j(M)}) = \operatorname{Ker}(D^2|_{\Omega^j(M)}) \simeq H^j(M,\mathbb{R}), \text{ for } j = 0, 1, \ldots, m. \tag{2.12}$$

We give here a baby example to explain the heat equation proof of the Atiyah-Singer index theorem (cf. [**3**]).

Let H be a compact Lie group acting on M on the left. Since the exterior differential commutes with the action of H on $\Omega^\bullet(M)$, H acts naturally on $H^j(M,\mathbb{R})$ for any j. The Lefschetz number for $h \in H$ is given by

$$\chi_h(M) = \sum_{j=0}^{m} (-1)^j \operatorname{Tr}[h|_{H^j(M,\mathbb{R})}] = \operatorname{Tr}_s[h|_{H^\bullet(M,\mathbb{R})}]. \tag{2.13}$$

The Lefschetz fixed-point formula computes $\chi_h(M)$ in term of geometric data on the fixed-point set of h.

Instead of working on $H^j(M,\mathbb{R})$, we will work on the much larger space $\Omega^\bullet(M)$ to establish the Lefschetz fixed-point formulas.

Since H is compact, by an averaging argument on H, we can assume that the metric g^{TM} is H-invariant. Then the operator D defined above is also H-invariant. We have the following result (cf. [**3**, Theorem 3.50, Proposition 6.3]),

Theorem 2.2 (McKean-Singer formula). — *For any $t > 0$,*

$$\chi_h(M) = \operatorname{Tr}_s[he^{-tD^2}]. \tag{2.14}$$

Proof. — For any $t > 0$, we have

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{Tr}_s[he^{-tD^2}] &= -\operatorname{Tr}_s[hD^2e^{-tD^2}] \\ &= -\frac{1}{2}\operatorname{Tr}_s[[D, hDe^{-tD^2}]] = 0. \end{aligned} \tag{2.15}$$

Here $[\cdot,\cdot]$ is a supercommutator defined as in (0.9), and as in the case of matrices, the supertrace of a supercommutator vanishes by a simple algebraic argument.

By (2.6) and (2.12), we have

$$\lim_{t\to+\infty} \operatorname{Tr}_s[he^{-tD^2}] = \chi_h(M). \tag{2.16}$$

Combining (2.15) and (2.16), we get (2.14). □

A simple analysis shows that only the fixed-points of h contribute to the limit of $\mathrm{Tr}_s[he^{-tD^2}]$ as $t \to 0$. Further simple work then leads to the Lefschetz fixed-point formulas.

Even though we will work on a more refined object the trace of a heat operator, the above philosophy still applies.

3. BISMUT'S EXPLICIT FORMULA FOR THE ORBITAL INTEGRALS

In this section, we give an introduction to orbital integrals and to Selberg trace formula, and we present the main result of this article: Bismut's explicit evaluation of the orbital integrals. Also, we compare Harish-Chandra's Plancherel theory with Bismut's explicit formula for the orbital integrals.

This section is organized as follows. In Section 3.1, we recall some basic facts on symmetric spaces, and we explain how the Casimir operator for a reductive Lie group induces a Bochner Laplacian on the associated symmetric space. In Section 3.2, we give an introduction to orbital integrals and to Selberg trace formula, and in Section 3.3, we describe the geometric definition of orbital integrals given by Bismut. In Section 3.4, we present the main result of this article, Bismut's explicit evaluation of the orbital integrals, and give some examples. Finally in Section 3.5, we present briefly Harish-Chandra's Plancherel theory for comparison with Bismut's result.

3.1. Casimir operator and Bochner Laplacian

Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}$ and Lie bracket $[\cdot,\cdot]$. Let $\theta \in \mathrm{Aut}(G)$ be its Cartan involution. Let K be the subgroup of G fixed by θ, with Lie algebra $\mathfrak{k}$. Then K is a maximal compact subgroup of G, and K is connected.

The Cartan involution θ acts naturally as a Lie algebra automorphism of $\mathfrak{g}$. Then the Cartan decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \text{ with } \mathfrak{p} = \{a \in \mathfrak{g} \,:\, \theta a = -a\},\ \mathfrak{k} = \{a \in \mathfrak{g} \,:\, \theta a = a\}. \tag{3.1}$$

From (3.1), we get

$$[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{p},\mathfrak{k}] \subset \mathfrak{p}. \tag{3.2}$$

Put $n = \dim \mathfrak{k}, m = \dim \mathfrak{p}$. Then $\dim \mathfrak{g} = m + n$.

If $g, h \in G,\ u \in \mathfrak{g}$, let $\mathrm{Ad}(g)h = ghg^{-1}$ be the adjoint action of g on h, and let $\mathrm{Ad}(g)u \in \mathfrak{g}$ denote the action of g on u via the adjoint representation. If $u, v \in \mathfrak{g}$, set

$$\mathrm{ad}(u)v = [u,v], \tag{3.3}$$

then ad is the derivative of the map $g \in G \to \mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$.

Let B be a real-valued nondegenerate symmetric bilinear form on $\mathfrak{g}$ which is invariant under the adjoint action of G on $\mathfrak{g}$, and also under the action of θ. Then (3.1) is an orthogonal splitting of $\mathfrak{g}$ with respect to B. We assume that B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Put $\langle\cdot,\cdot\rangle = -B(\cdot,\theta\cdot)$ the associated scalar product on $\mathfrak{g}$, which is invariant under the adjoint action of K. Let $|\cdot|$ be the corresponding norm on $\mathfrak{g}$. The splitting (3.1) is also orthogonal with respect to $\langle\cdot,\cdot\rangle$.

Remark 3.1. — For $G = \mathrm{GL}^+(\mathbf{q},\mathbb{R}) = \{A \in \mathrm{GL}(\mathbf{q},\mathbb{R}), \det A > 0\}$, the Cartan involution is given by $\theta(g) = {}^t g^{-1}$, where ${}^t\cdot$ denotes the transpose of a matrix. Then $K = \mathrm{SO}(\mathbf{q})$, the special orthogonal group, and $\mathfrak{k}$ is the vector space of anti-symmetric matrices and $\mathfrak{p}$ is the vector space of symmetric matrices. We can take $B(u,v) = 2\,\mathrm{Tr}^{\mathbb{R}^{\mathfrak{q}}}[uv]$ for $u,v \in \mathfrak{g} = \mathfrak{gl}(\mathbf{q},\mathbb{R}) = \mathrm{End}(\mathbb{R}^{\mathfrak{q}})$.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ which will be identified with the algebra of left-invariant differential operators on G. Let $C^{\mathfrak{g}} \in U(\mathfrak{g})$ be the Casimir element. If $\{e_i\}_{i=1}^m$ is an orthonormal basis of $(\mathfrak{p},\langle\cdot,\cdot\rangle)$ and if $\{e_i\}_{i=m+1}^{m+n}$ is an orthonormal basis of $(\mathfrak{k},\langle\cdot,\cdot\rangle)$, then

$$C^{\mathfrak{g}} = C^{\mathfrak{p}} + C^{\mathfrak{k}}, \text{ with } C^{\mathfrak{p}} = -\sum_{i=1}^{m} e_i^2,\ C^{\mathfrak{k}} = \sum_{i=m+1}^{m+n} e_i^2. \tag{3.4}$$

Then $C^{\mathfrak{k}}$ is the Casimir element of $\mathfrak{k}$ with respect to the bilinear form induced by B on $\mathfrak{k}$. Note that $C^{\mathfrak{g}}$ lies in the center of $U(\mathfrak{g})$.

Let $\rho^V : K \to \mathrm{Aut}(V)$ be an orthogonal or unitary representation of K on a finite dimensional Euclidean or Hermitian vector space V. We denote by $C^{\mathfrak{k},V} \in \mathrm{End}(V)$ the corresponding Casimir operator acting on V, given by

$$C^{\mathfrak{k},V} = \sum_{i=m+1}^{m+n} \rho^{V,2}(e_i). \tag{3.5}$$

Let

$$p : G \to X = G/K \tag{3.6}$$

be the quotient space. Then X is contractible. More precisely, X is a symmetric space and the exponential map $\exp : \mathfrak{p} \to G/K,\ a \mapsto pe^a$ is a diffeomorphism. We have a natural identification of vector bundles on X:

$$TX = G \times_K \mathfrak{p}, \tag{3.7}$$

where K acts on $\mathfrak{p}$ via the adjoint representation. The scalar product of $\mathfrak{p}$ descends to a Riemannian metric g^{TX} on TX. Let $\omega^{\mathfrak{g}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}$, and let $\omega^{\mathfrak{k}}$ be the $\mathfrak{k}$-component of $\omega^{\mathfrak{g}}$. Then $\omega^{\mathfrak{k}}$ defines a connection on the K-principal bundle $G \to G/K$. The connection ∇^{TX} on TX induced by $\omega^{\mathfrak{k}}$ and by (3.7) is precisely the Levi-Civita connection on (TX, g^{TX}).

Note since the adjoint representation of K preserves $\mathfrak{p}$ and $\mathfrak{k}$, we obtain $C^{\mathfrak{k},\mathfrak{p}} \in \mathrm{End}(\mathfrak{p})$, $C^{\mathfrak{k},\mathfrak{k}} \in \mathrm{End}(\mathfrak{k})$. In fact, $\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]$ is the scalar curvature of X, and $-\frac{1}{4}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]$ is the scalar curvature of K for the Riemannian structure induced by B (cf. [**12**, (2.6.8) and (2.6.9)]).

Let $\rho^E : K \to \mathrm{Aut}(E)$ be a unitary representation of K. Then the vector space E descends to a Hermitian vector bundle $F = G \times_K E$ on X, and $\omega^{\mathfrak{k}}$ induces a Hermitian connection ∇^F on F. Then $C^\infty(X, F)$ can be identified to $C^\infty(G, E)^K$, the K-invariant part of $C^\infty(G, E)$. The Casimir operator $C^{\mathfrak{g}}$, acting on $C^\infty(G, E)$, descends to an operator acting on $C^\infty(X, F)$, which will still be denoted by $C^{\mathfrak{g}}$.

Let A be a self-adjoint endomorphism of E which is K-invariant. Then A descends to a parallel self-adjoint section of $\mathrm{End}(F)$ over X.

Definition 3.2. — *Let $\mathcal{L}^X, \mathcal{L}^X_A$ act on $C^\infty(X, F)$ by the formulas,*

$$\begin{aligned} \mathcal{L}^X &= \frac{1}{2}C^{\mathfrak{g}} + \frac{1}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]; \\ \mathcal{L}^X_A &= \mathcal{L}^X + A. \end{aligned} \tag{3.8}$$

From (3.4), $-C^{\mathfrak{p}}$ descends to the Bochner Laplacian Δ^F on $C^\infty(X, F)$, the operator $C^{\mathfrak{k}}$ descends to a parallel section $C^{\mathfrak{k},F}$ of $\mathrm{End}(F)$ on X. If the representation ρ^E above is irreducible, then $C^{\mathfrak{k},F}$ acts as $c\,\mathrm{Id}_F$, where c is a constant function on X. Thus from (2.3) and (3.8), we have

$$\mathcal{L}^X = -\frac{1}{2}\Delta^F_\phi \ \text{ with } \ \phi = -C^{\mathfrak{k},F} - \frac{1}{8}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] - \frac{1}{24}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{3.9}$$

The group G acts on X on the left. This action lifts to F. More precisely, for any $h \in G$ and $[g, v] \in F$, the left action of h is given by

$$h.[g, v] = [hg, v] \in G \times_K E = F. \tag{3.10}$$

Then the operators $\mathcal{L}^X$, $\mathcal{L}^X_A$ commute with G.

Let $\Gamma \subset G$ be a discrete subgroup of G such that the quotient space $\Gamma\backslash G$ is compact. Set

$$Z = \Gamma\backslash X = \Gamma\backslash G/K. \tag{3.11}$$

Then Z is a compact locally symmetric space. In general Z is an orbifold. If Γ is torsion-free (i.e., if $\gamma \in \Gamma$, $k \in \mathbb{N}^*$, then $\gamma^k = 1$ implies $\gamma = 1$), then Z is a smooth manifold.

From now on, we assume that Γ is torsion free, so that $\Gamma = \pi_1(Z)$ and X is just the universal cover of Z.

A vector bundle like F on X descends to a vector bundle on Z, which we still denote by F. Then the operators $\mathscr{L}^X$, $\mathscr{L}^X_A$ descend to operators $\mathscr{L}^Z$, $\mathscr{L}^Z_A$ acting on $C^\infty(Z,F)$.

For $t>0$, let $e^{-t\mathscr{L}^X_A}(x,x')$ $(x,x'\in X)$, $e^{-t\mathscr{L}^Z_A}(z,z')$ $(z,z'\in Z)$ be the smooth kernels of the heat operators $e^{-t\mathscr{L}^X_A}$, $e^{-t\mathscr{L}^Z_A}$ with respect to the Riemannian volume forms dx', dz' respectively. By (2.7), we get

$$\begin{aligned}\operatorname{Tr}[e^{-t\mathscr{L}^Z_A}] &= \int_Z \operatorname{Tr}[e^{-t\mathscr{L}^Z_A}(z,z)]dz\\ &= \int_{\Gamma\backslash X}\sum_{\gamma\in\Gamma}\operatorname{Tr}[\gamma e^{-t\mathscr{L}^X_A}(\gamma^{-1}\widetilde{z},\widetilde{z})]dz.\end{aligned}\tag{3.12}$$

3.2. Orbital integrals and Selberg trace formula

Let $C^b(X,F)$ be the vector space of continuous bounded sections of F over X. Let Q be an operator acting on $C^b(X,F)$ with a continuous kernel $q(x,x')$ with respect to the volume form dx'. It is convenient to view q as a continuous function $q(g,g')$ defined on $G\times G$ with values in $\operatorname{End}(E)$ which satisfies for any $k,k'\in K$,

$$q(gk,g'k')=\rho^E(k^{-1})q(g,g')\rho^E(k').\tag{3.13}$$

Now we assume that the operator Q commutes with the left action of G on $C^b(X,F)$ defined in (3.10). This is equivalent to

$$q(gx,gx')=gq(x,x')g^{-1}\quad\text{for any } x,x'\in X,\ g\in G,\tag{3.14}$$

where the action of g^{-1} maps $F_{gx'}$ to $F_{x'}$, the action of g maps F_x to F_{gx}.

If we consider instead the kernel $q(g,g')$, then this implies that for all $g''\in G$,

$$q(g''g,g''g')=q(g,g')\in\operatorname{End}(E).\tag{3.15}$$

Thus the kernel q is determined by $q(1,g)$. Set

$$q(g)=q(1,g).\tag{3.16}$$

Then we obtain from (3.13) and (3.15) that for $g\in G, k\in K$,

$$q(k^{-1}gk)=\rho^E(k^{-1})q(g)\rho^E(k).\tag{3.17}$$

This implies that $\operatorname{Tr}^E[q(g)]$ is invariant when replacing g by $k^{-1}gk$.

In the sequel, we will use the same notation q for the various versions of the corresponding kernel Q.

Definition 3.3. — *The element $\gamma\in G$ is said to be elliptic if it is conjugate in G to an element of K. We say that γ is hyperbolic if it is conjugate in G to e^a, $a\in\mathfrak{p}$.*

For $\gamma\in G$, γ is semisimple if there exist $g\in G$, $a\in\mathfrak{p}$, $k\in K$ such that

$$\operatorname{Ad}(k)a=a,\quad \gamma=\operatorname{Ad}(g)\left(e^ak^{-1}\right).\tag{3.18}$$

By [**27**, Theorem 2.19.23], if $\gamma \in G$ is a semisimple element, $\mathrm{Ad}(g)e^a$ and $\mathrm{Ad}(g)k^{-1}$ are uniquely determined by γ (i.e., they do not depend on $g \in G$ such that (3.18) holds), and

$$Z(\gamma) = Z\left(\mathrm{Ad}(g)e^a\right) \cap Z\left(\mathrm{Ad}(g)k^{-1}\right), \tag{3.19}$$

where $Z(\gamma) \subset G$ is the centralizer of γ in G.

Let dk be the Haar measure on K that gives volume 1 to K. Let dg be measure on G (as a K-principal bundle on $X = G/K$) given by

$$dg = dx\, dk. \tag{3.20}$$

Then dg is a left-invariant Haar measure on G. Since G is unimodular, it is also a right-invariant Haar measure.

For $\gamma \in G$ semisimple, $Z(\gamma)$ is reductive and $K(\gamma)$, the fixed-points set of $\mathrm{Ad}(g)\theta\,\mathrm{Ad}(g)^{-1}$ in $Z(\gamma)$ (cf. (3.19)), is a maximal compact subgroup. Let dy be the volume element on the symmetric space $X(\gamma) = Z(\gamma)/K(\gamma)$ induced by B. Let dk' be the Haar measure on $K(\gamma)$ that gives volume 1 to $K(\gamma)$. Then $dz = dydk'$ is a left and right Haar measure on $Z(\gamma)$. Let dv be the canonical measure on $Z(\gamma)\backslash G$ that is canonically associated with dg and dz so that

$$dg = dzdv. \tag{3.21}$$

Definition 3.4 (Orbital integral). — *For $\gamma \in G$ semisimple, we define the orbital integral associated with Q and γ by*

$$\mathrm{Tr}^{[\gamma]}[Q] = \int_{Z(\gamma)\backslash G} \mathrm{Tr}^E[q(v^{-1}\gamma v)]dv, \tag{3.22}$$

once the integral converges.

Note that the map

$$Z(\gamma)\backslash G \to \mathcal{O}_\gamma = \mathrm{Ad}_G\,\gamma \ \text{ given by } \ v \to v^{-1}\gamma v \tag{3.23}$$

identifies $Z(\gamma)\backslash G$ as the orbit $\mathcal{O}_\gamma$ of γ with the adjoint action of G on G. This justifies the name "orbital integral" for (3.22).

Let $\Gamma \subset G$ be a discrete torsion free cocompact subgroup as in Section 3.1. Since the operator Q commutes with the left action of G, Q descends to an operator Q^Z acting on $C^\infty(Z, F)$. We assume that the sum $\sum_{\gamma\in\Gamma} q(g^{-1}\gamma g')$ is uniformly and absolutely convergent on $G \times G$.

Let $[\Gamma]$ be the set of conjugacy classes in Γ. If $[\gamma] \in [\Gamma]$, set

$$q^{X,[\gamma]}(g, g') = \sum_{\gamma'\in[\gamma]} q(g^{-1}\gamma' g'). \tag{3.24}$$

Then from (3.15)–(3.24), we get

$$q^Z(z,z') = \sum_{[\gamma]\in[\Gamma]} q^{X,[\gamma]}(g,g'), \tag{3.25}$$

with $g, g' \in G$ fixed lift of $z, z' \in Z$. Thus as in (3.12),

$$\operatorname{Tr}[Q^Z] = \sum_{[\gamma]\in[\Gamma]} \operatorname{Tr}[Q^{Z,[\gamma]}] \ \text{ with } \ \operatorname{Tr}[Q^{Z,[\gamma]}] = \int_Z \operatorname{Tr}[q^{X,[\gamma]}(z,z)]dz. \tag{3.26}$$

From (3.20), (3.24), (3.26), and the fact that $[\gamma] \simeq \Gamma \cap Z(\gamma)\backslash\Gamma$, we have

$$\begin{aligned} \operatorname{Tr}[Q^{Z,[\gamma]}] &= \int_{\Gamma\cap Z(\gamma)\backslash G} \operatorname{Tr}^E[q(g^{-1}\gamma g)]dg \\ &= \operatorname{Vol}\Big(\Gamma\cap Z(\gamma)\backslash Z(\gamma)\Big) \operatorname{Tr}^{[\gamma]}[Q] \\ &= \operatorname{Vol}\Big(\Gamma\cap Z(\gamma)\backslash X(\gamma)\Big) \operatorname{Tr}^{[\gamma]}[Q]. \end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we get

Theorem 3.5 (Selberg trace formula). — *We have*

$$\operatorname{Tr}\left[Q^Z\right] = \sum_{[\gamma]\in[\Gamma]} \operatorname{Vol}\Big(\Gamma\cap Z(\gamma)\backslash X(\gamma)\Big) \operatorname{Tr}^{[\gamma]}[Q]. \tag{3.28}$$

Selberg [**59**, (3.2)] was the first to give a closed formula for the trace of the heat operator on a compact hyperbolic Riemann surface via (3.28), which is the original Selberg trace formula. Harish-Chandra's Plancherel theory, developed from the 1950s until the 1970s, is an algorithm to reduce the computation of an orbital integral to a lower dimensional group by the discrete series method, cf. Section 3.5.

To understand better the structure of each integral in (3.22), we first reformulate it in more geometric terms.

3.3. Geometric orbital integrals

Let $d(\cdot,\cdot)$ be the Riemannian distance on X. If $\gamma \in G$, the displacement function d_γ is given by for $x \in X$,

$$d_\gamma(x) = d(x, \gamma x). \tag{3.29}$$

By [**1**, §6.1], the function d_γ is convex on X, i.e., for any geodesic $t \in \mathbb{R} \to x_t \in X$ with constant speed, the function $d_\gamma(x_t)$ is convex on $t \in \mathbb{R}$.

Recall that $p : G \to X = G/K$ is the natural projection in (3.6). We have the following geometric description on the semisimple elements in G.

Theorem 3.6 ([**12**], Theorem 3.1.2). — *The element $\gamma \in G$ is semisimple if and only if the function d_γ attains its minimum in X. If $\gamma \in G$ is semisimple, and*

$$X(\gamma) = \{x \in X : d_\gamma(x) = m_\gamma := \inf_{y\in X} d_\gamma(y)\}, \tag{3.30}$$

for $g \in G$, $x = pg \in X$, then $x \in X(\gamma)$ if and only if there exist $a \in \mathfrak{p}$, $k \in K$ such that

$$\gamma = \mathrm{Ad}(g)(e^a k^{-1}) \quad \textit{and} \;\; \mathrm{Ad}(k)a = a. \tag{3.31}$$

If $g_t = ge^{ta}$, then $t \in [0,1] \to x_t = pg_t$ is the unique geodesic connecting $x \in X(\gamma)$ and γx in X. Moreover, we have

$$m_\gamma = |a|. \tag{3.32}$$

Since the integral (3.27) depends only on the conjugacy class of γ, from Theorem 3.6 or (3.18), we may and we will assume that

$$\gamma = e^a k^{-1}, \quad \mathrm{Ad}(k)a = a, \quad a \in \mathfrak{p},\ k \in K. \tag{3.33}$$

Furthermore, by (3.19), we have

$$Z(\gamma) = Z(e^a) \cap Z(k), \quad \mathfrak{z}(\gamma) = \mathfrak{z}(e^a) \cap \mathfrak{z}(k), \tag{3.34}$$

where we use the symbol $\mathfrak{z}$ to denote the corresponding Lie algebras of the centralizers.

Put

$$\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}, \quad \mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}. \tag{3.35}$$

From (3.2) and (3.34), we get

$$\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{3.36}$$

Thus the restriction of B to $\mathfrak{z}(\gamma)$ is non-degenerate. Let $\mathfrak{z}^\perp(\gamma)$ be the orthogonal space to $\mathfrak{z}(\gamma)$ in $\mathfrak{g}$ with respect to B. Then $\mathfrak{z}^\perp(\gamma)$ splits as

$$\mathfrak{z}^\perp(\gamma) = \mathfrak{p}^\perp(\gamma) \oplus \mathfrak{k}^\perp(\gamma), \tag{3.37}$$

where $\mathfrak{p}^\perp(\gamma) \subset \mathfrak{p}$, $\mathfrak{k}^\perp(\gamma) \subset \mathfrak{k}$ are the orthogonal spaces to $\mathfrak{p}(\gamma)$, $\mathfrak{k}(\gamma)$ in $\mathfrak{p}$, $\mathfrak{k}$ with respect to the scalar product induced by B.

Set

$$K(\gamma) = K \cap Z(\gamma), \tag{3.38}$$

then from (3.34) and (3.35), $\mathfrak{k}(\gamma)$ is just the Lie algebra of $K(\gamma)$.

THEOREM 3.7 ([**12**], Theorems 3.3.1, 3.4.1, 3.4.3). — *The set $X(\gamma)$ is a submanifold of X. In the geodesic coordinate system centered at $p1$, we have the identification*

$$X(\gamma) = \mathfrak{p}(\gamma). \tag{3.39}$$

The action of $Z(\gamma)$ on $X(\gamma)$ is transitive and we have the identification of $Z(\gamma)$-manifolds,

$$X(\gamma) \simeq Z(\gamma)/K(\gamma). \tag{3.40}$$

The map

$$\rho_\gamma : (g, f, k') \in Z(\gamma) \times_{K(\gamma)} (\mathfrak{p}^\perp(\gamma) \times K) \to g e^f k' \in G \tag{3.41}$$

is a diffeomorphism of left $Z(\gamma)$-spaces, and of right K-spaces. The map $(g, f, k') \mapsto (g, f)$ corresponds to the projection $p : G \to X = G/K$. In particular, the map

$$\rho_\gamma : (g, f) \in Z(\gamma) \times_{K(\gamma)} \mathfrak{p}^\perp(\gamma) \to p(g e^f) \in X \tag{3.42}$$

is a diffeomorphism.

Moreover, under the diffeomorphism (3.41), *we have the identity of right K-spaces,*

$$\mathfrak{p}^\perp(\gamma)_{K(\gamma)} \times K = Z(\gamma)\backslash G. \tag{3.43}$$

Finally, there exists $C_\gamma > 0$ such that if $f \in \mathfrak{p}^\perp(\gamma)$, $|f| > 1$,

$$d_\gamma(\rho_\gamma(1, f)) \geq |a| + C_\gamma |f|. \tag{3.44}$$

The map ρ_γ in (3.42) is the normal coordinate system on X based at $X(\gamma)$.

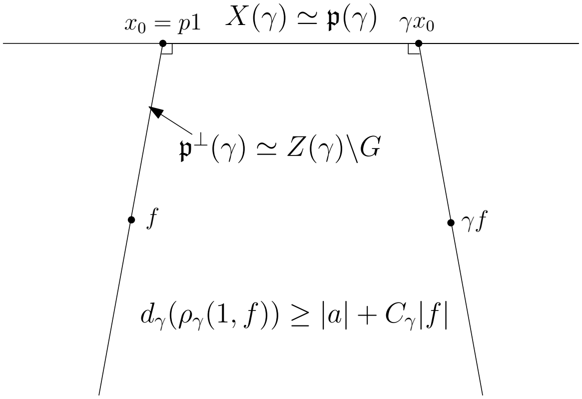

FIGURE 1. Normal coordinate

Recall that dy is the volume element on $X(\gamma)$ (cf. Section 3.2). Let df be the volume element on $\mathfrak{p}^\perp(\gamma)$. Then $dydf$ is a volume form on $Z(\gamma) \times_{K(\gamma)} \mathfrak{p}^\perp(\gamma)$ that is

$Z(\gamma)$-invariant. Let $r(f)$ be the smooth function on $\mathfrak{p}^\perp(\gamma)$ that is $K(\gamma)$-invariant such that we have the identity of volume element on X via (3.42),

$$dx = r(f)dydf, \text{ with } r(0) = 1. \tag{3.45}$$

In view of (3.43), (3.45), Bismut could reformulate geometrically the orbital integral (3.22) as an integral along the normal direction of $X(\gamma)$ in X.

Proposition 3.8 (Geometric orbital integral). — *The orbital integral for the operator Q in Section 3.2 and a semisimple element $\gamma \in G$ is given by*

$$\mathrm{Tr}^{[\gamma]}[Q] = \int_{\mathfrak{p}^\perp(\gamma)} \mathrm{Tr}^E[q(e^{-f}\gamma e^f)]r(f)df. \tag{3.46}$$

Equation (3.46) gives a geometric interpretation for orbital integrals. It is remarkable that even before its explicit computation, the variational problem connected with the minimization of the displacement function d_γ is used in (3.46).

We need the following criterion for the semisimplicity of an element.

Proposition 3.9 (Selberg **[60]**, Lemmas 1, 2). — *If $\Gamma \subset G$ is a discrete cocompact subgroup, then for any $\gamma \in \Gamma$, γ is semisimple, and $\Gamma \cap Z(\gamma)$ is cocompact in $Z(\gamma)$.*

Proof. — Let U be a compact subset of G such that $G = \Gamma \cdot U$. Let $\gamma \in \Gamma$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a family of points in X such that $d(x_k, \gamma x_k) \to m_\gamma = \inf_{x\in X} d(x, \gamma x)$ as $k \to +\infty$.

Then there exists $\gamma_k \in \Gamma$, $x'_k \in U$ such that $\gamma_k x'_k = x_k$. Since U is compact, there is a subsequence $\{x'_{k_j}\}_{j\in\mathbb{N}}$ of $\{x'_k\}_{k\in\mathbb{N}}$ such that as $j \to +\infty$, $x'_{k_j} \to y \in U$. Then

$$\begin{aligned} d(y, \gamma_{k_j}^{-1}\gamma\gamma_{k_j}y) &\le d(x'_{k_j}, y) + d(x'_{k_j}, \gamma_{k_j}^{-1}\gamma\gamma_{k_j}x'_{k_j}) + d(\gamma_{k_j}^{-1}\gamma\gamma_{k_j}x'_{k_j}, \gamma_{k_j}^{-1}\gamma\gamma_{k_j}y) \\ &= 2d(x'_{k_j}, y) + d(x_{k_j}, \gamma x_{k_j}), \end{aligned} \tag{3.47}$$

where the right side tends to m_γ as $j \to +\infty$.

Since Γ is discrete and each $\gamma_{k_j}^{-1}\gamma\gamma_{k_j} \in \Gamma$, the set of such $\gamma_{k_j}^{-1}\gamma\gamma_{k_j}$ is bounded, so that there exist infinitely many j such that $\gamma_{k_j}^{-1}\gamma\gamma_{k_j} = \gamma' \in \Gamma$. Then

$$m_\gamma = d(y, \gamma' y) = d(\gamma_{k_j}y, \gamma\gamma_{k_j}y). \tag{3.48}$$

This means that d_γ reaches its minimum in X. Therefore γ is semisimple.

Since Γ is discrete, $[\gamma]$ is closed in G, thus $\Gamma \cdot Z(\gamma)$ as the inverse image of $[\gamma]$ of the continuous map $g \in G \to g\gamma g^{-1} \in G$, is closed in G. This implies $\Gamma \cap Z(\gamma)\backslash Z(\gamma) = \Gamma\backslash\Gamma \cdot Z(\gamma)$ is a closed subset of the compact quotient $\Gamma\backslash G$. Thus $\Gamma \cap Z(\gamma)$ is cocompact in $Z(\gamma)$. □

Let $\Gamma \subset G$ be a discrete torsion free cocompact subgroup as in Section 3.2. Set $Z = \Gamma\backslash X$, then $\Gamma = \pi_1(Z)$. For $x \in X(\gamma)$, the unique geodesic from x to γx descends to the closed geodesic in Z in the homotopy class $\gamma \in \Gamma$ which has the shortest length m_γ. Thus the Selberg trace formula (3.28) relates the trace of an operator Q to the dynamical properties of the geodesic flow on Z via orbital integrals.

3.4. Bismut's explicit formula for orbital integrals

By the standard heat kernel estimate, for the heat operator $e^{-t\mathscr{L}_A^X}$ on X, there exist $c > 0$, $\lambda, C > 0$, $M > 0$ such that for any $t > 0$, $x, x' \in X$, we have (cf. for instance [**49**, (3.1)])

$$\left|e^{-t\mathscr{L}_A^X}(x, x')\right| \leq Ct^{-M} e^{\lambda t - c\, d^2(x,x')/t}. \tag{3.49}$$

Note also that by Rauch's comparison theorem, there exist $C_0, C_1 > 0$ such that for all $f \in \mathfrak{p}^\perp(\gamma)$,

$$|r(f)| \leq C_0 e^{C_1|f|}. \tag{3.50}$$

From (3.44), (3.49) and (3.50), the orbital integral $\mathrm{Tr}^{[\gamma]}[e^{-t\mathscr{L}_A^X}]$ is well-defined for any semisimple element $\gamma \in G$.

Let $\gamma \in G$ be the semisimple element as in (3.33). Set

$$\mathfrak{p}_0 = \mathfrak{z}(a) \cap \mathfrak{p}, \quad \mathfrak{k}_0 = \mathfrak{z}(a) \cap \mathfrak{k}, \quad \mathfrak{z}_0 = \mathfrak{z}(a) = \mathfrak{p}_0 \oplus \mathfrak{k}_0. \tag{3.51}$$

Let $\mathfrak{z}_0^\perp$ be the orthogonal space to $\mathfrak{z}_0$ in $\mathfrak{g}$ with respect to B.

Let $\mathfrak{p}_0^\perp(\gamma)$ be the orthogonal to $\mathfrak{p}(\gamma)$ in $\mathfrak{p}_0$, and let $\mathfrak{k}_0^\perp(\gamma)$ be the orthogonal space to $\mathfrak{k}(\gamma)$ in $\mathfrak{k}_0$. Then the orthogonal space to $\mathfrak{z}(\gamma)$ in $\mathfrak{z}_0$ is

$$\mathfrak{z}_0^\perp(\gamma) = \mathfrak{p}_0^\perp(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma). \tag{3.52}$$

For $Y_0^\mathfrak{k} \in \mathfrak{k}(\gamma)$, we claim that

$$\det\Big(1 - \exp(-i\theta\, \mathrm{ad}(Y_0^\mathfrak{k}))\, \mathrm{Ad}(k^{-1})\Big)\Big|_{\mathfrak{z}_0^\perp(\gamma)} \det\Big(1 - \mathrm{Ad}(k^{-1})\Big)\Big|_{\mathfrak{z}_0^\perp(\gamma)} \tag{3.53}$$

has a natural square root, which depends analytically on $Y_0^\mathfrak{k}$. Indeed, $\mathrm{ad}(Y_0^\mathfrak{k})$ commutes with $\mathrm{Ad}(k^{-1})$, and no eigenvalue of $\mathrm{Ad}(k)$ acting on $\mathfrak{z}_0^\perp(\gamma)$ is equal to 1. If $\mathfrak{z}_0^\perp(\gamma)$ is 1-dimensional, then $\mathrm{Ad}(k)|_{\mathfrak{z}_0^\perp(\gamma)} = -1$ and $\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{z}_0^\perp(\gamma)} = 0$, the square root is just 2. If $\mathfrak{z}_0^\perp(\gamma)$ is 2-dimensional, if $\mathrm{Ad}(k)|_{\mathfrak{z}_0^\perp(\gamma)}$ is a rotation of angle ϕ and $\theta\, \mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{z}_0^\perp(\gamma)}$ acts by an infinitesimal rotation of angle ϕ', such a square root is given by (cf. [**12**, (5.4.10)])

$$4 \sin\Big(\frac{\phi}{2}\Big) \sin\Big(\frac{\phi + i\phi'}{2}\Big). \tag{3.54}$$

If V is a finite dimensional Hermitian vector space and if $\Theta \in \operatorname{End}(V)$ is self-adjoint, then $\dfrac{\Theta/2}{\sinh(\Theta/2)}$ is a self-adjoint positive endomorphism. Set

$$\widehat{A}(\Theta) = \det{}^{1/2}\Big[\frac{\Theta/2}{\sinh(\Theta/2)}\Big]. \tag{3.55}$$

In (3.55), the square root is taken to be the positive square root.

For $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, set

(3.56)

$$J_\gamma(Y_0^{\mathfrak{k}}) = \frac{1}{\big|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\big|^{1/2}} \cdot \frac{\widehat{A}\big(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma)}\big)}{\widehat{A}\big(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(\gamma)}\big)} \cdot \left[\frac{1}{\det(1-\operatorname{Ad}(k^{-1}))|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\Big(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k^{-1})\Big)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\Big(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k^{-1})\Big)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

From (3.53), we know that (3.56) is well-defined. Moreover, there exist c_γ, $C_\gamma > 0$ such that for any $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$

$$|J_\gamma(Y_0^{\mathfrak{k}})| \leq c_\gamma\, e^{C_\gamma |Y_0^{\mathfrak{k}}|}. \tag{3.57}$$

We note that $p = \dim \mathfrak{p}(\gamma)$, $q = \dim \mathfrak{k}(\gamma)$ and $r = \dim \mathfrak{z}(\gamma) = p+q$. Now we can restate Theorem 0.1 as follows.

THEOREM 3.10 ([**12**], Theorem 6.1.1). — *For any $t > 0$, we have*

(3.58)

$$\operatorname{Tr}^{[\gamma]}[e^{-t\mathcal{L}_A^X}] = \frac{e^{-|a|^2/2t}}{(2\pi t)^{p/2}} \int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^{\mathfrak{k}}) \operatorname{Tr}^E\Big[\rho^E(k^{-1}) e^{-i\rho^E(Y_0^{\mathfrak{k}}) - tA}\Big] e^{-|Y_0^{\mathfrak{k}}|^2/2t} \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

Remark 3.11. — For $\gamma = 1$, we have $\mathfrak{k}(1) = \mathfrak{k}$, $\mathfrak{p}(1) = \mathfrak{p}$, and for $Y_0^{\mathfrak{k}} \in \mathfrak{k}$, by (3.56),

$$J_1(Y_0^{\mathfrak{k}}) = \frac{\widehat{A}\big(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}}\big)}{\widehat{A}\big(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}}\big)}. \tag{3.59}$$

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$. Let $\operatorname{Tr}^{[\gamma]}[\cos(s\sqrt{\mathcal{L}_A^X})]$ be the even distribution on $\mathbb{R}$ determined by the condition that for any even function $\mu \in \mathcal{S}(\mathbb{R})$ with compactly supported Fourier transformation $\widehat{\mu}$, we have

$$\operatorname{Tr}^{[\gamma]}\left[\mu\left(\sqrt{\mathcal{L}_A^X}\right)\right] = \int_{\mathbb{R}} \widehat{\mu}(s) \operatorname{Tr}^{[\gamma]}\left[\cos\left(2\pi s\sqrt{\mathcal{L}_A^X}\right)\right] ds. \tag{3.60}$$

The wave operator $\cos(\sqrt{2}\pi s\sqrt{\mathcal{L}_A^X})$ defines a distribution on $\mathbb{R} \times X \times X$.

Let $\Delta^{\mathfrak{z}(\gamma)}$ be the standard Laplacian on $\mathfrak{z}(\gamma)$ with respect to the scalar product $\langle\cdot,\cdot\rangle = -B(\cdot,\theta\cdot)$. Now we can state the following microlocal version of Theorem 3.10 for the wave operator.

THEOREM 3.12 (**[12]**, Theorem 6.3.2). — *We have the following identity of even distributions on $\mathbb{R}$ supported on $\{|s| \geq \sqrt{2}|a|\}$ and with singular support in $\pm\sqrt{2}|a|$,*

(3.61)

$$\operatorname{Tr}^{[\gamma]}\left[\cos\left(s\sqrt{\mathscr{L}_A^X}\right)\right] = \int_{H^\gamma} \operatorname{Tr}^E\left[\cos\left(s\sqrt{-\frac{1}{2}\Delta^{\mathfrak{z}(\gamma)} + A}\right) J_\gamma(Y_0^{\mathfrak{k}})\rho^E(k^{-1})e^{-i\rho^E(Y_0^{\mathfrak{k}})}\right],$$

where $H^\gamma = \{0\} \times (a, \mathfrak{k}(\gamma)) \subset \mathfrak{z}(\gamma) \times \mathfrak{z}(\gamma)$.

Remark 3.13. — We assume that the semisimple element γ is nonelliptic, i.e., $a \neq 0$. We also assume that

$$[\mathfrak{k}(\gamma), \mathfrak{p}_0] = 0. \tag{3.62}$$

Then for $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, $\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma)} = 0$, $\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}_0^\perp(\gamma)} = 0$.

Now from (3.58), we have [**12**, Theorem 8.2.1]: for $t > 0$,

$$\begin{aligned}\operatorname{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}_A^X}\right] &= \frac{e^{-|a|^2/2t}}{\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{1}{\det(1-\operatorname{Ad}(k^{-1}))|_{\mathfrak{p}_0^\perp(\gamma)}} \\ &\quad \frac{1}{(2\pi t)^{p/2}} \cdot \operatorname{Tr}^E\left[\rho^E(k^{-1})\exp\Big(-t\Big(A + \frac{1}{48}\operatorname{Tr}^{\mathfrak{k}_0}[C^{\mathfrak{k}_0,\mathfrak{k}_0}] + \frac{1}{2}C^{\mathfrak{k}_0,E}\Big)\Big)\right].\end{aligned} \tag{3.63}$$

Note that if G is of real rank 1, then $\mathfrak{p}_0$ is the vector subspace generated by a, so that (3.62) holds. Thus (3.63) recovers the result of Sally-Warner [**58**] where they assume that the real rank of G is 1.

From (3.28) and (3.58), we obtain a refined version of the Selberg trace formula for the Casimir operator :

$$\operatorname{Tr}[e^{-t\mathscr{L}_A^Z}] = \sum_{[\gamma]\in[\Gamma]} \operatorname{Vol}\Big(\Gamma\cap Z(\gamma)\backslash X(\gamma)\Big) \operatorname{Tr}^{[\gamma]}[e^{-t\mathscr{L}_A^X}], \tag{3.64}$$

and each term $\operatorname{Tr}^{[\gamma]}[\cdot]$ is given by the closed formula (3.58).

We give two examples here to explain the explicit version of the Selberg trace formula (3.64).

Example 3.14 (Poisson summation formula). — Take $G = \mathbb{R}$ and $A = 0$. Then $K = \{0\}$. We have $X = \mathbb{R}$ and $\mathscr{L}_A^X = -\frac{1}{2}\Delta^{\mathbb{R}} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$, where x is the coordinate on $\mathbb{R}$. Let $p_t(x,x')$ be the heat kernel associated with $e^{t\Delta^{\mathbb{R}}/2}$.

For $a \in \mathbb{R}$, we have $Z(a) = \mathbb{R}, \mathfrak{k}(a) = \{0\}$. By (3.22) or (3.46), we have

$$\operatorname{Tr}^{[a]}\left[e^{-t\mathcal{L}_A^X}\right] = p_t(0, a). \tag{3.65}$$

From (3.58), we get

$$\operatorname{Tr}^{[a]}\left[e^{-t\mathcal{L}_A^X}\right] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}}. \tag{3.66}$$

Thus (3.58) gives simply an evaluation of the heat kernel on $\mathbb{R}$ which is well-known that

$$p_t(x, x') = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x')^2}{2t}}. \tag{3.67}$$

Take $\Gamma = \mathbb{Z} \subset \mathbb{R}$, then $Z = \mathbb{Z}\backslash\mathbb{R} = \mathbb{S}^1$. For any $\gamma \in \Gamma$, $X(\gamma) = Z(\gamma)/K(\gamma) = Z(\gamma) = \mathbb{R}$. Thus $\Gamma \cap Z(\gamma)\backslash Z(\gamma) = \mathbb{Z}\backslash\mathbb{R} = \mathbb{S}^1$ and $\operatorname{Vol}(\mathbb{S}^1) = 1$. The Selberg trace formula (3.64) reduces to the Poisson summation formula:

$$\sum_{k\in\mathbb{Z}} e^{-2\pi^2 k^2 t} = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{k^2}{2t}} \quad \text{for any} \quad t > 0. \tag{3.68}$$

Example 3.15. — Let $G = \mathrm{SL}_2(\mathbb{R})$ be the 2×2 real special linear group with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. The Cartan involution is given by $\theta : G \to G$, $g \mapsto {}^t g^{-1}$. Then $K = \mathrm{SO}(2) = \left\{ \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} : \beta \in \mathbb{R} \right\} \simeq \mathbb{S}^1$ is the corresponding maximal compact subgroup and $X = G/K$ is the Poincaré upper half-plane defined as $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0, x \in \mathbb{R}\}$. Precisely, an element $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by

$$gz = \frac{az + b}{cz + d} \in \mathbb{H} \quad \text{for } z \in \mathbb{H}. \tag{3.69}$$

The Cartan decomposition of $\mathfrak{sl}_2(\mathbb{R})$ is

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \tag{3.70}$$

where $\mathfrak{k}$ is the set of real antisymmetric matrices, and $\mathfrak{p}$ is the set of traceless symmetric matrices. Let B be the bilinear form on $\mathfrak{g}$ defined for $u, v \in \mathfrak{g}$ by

$$B(u, v) = 2 \operatorname{Tr}^{\mathbb{R}^2}[uv]. \tag{3.71}$$

Set

$$e_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}. \tag{3.72}$$

Then $\{e_1, e_2\}$ is a basis of $\mathfrak{p}$, and e_3 is a basis of $\mathfrak{k}$. They together form an orthonormal basis of the Euclidean space $(\mathfrak{g}, \langle\cdot,\cdot\rangle = -B(\cdot, \theta\cdot))$. Moreover,we have the relations,

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = -e_2. \tag{3.73}$$

The metric on X is given by $\frac{1}{y^2}(dx^2+dy^2)$. The scalar curvature of X is

$$\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] = -2|[e_1,e_2]|^2 = -2. \tag{3.74}$$

Let Δ^X be the Bochner Laplacian acting on $C^\infty(X,\mathbb{C})$. Then $\Delta^X = y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$. Since $\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]=0$ here, we have on $C^\infty(X,\mathbb{C})$,

$$\mathscr{L}^X = \frac{1}{2}C^{\mathfrak{g}} + \frac{1}{16}\,\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{48}\,\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] = -\frac{1}{2}\Delta^X - \frac{1}{8}. \tag{3.75}$$

From (3.73), we see that a semisimple nonelliptic element $\gamma\in G$ is hyperbolic. Thus such γ is conjugate to e^{ae_1} with some $a\in\mathbb{R}\backslash\{0\}$. Note that the orbital integral depends only on the conjugacy class of γ in G

If $\gamma = e^{ae_1}$ with $a\in\mathbb{R}\backslash\{0\}$, then by (3.73), $\mathfrak{k}(\gamma)=0$, $\mathfrak{z}_0=\mathfrak{z}(\gamma)=\mathbb{R}e_1$, and we have

$$\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp} = -(e^{a/2}-e^{-a/2})^2. \tag{3.76}$$

From Theorem 3.10, (3.75) and (3.76), we get

$$\mathrm{Tr}^{[\gamma]}\left[e^{t\Delta^X/2}\right] = \frac{1}{\sqrt{2\pi t}}\frac{\exp(-\frac{a^2}{2t}-\frac{t}{8})}{2\sinh(\frac{|a|}{2})}. \tag{3.77}$$

For $Y_0^{\mathfrak{k}} = y_0e_3\in\mathfrak{k}$, the relations (3.73) imply that

$$\widehat{A}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}}) = \frac{y_0/2}{\sinh(y_0/2)}. \tag{3.78}$$

From Theorem 3.10, (3.59) and (3.78), we get

$$\mathrm{Tr}^{[1]}[e^{t\Delta^X/2}] = \frac{e^{-t/8}}{2\pi t}\int_{\mathbb{R}} e^{-y_0^2/2t}\frac{y_0/2}{\sinh(y_0/2)}\frac{dy_0}{\sqrt{2\pi t}}. \tag{3.79}$$

By taking the derivative with respect to y_0 in both sides of $\frac{1}{\sqrt{2\pi t}}e^{-y_0^2/2t} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t\rho^2/2-i\rho y_0}d\rho$, we get

$$\frac{1}{\sqrt{2\pi t}}e^{-y_0^2/2t}\frac{y_0}{t} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t\rho^2/2}\rho\sin(\rho y_0)d\rho. \tag{3.80}$$

Thus

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}} e^{-y_0^2/2t}\frac{y_0/2}{\sinh(y_0/2)}\frac{dy_0}{\sqrt{2\pi t}} &= \frac{1}{4\pi}\int_{\mathbb{R}} e^{-t\rho^2/2}\rho\left(\int_{-\infty}^{+\infty}\frac{\sin(\rho y_0)}{\sinh(y_0/2)}dy_0\right)d\rho\\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-t\rho^2/2}\rho\tanh(\pi\rho)d\rho,\end{aligned} \tag{3.81}$$

where we use the identity $\int_{-\infty}^{+\infty}\frac{\sin(\rho y_0)}{\sinh(y_0/2)}dy_0 = 2\pi\tanh(\pi\rho)$.

Let $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ be a discrete torsion-free cocompact subgroup. Then $Z=\Gamma\backslash X$ is a compact Riemann surface. We say that $\gamma\in\Gamma$ is primitive if there does not exist $\beta\in\Gamma$ and $k\in\mathbb{N}$, $k\geq 2$ such that $\gamma=\beta^k$.

If $\gamma = e^{ae_1} \in \Gamma$ is primitive, then $|a|$ is the length of the corresponding closed geodesic in Z and for any $k \in \mathbb{Z}$, $k \neq 0$, $Z(\gamma^k) = Z(\gamma) = e^{\mathbb{R}e_1}$, and moreover,

$$\mathrm{Vol}(Z(\gamma^k) \cap \Gamma \backslash Z(\gamma^k)) = |a|. \tag{3.82}$$

Thus by (3.64), (3.77), (3.79), (3.81) and (3.82), we get

$$\begin{aligned}\mathrm{Tr}[e^{t\Delta^Z/2}] &= \sum_{\substack{\gamma\in\Gamma \text{ primitive},\\ [\gamma]=[e^{ae_1}],\, a\neq 0}} |a| \sum_{k\in\mathbb{N},\, k\neq 0} \mathrm{Tr}^{[e^{kae_1}]}[e^{t\Delta^X/2}] + \mathrm{Vol}(Z)\,\mathrm{Tr}^{[1]}[e^{t\Delta^X/2}] \\ &= \sum_{\substack{\gamma\in\Gamma \text{ primitive},\\ [\gamma]=[e^{ae_1}],\, a\neq 0}} |a| \sum_{k\in\mathbb{N},\, k\neq 0} \frac{1}{\sqrt{2\pi t}} \frac{1}{2\sinh(\frac{k|a|}{2})} e^{-\frac{k^2a^2}{2t}-\frac{t}{8}} \\ &\quad + \frac{\mathrm{Vol}(Z)}{4\pi} e^{-t/8} \int_{\mathbb{R}} e^{-t\rho^2/2} \rho \tanh(\pi\rho) d\rho.\end{aligned} \tag{3.83}$$

Formula (3.83) is exactly the original Selberg trace formula in [**59**, (3.2)] (cf. also [**52**, p. 233]).

3.5. Harish-Chandra's Plancherel Theory

In this subsection, we briefly describe Harish-Chandra's approach to orbital integrals. This approach can be used to evaluate the orbital integrals of arbitrary test function, for sufficiently regular semisimple elements. This formula contains complicated expressions involving infinite sums which do not converge absolutely, and have no obvious closed form except for some special groups. An useful reference on Harish-Chandra's work on orbital integrals is Varadarajan's book [**64**].

Recall that G is a connected reductive group. Denote by $G' \subset G$ the space of regular elements. Let $C_c^\infty(G)$ be the vector space of smooth functions with compact support on G. For $f \in C_c^\infty(G)$, attached to each θ-invariant Cartan subgroup H of G, Harish-Chandra introduce a smooth function $'F_f^H$ (cf. [**34**, §17]), as an orbital integral of f in a certain sense, defined on $H \cap G'$, which has reasonable limiting behavior on the singular set in H.

Let γ be a semisimple element such that (3.33) holds. If γ is regular, then up to conjugation there exists a unique θ-invariant Cartan subgroup H which contains γ. In this case, $'F_f^H(\gamma)$ is equal to a product of $\mathrm{Tr}^{[\gamma]}[f]$ and an explicit Lefschetz like denominator of γ. Now if γ is a singular semisimple element, let H be the unique (up to conjugation) θ-invariant Cartan subgroup with maximal compact dimension, which contains γ. Following Harish-Chandra [**33**], there is an explicit differential operator D defined on H such that

$$\mathrm{Tr}^{[\gamma]}[f] = \lim_{\gamma'\in H\cap G' \to \gamma} D\, {'F_f^H}(\gamma'). \tag{3.84}$$

Thus, to determine the orbital integral $\mathrm{Tr}^{[\gamma]}[f]$, it is enough to calculate $'F_f^H$ on the regular set $H \cap G'$.

Take $\gamma \in H \cap G'$ a regular element in H. Harish-Chandra developed certain techniques to calculate $'F_f^H$, obtaining formulas which are known as *Fourier inverse formula*. Indeed, $f \in C_c^\infty(G) \to {'F_f^H}(\gamma)$ defines an invariant distribution on G. The idea is to write $'F_f^H(\gamma)$ as a combination of invariant eigendistributions (i.e., a distribution on G which is invariant under the adjoint action of G, and which is an eigenvector of the center of $U(\mathfrak{g})$), like the global character of the discrete series representations and the unitary principal series representations of G, as well as certain singular invariant eigendistributions. More precisely, let $H = H_I H_R$ be Cartan decomposition of H (cf. [**34**, §8]), where H_I is a compact Abelian group and H_R is a vector space. Denote by $\widehat{H}, \widehat{H}_I, \widehat{H}_R$ the set of irreducible unitary representations of H, H_I, H_R. Then $\widehat{H} = \widehat{H}_I \times \widehat{H}_R$. Following [**36, 41**], for $a^* = (a_I^*, a_R^*) \in \widehat{H}$, we can associate an invariant eigendistribution $\Theta_{a^*}^H$ on G. Note that if H is compact and if a_I^* is regular, then $\Theta_{a^*}^H$ is the global character of the discrete series representations of G, and that if H is noncompact and if a_I^* is regular, then $\Theta_{a^*}^H$ is the global character of the unitary principal series representations of G. When a_I^* is singular, $\Theta_{a^*}^H$ is much more complicated. It is an alternating sum of some unitary characters, which in general are reducible.

In [**36**], Harish-Chandra announced the following theorem.

Theorem 3.16 ([**36**], Theorem 15). — *Let $\{H_1, \ldots, H_l\}$ be the complete set of non conjugated θ-invariant Cartan subgroups of G. Then there exist computable continuous functions Φ_{ij} on $H_i \times \widehat{H}_j$ such that for any regular element $\gamma \in H_i \cap G'$,*

$$'F_f^{H_i}(\gamma) = \sum_{j=1}^{l} \sum_{a_I^* \in \widehat{H}_{jI}} \int_{a_R^* \in \widehat{H}_{jR}} \Phi_{ij}(\gamma, a_I^*, a_R^*) \Theta_{a^*}^{H_j}(f) da_R^*. \tag{3.85}$$

In [**36**], Harish-Chandra only explained the idea of a proof by induction on $\dim G$. A more explicit version is obtained by Sally-Warner [**58**] when G is of real rank one (cf. Remark 3.13), and by Herb [**40**] (cf. also Bouaziz [**25**]) for general G. However, Herb's formula only holds for γ in an open dense subset of $H_i \cap G'$ and involves certain infinite sum of integrals which converges, but cannot be directly differentiated, term by term. In particular, the orbital integral of singular semisimple elements could not be obtained from Herb's formula by applying term by term the differential operator D in (3.84). When $\gamma = 1$, much more is known:

THEOREM 3.17 (Harish-Chandra [**35**]). — *There exists computable real analytic elementary functions $p^{H_j}(a^*)$ defined on $\widehat{H}_j$ such that for $f \in C_c^\infty(G)$, we have*

$$\operatorname{Tr}^{[1]}[f] = f(1) = \sum_{j=1}^{l} \sum_{a_I^* \in \widehat{H}_{jI}, \text{regular}} \int_{a_R^* \in \widehat{H}_{jR}} \Theta_{a^*}^{H_j}(f) p^{H_j}(a_I^*, a_R^*) da_R^*. \tag{3.86}$$

Theorem 3.17 can be applied to more general functions such as Harish-Chandra Schwartz functions, e.g., the trace of the heat kernel $q_t \in C^\infty(G, \operatorname{End}(E))$ of $e^{-t\mathcal{L}_A^X}$. Thus,

$$\operatorname{Tr}^{[1]}\left[e^{-t\mathcal{L}_A^X}\right] = \sum_{j=1}^{l} \sum_{a_I^* \in \widehat{H}_{jI}, \text{regular}} \int_{a_R^* \in \widehat{H}_{jR}} \Theta_{a^*}^{H_j}\left(\operatorname{Tr}^E[q_t]\right) p^{H_j}(a_I^*, a_R^*) da_R^*. \tag{3.87}$$

For H_j, we can associate a cuspidal parabolic subgroup P_j with Langlands decomposition $P_j = M_j H_{jR} N_j$ such that $H_{jI} \subset M_j$ is a compact Cartan subgroup of M_j. For $a^* = (a_I^*, a_R^*) \in \widehat{H}_j$ with a_I^* regular, denote by $(\varsigma_{a_I^*}, V_{a_I^*})$ the discrete series representations of M_j associated to a_I^*, and denote by (π_{a^*}, V_{a^*}) the associated principal series representations of G associated to $\varsigma_{a_I^*}$ and a_R^*. We have

$$\Theta_{a^*}^{H_j}\left(\operatorname{Tr}^E[q_t]\right) = \operatorname{Tr}^{V_{a^*}\otimes E}\left[\pi_{a^*}(q_t)\right] \text{ with } \pi_{a^*}(q_t) = \int_G q_t(g)\pi_{a^*}(g)dg. \tag{3.88}$$

It is not difficult to see that the image of the operator $\pi_{a^*}(q_t)$ is $(V_{a^*} \otimes E)^K \simeq (V_{a_I^*} \otimes E)^{K\cap M_j}$, and $\pi_{a^*}(q_t)$ acts as $e^{-t(\frac{1}{2}C^{\mathfrak{g},\pi_{a^*}} + \frac{1}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + A)}$ on its image. We get

$$\operatorname{Tr}^{V_{a^*}\otimes E}\left[\pi_{a^*}(q_t)\right] = e^{-t(\frac{1}{2}C^{\mathfrak{g},\pi_{a^*}} + \frac{1}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}])} \operatorname{Tr}^{(V_{a_I^*}\otimes E)^{K\cap M_j}}[e^{-tA}]. \tag{3.89}$$

Thus,

$$\begin{aligned}\operatorname{Tr}^{[1]}\left[e^{-t\mathcal{L}_A^X}\right] = \sum_{j=1}^{l} \sum_{a_I^* \in \widehat{H}_{jI}, \text{regular}} \int_{a_R^* \in \widehat{H}_{jR}} & e^{-t(\frac{1}{2}C^{\mathfrak{g},\pi_{a^*}} + \frac{1}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}])} \\ & \operatorname{Tr}^{(V_{a_I^*}\otimes E)^{K\cap M_j}}[e^{-tA}] p^{H_j}(a_I^*, a_R^*) da_R^*.\end{aligned} \tag{3.90}$$

Remark 3.18. — Equation (3.90) is not as explicit as (3.58), because in general it is not easy to determine all parabolic subgroups, all the discrete series of M, and the Plancherel densities $p^{H_j}(a^*)$.

We hope that from these descriptions, the readers got an idea on Harish-Chandra's Plancherel theory as an algorithm to compute orbital integrals. These results use the full force of the unitary representation theory (harmonic analysis) of reductive Lie groups, both at the technical and the representation level.

Bismut's explicit formula of the orbital integrals associated with the Casimir operator gives a closed formula in full generality for any semisimple element and any reductive Lie group. Bismut avoided completely the use of the harmonic analysis on reductive Lie groups. The hypoelliptic deformation allows him to localize the orbital integral for γ to any neighborhood of the family of shortest geodesic associated with γ, i.e., $X(\gamma)$.

There is a mysterious connection between Harish-Chandra's Plancherel theory and Theorem 0.1: in Harish-Chandra's Plancherel theory, the integral are taken on the $\mathfrak{p}$ part, but in Theorem 0.1, the integral is on the $\mathfrak{k}$ part. In particular, in Example 3.15 for $G = \mathrm{SL}_2(\mathbb{R})$, we obtain the contribution $\int_{\mathbb{R}} e^{-t\rho^2/2}\rho\tanh(\pi\rho)d\rho$ from the Plancherel theory for $\gamma = 1$. This coincide with (3.79) by using a Fourier transformation argument as explained in (3.81).

Remark 3.19. — Assume $G = \mathrm{SO}^0(m,1)$ with m odd. There exists only one Cartan subgroup H, and $p^H(a_I^*, \cdot)$ is an explicit polynomial. In this case, (3.90) becomes completely explicit.

4. GEOMETRIC HYPOELLIPTIC OPERATOR AND DYNAMICAL SYSTEMS

In this section, we explain how to construct geometrically the hypoelliptic Laplacians for a symmetric space, with the goal to prove Theorem 0.1 in the spirit of the heat kernel proof of the Lefschetz fixed-point formula (cf. Section 2.2). We introduce a hypoelliptic version of the orbital integral that depends on b. The analog of the methods of local index theory are needed to evaluate the limit. Theorem 4.13 identifies the orbital integral associated with the Casimir operator to the hypoelliptic orbital integral for the parameter $b > 0$. As $b \to +\infty$, the hypoelliptic orbital integral localizes near $X(\gamma)$.

This section is organized as follows. In Section 4.1, we explain how to compute the cohomology of a vector space by using algebraic de Rham complex and its Bargmann transformation, whose Hodge Laplacian is a harmonic oscillator. In Section 4.2, we recall the construction of the Dirac operator of Kostant, and in Sections 4.3, we construct the geometric hypoelliptic Laplacian by combining the constructions in Sections 4.1 and 4.2. In Section 4.4, we introduce the hypoelliptic orbital integrals and a hypoelliptic version of the McKean-Singer formula for these orbital integrals. In Section 4.5, we describe the limit of the hypoelliptic orbital integrals as $b \to +\infty$. Finally, in Section 4.6, we explain some relations of the hypoelliptic heat equation to the wave equation on the base manifold, which plays an important role in the proof of uniform Gaussian-like estimates for the hypoelliptic heat kernel.

4.1. Cohomology of a vector space and harmonic oscillator

Let V be a real vector space of dimension n, and let V^* be its dual. Let Y be the tautological section of V over V. Then Y can be identified with the corresponding radial vector field. Let d^V denote the de Rham operator.

Let L_Y be the Lie derivative associated with Y, and let i_Y be the contraction of Y. By Cartan's formula, we have the identity

$$L_Y = [d^V, i_Y]. \tag{4.1}$$

Let $S^\bullet(V^*) = \bigoplus_{j=0}^{\infty} S^j(V^*)$ be the symmetric algebra of V^*, which can be canonically identified with the polynomial algebra of V. Then $\Lambda^\bullet(V^*) \otimes S^\bullet(V^*)$ is the vector space of polynomial forms on V. Let $N^{S^\bullet(V^*)}$, $N^{\Lambda^\bullet(V^*)}$ be the number operators on $S^\bullet(V^*)$, $\Lambda^\bullet(V^*)$, which act by multiplication by k on $S^k(V^*)$, $\Lambda^k(V^*)$. Then

$$L_Y|_{\Lambda^\bullet(V^*)\otimes S^\bullet(V^*)} = N^{S^\bullet(V^*)} + N^{\Lambda^\bullet(V^*)}. \tag{4.2}$$

By (4.1) and (4.2), the cohomology of the polynomial forms $(\Lambda^\bullet(V^*) \otimes S^\bullet(V^*), d^V)$ on V is equal to $\mathbb{R}1$.

Assume that V is equipped with a scalar product. Then $\Lambda^\bullet(V^*)$, $S^\bullet(V^*)$ inherit associated scalar products. For instance, if $V = \mathbb{R}$, then $\|1^{\otimes j}\|^2 = j!$. With respect to this scalar product on $\Lambda^\bullet(V^*) \otimes S^\bullet(V^*)$, i_Y is the adjoint of d^V. Therefore L_Y is the associated Hodge Laplacian on $\Lambda^\bullet(V^*)\otimes S^\bullet(V^*)$. Remarkably enough, it does not depend on g^V. By (4.2), we get

$$\operatorname{Ker}(L_Y) = \mathbb{R}1. \tag{4.3}$$

We have given a Hodge theoretic interpretation to the proof that the cohomology of the complex of polynomial forms is concentrated in degree 0.

Let Δ^V denote the (negative) Laplacian on V. Let $L_2(V)$ be the corresponding Hilbert space of square integrable real-valued functions on V.

Definition 4.1. — *Let $T : S^\bullet(V^*) \to L_2(V)$ be the map such that given $P \in S^\bullet(V^*)$, then*

$$(TP)(Y) = \pi^{-n/4} e^{-\frac{|Y|^2}{2}} (e^{-\Delta^V/2} P)(\sqrt{2}Y). \tag{4.4}$$

Since P is a polynomial, $e^{-\Delta^V/2}P$ is defined by taking the obvious formal expansion of $e^{-\Delta^V/2}$. Its inverse, the Bargmann kernel, is given by

$$(Bf)(Y) = \pi^{n/4} e^{\Delta^V/2}\Big(e^{|Y|^2/4} f(\frac{Y}{\sqrt{2}})\Big). \tag{4.5}$$

Here the operator $e^{\Delta^V/2}$ is defined via the standard heat kernel of V.

Set

$$\overline{d} = T d^V B, \quad \overline{d}^* = T i_Y B : \Lambda^\bullet(V^*) \otimes L_2(V) \to \Lambda^\bullet(V^*) \otimes L_2(V). \tag{4.6}$$

Then by (4.4) and (4.5), we get

$$\overline{d} = \frac{1}{\sqrt{2}}(d^V + Y^*\wedge)\ , \quad \overline{d}^* = \frac{1}{\sqrt{2}}(d^{V*} + i_Y). \tag{4.7}$$

Here Y^* is the metric dual of Y in V^*, and d^{V*} is the usual formal L_2 adjoint of d^V.

Let $\{e_j\}$ be an orthonormal basis of V and let $\{e^j\}$ be its dual basis. For $U \in V$, let ∇_U be the usual differential along the vector U. Put $Y = \sum_{j=1}^n Y_j e_j$, then

$$\begin{aligned} d^V &= \sum_{j=1}^n e^j \wedge \nabla_{e_j}, \quad & d^{V*} &= -\sum_{j=1}^n i_{e_j} \nabla_{e_j}; \\ Y^*\wedge &= \sum_{j=1}^n Y_j e^j \wedge, & i_Y &= \sum_{j=1}^n Y_j i_{e_j}. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we get

$$\overline{d}^2 = (\overline{d}^*)^2 = 0, \quad T L_Y T^{-1} = [\overline{d}, \overline{d}^*] = \frac{1}{2}\Big(- \Delta^V + |Y|^2 - n\Big) + N^{\Lambda^\bullet(V^*)}. \tag{4.9}$$

Note that $\frac{1}{2}(-\Delta^V + |Y|^2 - n)$ is the harmonic oscillator on V already appeared in (1.3). In (4.3), we saw that the kernel of $[d^V, i_Y]$ in $\Lambda^\bullet(V^*) \otimes S^\bullet(V^*)$ is generated by 1 and so it is 1-dimensional and is concentrated in total degree 0. Equivalently the kernel of the unbounded operator $[\overline{d}, \overline{d}^*]$ acting on $\Lambda^\bullet(V^*) \otimes L_2(V)$ is 1-dimensional and is generated by the function $e^{-|Y|^2/2}/\pi^{n/4}$.

4.2. The Dirac operator of Kostant

Let V be a finite dimensional real vector space of dimension n and let B be a real valued symmetric bilinear form on V.

Let $c(V)$ be the Clifford algebra associated to (V, B). Namely, $c(V)$ is the algebra generated over $\mathbb{R}$ by $1, u \in V$ and the commutation relations for $u, v \in V$,

$$uv + vu = -2B(u, v). \tag{4.10}$$

We denote by $\widehat{c}(V)$ the Clifford algebra associated to $-B$. Then $c(V), \widehat{c}(V)$ are filtered by length, and their corresponding $\mathrm{Gr}^{\cdot}$ is just $\Lambda^\bullet(V)$. Also they are $\mathbb{Z}_2$-graded by length.

In the sequel, we assume that B is nondegenerate. Let $\varphi : V \to V^*$ be the isomorphism such that if $u, v \in V$,

$$(\varphi u, v) = B(u, v). \tag{4.11}$$

If $u \in V$, let $c(u), \widehat{c}(u)$ act on $\Lambda^\bullet(V^*)$ by

$$c(u) = \varphi u \wedge - i_u, \qquad \widehat{c}(u) = \varphi u \wedge + i_u. \tag{4.12}$$

Here i_u is the contraction operator by u.

Using supercommutators as in (0.9), from (4.12), we find that for $u, v \in V$,

$$[c(u), c(v)] = -2B(u,v), \quad [\widehat{c}(u), \widehat{c}(v)] = 2B(u,v), \quad [c(u), \widehat{c}(v)] = 0. \tag{4.13}$$

Equation (4.13) shows that $c(\cdot), \widehat{c}(\cdot)$ are representations of the Clifford algebras $c(V)$, $\widehat{c}(V)$ on $\Lambda^\bullet(V^*)$.

We will apply now the above constructions to the vector space $(\mathfrak{g}, B)$ of Section 3.1.

If $\{e_i\}_{i=1}^{m+n}$ is a basis of $\mathfrak{g}$, we denote by $\{e_i^*\}_{i=1}^{m+n}$ its dual basis of $\mathfrak{g}$ with respect to B (i.e., $B(e_i, e_j^*) = \delta_{ij}$), and by $\{e^i\}_{i=1}^{m+n}$ the dual basis of $\mathfrak{g}^*$.

Let $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ be such that if $a, b, c \in \mathfrak{g}$,

$$\kappa^{\mathfrak{g}}(a,b,c) = B([a,b],c). \tag{4.14}$$

Let $\widehat{c}(\kappa^{\mathfrak{g}}) \in \widehat{c}(V)$ correspond to $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ defined by

$$\widehat{c}(\kappa^{\mathfrak{g}}) = \frac{1}{6}\kappa^{\mathfrak{g}}(e_i^*, e_j^*, e_k^*)\,\widehat{c}(e_i)\,\widehat{c}(e_j)\,\widehat{c}(e_k). \tag{4.15}$$

Definition 4.2. — *Let $\widehat{D}^{\mathfrak{g}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the Dirac operator*

$$\widehat{D}^{\mathfrak{g}} = \sum_{i=1}^{m+n} \widehat{c}(e_i^*) e_i - \frac{1}{2}\widehat{c}(\kappa^{\mathfrak{g}}). \tag{4.16}$$

Note that $\widehat{c}(\kappa^{\mathfrak{g}}), \widehat{D}^{\mathfrak{g}}$ are G-invariant. The operator $\widehat{D}^{\mathfrak{g}}$ acts naturally on $C^\infty(G, \Lambda^\bullet(\mathfrak{g}^*))$.

Theorem 4.3 (Kostant formula, **[45]**, **[12**, Theorem 2.7.2, (2.6.11)**]**)

$$\widehat{D}^{\mathfrak{g},2} = -C^{\mathfrak{g}} - \frac{1}{8}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] - \frac{1}{24}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{4.17}$$

4.3. Construction of geometric hypoelliptic operators

The operator $\widehat{D}^{\mathfrak{g}}$ acts naturally on $C^\infty(G, \Lambda^\bullet(\mathfrak{g}^*))$ and also on $C^\infty(G, \Lambda^\bullet(\mathfrak{g}^*) \otimes S^\bullet(\mathfrak{g}^*))$. As we saw in Section 4.1, from a cohomological point of view, $\Lambda^\bullet(\mathfrak{g}^*) \otimes S^\bullet(\mathfrak{g}^*) \simeq \mathbb{R}$. This is how ultimately $C^\infty(G, \mathbb{R})$ (and $C^\infty(X, \mathbb{R})$) will reappear.

We denote by $\Delta^{\mathfrak{p}\oplus\mathfrak{k}}$ the standard Euclidean Laplacian on the Euclidean vector space $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. If $Y \in \mathfrak{g}$, we split Y in the form

$$Y = Y^{\mathfrak{p}} + Y^{\mathfrak{k}} \quad \text{with } Y^{\mathfrak{p}} \in \mathfrak{p}, Y^{\mathfrak{k}} \in \mathfrak{k}. \tag{4.18}$$

If $U \in \mathfrak{g}$, we use the notation

$$\begin{aligned} \widehat{c}(\operatorname{ad}(U)) &= -\frac{1}{4}B([U, e_i^*], e_j^*)\,\widehat{c}(e_i)\,\widehat{c}(e_j), \\ c(\operatorname{ad}(U)) &= \frac{1}{4}B([U, e_i^*], e_j^*)\,c(e_i)\,c(e_j). \end{aligned} \tag{4.19}$$

Here is the operator $\mathfrak{D}_b$ appeared in **[12**, Definition 2.9.1**]** which acts on

$$C^\infty(G, \Lambda^\bullet(\mathfrak{g}^*) \otimes S^\bullet(\mathfrak{g}^*)) \text{ “}{\simeq}\text{” } C^\infty(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*)). \tag{4.20}$$

Definition 4.4. — *Set*

$$\mathfrak{D}_b = \widehat{D}^{\mathfrak{g}} + ic\Big([Y^{\mathfrak{k}}, Y^{\mathfrak{p}}]\Big) + \frac{\sqrt{2}}{b}\Big(\overline{d}^{\mathfrak{p}} - i\overline{d}^{\mathfrak{k}} + \overline{d}^{\mathfrak{p}*} + i\overline{d}^{\mathfrak{k}*}\Big). \tag{4.21}$$

The introduction of i in the third term in the right-hand side of (4.21) is made so that its principal symbol anticommutes with the principal symbol of $\widehat{D}^{\mathfrak{g}}$.

Let $\{e_j\}_{j=1}^m$ be an orthonormal basis of $\mathfrak{p}$, and let $\{e_j\}_{j=m+1}^{m+n}$ be an orthonormal basis of $\mathfrak{k}$. If $U \in \mathfrak{k}$, $\mathrm{ad}(U)|_{\mathfrak{p}}$ acts as an antisymmetric endomorphism of $\mathfrak{p}$ and by (4.19), we have

$$c(\mathrm{ad}(U)|_{\mathfrak{p}}) = \frac{1}{4}\sum_{1\le i,j\le m}\langle [U, e_i], e_j\rangle c(e_i)c(e_j). \tag{4.22}$$

Finally, if $v \in \mathfrak{p}$, $\mathrm{ad}(v)$ exchanges $\mathfrak{k}$ and $\mathfrak{p}$ and is antisymmetric with respect to B, i.e., it is symmetric with respect to the scalar product on $\mathfrak{g}$. Moreover, by (4.19)

$$c(\mathrm{ad}(v)) = -\frac{1}{2}\sum_{\substack{m+1\le i\le m+n\\ 1\le j\le m}}\langle [v, e_i], e_j\rangle c(e_i)c(e_j). \tag{4.23}$$

If $v \in \mathfrak{g}$, we denote by ∇^V_v the corresponding differential operator along $\mathfrak{g}$. In particular, $\nabla^V_{[Y^{\mathfrak{k}},Y^{\mathfrak{p}}]}$ denotes the differentiation operator in the direction $[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}] \in \mathfrak{p}$. If $Y \in \mathfrak{g}$, we denote by $\underline{Y}^{\mathfrak{p}} + i\underline{Y}^{\mathfrak{k}}$ the section of $U(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$ associated with $Y^{\mathfrak{p}} + iY^{\mathfrak{k}} \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

Theorem 4.5 ([**12**], Theorem 2.11.1). — *The following identity holds:*

$$\begin{aligned}\frac{\mathfrak{D}_b^2}{2} = \frac{\widehat{D}^{\mathfrak{g},2}}{2} + \frac{1}{2}\Big|[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}]\Big|^2 + \frac{1}{2b^2}\Big(-\Delta^{\mathfrak{p}\oplus\mathfrak{k}} + |Y|^2 - m - n\Big) + \frac{N^{\Lambda^\bullet(\mathfrak{g}^*)}}{b^2}\\ + \frac{1}{b}\Big(\underline{Y}^{\mathfrak{p}} + i\underline{Y}^{\mathfrak{k}} - i\nabla^V_{[Y^{\mathfrak{k}},Y^{\mathfrak{p}}]} + \widehat{c}(\mathrm{ad}(Y^{\mathfrak{p}} + iY^{\mathfrak{k}})) + 2ic(\mathrm{ad}(Y^{\mathfrak{k}})|_{\mathfrak{p}}) - c(\mathrm{ad}(Y^{\mathfrak{p}}))\Big).\end{aligned} \tag{4.24}$$

By (3.7),

$$G \times_K \mathfrak{g} = TX \oplus N, \text{ with } N = G \times_K \mathfrak{k}. \tag{4.25}$$

Let $\widehat{\mathscr{X}}$ be the total space of $TX \oplus N$ over X, and let $\widehat{\pi} : \widehat{\mathscr{X}} \to X$ be the natural projection. Let $Y = Y^{TX} + Y^N$, $Y^{TX} \in TX$, $Y^N \in N$ be the canonical sections of $\widehat{\pi}^*(TX \oplus N)$, $\widehat{\pi}^*(TX)$, $\widehat{\pi}^*(N)$ over $\widehat{\mathscr{X}}$.

Note that the natural action of K on $C^\infty(\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*) \otimes E)$ is given by

$$(k \cdot \phi)(Y) = \rho^{\Lambda^\bullet(\mathfrak{g}^*)\otimes E}(k)\,\phi(\mathrm{Ad}(k^{-1})Y), \text{ for } \phi \in C^\infty(\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*) \otimes E). \tag{4.26}$$

Therefore

$$S^\bullet(T^*X \oplus N^*) \otimes \Lambda^\bullet(T^*X \oplus N^*) \otimes F = G \times_K (S^\bullet(\mathfrak{g}^*) \otimes \Lambda^\bullet(\mathfrak{g}^*) \otimes E), \tag{4.27}$$

and the bundle $G\times_K C^\infty(\mathfrak{g},\Lambda^\bullet(\mathfrak{g}^*)\otimes E)$ over X is just

$$C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F)).$$

By (4.26), the K action on $C^\infty(G\times\mathfrak{g},\Lambda^\bullet(\mathfrak{g}^*)\otimes E)$ is given by

$$(k\cdot s)(g,Y)=\rho^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}(k)s\left(gk,\mathrm{Ad}(k^{-1})Y\right). \tag{4.28}$$

If a vector space W is a K-representation, we denote by W^K its K-invariant subspace. Then

$$\begin{aligned} &C^\infty(G,S^\bullet(\mathfrak{g}^*)\otimes\Lambda^\bullet(\mathfrak{g}^*)\otimes E)^K\\ &\qquad=C^\infty(X,S^\bullet(T^*X\oplus N^*)\otimes\Lambda^\bullet(T^*X\oplus N^*)\otimes F)\\ &\qquad\text{“}\simeq\text{”}\ C^\infty(X,C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F)))\\ &\qquad=C^\infty(\widehat{\mathscr{X}},\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F)). \end{aligned} \tag{4.29}$$

As we saw in Section 3.1, the connection form $\omega^{\mathfrak{k}}$ on K-principal bundle $p:G\to X=G/K$ also induces a connection on $C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))$ over X, which is denoted by $\nabla^{C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}$. In particular, for the canonical section Y^{TX} of $\widehat{\pi}^*(TX)$ over $\widehat{\mathscr{X}}$, the covariant differentiation with respect to the given canonical connection in the horizontal direction corresponding to Y^{TX} is

$$\nabla^{C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}_{Y^{TX}}. \tag{4.30}$$

Since the operator $\mathfrak{D}_b$ is K-invariant, by (4.29), it descends to an operator $\mathfrak{D}_b^X$ acting on $C^\infty(\widehat{\mathscr{X}},\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))$. It is the same for the operator $\widehat{D}^{\mathfrak{g}}$, which descends to an operator $\widehat{D}^{\mathfrak{g},X}$ over X.

Recall that A is the self-adjoint K-invariant endomorphism of E in Section 3.1. For $b>0$, let $\mathscr{L}_b^X,\mathscr{L}_{A,b}^X$ act on $C^\infty(\widehat{\mathscr{X}},\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))$ by

$$\begin{aligned} \mathscr{L}_b^X&=-\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2}+\frac{1}{2}\mathfrak{D}_b^{X,2},\\ \mathscr{L}_{A,b}^X&=\mathscr{L}_b^X+A. \end{aligned} \tag{4.31}$$

Let $\langle\cdot,\cdot\rangle$ be the usual L_2 Hermitian product on the vector space of smooth compactly supported sections of $\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F)$ over $\widehat{\mathscr{X}}$. Set

$$\begin{aligned} \alpha&=\frac{1}{2}\Big(-\Delta^{TX\oplus N}+|Y|^2-m-n\Big)+N^{\Lambda^\bullet(T^*X\oplus N^*)},\\ \beta&=\nabla^{C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}_{Y^{TX}}+\widehat{c}(\mathrm{ad}(Y^{TX}))\\ &\quad-c(\mathrm{ad}(Y^{TX})+i\theta\mathrm{ad}(Y^N))-i\rho^E(Y^N),\\ \vartheta&=\frac{1}{2}\Big|[Y^N,Y^{TX}]\Big|^2. \end{aligned} \tag{4.32}$$

THEOREM 4.6 ([**12**], Theorems 2.12.5, 2.13.2). — *We have*

$$\mathscr{L}_b^X = \frac{\alpha}{b^2} + \frac{\beta}{b} + \vartheta. \tag{4.33}$$

The operator $\frac{\partial}{\partial t} + \mathscr{L}_b^X$ *is hypoelliptic.*

Also $\frac{1}{b}\nabla^{C^\infty(TX\oplus N, \widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}_{Y^{TX}}$ *is formally skew-adjoint with respect to* $\langle\cdot,\cdot\rangle$ *and* $\mathscr{L}_b^X - \frac{1}{b}\nabla^{C^\infty(TX\oplus N, \widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}_{Y^{TX}}$ *is formally self-adjoint with respect to* $\langle\cdot,\cdot\rangle$.

Remark 4.7. — We will now explain the presence of the term $ic([Y^{\mathfrak{k}}, Y^{\mathfrak{p}}])$ in the right-hand side of (4.21). Instead of $\mathfrak{D}_b$, we could consider the operator

$$D_b = \widehat{D}^{\mathfrak{g}} + \frac{1}{b}(\overline{d}^{\mathfrak{p}} - i\overline{d}^{\mathfrak{k}} + \overline{d}^{\mathfrak{p}*} + i\overline{d}^{\mathfrak{k}*}). \tag{4.34}$$

From (4.7), (4.9), (4.16) and (4.34), we get

$$D_b^2 = \widehat{D}^{\mathfrak{g},2} + \frac{1}{2b^2}(-\Delta^{\mathfrak{p}\oplus\mathfrak{k}}) + \frac{\sqrt{2}}{b}(\underline{Y}^{\mathfrak{p}} + i\underline{Y}^{\mathfrak{k}}) + \text{zero order terms}. \tag{4.35}$$

If $e \in \mathfrak{k}$, let $\nabla_{e,l}$ be the differentiation operator with respect to the left invariant vector field e, by (4.28), for $s \in C^\infty(G, C^\infty(\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*)\otimes E))^K$,

$$\nabla_{e,l}s = (L^V_{[e,Y]} - \rho^E(e))s. \tag{4.36}$$

Here $[e, Y]$ is a Killing vector field on $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ and the corresponding Lie derivative $L^V_{[e,Y]}$ acts on $C^\infty(\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*))$. By [**12**, (2.12.4)], we have the formula

$$L^V_{[e,Y]} = \nabla^V_{[e,Y]} - (c + \widehat{c})(\mathrm{ad}(e)). \tag{4.37}$$

When we use the identification (4.29), the operator $i\underline{Y}^{\mathfrak{k}}$ contributes the first order differential operator $i\nabla^V_{[Y^N, Y^{TX}]}$ along TX. This term is very difficult to control analytically.

The miraculous fact is that after adding $ic([Y^{\mathfrak{k}}, Y^{\mathfrak{p}}])$ to D_b, in the operator $\mathscr{L}_b^X$, we have eliminated $i\nabla^V_{[Y^N, Y^{TX}]}$ and we add instead the term $\vartheta = \frac{1}{2}|[Y^N, Y^{TX}]|^2$, which is nonnegative. This ensures that the operator $\frac{\alpha}{b^2} + \vartheta$ is bounded below. The operator $\mathscr{L}_b^X$ is a nice operator.

PROPOSITION 4.8 ([**12**], Proposition 2.15.1). — *We have the identity*

$$\left[\mathfrak{D}_b^X, \mathscr{L}_{A,b}^X\right] = 0. \tag{4.38}$$

Proof. — The classical Bianchi identity say that

$$\left[\mathfrak{D}_b^X, \mathfrak{D}_b^{X,2}\right] = 0. \tag{4.39}$$

By (4.17), $\widehat{D}^{\mathfrak{g},X,2}$ is the Casimir operator (up to a constant), so that

$$\left[\mathfrak{D}_b^X, \widehat{D}^{\mathfrak{g},X,2}\right] = 0. \tag{4.40}$$

We have the trivial $[\mathfrak{D}_b^X, A] = 0$. From (4.31), (4.39) and (4.40), we get (4.38). □

By analogy with (2.16), we will need to show that as $b \to 0$, in a certain sense,

$$e^{-t\mathscr{L}_{A,b}^X} \to e^{-t\mathscr{L}_A^X}. \tag{4.41}$$

We explain here an algebraic argument which gives evidence for (4.41). This will be the analog of (1.4). We denote by H the fiberwise kernel of α, so that

$$H = e^{-|Y|^2/2} \otimes F. \tag{4.42}$$

Let $H^\perp$ be the orthogonal to H in $L_2(\widehat{\mathscr{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$.

Note that β maps H to $H^\perp$. Let α^{-1} be the inverse of α restricted to $H^\perp$. Let $P, P^\perp$ be the orthogonal projections on H and $H^\perp$ respectively. We embed $L_2(X,F)$ into $L_2(\widehat{\mathscr{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$ isometrically via $s \to \widehat{\pi}^* s\, e^{-|Y|^2/2}/\pi^{(m+n)/4}$.

Theorem 4.9 ([**12**], Theorem 2.16.1). — *The following identify holds:*

$$P(\vartheta - \beta\alpha^{-1}\beta)P = \mathscr{L}^X. \tag{4.43}$$

Proof. — From (4.21), we can write

$$\frac{1}{\sqrt{2}}\mathfrak{D}_b^X = E_1 + \frac{F_1}{b}, \quad \text{with } E_1 = \frac{1}{\sqrt{2}}\left(\widehat{D}^{\mathfrak{g},X} + ic([Y^N, Y^{TX}])\right). \tag{4.44}$$

Then comparing (4.31), (4.33) and (4.44), we get

$$\alpha = F_1^2, \quad \beta = [E_1, F_1], \quad \vartheta = E_1^2 - \frac{1}{2}\widehat{D}^{\mathfrak{g},X,2}. \tag{4.45}$$

Since H is the kernel of F_1, we have $PF_1 = F_1P = 0$. We obtain thus

(4.46)
$$P(\vartheta - \beta\alpha^{-1}\beta)P = P\left(E_1^2 - E_1P^\perp E_1 - \frac{1}{2}\widehat{D}^{\mathfrak{g},X,2}\right)P = (PE_1P)^2 - \frac{1}{2}P\widehat{D}^{\mathfrak{g},X,2}P.$$

But H is of degree 0 in $\Lambda^\bullet(\mathfrak{g}^*)$, $\widehat{D}^{\mathfrak{g}} + ic([Y^{\mathfrak{k}}, Y^{\mathfrak{p}}])$ is of odd degree, we know that $PE_1P = 0$. Thus, (4.43) holds. □

4.4. Hypoelliptic orbital integrals

Under the formalism of Section 3.2, we replace now the finite dimensional vector space E by the infinite dimensional vector space

$$\mathscr{E} = \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes S^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E. \tag{4.47}$$

Using (4.29), from now on, we will work systematically on $C^\infty(\widehat{\mathscr{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$.

Let dY be the volume element of $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with respect to the scalar product $\langle\cdot,\cdot\rangle = -B(\cdot, \theta\cdot)$. It defines a fiberwise volume element on the fiber $TX \oplus N$, which we still denote by dY. Our kernel $q(g)$ now acts as an endomorphism of $\mathscr{E}$ and verifies (3.15) and (3.17). In what follows, the operator $q(g)$ is given by continuous kernels $q(g, Y, Y')$,

$Y, Y' \in \mathfrak{g}$. Let $q((x,Y),(x',Y'))$, $(x,Y),(x',Y') \in \widehat{\mathscr{X}}$ be the corresponding kernel on $\widehat{\mathscr{X}}$.

DEFINITION 4.10 ([**12**], Definition 4.3.3). — *For a semisimple element $\gamma \in G$, we define $\mathrm{Tr}_s^{[\gamma]}[Q]$ as in* (3.46),

$$\mathrm{Tr}_s^{[\gamma]}[Q] = \int_{\mathfrak{p}^\perp(\gamma)\times\mathfrak{g}} \mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}\left[q(e^{-f}\gamma e^f, Y, Y)\right] r(f) df dY \tag{4.48}$$

once it is well-defined. Note here $\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\cdot] = \mathrm{Tr}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[(-1)^{N^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)}}\cdot]$, *i.e., we use the natural $\mathbb{Z}_2$-grading on $\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)$.*

DEFINITION 4.11. — *Let* $\mathbf{P}$ *be the projection from* $\Lambda^\bullet(T^*X \oplus N^*) \otimes F$ *on* $\Lambda^0(T^*X \oplus N^*) \otimes F$.

Recall that $e^{-t\mathscr{L}_A^X}(x,x')$ is the heat kernel of $\mathscr{L}_A^X$ in Section 3.1. For $t > 0$, $(x,Y),(x',Y') \in \widehat{\mathscr{X}}$, put

$$q_{0,t}^X\big((x,Y),(x',Y')\big) = \mathbf{P} e^{-t\mathscr{L}_A^X}(x,x')\pi^{-(m+n)/2} e^{-\frac{1}{2}\left(|Y|^2+|Y'|^2\right)}\mathbf{P}. \tag{4.49}$$

Let $e^{-t\mathscr{L}_{A,b}^X}$ be the heat operator of $\mathscr{L}_{A,b}^X$ and $q_{b,t}^X\big((x,Y),(x',Y')\big)$ be the kernel of the heat operator $e^{-t\mathscr{L}_{A,b}^X}$ associated with the volume form $dx'dY'$. In [**12**, §11.5, 11.7], Bismut studied in detail the smoothness of $q_{b,t}^X((x,Y),(x',Y'))$ for $t > 0, b > 0$, $(x,Y),(x',Y') \in \widehat{\mathscr{X}}$. In particular, he showed that it is rapidly decreasing in the variables Y, Y'.

Now we state an important result [**12**, Theorem 4.5.2] whose proof was given in [**12**, §14] where Theorem 4.9 plays an important role. It ensures that the hypoelliptic orbital integral is well-defined for $e^{-t\mathscr{L}_{A,b}^X}$ and that the analog of Theorem 2.2 holds for $h = e^{-t\mathscr{L}_A^X}$ and $\mathfrak{D}_b^X$.

THEOREM 4.12. — *Given $0 < \epsilon \leq M$, there exist $C, C' > 0$ such that for $0 < b \leq M$, $\epsilon \leq t \leq M$, $(x,Y),(x',Y') \in \widehat{\mathscr{X}}$,*

$$\left|q_{b,t}^X\big((x,Y),(x',Y')\big)\right| \leq C \exp\Big(-C'\Big(d^2(x,x') + |Y|^2 + |Y'|^2\Big)\Big). \tag{4.50}$$

Moreover, as $b \to 0$,

$$q_{b,t}^X\big((x,Y),(x',Y')\big) \to q_{0,t}^X\big((x,Y),(x',Y')\big). \tag{4.51}$$

The formal analog of Theorem 2.2 is as follows.

THEOREM 4.13 ([**12**], Theorem 4.6.1). — *For any $b > 0, t > 0$, we have*

$$\mathrm{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}_A^X}\right] = \mathrm{Tr}_s^{[\gamma]}\left[e^{-t\mathscr{L}_{A,b}^X}\right]. \tag{4.52}$$

Proof. — In [**12**, §4.3], Bismut showed that the hypoelliptic orbital integral (4.48) is a trace on certain algebras of operators given by smooth kernels which exhibit a Gaussian decay like in (4.50). By Theorem 4.12, the kernel function $q^X_{b,t}$ is in this algebra. As in (2.15), by Proposition 4.8,

$$\begin{aligned}\frac{\partial}{\partial b}\operatorname{Tr}_s^{[\gamma]}\left[e^{-t\mathscr{L}^X_{A,b}}\right] &= \operatorname{Tr}_s^{[\gamma]}\left[-t\Big(\frac{\partial}{\partial b}\mathscr{L}^X_{A,b}\Big)e^{-t\mathscr{L}^X_{A,b}}\right]\\ &= -t\operatorname{Tr}_s^{[\gamma]}\left[\frac{1}{2}\Big[\mathfrak{D}^X_b,\frac{\partial}{\partial b}\mathfrak{D}^X_b\Big]e^{-t\mathscr{L}^X_{A,b}}\right]\\ &= -\frac{t}{2}\operatorname{Tr}_s^{[\gamma]}\left[\Big[\mathfrak{D}^X_b,(\frac{\partial}{\partial b}\mathfrak{D}^X_b)e^{-t\mathscr{L}^X_{A,b}}\Big]\right]=0.\end{aligned}\tag{4.53}$$

By Theorem 4.12, we have

$$\lim_{b\to 0}\operatorname{Tr}_s^{[\gamma]}\left[e^{-t\mathscr{L}^X_{A,b}}\right] = \operatorname{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}^X_{A}}\right].\tag{4.54}$$

From (4.53), (4.54), we get (4.52). □

4.5. Proof of Theorem 3.10

For $b>0$, $s(x,Y)\in C^\infty(\widehat{\mathscr{X}},\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))$, set

$$F_b s(x,Y) = s(x,-bY).\tag{4.55}$$

Put

$$\underline{\mathscr{L}}^X_{A,b} = F_b\mathscr{L}^X_{A,b}F_b^{-1}.\tag{4.56}$$

Let $\underline{q}^X_{b,t}((x,Y),(x',Y'))$ be the kernel associated with $e^{-t\underline{\mathscr{L}}^X_{A,b}}$. When $t=1$, we will write $\underline{q}^X_b$ instead of $\underline{q}^X_{b,1}$. Then from (4.56), we have

$$\underline{q}^X_{b,t}\big((x,Y),(x',Y')\big) = (-b)^{m+n}q^X_{b,t}\big((x,-bY),(x',-bY')\big).\tag{4.57}$$

Let a^{TX} be the vector field on X associated with a in (3.33) induced by the left action of G on X (cf. (3.10)). Let $d(\cdot,X(\gamma))$ be the distance function to $X(\gamma)$.

THEOREM 4.14 ([**12**], Theorem 9.1.1, (9.1.6)). — *Given $0<\epsilon\le M$, there exist $C,C'>0$, such that for any $b\ge 1$, $\epsilon\le t\le M$, $(x,Y),(x',Y')\in\widehat{\mathscr{X}}$,*

$$\left|\underline{q}^X_{b,t}\big((x,Y),(x',Y')\big)\right| \le Cb^{4m+2n}\exp\Big(-C\Big(d^2(x,x')+|Y|^2+|Y'|^2\Big)\Big).\tag{4.58}$$

Given $\delta>1,\beta>0$, $0<\epsilon\le M$, there exist $C,C'>0$, such that for any $b\ge 1$, $\epsilon\le t\le M$, $(x,Y)\in\widehat{\mathscr{X}}$, if $d(x,X(\gamma))\ge\beta$,

$$\left|\underline{q}^X_{b,t}\big((x,Y),\gamma(x,Y)\big)\right| \le Cb^{-\delta}\exp\Big(-C'\Big(d^2_\gamma(x)+|Y|^2\Big)\Big).\tag{4.59}$$

Given $\delta > 1, \beta > 0, \mu > 0$, there exist $C, C' > 0$ such that for any $b \geq 1$, $(x, Y) \in \widehat{\mathscr{X}}$, if $d(x, X(\gamma)) \leq \beta$, and $|Y^{TX} - a^{TX}(x)| \geq \mu$,

$$\left| \underline{q}^X_{b,t}\big((x,Y), \gamma(x,Y)\big)\right| \leq C b^{-\delta} e^{-C'|Y|^2}. \tag{4.60}$$

In view of Theorem 4.14, the proof of Theorem 3.10 consists in obtaining the asymptotics of $\mathrm{Tr}_s^{[\gamma]}[e^{-t\mathscr{L}^X_{A,b}}]$ as $b \to +\infty$. By [**12**, (2.14.4)], the operator in (4.31) associated with B/t is up to conjugation, $t\mathscr{L}^X_{\sqrt{t}b}$. Observe that $J_\gamma(Y_0^{\mathfrak{k}})$ is unchanged when replacing the bilinear form B by B/t, $t > 0$. Thus we only need to establish the corresponding result for $t = 1$.

When $f \in \mathfrak{p}^\perp(\gamma)$, we identify e^f with $e^f p1$. For $f \in \mathfrak{p}^\perp(\gamma)$, $Y \in (TX \oplus N)_{e^f}$, set

$$\underline{Q}^X_b(e^f, Y) = \mathrm{Tr}_s^{\Lambda^\bullet(T^*X \oplus N^*)\otimes F}\left[\gamma \underline{q}^X_b\big((e^f, Y), \gamma(e^f, Y)\big)\right]. \tag{4.61}$$

Then

$$\mathrm{Tr}_s^{[\gamma]}\left[e^{-\mathscr{L}^X_{A,b}}\right] = \int_{(e^f,Y)\in \widehat{\pi}^{-1}\mathfrak{p}^\perp(\gamma)} \underline{Q}^X_b(e^f, Y) r(f) df dY. \tag{4.62}$$

Take $\beta \in]0,1]$. By Theorem 4.14, as $b \to +\infty$,

$$\begin{aligned} &\int_{(e^f,Y)\in \widehat{\pi}^{-1}\mathfrak{p}^\perp(\gamma), |f|\geq \beta} \underline{Q}^X_b(e^f, Y) r(f) df dY \to 0, \\ &\int_{(e^f,Y)\in \widehat{\pi}^{-1}\mathfrak{p}^\perp(\gamma), |f|<\beta, |Y^{TX} - a^{TX}(e^f)|\geq \mu} \underline{Q}^X_b(e^f, Y) r(f) df dY \to 0. \end{aligned} \tag{4.63}$$

We need to understand the integral on the domain $|f| < \beta$, $|Y^{TX} - a^{TX}(e^f)| < \mu$, when $b \to +\infty$.

Let $\pi : \mathscr{X} \to X$ be the total space of the tangent bundle TX to X. Let $\varphi_t|_{t\in\mathbb{R}}$ be the group of diffeomorphisms of $\mathscr{X}$ induced by the geodesic flow. By [**12**, Proposition 3.5.1], $\varphi_1(x, Y^{TX}) = \gamma \cdot (x, Y^{TX})$ is equivalent to $x \in X(\gamma)$ and $Y^{TX} = a^{TX}(x)$. Equation (4.63) shows that as $b \to +\infty$, the right-hand side of (4.62) localizes near the minimizing geodesic x_t connecting x and γx so that $\dot{x} = a^{TX}$.

Let $N(\gamma)$ be the vector bundle on $X(\gamma)$ which is the analog of the vector bundle N on X in (4.25). Then $N(\gamma) \subset N|_{X(\gamma)}$. Let $N^\perp(\gamma)$ be the orthogonal to $N(\gamma)$ in $N|_{X(\gamma)}$. Clearly,

$$N^\perp(\gamma) = Z(\gamma) \times_{K(\gamma)} \mathfrak{k}^\perp(\gamma). \tag{4.64}$$

Let $p_\gamma : X \to X(\gamma)$ be the projection defined by (3.40) and (3.42). We trivialize the vector bundles TX, N by parallel transport along the geodesics orthogonal to $X(\gamma)$ with respect to the connection ∇^{TX}, ∇^N, so that TX, N can be identified with $p_\gamma^* TX|_{X(\gamma)}, p_\gamma^* N|_{X(\gamma)}$. At $x = p1$, we have

$$N(\gamma) = \mathfrak{k}(\gamma), \qquad N^\perp(\gamma) = \mathfrak{k}^\perp(\gamma). \tag{4.65}$$

Therefore at $\rho_\gamma(1,f)$, we may write $Y^N \in N$ in the form

$$Y^N = Y_0^{\mathfrak{k}} + Y^{N,\perp}, \qquad \text{with } Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma), Y^{N,\perp} \in \mathfrak{k}^\perp(\gamma). \tag{4.66}$$

Let $dY_0^{\mathfrak{k}}$, $dY^{N,\perp}$ be the volume elements on $\mathfrak{k}(\gamma)$, $\mathfrak{k}^\perp(\gamma)$, so that

$$dY^N = dY_0^{\mathfrak{k}}\, dY^{N,\perp}. \tag{4.67}$$

To evaluate the limit of (4.62) as $b \to +\infty$ for $\beta > 0$, we may by (4.63), as well consider the integral

$$\begin{aligned}\int_{|f|<\beta, |Y^{TX} - a^{TX}(e^f)|<\mu} &\underline{Q}_b^X\left(e^f, Y\right) r(f) df dY^{TX} dY_0^{\mathfrak{k}} dY^{N,\perp} \\ = b^{-4m-2n+2r} \int_{|f|<b^2\beta, |Y^{TX}|<b^2\mu} &\underline{Q}_b^X\left(e^{f/b^2}, \frac{Y^{TX}}{b^2} + a^{TX}(e^{f/b^2}), Y_0^{\mathfrak{k}} + \frac{Y^{N,\perp}}{b^2}\right) \\ &\qquad r(f/b^2) df dY^{TX} dY_0^{\mathfrak{k}} dY^{N,\perp}.\end{aligned} \tag{4.68}$$

Let $\underline{\mathfrak{z}}(\gamma)$ be the another copy of $\mathfrak{z}(\gamma)$, and let $\underline{\mathfrak{z}}(\gamma)^*$ be the corresponding copy of the dual of $\underline{\mathfrak{z}}(\gamma)$. Also, for $u \in \mathfrak{z}(\gamma)^*$, we denote by $\underline{u}$ the corresponding element in $\underline{\mathfrak{z}}(\gamma)^*$. Let $e_1, \ldots, e_r$ be a basis of $\mathfrak{z}(\gamma)$, let $e^1, \ldots, e^r$ be the corresponding dual basis of $\mathfrak{z}(\gamma)^*$.

Put $\mathcal{G} = \mathrm{End}(\Lambda^\bullet(\mathfrak{g}^*))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}}(\gamma)^*)$. Let $e_{r+1}, \ldots, e_{m+n}$ be a basis of $\mathfrak{z}^\perp(\gamma)$, and let $e_{r+1}^*, \ldots, e_{m+n}^*$ be the dual basis to $e_{r+1}, \ldots, e_{m+n}$ with respect to $B|_{\mathfrak{z}^\perp(\gamma)}$. Then $\mathcal{G}$ is generated by all the monomials in $c(e_i), \widehat{c}(e_i), 1 \le i \le m+n$, $\underline{e}^j$, $1 \le j \le r$. Let $\widehat{\mathrm{Tr}}_s$ be the linear map from $\mathcal{G}$ into $\mathbb{R}$ that, up to permutation, vanishes on all monomials except those of the following form:

(4.69)

$$\widehat{\mathrm{Tr}}_s\left[c(e_1)\underline{e}^1 \cdots c(e_r)\underline{e}^r c(e_{r+1}^*)\widehat{c}(e_{r+1}) \cdots c(e_{m+n}^*)\widehat{c}(e_{m+n})\right] = (-1)^r(-2)^{m+n-r}.$$

For $u \in \mathcal{G}$, $v \in \mathrm{End}(E)$, we define

$$\widehat{\mathrm{Tr}}_s[uv] = \widehat{\mathrm{Tr}}_s[u]\,\mathrm{Tr}^E[v]. \tag{4.70}$$

Set

$$\alpha = \sum_{i=1}^r c(e_i)\underline{e}^i \in c(\mathfrak{z}(\gamma))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}}(\gamma)^*). \tag{4.71}$$

Definition 4.15. — *Let $\mathfrak{q}_b^X((x,Y),(x',Y'))$ denote the smooth kernel associated with $e^{-\underline{\mathcal{L}}_{A,b}^X - \alpha}$, and*

$$\underline{Q}_b^X(x,Y) = \gamma \mathfrak{q}_b^X\big((x,Y),\gamma(x,Y)\big). \tag{4.72}$$

Since $\underline{\mathcal{L}}_{A,b}^X + \alpha$ can be obtained from $\underline{\mathcal{L}}_{A,b}^X$ by a conjugation, by a simple argument on Clifford algebras, we get:

Proposition 4.16 ([**12**], Proposition 9.5.4). — *For $b > 0$, the following identity holds:*

$$Q_b^X(x,Y) = b^{-2r}\widehat{\mathrm{Tr}}_s\left[\mathcal{Q}_b^X(x,Y)\right]. \tag{4.73}$$

Now we define a limit operator acting on $C^\infty(\mathfrak{p}\times\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*)\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}}(\gamma)^*)\otimes E)$. We denote by y the tautological section of the first component of $\mathfrak{p}$ in $\mathfrak{p}\times\mathfrak{g}$, and by $Y = Y^{\mathfrak{p}} + Y^{\mathfrak{k}}$ the tautological section of $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$. Let dy the volume form on $\mathfrak{p}$ and let dY the volume form on $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$.

Definition 4.17. — *Given $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, set*

$$\begin{aligned}\mathscr{P}_{a,Y_0^{\mathfrak{k}}} = \frac{1}{2}\left|[Y^{\mathfrak{k}},a] + [Y_0^{\mathfrak{k}}, Y^{\mathfrak{p}}]\right|^2 - \frac{1}{2}\Delta^{\mathfrak{p}\oplus\mathfrak{k}} + \sum_{i=1}^r c(e_i)\underline{e}^i - \nabla^H_{Y^{\mathfrak{p}}} - \nabla^V_{[a+Y_0^{\mathfrak{k}},[a,y]]}\\ - \widehat{c}(\mathrm{ad}(a)) + c(\mathrm{ad}(a) + i\theta\,\mathrm{ad}(Y_0^{\mathfrak{k}}))\end{aligned} \tag{4.74}$$

acting on $C^\infty(\mathfrak{p}\times\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*)\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}}(\gamma)^*)\otimes E)$.

Let $R_{Y_0^{\mathfrak{k}}}((y,Y),(y',Y'))$ be the smooth kernel of $e^{-\mathscr{P}_{a,Y_0^{\mathfrak{k}}}}$ with respect to the volume form $dydY$ on $\mathfrak{p}\times\mathfrak{g}$. Then

$$R_{Y_0^{\mathfrak{k}}}((y,Y),(y',Y')) \in \mathrm{End}(\Lambda^\bullet(\mathfrak{z}^\perp(\gamma)^*))\widehat{\otimes}c(\mathfrak{z}(\gamma))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}}(\gamma)^*). \tag{4.75}$$

The following result gives an estimate and pointwise asymptotics of $\mathcal{Q}_b^X$.

Theorem 4.18 ([**12**], Theorems 9.5.6, 9.6.1). — *Given $\beta > 0$, there exist $C, C'_\gamma > 0$ such that for $b \geq 1$, $f \in \mathfrak{p}^\perp(\gamma)$, $|f| \leq \beta b^2$, and $|Y^{TX}| \leq \beta b^2$,*

$$\begin{aligned}&b^{-4m-2n}\left|\mathcal{Q}_b^X\left(e^{f/b^2}, a^{TX}(e^{f/b^2}) + Y^{TX}/b^2, Y_0^{\mathfrak{k}} + Y^{N,\perp}/b^2\right)\right|\\ &\leq C\exp\left(-C'\left|Y_0^{\mathfrak{k}}\right|^2 - C'_\gamma\left(|f|^2 + \left|Y^{TX}\right|^2 + \left|(\mathrm{Ad}(k^{-1})-1)Y^{N,\perp}\right| + \left|[a,Y^{N,\perp}]\right|\right)\right).\end{aligned} \tag{4.76}$$

As $b \to +\infty$,

$$\begin{aligned}&b^{-4m-2n}\mathcal{Q}_b^X\left(e^{f/b^2}, a^{TX}(e^{f/b^2}) + Y^{TX}/b^2, Y_0^{\mathfrak{k}} + Y^{N,\perp}/b^2\right)\\ &\to e^{-(|a|^2+|Y_0^{\mathfrak{k}}|^2)/2}\,\mathrm{Ad}\left(k^{-1}\right)R_{Y_0^{\mathfrak{k}}}\left((f,Y),\mathrm{Ad}(k^{-1})(f,Y)\right)\rho^E\left(k^{-1}\right)e^{-i\rho^E(Y_0^{\mathfrak{k}})-A}.\end{aligned} \tag{4.77}$$

A crucial computation in [**12**, Theorem 5.5.1, (5.1.11)] gives the following key result.

Theorem 4.19. — *For $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, we have the identity*

$$(2\pi)^{r/2}\int_{\mathfrak{p}^\perp(\gamma)\times(\mathfrak{p}\oplus\mathfrak{k}^\perp(\gamma))}\widehat{\mathrm{Tr}}_s\left[\mathrm{Ad}(k^{-1})R_{Y_0^{\mathfrak{k}}}\left((y,Y),\mathrm{Ad}(k^{-1})(y,Y)\right)\right]dydY = J_\gamma\left(Y_0^{\mathfrak{k}}\right). \tag{4.78}$$

From Theorems 4.18, 4.19, (4.61)-(4.63), and (4.73), we obtain Theorem 3.10.

Example 4.20 ([**12**], §10.6; [**13**], §5.1). — In Example 3.14, we have N equal to 0, $\widehat{\mathcal{X}} = TX \oplus N = T\mathbb{R} = \mathbb{R} \oplus \mathbb{R}$. Using the coordinates $(x, y) \in \mathbb{R} \oplus \mathbb{R}$, we get

$$\mathcal{L}_b^X = M_b + \frac{N^{\Lambda^\bullet(\mathbb{R})}}{b^2}, \qquad \text{with } M_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) + \frac{y}{b}\frac{\partial}{\partial x}. \tag{4.79}$$

The heat kernel $p_{b,t}((x,y),(x',y'))$ associated with e^{-tM_b} depends only on $x'-x, y, y'$. The heat kernel of the operator $-\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial x}$ was first calculated by Kolmogorov [**44**], and $p_{b,t}((x,y),(x',y'))$ has been computed explicitly in [**12**, Proposition 10.5.1].

Let $a \in \mathfrak{p} = \mathbb{R}$. Then a acts as translation by a on the first component of $\mathbb{R} \oplus \mathbb{R}$. From (4.48) and (4.79), we deduce that

$$\operatorname{Tr}_s^{[a]}\left[e^{-t\mathcal{L}_b^X}\right] = \left(1 - e^{-t/b^2}\right)\int_{\mathbb{R}} p_{b,t}\big((0,Y),(a,Y)\big)dY. \tag{4.80}$$

Theorem 4.13 can be stated in the special case of this example as follows.

Theorem 4.21. — *For any $t > 0, b > 0$, we have*

$$\operatorname{Tr}^{[a]}\left[e^{t\Delta^{\mathbb{R}}/2}\right] = \operatorname{Tr}_s^{[a]}\left[e^{-t\mathcal{L}_b^X}\right]. \tag{4.81}$$

Proof. — We give a simple direct proof which can be ultimately easily justified. Note that

$$M_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + \left(y + b\frac{\partial}{\partial x}\right)^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{4.82}$$

By (4.82), we get

$$e^{-b\frac{\partial^2}{\partial x\partial y}} M_b e^{b\frac{\partial^2}{\partial x\partial y}} = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{4.83}$$

Using the fact that $p_{b,t}((x,y),(x',y'))$ only depends on $x'-x$, y, y', we deduce from (4.83) that

$$\begin{aligned}\int_{\mathbb{R}} p_{b,t}\big((0,y),(a,y)\big)dy &= \operatorname{Tr}\left[e^{-\frac{t}{2b}\left(-\frac{\partial^2}{\partial y^2}+y^2-1\right)}\right]\operatorname{Tr}^{[a]}\left[e^{t\Delta^{\mathbb{R}}/2}\right]\\ &= \frac{1}{1-e^{-t/b^2}}\operatorname{Tr}^{[a]}\left[e^{t\Delta^{\mathbb{R}}/2}\right],\end{aligned} \tag{4.84}$$

since the spectrum of the harmonic oscillator $\frac{1}{2}(-\frac{\partial^2}{\partial y^2} + y^2 - 1)$ is $\mathbb{N}$. By (4.80) and (4.84), we get (4.81). □

By (4.81), we can compute the limit as $b \to +\infty$ of the right-hand side of (4.80) from the explicit formula of $p_{b,t}((x,y),(x',y'))$, and in this way we get (3.66). In other words, we interpret (3.66) as a consequence of a local index theorem.

4.6. A brief idea on the proof of Theorems 4.12, 4.14, 4.18

The wave operator for the elliptic Laplacian has the property of finite propagation speed, which explain the Gaussian decay of the elliptic heat kernel. The hypoelliptic Laplacian does not have a wave equation.

One difficult point in Theorems 4.12, 4.14 and 4.18 is to get the uniform Gaussian-like estimate. Let us give an argument back to [**12**, §12.3] which explains some heuristic relations of the hypoelliptic heat equation to the wave equation on X. Here $q^X_{b,t}$ will denote scalar hypoelliptic heat kernel on the total space $\mathscr{X}$ of the tangent bundle TX. Put

$$\begin{aligned}\sigma_{b,t}((x,Y),x') &= \int_{Y'\in T_{x'}X} q^X_{b,t}\big((x,Y),(x',Y')\big)dY',\\ M_{b,t}((x,Y),x') &= \frac{1}{\sigma_{b,t}((x,Y),x')}\int_{Y'\in T_{x'}X} q^X_{b,t}\big((x,Y),(x',Y')\big)(Y'\otimes Y')dY'.\end{aligned}\tag{4.85}$$

Then $M_{b,t}((x,Y),x')$ takes its values in symmetric positive endomorphisms of $T_{x'}X$. We can associate to $M_{b,t}$ the second order elliptic operator acting on $C^\infty(X,\mathbb{R})$,

$$\mathbf{M}_{b,t}(x,Y)g(x') = \left\langle \nabla^{TX}_{\cdot}\nabla_{\cdot}, M_{b,t}((x,Y),x')g(x')\right\rangle,\tag{4.86}$$

where the operator $\nabla^{TX}_{\cdot}\nabla_{\cdot}$ acts on the variable x'. Then we have [**12**, (12.3.12)]

$$\left(b^2\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}-\mathbf{M}_{b,t}(x,Y)\right)\sigma_{b,t}((x,Y),\cdot)=0.\tag{4.87}$$

This is a hyperbolic equation. As $b\to 0$, it converges in the proper sense to the standard parabolic heat operator

$$\Big(\frac{\partial}{\partial t}-\frac{1}{2}\Delta\Big)p_t(x,\cdot)=0.\tag{4.88}$$

The above consideration plays an important role in the proof given in [**12**] of the estimates (4.50), (4.58), (4.59) and (4.76).

5. ANALYTIC TORSION AND DYNAMICAL ZETA FUNCTION

Recall that a flat vector bundle (F,∇) with flat connection ∇ over a smooth manifold M comes from a representation $\rho:\pi_1(M)\to \mathrm{GL}(\mathbf{q},\mathbb{C})$ so that if $\widetilde{M}$ is the universal cover of M, then $F=\widetilde{M}\times_\rho\mathbb{C}^{\mathbf{q}}$. The analytic torsion associated with a flat vector bundle on a smooth compact Riemannian manifold M is a classical spectral invariant defined by Ray and Singer [**56**] in 1971. It is a regularized determinant of the Hodge Laplacian for the de Rham complex associated with this flat vector bundle.

For $\Gamma\subset G$ a discrete cocompact torsion free subgroup of a connected reductive Lie group G, if $Z=\Gamma\backslash G/K$ is the locally symmetric space as in (3.11), then $\Gamma=\pi_1(Z)$.

By the superrigidity theorem of Margulis [**51**, Chap. VII, §5], if the real rank of G is ≥ 2, a general representation of Γ is not too far from a unitary representation of Γ or the restriction to Γ of a representation of $G_{\mathbb{C}}$, the complexification of G. See [**24**, Chap. XIII, 4.6] for more details.

Assume that the difference of the complex ranks of G and K is different from 1. For a flat vector bundle induced by a $G_{\mathbb{C}}$-representation, as an application of Theorem 0.1, we obtain a vanishing result of individual orbital integrals that appear in the supertrace of the heat kernel from which the analytic torsion can be obtained. In particular, this implies that the associated analytic torsion is equal to 1 (cf. Theorem 5.5).

We explain finally Shen's recent solution on Fried's conjecture for locally symmetric spaces: for any unitary representation of Γ such that the cohomology of the associated flat vector bundle on $Z = \Gamma\backslash G/K$ vanishes, the value at zero of a Ruelle dynamical zeta function identifies to the associated analytic torsion.

This section is organized as follows. In Section 5.1, we introduce the Ray-Singer analytic torsion. In Section 5.2, we study the analytic torsion on locally symmetric spaces for flat vector bundles induced by a representation of $G_{\mathbb{C}}$. Finally in Section 5.3, we describe Shen's solution of Fried's conjecture in the case of locally symmetric spaces. In Section 5.4, we make some remarks on related research directions.

5.1. Analytic torsion

Let M be a compact manifold of dimension m. Let (F, ∇) be a flat complex vector bundle on M with flat connection ∇ (i.e., its curvature $(\nabla)^2 = 0$). The flat connection ∇ induces an exterior differential operator d on $\Omega^\bullet(M, F)$, the vector space of differential forms on M with values in F, and $d^2 = 0$. Let $H^\bullet(M, F)$ be the cohomology group of the complex $(\Omega^\bullet(M, F), d)$ as in (2.8).

Let h^F be a Hermitian metric on F. Then as explained in Section 2.2, g^{TM} and h^F induce naturally a Hermitian product on $\Omega^\bullet(M, F)$. Let D be as in (2.10).

We introduce here a refined spectral invariant of D^2 which is particularly interesting.

Let P be the orthogonal projection from $\Omega^\bullet(M, F)$ onto $\mathrm{Ker}(D)$ and let $P^\perp = 1 - P$. Let N be the number operator acting on $\Omega^\bullet(M, F)$, i.e., multiplication by j on $\Omega^j(M, F)$. For $s \in \mathbb{C}$ and $\mathrm{Re}(s) > \frac{m}{2}$, set

$$\begin{aligned} \theta(s) &= -\sum_{j=0}^{m} (-1)^j j \,\mathrm{Tr}\,|_{\Omega^j(M,F)}[(D^2)^{-s} P^\perp] \\ &= -\frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}_s[N e^{-tD^2} P^\perp] t^s \frac{dt}{t}, \end{aligned} \tag{5.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

From the small time heat kernel expansion (cf. [3, Theorem 2.30]), we know that $\theta(s)$ is well-defined for $\mathrm{Re}(s) > \frac{m}{2}$ and extends holomorphically near $s = 0$.

Definition 5.1 ([56]). — *The (Ray-Singer) analytic torsion is defined as*

$$T(g^{TM}, h^F) = \exp\Big(\frac{1}{2}\frac{\partial\theta}{\partial s}(0)\Big). \tag{5.2}$$

We have the formal identity,

$$T(g^{TM}, h^F) = \prod_{j=0}^{m} \det(D^2|_{\Omega^j(M,F)})^{(-1)^j j/2}. \tag{5.3}$$

Remark 5.2. — a) If h^F is parallel with respect to ∇, then F is induced by a unitary representation of $\pi_1(M)$, and we say that (F, ∇, h^F) is a unitary flat vector bundle. In this case, if m is even and M is orientable, by a Poincaré duality argument, we have $T(g^{TM}, h^F) = 1$.

b) If m is odd, and $H^\bullet(M, F) = 0$, then $T(g^{TM}, h^F)$ does not depend on the choice of g^{TM}, h^F, thus it is a topological invariant (cf. [23, Theorem 0.1]).

5.2. Analytic torsion for locally symmetric spaces

We use the same notation and assumptions as in Section 3. Recall that $\rho^E : K \to \mathrm{U}(E)$ is a finite dimensional unitary representation of K, and $F = G\times_K E$ is the induced Hermitian vector bundle on the symmetric space $X = G/K$. Assume form now on that the complexification $G_{\mathbb{C}}$ of G exists, and the representation ρ^E is induced by a holomorphic representation of $G_{\mathbb{C}} \to \mathrm{Aut}(E)$, that is still denoted by ρ^E. We have the canonical identification of $G \times_K E$ as a trivial bundle E on X:

$$F = G \times_K E \to X \times E, \quad (g, v) \to \rho^E(g)v. \tag{5.4}$$

This induces a canonical flat connection ∇ on F such that

$$\nabla = \nabla^F + \rho^E \omega^{\mathfrak{p}}. \tag{5.5}$$

Remark 5.3. — Let U be a maximal compact subgroup of $G_{\mathbb{C}}$. Then U is the compact form of G and $\mathfrak{u} = i\mathfrak{p}\oplus\mathfrak{k}$ is its Lie algebra. By Weyl's unitary trick [43, Proposition 5.7], if U is simply connected, it is equivalent to consider representations of G, of U on E, or holomorphic representations of the complexification $G_{\mathbb{C}}$ of G on E, or representations of $\mathfrak{g}$, or $\mathfrak{u}$ on E.

We fix a U-invariant Hermitian metric on E. This implies in particular it is K-invariant, and $\rho^E(v) \in \mathrm{End}(E)$ is symmetric for $v \in \mathfrak{p}$. This induces a Hermitian metric h^F on F. As in Section 5.1, we consider now the operator D acting on $\Omega^\bullet(X, F)$ induced by g^{TX}, h^F.

Let $C^{\mathfrak{g},X}$ be the Casimir operator of G acting on $C^\infty(X, \Lambda^\bullet(T^*X)\otimes F)$ as in (3.8). Then by [**12**, (2.6.11)] and [**22**, Proposition 8.4], we have

$$D^2 = C^{\mathfrak{g},X} - C^{\mathfrak{g},E}. \tag{5.6}$$

Let T be a maximal torus in K and let $\mathfrak{t}\subset\mathfrak{k}$ be its Lie algebra. Set

$$\mathfrak{b} = \{v\in\mathfrak{p} : [v,\mathfrak{t}] = 0\}. \tag{5.7}$$

Put

$$\mathfrak{h} = \mathfrak{b}\oplus\mathfrak{t}. \tag{5.8}$$

By [**43**, p. 129], we know that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and that $\dim\mathfrak{t}$ is the complex rank of K and $\dim\mathfrak{h}$ is the complex rank of G. Also, m and $\dim\mathfrak{b}$ have the same parity.

For $\gamma = e^a k^{-1}\in G$ a semisimple element as in (3.33), let $K^0(\gamma)\subset K(\gamma)$ be the connected component of the identity. Let $T(\gamma)\subset K^0(\gamma)$ be a maximal torus in $K^0(\gamma)$, and let $\mathfrak{t}(\gamma)\subset\mathfrak{k}(\gamma)$ be its Lie algebra. By (3.34) and (3.38), k commutes with $T(\gamma)$, thus by [**43**, Theorem 4.21], there exists $k_1\in K$ such that $k_1T(\gamma)k_1^{-1}\subset T$, $k_1kk_1^{-1}\subset T$. By working on $k_1\gamma k_1^{-1} = e^{\mathrm{Ad}(k_1)a}(\mathrm{Ad}(k_1)k)^{-1}$ instead of γ, we may and we will assume that $T(\gamma)\subset T$, $k\in T$. In particular $\mathfrak{t}(\gamma)\subset\mathfrak{t}$. Set

$$\mathfrak{b}(\gamma) = \{v\in\mathfrak{p} : [v,\mathfrak{t}(\gamma)] = 0,\ \mathrm{Ad}(k)v = v\}. \tag{5.9}$$

Then

$$\mathfrak{b}\subset\mathfrak{b}(\gamma) \quad\text{and}\quad \mathfrak{b}(1) = \mathfrak{b}. \tag{5.10}$$

Recall that $N^{\Lambda^\bullet(T^*X)}$ is the number operator on $\Lambda^\bullet(T^*X)$.

THEOREM 5.4 ([**12**], Theorem 7.9.1, [**21**], [**22**], Theorem 8.6, Remark 8.7)

For any semisimple element $\gamma\in G$, if m is even, or if m is odd and $\dim\mathfrak{b}(\gamma)\geq 2$, then for any $t>0$, we have

$$\mathrm{Tr}_s^{[\gamma]}\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\right)e^{-\frac{t}{2}D^2}\right] = 0. \tag{5.11}$$

Proof. — By Theorem 3.10, (3.8) and (5.6), for any $t>0$ and any semisimple element $\gamma\in G$,

$$\mathrm{Tr}_s^{[\gamma]}\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\right)e^{-\frac{t}{2}D^2}\right] = \frac{e^{-|a|^2/2t}}{(2\pi t)^{p/2}}e^{\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]}$$
$$\int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^{\mathfrak{k}})\,\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}\left[\left(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\right)\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(k^{-1})e^{-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(Y_0^{\mathfrak{k}})+\frac{t}{2}C^{\mathfrak{g},E}}\right]$$
$$e^{-|Y_0^{\mathfrak{k}}|^2/2t}\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}. \tag{5.12}$$

But

$$\text{(5.13)}\quad \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}\left[\left(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\right)\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(k^{-1})e^{-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(Y_0^{\mathfrak{k}})+\frac{t}{2}C^{\mathfrak{g},E}}\right]$$
$$=\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\left[\left(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\right)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1})e^{-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}})}\right]\operatorname{Tr}^E\left[\rho^E(k^{-1})e^{-i\rho^E(Y_0^{\mathfrak{k}})+\frac{t}{2}C^{\mathfrak{g},E}}\right].$$

If u is an isometry of $\mathfrak{p}$, we have

$$\text{(5.14)}\quad \begin{aligned}\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}[u]&=\det\left(1-u^{-1}\right),\\ \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\left[N^{\Lambda^\bullet(\mathfrak{p}^*)}u\right]&=\frac{\partial}{\partial s}\det\left(1-u^{-1}e^s\right)(0).\end{aligned}$$

If the eigenspace associated with the eigenvalue 1 is of dimension ≥ 1, the fist quantity in (5.14) vanishes. If it is of dimension ≥ 2, the second expression in (5.14) also vanishes. Also, if m is even and u preserves the orientation, then

$$\text{(5.15)}\quad \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\left[\left(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\right)u\right]=0.$$

From (5.12), (5.14) and (5.15), we get (5.11). □

Now let Γ be a discrete torsion free cocompact subgroup of G. Set $Z=\Gamma\backslash X$. Then $\pi_1(Z)=\Gamma$ and the flat vector bundle F descents as a flat vector bundle F over Z.

THEOREM 5.5 ([**22**], Remark 8.7). — *For a flat vector bundle F on $Z=\Gamma\backslash X$ induced by a holomorphic representation of $G_{\mathbb{C}}$, if m is even, or if m is odd and $\dim\mathfrak{b}\geq 3$, then $T(g^{TZ},h^F)=1$.*

Proof. — Under the condition of Theorem 5.5, from Theorem 5.4, (3.64) and (5.10), we get

$$\text{(5.16)}\quad \operatorname{Tr}_s\left[\left(N^{\Lambda^\bullet(T^*Z)}-\frac{m}{2}\right)e^{-\frac{t}{2}D^{Z,2}}\right]=0.$$

Now Theorem 5.5 is a direct consequence of (2.14) for $h=1$, (5.1) and (5.16). □

Remark 5.6. — a) If F is trivial, i.e., it is induced by the trivial representation of G, then Theorem 5.4 under the condition of Theorem 5.5 was first obtained by Moscovici-Stanton [**54**, Theorem 2.1].

b) Assume G is semisimple, then the induced metric h^F on F is unimodular, i.e., the metric $h^{\det F}$ on $\det F:=\Lambda^{\max}F$ induced by h^F is parallel with respect to the flat connection on $\det F$. In this case, Theorem 5.4 for $\gamma=1$ was first obtained by Bergeron-Venkatesh [**2**, Theorem 5.2], and Müller and Pfaff [**55**] gave a new proof of Theorem 5.5.

c) We can drop the condition on torsion freeness of Γ in (5.16).

Remark 5.7. — For $p, q \in \mathbb{N}$, let $\mathrm{SO}^0(p,q)$ be the connected component of the identity in the real group $\mathrm{SO}(p,q)$. By [**39**, Table V p. 5.18] and [**43**, Table C1 p. 713, Table C2 p. 714], among the noncompact simple connected complex groups such that m is odd and $\dim \mathfrak{b} = 1$, there is only $\mathrm{SL}_2(\mathbb{C})$, and among the noncompact simple real connected groups, there are only $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{H})$, and $\mathrm{SO}^0(p,q)$ with pq odd > 1. Also, by [**39**, p. 519, 520], $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{so}(3,1)$, $\mathfrak{sl}_4(\mathbb{R}) = \mathfrak{so}(3,3)$, $\mathfrak{sl}_2(\mathbb{H}) = \mathfrak{so}(5,1)$. Therefore the above list can be reduced to $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SO}^0(p,q)$ with pq odd > 1.

Assume from now on that $\rho : \Gamma \to U(\mathbf{q})$ be a unitary representation. Then $F = X \times_\Gamma \mathbb{C}^{\mathfrak{q}}$ is a flat vector bundle on $Z = \Gamma \backslash X$ with metric h^F induced by the canonical metric on $\mathbb{C}^{\mathfrak{q}}$, i.e., F is a unitary flat vector bundle on Z with holonomy ρ. By Remark 5.2, if m is even, then $T(g^{TZ}, h^F) = 1$. Thus we can simply assume that m is odd.

Observe that the pull back of (F, h^F) over X is $\mathbb{C}^{\mathfrak{q}}$ with canonical metric, thus the heat kernel on X is given by

$$e^{-tD^2}(x,x') = e^{-tD_0^2}(x,x') \otimes \mathrm{Id}_{\mathbb{C}^{\mathfrak{q}}} \tag{5.17}$$

where $e^{-tD_0^2}(x,x') \in \Lambda^\bullet(T_x^*X) \otimes \Lambda^\bullet(T_{x'}^*X)^*$ is the heat kernel on X for the trivial representation $G \to \mathrm{Aut}(\mathbb{C})$. Thus,

$$\begin{aligned}\mathrm{Tr}_s\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\right)\gamma e^{-tD^2}(\gamma^{-1}\widetilde{z}, \widetilde{z})\right] \\ = \mathrm{Tr}[\rho(\gamma)]\, \mathrm{Tr}_s\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\right)\gamma e^{-tD_0^2}(\gamma^{-1}\widetilde{z}, \widetilde{z})\right].\end{aligned} \tag{5.18}$$

Note that for $\gamma \in \Gamma$, $\mathrm{Tr}[\rho(\gamma)]$ depends only on the conjugacy class of γ, thus form (3.12), (3.27) and (5.18), we get the analog of Theorem 3.5,

$$\begin{aligned}\mathrm{Tr}_s\left[\left(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\right) e^{-tD^{Z,2}}\right] \\ = \sum_{[\gamma]\in[\Gamma]} \mathrm{Vol}\left(\Gamma \cap Z(\gamma)\backslash X(\gamma)\right) \mathrm{Tr}[\rho(\gamma)]\, \mathrm{Tr}_s^{[\gamma]}\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\right) e^{-tD_0^2}\right].\end{aligned} \tag{5.19}$$

Since the metric h^F is given by the unitary representation ρ and g^{TZ} is induced by the bilinear form B on $\mathfrak{g}$, we denote the analytic torsion in Section 5.1 by $T(F)$.

By Theorem 5.4 for the trivial representation $G \to \mathrm{Aut}(\mathbb{C})$ and (5.19), we get a result similar to Theorem 5.5.

THEOREM 5.8 ([**54**], Corol. 2.2). — *For a unitary flat vector bundle F on $Z = \Gamma\backslash X$, if m is odd and* $\dim \mathfrak{b} \geq 3$, *then for $t > 0$, we have*

$$\mathrm{Tr}_s\left[\left(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\right) e^{-tD^{Z,2}/2}\right] = 0. \tag{5.20}$$

In particular, $T(F) = 1$.

5.3. Fried's conjecture for locally symmetric spaces

The possible relation of the topological torsion to the dynamical systems was first observed by Milnor [**53**] in 1968. A quantitative description of their relation was formulated by Fried [**30**] when Z is a closed oriented hyperbolic manifold. Namely, he showed that for an acyclic unitary flat vector bundle F, the value at zero of the Ruelle dynamical zeta function, constructed using the closed geodesics in Z and the holonomy of F, is equal to the associated analytic torsion. In [**31**, p.66], Fried's conjectured that a similar result still holds for general closed locally homogenous manifolds. In 1991, Moscovici-Stanton [**54**] made an important contribution to the proof of Fried's conjecture for locally symmetric spaces.

Let $\Gamma \subset G$ be a discrete cocompact torsion free subgroup of a connected reductive Lie group G. Then we get the symmetric space $X = G/K$ and the locally symmetric space $Z = \Gamma\backslash X$. By Remark 5.2, we may assume that $\dim Z = m$ is odd.

Recall that $[\Gamma]$ is the set of conjugacy classes of Γ. For $[\gamma] \in [\Gamma]\backslash\{1\}$, denote by $B_{[\gamma]}$ the space of closed geodesics in $[\gamma]$. As a subset of the loop space LZ, we equipped $B_{[\gamma]}$ the induced topology and smooth structure. By Proposition 3.9, $B_{[\gamma]} \simeq \Gamma \cap Z(\gamma)\backslash X(\gamma)$ is a compact locally symmetric space, and the elements of $B_{[\gamma]}$ have the same length $l_{[\gamma]} > 0$. The group $\mathbb{S}^1$ acts on $B_{[\gamma]}$ by rotation. This action is locally free. Denote by $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) \in \mathbb{Q}$ the orbifold Euler characteristic number for the quotient orbifold $\mathbb{S}^1\backslash B_{[\gamma]}$. Recall that if $e(\mathbb{S}^1\backslash B_{[\gamma]}, \nabla^{T(\mathbb{S}^1\backslash B_{[\gamma]})}) \in \Omega^\bullet(\mathbb{S}^1\backslash B_{[\gamma]}, o(T(\mathbb{S}^1\backslash B_{[\gamma]})))$ is the Euler form defined using Chern-Weil theory for the Levi-Civita connection $\nabla^{T(\mathbb{S}^1\backslash B_{[\gamma]})}$, then

$$\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) = \int_{\mathbb{S}^1\backslash B_{[\gamma]}} e(\mathbb{S}^1\backslash B_{[\gamma]}, \nabla^{T(\mathbb{S}^1\backslash B_{[\gamma]})}). \tag{5.21}$$

Let

$$n_{[\gamma]} = \left|\mathrm{Ker}\left(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]})\right)\right| \tag{5.22}$$

be the generic multiplicity of $B_{[\gamma]}$.

Definition 5.9. — *Given a representation $\rho : \Gamma \to U(\mathbf{q})$, we say that the dynamical zeta function $R_\rho(\sigma)$ is well-defined if the following properties hold:*

1. *For $\sigma \in \mathbb{C}$, $\mathrm{Re}(\sigma) \gg 1$, the sum*

$$\Xi_\rho(\sigma) = \sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{n_{[\gamma]}} e^{-\sigma l_{[\gamma]}} \tag{5.23}$$

defines to a holomorphic function.

2. *The function $R_\rho(\sigma) = \exp(\Xi_\rho(\sigma))$ has a meromorphic extension to $\sigma \in \mathbb{C}$.*

Note that $\gamma \in \Gamma$ is primitive means if $\gamma = \beta^k$, $\beta \in \Gamma$, $k \in \mathbb{N}^*$, then $\gamma = \beta$.

Remark 5.10. — If Z is a compact oriented hyperbolic manifold, then $\mathbb{S}^1\backslash B_{[\gamma]}$ is a point. Moreover, if ρ is the trivial representation, then

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \frac{1}{n_{[\gamma]}} e^{-\sigma l_{[\gamma]}}\right) = \prod_{[\gamma]\ \text{primitive},\gamma\neq 1} \left(1 - e^{-\sigma l_{[\gamma]}}\right)^{-1}. \tag{5.24}$$

THEOREM 5.11 ([**62**]). — *For any unitary flat vector bundle F on Z with holonomy ρ, the dynamical zeta function $R_\rho(\sigma)$ is a well-defined meromorphic function on $\mathbb{C}$ which is holomorphic for* $\mathrm{Re}(\sigma) \gg 1$*. Moreover, there exist explicit constants $C_\rho \in \mathbb{R}$ and $r_\rho \in \mathbb{Z}$ such that, when $\sigma \to 0$, we have*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathscr{O}(\sigma^{r_\rho+1}). \tag{5.25}$$

If $H^\bullet(Z,F) = 0$, then

$$C_\rho = 1, \qquad r_\rho = 0, \tag{5.26}$$

so that

$$R_\rho(0) = T(F)^2. \tag{5.27}$$

Proof of Theorem 5.11. — The most difficult part of the proof is to express the $R_\rho(\sigma)$ as a product of determinant of shifted Casimir operators, [(2)] in fact being the analytic torsion. Shen's idea is to interpret the right-hand side of (5.23) as the Selberg trace formula (by eliminating the term $\mathrm{Tr}^{[1]}$) of the heat kernel for some representations of K by using Theorems 3.5 and 3.10.

By Theorem 5.8, we can concentrate on the proof in the case $\dim \mathfrak{b} = 1$. From now on, we assume $\dim \mathfrak{b} = 1$.

Up to conjugation, there exists a unique standard parabolic subgroup $Q \subset G$ with Langlands decomposition $Q = M_Q A_Q N_Q$ such that $\dim A_Q = 1$. Let $\mathfrak{m}, \mathfrak{b}, \mathfrak{n}$ be the respective Lie algebras of M_Q, A_Q, N_Q. Let M be the connected component of identity of M_Q. Then M is a connected reductive group with maximal compact subgroup $K_M = M \cap K$ and with Cartan decomposition $\mathfrak{m} = \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{k}_\mathfrak{m}$, and K_M acts on $\mathfrak{p}_\mathfrak{m}$, M acts on $\mathfrak{n}$ and M acts trivially on $\mathfrak{b}$. We have an identity of real K_M-representations

$$\mathfrak{p} \simeq \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{b} \oplus \mathfrak{n}, \tag{5.28}$$

and $\dim \mathfrak{n}$ is even. Moreover there exists $\nu \in \mathfrak{b}^*$ such that (cf. [**62**, Proposition 6.2])

$$[a,f] = \langle \nu, a\rangle f \quad \text{for any } f \in \mathfrak{n}, a \in \mathfrak{b}. \tag{5.29}$$

For $\gamma \in G$ semisimple, Shen [**62**, Proposition 4.11] observes that

$$\gamma \text{ can be conjugated into } H := \exp(\mathfrak{b})T \ (cf. (5.8)) \text{ iff } \dim \mathfrak{b}(\gamma) = 1. \tag{5.30}$$

(2) The gap in Moscovici-Stanton's paper comes from using an operator $\Delta_\phi^{j,l}$ [**54**, p. 206] which could not be defined on Z.

Let $R(K,\mathbb{R}), R(K_M,\mathbb{R})$ be the real representation rings of K and K_M. We can prove that the restriction $R(K,\mathbb{R}) \to R(K_M,\mathbb{R})$ is injective. The key result [**62**, Theorem 6.11] is that the K_M-representation on $\mathfrak{n}$ has a unique lift in $R(K,\mathbb{R})$. As $\mathfrak{p}$ is a K-representation, K_M-action on $\mathfrak{p}_\mathfrak{m}$ also lifts to K by lifting $\mathfrak{b}$ as a trivial K-representation in (5.28).

For a real finite dimensional representation ς of M on the vector space E_ς, such that $\varsigma|_{K_M}$ can be lifted into $R(K,\mathbb{R})$, this implies that there exists a real finite dimensional $\mathbb{Z}_2$-representation $\widehat{\varsigma} = \widehat{\varsigma}^+ - \widehat{\varsigma}^-$ of K on $E_{\widehat{\varsigma}} = E^+_{\widehat{\varsigma}} - E^-_{\widehat{\varsigma}}$ such that we have the equality in $R(K_M,\mathbb{R})$,

$$E_{\widehat{\varsigma}}|_{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_\mathfrak{m}} (-1)^i \Lambda^i(\mathfrak{p}^*_\mathfrak{m}) \otimes (E_\varsigma|_{K_M}). \tag{5.31}$$

Let $\mathscr{E}_{\widehat{\varsigma}} = G \times_K E_{\widehat{\varsigma}}$ be the induced $\mathbb{Z}_2$-graded vector bundle on X, and $\mathscr{F}_{\widehat{\varsigma}} = \Gamma\backslash\mathscr{E}_{\widehat{\varsigma}}$. Let $C^{\mathfrak{g},Z,\widehat{\varsigma},\rho}$ be the Casimir element of G acting on $C^\infty(Z, \mathscr{F}_{\widehat{\varsigma}} \otimes F)$. Modulo some technical conditions, with the help of Theorem 5.4 and (5.30), Shen [**62**, Theorems 5.3, 7.3] obtains the identity

$$\begin{aligned}\operatorname{Tr}_s\left[e^{-tC^{\mathfrak{g},Z,\widehat{\varsigma},\rho}/2}\right] &= \mathbf{q}\operatorname{Vol}(Z)\operatorname{Tr}^{[1]}\left[e^{-tC^{\mathfrak{g},X,\widehat{\varsigma}}/2}\right]\\ &+ \frac{1}{\sqrt{2\pi t}}e^{-c_\varsigma t}\sum_{[\gamma]\in[\Gamma]\backslash\{1\}}\operatorname{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{n_{[\gamma]}}\frac{\operatorname{Tr}^{E_\varsigma}[\varsigma(k^{-1})]}{\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{1/2}}|a|e^{-|a|^2/2t},\end{aligned} \tag{5.32}$$

where c_ς is some explicit constant, and k^{-1} is defined in (3.33). We do not write here the exact formula for $\operatorname{Tr}^{[1]}[e^{-tC^{\mathfrak{g},X,\widehat{\varsigma}}/2}]$.

By (5.32), if we set

$$\Xi_{\varsigma,\rho}(\sigma) = \sum_{[\gamma]\in[\Gamma]\backslash\{1\}}\operatorname{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{n_{[\gamma]}}\frac{\operatorname{Tr}^{E_\varsigma}[\varsigma(k^{-1})]}{\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{1/2}}e^{-|a|\sigma}, \tag{5.33}$$

we need to eliminate the denominator $\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{1/2}$ to relate $\Xi_{\varsigma,\rho}$ to Ξ_ρ, the logarithm of the dynamical zeta function $R_\rho(\sigma)$.

Set $2l = \dim \mathfrak{n}$. The following observation is crucial.

Proposition 5.12. — [**62**, Proposition 6.5] *For $\gamma = e^a k^{-1} \in H := \exp(\mathfrak{b})T$, $a \neq 0$, with $\nu \in \mathfrak{b}^*$ in (5.29), we have*

$$\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{1/2} = \sum_{j=0}^{2l}(-1)^j\operatorname{Tr}^{\Lambda^j(\mathfrak{n}^*)}\left[\operatorname{Ad}\left(k^{-1}\right)\right]e^{(l-j)|\nu||a|}, \tag{5.34}$$

Let ς_j be the representation of M on $\Lambda^j(\mathfrak{n}^*)$. By (5.33) and (5.34), we have

$$\Xi_\rho(\sigma) = \sum_{j=0}^{2l} (-1)^j \Xi_{\varsigma_j,\rho}(\sigma + (j-l)|\nu|). \tag{5.35}$$

On the other hand, since $\dim \mathfrak{b} = 1$, from (5.28), we have the following identity in $R(K_M, \mathbb{R})$,

$$\sum_{i=0}^{\dim \mathfrak{p}} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{\dim \mathfrak{n}} (-1)^j \Big(\sum_{i=0}^{\dim \mathfrak{p}_\mathfrak{m}} (-1)^i \Lambda^i(\mathfrak{p}^*_\mathfrak{m}) \Big) \otimes \Lambda^j(\mathfrak{n}^*). \tag{5.36}$$

Note that $\sum_{i=0}^{\dim \mathfrak{p}} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) \in R(K, \mathbb{R})$ is used to define the analytic torsion. From (5.35) and (5.36), Shen obtains a very interesting expression for $R_\rho(\sigma)$ in term of determinants of shifted Casimir operators, from which he could obtain Equation (5.25).

For a representation ς of M in (5.31), set

$$r_{\varsigma,\rho} = \dim_\mathbb{C} \operatorname{Ker}(C^{\mathfrak{g}, Z, \widehat{\varsigma^+}, \rho}) - \dim_\mathbb{C} \operatorname{Ker}(C^{\mathfrak{g}, Z, \widehat{\varsigma^-}, \rho}). \tag{5.37}$$

Then Shen [**62**, (5.12), (7.75)] obtains the formula

$$C_\rho = \prod_{j=0}^{l-1} \big(-4|l-j|^2|\nu|^2\big)^{(-1)^{j-1} r_{\varsigma_j,\rho}}, \qquad r_\rho = 2\sum_{j=0}^{l} (-1)^{j-1} r_{\varsigma_j,\rho}. \tag{5.38}$$

Shen shows that if $H^\bullet(Z,F) = 0$, then $r_{\varsigma_j,\rho} = 0$ for any $0 \le j \le l$ by using the spectral aspect of the Selberg trace formula (Theorem 0.1 and (0.5)), and some deep results on the representation theory of reductive groups.

For a G-representation $\pi : G \to \operatorname{Aut}(V)$ and $v \in V$, recall that v is said to be differentiable if $c_v : G \to V$, $c_v(g) = \pi(g)v$ is C^∞, that v is said to be K-finite if it is contained in a finite dimensional subspace stable under K. We denote by $V_{(K)}$ the subspace of differentiable and K-finite vectors in V. Let $H^\bullet(\mathfrak{g}, K; V_{(K)})$ be the $(\mathfrak{g}, K)$-cohomology of $V_{(K)}$.

We denote by $\widehat{G}_u$ the unitary dual of G, that is the set of equivalence classes of complex irreducible unitary representation π of G on the Hilbert space V_π. Let χ_π be the corresponding infinitesimal character.

Let $\widehat{p} : \Gamma\backslash G \to Z$ be the natural projection. The group G acts unitarily on the right on $L_2(\Gamma\backslash G, \widehat{p}^* F)$, then $L_2(\Gamma\backslash G, \widehat{p}^* F)$ decomposes into a discrete Hilbert direct sum with finite multiplicities of irreducible unitary representations of G,

$$L_2(\Gamma\backslash G, \widehat{p}^* F) = \widetilde{\bigoplus_{\pi \in \widehat{G}_u}} n_\rho(\pi) V_\pi \quad \text{with} \quad n_\rho(\pi) < +\infty, \tag{5.39}$$

here $\widetilde{\ }$ means the Hilbert completion.

Set

$$W = \widetilde{\bigoplus_{\pi \in \widehat{G}_u, \chi_\pi \text{ is trivial}}} n_\rho(\pi) V_\pi, \tag{5.40}$$

then W is the closure in $L_2(\Gamma\backslash G, \widehat{p}^* F)$ of W^∞, the subspace of $C^\infty(\Gamma\backslash G, \widehat{p}^* F)$ on which the center of $U(\mathfrak{g})$ acts by the same scalar as in the trivial representation of $\mathfrak{g}$. By standard arguments [**24**, Chap. VII, Theorem 3.2, Corollary 3.4], the cohomology $H^\bullet(Z, F)$ is canonically isomorphic to the $(\mathfrak{g}, K)$-cohomology $H^\bullet(\mathfrak{g}, K; W_{(K)})$ of $W_{(K)}$, the vector space of differentiable and K-finite vectors of W, i.e.,

$$H^\bullet(Z, F) = \bigoplus_{\pi \in \widehat{G}_u, \chi_\pi \text{ is trivial}} n_\rho(\pi)\, H^\bullet(\mathfrak{g}, K; V_{\pi,(K)}). \tag{5.41}$$

Vogan-Zuckerman [**66**, Theorem 1.4] and Vogan [**65**, Theorem 1.3] classified all irreducible unitary representations of G with nonzero $(\mathfrak{g}, K)$-cohomology. On the other hand, in [**57**, Theorem 1.8], Salamanca-Riba showed that any irreducible unitary representation of G with trivial infinitesimal character is in the class specified by Vogan and Zuckerman, which means that it possesses nonzero $(\mathfrak{g}, K)$-cohomology. In summary, if $(\pi, V_\pi) \in \widehat{G}_u$, then

$$\chi_\pi \text{ is non-trivial if and only if } H^\bullet(\mathfrak{g}, K; V_{\pi,(K)}) = 0. \tag{5.42}$$

By the above considerations, $H^\bullet(Z, F) = 0$ is equivalent to $W = 0$. This is the main algebraic ingredient in the proof of (5.26).

Shen's contribution [**62**, Corollary 8.15] is to give a formula for $r_{\varsigma_j,\rho}$ using Hecht-Schmid's work [**37**, Theorem 3.6, Corollary 3.32] on the $\mathfrak{n}$-homology of W, Theorem 3.10 and (3.64). From this formula, we see immediately that $W = 0$ implies $r_{\varsigma_j,\rho} = 0$ for $0 \leq j \leq l$. □

5.4. Final remarks

1. Theorem 3.10 gives an explicit formula for the orbital integrals for the heat kernel of the Casimir operator and it holds for any semisimple element $\gamma \in G$. A natural question is how to evaluate or define the weighted orbital integrals that appear in Selberg trace formula for a discrete subgroup $\Gamma \subset G$ such that $\Gamma\backslash G$ has a finite volume.

2. Bismut-Goette [**19**] introduced a local topological invariant for compact manifolds with a compact Lie group action: the V-invariant. It appears as an exotic term in the difference between two natural versions of equivariant analytic torsion. The V-invariant shares formally many similarities with the analytic torsion, and if we apply formally the construction of the V-invariant to the associated loop space equipped with its natural $\mathbb{S}^1$ action, then we get the analytic torsion.

In Shen's proof of Fried's conjecture for locally symmetric spaces, Shen observed that the V-invariant for the $\mathbb{S}^1$-action on $B_{[\gamma]}$ is exactly

$$-\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{2n_{[\gamma]}}. \tag{5.43}$$

This suggests a general definition of the Ruelle dynamical zeta function for any compact manifold by replacing $-\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{2n_{[\gamma]}}$ in Definition 5.9 by the associated V-invariant. One could then compare it with the analytic torsion, and obtain a generalized version of Fried's conjecture for any manifold with non positive curvature. Note that for a strictly negative curvature manifold, $B_{[\gamma]}$ is a circle and the V-invariant is $-\frac{1}{2n_{[\gamma]}}$. Recently, Giulietti-Liverani-Pollicott **[32]**, Dyatlov-Zworski **[26]**, Faure-Tsujii **[28]** established that for the trivial representation of $\pi_1(Z)$, and when Z has strictly negative curvature, the Ruelle dynamical zeta function is a well-defined meromorphic function on $\mathbb{C}$.

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Xiaonan MA

Université Paris-Diderot Paris 7
Institut de Mathématiques de Jussieu–Paris Rive Gauche
UMR CNRS 7586
Bâtiment Sophie Germain
Case 7012
F-75205 Paris Cedex 13
E-mail : xiaonan.ma@imj-prg.fr