

ON CLUSTER ALGEBRAS WITH COEFFICIENTS AND 2-CALABI-YAU CATEGORIES

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ABSTRACT. Building on work by Geiss-Leclerc-Schröer and by Buan-Iyama-Reiten-Scott we investigate the link between cluster algebras with coefficients and suitable 2-Calabi-Yau categories. These include the cluster-categories associated with acyclic quivers and certain Frobenius subcategories of module categories over preprojective algebras. Our motivation comes from the conjectures formulated by Fomin and Zelevinsky in ‘Cluster algebras IV: Coefficients’. We provide new evidence for Conjectures 5.4, 6.10, 7.2, 7.10 and 7.12 and show by an example that the statement of Conjecture 7.17 does not always hold.

1. INTRODUCTION

In this article, we pursue the representation-theoretic approach to Fomin-Zelevinsky’s cluster algebras [18] [19] [2] [20] developed by Marsh-Reineke-Zelevinsky [28], Buan-Marsh-Reineke-Reiten-Todorov [5] [8], Caldero-Chapoton and Caldero-Keller [11] [9] [10], Geiss-Leclerc-Schröer [21] [22] [23], Buan-Iyama-Reiten-Scott [3] and many others, *cf.* the surveys [4] [32] [31].

Our emphasis here is on cluster algebras with coefficients. Coefficients are of great importance both in geometric examples of cluster algebras [24] [25] [2] [33] [23] and in the study of duality phenomena [17] as shown in [20]. Following [3], we consider two types of categories which allow us to incorporate coefficients into the representation-theoretic model:

- 1) 2-Calabi-Yau Frobenius categories;
- 2) 2-Calabi-Yau ‘subtriangulated’ categories, *i.e.* full subcategories of the form $(\Sigma\mathcal{D})^\perp$ of a 2-Calabi-Yau triangulated category \mathcal{C} , where \mathcal{D} is a rigid functorially finite subcategory of \mathcal{C} and Σ the suspension functor of \mathcal{C} .

In both cases, we establish the link between the category and its associated cluster algebra using (variants of) cluster characters in the sense of Palu [30]. For subtriangulated categories, we use the restriction of the cluster characters constructed in [30]. For Frobenius categories, we construct a suitable variant in section 2 (Theorem 2.2).

The work of Geiss-Leclerc-Schröer [22] [23] and Buan-Iyama-Reiten-Scott [3] provides us with large classes of 2-Calabi-Yau Frobenius categories and of 2-Calabi-Yau subtriangulated categories which admit cluster structures in the sense of [3]. Our general results imply that for these classes, the 2-Calabi-Yau categories do yield ‘categorifications’ of the corresponding cluster algebras with coefficients (Theorems 4.3 and 5.3). As an application, we show that Conjectures 7.2, 7.10 and 7.12 of [20] hold for these cluster algebras (Proposition 4.4 and Theorem 5.3). Let us recall the statements of these conjectures:

7.2 cluster monomials are linearly independent;

Date: October 16, 2007. Last modified on October 16, 2007.

1991 Mathematics Subject Classification. 18E30, 16D90, 18G10.

Key words and phrases. Cluster algebra, Tilting, 2-Calabi-Yau category.

- 7.10 different cluster monomials have different \mathbf{g} -vectors and the \mathbf{g} -vectors of the cluster variables in any given cluster form a basis of the ambient lattice;
- 7.12 the \mathbf{g} -vectors of a cluster variable with respect to two neighbouring clusters are related by a certain piecewise linear transformation (so that the \mathbf{g} -vectors equal the \mathbf{g}^\dagger -vectors of [13]).

In the case of cluster algebras with principal coefficients admitting a categorification by a 2-Calabi-Yau subtriangulated category, we obtain a representation-theoretic interpretation of the F -polynomial defined in section 3 of [20], cf. Theorem 5.5. This interpretation implies in particular that Conjecture 5.4 of [20] holds in this case: The constant coefficient of the F -polynomial equals 1. We also deduce that the multidegree of the F -polynomial associated with a rigid indecomposable equals the dimension vector of the corresponding module (Proposition 5.6). By combining this with recent work by Buan-Marsh-Reiten [7], cf. also [16], we obtain a counterexample to Conjecture 7.17 (and 6.11) of [20]. We point out that the corresponding computations were already present in G. Cerulli's work [12]. Following a suggestion by A. Zelevinsky, we show that, by assuming the existence of suitable categorifications, instead of the equality claimed in Conjecture 7.17, one does have an inequality: The multidegree of the F -polynomial is greater or equal to the denominator vector (Proposition 5.8). We also show in certain cases that the transformation rule for \mathbf{g} -vectors predicted by Conjecture 6.10 of [20] does hold (Proposition 5.9).

Let us emphasize that our proofs of some of the conjectures of [20] depend on the existence of suitable Hom-finite 2-Calabi-Yau categories with a cluster-tilting object. This hypothesis imposes a finiteness condition on the corresponding cluster algebra (to the best of our knowledge, it is not known how to express this condition in combinatorial terms). The construction of such 2-Calabi-Yau categories is a non trivial problem for which we rely on [5] in the acyclic case and on [22] [23] [3] and [1] in the non acyclic case. As A. Zelevinsky has kindly informed us, many of the conjectures of [20] will be proved in [15] in full generality building on [28] and [14].

Acknowledgments. We thank A. Zelevinsky for stimulating discussions and for informing us about the ongoing work in [15] and [12]. We are grateful to A. Buan and I. Reiten for sharing the results on dimension vectors obtained in [7]. We thank J. Schröer and O. Iyama for their interest and for motivating discussions. The first-named author gratefully acknowledges a 5-month fellowship of the Liegrits network (MRTN-CT 2003-505078) during which this research was carried out.

2. CLUSTER CHARACTERS FOR 2-CALABI-YAU FROBENIUS CATEGORIES

Let k be an algebraically closed field and \mathcal{E} a k -linear Frobenius category with split idempotents. We assume that

- \mathcal{E} is Hom-finite,
- the stable category $\mathcal{C} = \underline{\mathcal{E}}$ is 2-Calabi-Yau,
- \mathcal{E} contains a cluster-tilting object T .

Let B be the endomorphism algebra of T (in \mathcal{E}) and $\underline{B} = \text{End}_{\mathcal{C}}(T)$. Let

$$F = \text{Hom}_{\mathcal{E}}(T, ?) : \mathcal{E} \rightarrow \text{mod } B,$$

$$G = \text{Hom}_{\mathcal{C}}(T, ?) : \mathcal{E} \rightarrow \text{mod } \underline{B}.$$

Let T_i , $1 \leq i \leq n$, be the pairwise non isomorphic indecomposable direct summands of T . Assume that T_i is projective exactly for $r < i \leq n$. For $1 \leq i \leq n$, let S_i be the top of the indecomposable projective $P_i = FT_i$. Note that B and \underline{B} are finite dimensional k -algebras, so finitely presented modules are the same as finitely generated modules. As in section 4 of [26], we identify $\text{Mod } \underline{B}$ with the full subcategory of $\text{Mod } B$ formed by the modules without

S_i ($r < i \leq n$) as composition factors. Let $\text{per } B$ be the perfect derived category of B and $\mathcal{D}^b(\text{mod } B)$ the bounded derived category of $\text{mod } B$. We have the following embeddings

$$\text{mod } \underline{B} \hookrightarrow \text{per } B \hookrightarrow \mathcal{D}^b(\text{mod } B).$$

We have a bilinear form

$$\langle \cdot, \cdot \rangle : K_0(\text{per } B) \times K_0(\mathcal{D}^b(\text{mod } B)) \longrightarrow \mathbb{Z}$$

defined by

$$\langle [P], [X] \rangle = \sum (-1)^i \dim \text{Hom}_{\mathcal{D}^b(\text{mod } B)}(P, \Sigma^i X),$$

where $K_0(\text{per } B)$ (*resp.* $K_0(\mathcal{D}^b(\text{mod } B))$) is the Grothendieck group of $\text{per } B$ (*resp.* $\mathcal{D}^b(\text{mod } B)$) and Σ is the shift functor of $\mathcal{D}^b(\text{mod } B)$.

For arbitrary B -modules L and N , put

$$[L, N] = {}^0[L, N] = \dim_k \text{Hom}_B(L, N) \quad \text{and} \quad {}^i[L, N] = \dim_k \text{Ext}_B^i(L, N) \quad \text{for } i \geq 1.$$

Let

$$\langle L, N \rangle_\tau = [L, N] - {}^1[L, N] \quad \text{and} \quad \langle L, N \rangle_3 = \sum_{i=0}^3 (-1)^i {}^i[L, N]$$

be the truncated Euler forms on the split Grothendieck group $K_0^{sp}(\text{mod } B)$. By the proposition below, if L is a \underline{B} -module, then $\langle L, N \rangle_3$ only depends on the dimension vector $\underline{\dim} L$ in $K_0(\text{mod } B)$. We put

$$\langle \underline{\dim} L, N \rangle_3 = \langle L, N \rangle_3.$$

For $M \in \mathcal{E}$, define the Laurent polynomial

$$X'_M = \prod_{i=1}^n x_i^{\langle FM, S_i \rangle_\tau} \sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{i=1}^n x_i^{-\langle e, S_i \rangle_3}.$$

Proposition 2.1. a) *The restriction of the canonical map*

$$K_0(\text{per } B) \longrightarrow K_0(\text{mod } B)$$

to the subgroup generated by the $[S_i]$, $1 \leq i \leq r$, is injective.

b) *If L, N are two \underline{B} -modules such that $\underline{\dim} L = \underline{\dim} N$ in $K_0(\text{mod } B)$, then*

$$\langle L, Y \rangle_3 = \langle N, Y \rangle_3$$

for each finitely generated B -module Y .

Proof. a) We need to show that for arbitrary \underline{B} -modules L, N with $\underline{\dim} L = \underline{\dim} N$, we have $[L] = [N]$ in $K_0(\text{per } B)$. Let

$$0 = L_s \subset L_{s-1} \subset \cdots \subset L_0 = L$$

and

$$0 = N_s \subset N_{s-1} \subset \cdots \subset N_0 = N$$

be composition series of L and N respectively. By [26], we know that every \underline{B} -module has projective dimension at most 3 in $\text{mod } B$. Assume for simplicity that $L_{s-1} = S_1$,

$L_{s-2}/L_{s-1} = S_2$. Denote by P_i^* a minimal projective resolution of S_i . Then we have the following commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & P_1^3 & \longrightarrow & P_1^2 & \longrightarrow & P_1^1 & \longrightarrow & P_1^0 & \longrightarrow & L_{s-1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1^3 \oplus P_2^3 & \longrightarrow & P_1^2 \oplus P_2^2 & \longrightarrow & P_1^1 \oplus P_2^1 & \longrightarrow & P_1^0 \oplus P_2^0 & \longrightarrow & L_{s-2} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_2^3 & \longrightarrow & P_2^2 & \longrightarrow & P_2^1 & \longrightarrow & P_2^0 & \longrightarrow & L_{s-2}/L_{s-1} & \longrightarrow & 0
\end{array}$$

where the middle term is a projective resolution of L_{s-2} . In this way, we inductively construct projective resolutions for L and N . If m_i is the multiplicity of S_i in the composition factors of L and N , then we obtain projective resolutions of L and N of the form

$$\begin{aligned}
0 &\rightarrow \bigoplus_{i=1}^r (P_i^3)^{m_i} \xrightarrow{f_3} \bigoplus_{i=1}^r (P_i^2)^{m_i} \xrightarrow{f_2} \bigoplus_{i=1}^r (P_i^1)^{m_i} \xrightarrow{f_1} \bigoplus_{i=1}^r (P_i^0)^{m_i} \rightarrow L \rightarrow 0, \\
0 &\rightarrow \bigoplus_{i=1}^r (P_i^3)^{m_i} \xrightarrow{g_3} \bigoplus_{i=1}^r (P_i^2)^{m_i} \xrightarrow{g_2} \bigoplus_{i=1}^r (P_i^1)^{m_i} \xrightarrow{g_1} \bigoplus_{i=1}^r (P_i^0)^{m_i} \rightarrow N \rightarrow 0.
\end{aligned}$$

Let P^L (*resp.* P^N) be the projective resolution complex of L (*resp.* N). We have $L \cong P^L$ and $N \cong P^N$ in $\text{per } B$, which implies $[L] = [P^L] = [P^N] = [N]$ in $K_0(\text{per } B)$.

b) We have

$$\begin{aligned}
\langle L, Y \rangle_3 &= \langle P^L, Y \rangle = \langle [P^L], [Y] \rangle, \\
\langle N, Y \rangle_3 &= \langle P^N, Y \rangle = \langle [P^N], [Y] \rangle.
\end{aligned}$$

By a), we have $[P^L] = [P^N]$ in $K_0(\text{per } B)$, which implies the equality. \square

One should note that the truncated Euler form $\langle \cdot, \cdot \rangle_3$ does not descend to the Grothendieck group $K_0(\text{mod } B)$ in general (except if the global dimension of B is not greater than 3).

Recall the definitions of index and coindex in [29] for a 2-Calabi-Yau triangulated category \mathcal{T} admitting a cluster tilting object R . For any $X \in \mathcal{T}$, we have the minimal triangles (unique up to isomorphism)

$$\begin{aligned}
R_X^1 &\rightarrow R_X^0 \rightarrow X \rightarrow \Sigma R_X^1, \\
X &\rightarrow \Sigma^2 R_0^X \rightarrow \Sigma^2 R_1^X \rightarrow \Sigma X,
\end{aligned}$$

where $R_X^0, R_X^1, R_0^X, R_1^X$ in $\text{add } R$. The index and coindex of X with respect to R are defined to be the classes in $K_0(\text{add } R)$

$$\begin{aligned}
\mathbf{ind}_R(X) &= [R_X^0] - [R_X^1], \\
\mathbf{coind}_R(X) &= [R_0^X] - [R_1^X].
\end{aligned}$$

Since \mathcal{C} is 2-Calabi-Yau, $\underline{T} = \bigoplus_{i=1}^r T_i$ is a cluster tilting object of \mathcal{C} . Thus, we have the ‘generalized Caldero-Chapoton map’ on \mathcal{C} , which was introduced in [9] and studied in [29],

$$X_M = \prod_{i=1}^r x_i^{-[\mathbf{coind}_{\underline{T}}(M):T_i]} \sum_e \chi(\text{Gr}_e(GM)) \prod_{i=1}^r x_i^{\langle S_i, e \rangle_a},$$

where $\langle X, Y \rangle_a = \langle X, Y \rangle_\tau - \langle Y, X \rangle_\tau$ is the antisymmetric form. As shown in [29], this antisymmetric form descends to the Grothendieck group $K_0(\text{mod } \underline{B})$.

Theorem 2.2. a) We have $X_{T_i}^1 = x_i$ for $1 \leq i \leq n$.

- b) The specialization of X'_M at $x_{r+1} = x_{r+2} = \dots = 1$ is $X_{\Sigma M}$, where Σ is the suspension of \mathcal{C} .
- c) If L and M are objects of \mathcal{E} such that $\text{Ext}_{\mathcal{E}}^1(L, M)$ is one-dimensional and we have non split conflations

$$0 \rightarrow L \rightarrow C \rightarrow M \rightarrow 0 \text{ and } 0 \rightarrow M \rightarrow C' \rightarrow L \rightarrow 0,$$

then we have

$$X'_L X'_M = X'_C + X'_{C'}.$$

Proof. a) is straightforward.

b) We have

$$X_{\Sigma M} = \prod_{i=1}^r x_i^{-[\text{coind}_{\underline{T}}(\Sigma M):T_i]} \sum_e \chi(\text{Gr}_e(G\Sigma M)) \prod_{i=1}^r x_i^{\langle S_i, e \rangle_a}.$$

Now by the definition, we have

$$G\Sigma M = \text{Hom}_{\mathcal{C}}(T, \Sigma M) = \text{Ext}_{\mathcal{E}}(T, M).$$

Therefore, we only need to show that the exponents of x_i , $1 \leq i \leq r$, in the corresponding terms of $X_{\Sigma M}$ and X'_M are equal. There exists a triangle in \mathcal{C}

$$T_M^1 \rightarrow T_M^0 \rightarrow M \rightarrow \Sigma T_1$$

with T_M^0 and T_M^1 in $\text{add } \underline{T}$. We may and will assume that this triangle is minimal, *i.e.* does not admit a non zero direct factor of the form

$$T' \rightarrow T' \rightarrow 0 \rightarrow \Sigma T'.$$

Since \mathcal{E} is Frobenius, we can lift this triangle to a short exact sequence in \mathcal{E}

$$0 \rightarrow T_M^1 \rightarrow T_M^0 \oplus P \rightarrow M \rightarrow 0,$$

where P is a projective of \mathcal{E} . Applying the functor F to this short exact sequence, we get a projective resolution of FM as a B -module,

$$0 \rightarrow FT_M^1 \rightarrow F(T_M^0 \oplus P) \rightarrow FM \rightarrow 0.$$

Therefore, we have

$$\langle FM, S_i \rangle_{\tau} = [FT_M^0 \oplus FP, S_i] - [FT_M^1, S_i] = [FT_M^0, S_i] - [FT_M^1, S_i]$$

for $1 \leq i \leq r$.

On the other side, we have the following minimal triangle

$$\Sigma M \rightarrow \Sigma^2 T_M^1 \rightarrow \Sigma^2 T_M^0 \rightarrow \Sigma^2 M.$$

By the definition of the coindex, we get

$$-[\text{coind}_{\underline{T}}(\Sigma M) : T_i] = -[T_M^1 - T_M^0 : T_i] = \langle FM, S_i \rangle_{\tau}, \text{ for } 1 \leq i \leq r.$$

Next we will show that $\langle S_i, e \rangle_a = -\langle e, S_i \rangle_3$. Let N be a B -module such that $\underline{\dim} N = e$. Note that N and S_i , $1 \leq i \leq r$, are \underline{B} -modules and all of them are finitely presented B -modules. Therefore, they lie in the perfect derived category $\text{per}(B)$. Thus, we can use the relative 3-Calabi-Yau property of $\text{per}(M)$ (*cf.* [26]) to deduce that $\langle S_i, e \rangle_a = -\langle e, S_i \rangle_3$. We have

$$\begin{aligned} \text{Ext}_B^2(N, S_i) &= \text{Ext}_B(S_i, N) = \text{Ext}_{\underline{B}}(S_i, N), \\ \text{Ext}_B^3(N, S_i) &= \text{Hom}_B(S_i, N) = \text{Hom}_{\underline{B}}(S_i, N), \end{aligned}$$

for $1 \leq i \leq r$. By the definition of $\langle S_i, N \rangle_a$, we have

$$\begin{aligned} \langle S_i, N \rangle_a &= \dim_k \operatorname{Hom}_{\underline{B}}(S_i, N) - \dim_k \operatorname{Ext}_{\underline{B}}(S_i, N) + \dim_k \operatorname{Ext}_{\underline{B}}(N, S_i) - \dim_k \operatorname{Hom}_{\underline{B}}(N, S_i) \\ &= \dim_k \operatorname{Hom}_B(S_i, N) - \dim_k \operatorname{Ext}_B(S_i, N) + \dim_k \operatorname{Ext}_B(N, S_i) - \dim_k \operatorname{Hom}_B(N, S_i) \\ &= {}^3[N, S_i] - {}^2[N, S_i] + {}^1[N, S_i] - [N, S_i] \\ &= -\langle N, S_i \rangle_3. \end{aligned}$$

c) Let

$$0 \rightarrow L \xrightarrow{i} C \xrightarrow{p} M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow M \xrightarrow{i'} C' \xrightarrow{p'} L \rightarrow 0,$$

be the non-split conflations in \mathcal{E} , and

$$\begin{aligned} \Sigma L &\xrightarrow{G\Sigma i} \Sigma C \xrightarrow{G\Sigma p} \Sigma M \rightarrow \Sigma^2 L \\ \Sigma M &\xrightarrow{G\Sigma i'} \Sigma C' \xrightarrow{G\Sigma p'} \Sigma L \rightarrow \Sigma^2 N \end{aligned}$$

the associated triangles in \mathcal{C} . For any classes e, f, g in the Grothendieck group $K_0(\operatorname{mod} \underline{B})$, let $X_{e,f}$ be the variety whose points are the \underline{B} -submodules $E \subset G\Sigma C$ such that the dimension vector of $(G\Sigma i)^{-1}E$ equals e and the dimension vector of $(G\Sigma p)E$ equals f . Similarly, let $Y_{f,e}$ be the variety whose points are the \underline{B} -submodules $E \subset G\Sigma C'$ such that the dimension vector of $(G\Sigma i')^{-1}E$ equals f and the dimension vector of $(G\Sigma p')E$ equals e . Put

$$\begin{aligned} X_{e,f}^g &= X_{e,f} \cap \operatorname{Gr}_g(G\Sigma C), \\ Y_{f,e}^g &= Y_{f,e} \cap \operatorname{Gr}_g(G\Sigma C'). \end{aligned}$$

Since \mathcal{C} is a 2-CY triangulated category, by section 5.1 of [29] we also have

$$\chi(\operatorname{Gr}_e(G\Sigma L) \times \operatorname{Gr}_f(G\Sigma M)) = \sum_g \chi(X_{e,f}^g) + \chi(Y_{f,e}^g).$$

Therefore, part c) is a consequence of the following lemma.

Lemma 2.3. *If $X_{e,f}^g \neq \emptyset$, then we have the following equality*

$$-\langle g, S_i \rangle_3 + \langle FC, S_i \rangle_\tau = -\langle e + f, S_i \rangle_3 + \langle FL, S_i \rangle_\tau + \langle FM, S_i \rangle_\tau, \quad 1 \leq i \leq n.$$

Proof. We have the following commutative diagram as in section 4 of [29]

$$\begin{array}{ccccccc} (G\Sigma i)^{-1}E & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & (G\Sigma p)E & \longrightarrow & 0 \\ \downarrow i & & \downarrow j & & \downarrow k & & \\ G\Sigma L & \xrightarrow{G\Sigma i} & G\Sigma C & \xrightarrow{G\Sigma p} & G\Sigma M & \longrightarrow & G\Sigma^2 L \end{array}$$

where i, j, k are monomorphisms, β is an epimorphism and $[E] = g$, $[(G\Sigma i)^{-1}E] = e$, $[(G\Sigma p)E] = f$ in $K_0(\operatorname{mod} B)$. One can easily show that $\ker G\Sigma i = \ker \alpha$. We have an exact sequence

$$0 \rightarrow \ker \alpha \rightarrow (G\Sigma i)^{-1}E \rightarrow E \rightarrow (G\Sigma p)E \rightarrow 0.$$

If we apply $F = \operatorname{Hom}_{\mathcal{E}}(T, ?)$ to the short exact sequence

$$0 \rightarrow L \rightarrow C \rightarrow M \rightarrow 0,$$

we get the long exact sequences of B -modules

$$0 \rightarrow FL \rightarrow FC \rightarrow FM \rightarrow G\Sigma L \xrightarrow{G\Sigma i} G\Sigma C \rightarrow \dots,$$

and

$$0 \rightarrow FL \xrightarrow{Fi} FC \xrightarrow{Fp} FM \rightarrow \ker \alpha \rightarrow 0.$$

Since $\ker \alpha$, $(G\Sigma i)^{-1}E$, E , $(G\Sigma p)E$ are \underline{B} -modules, and the projective dimensions of FL , FC , FM are not greater than 1, we can use the method of Proposition 2.1 to construct the projective resolutions and compute the truncated Euler forms. We get that

$$\langle e, S_i \rangle_3 + \langle f, S_i \rangle_3 = \langle g, S_i \rangle_3 + \langle \ker \alpha, S_i \rangle_3,$$

and

$$\langle FL, S_i \rangle_3 + \langle FM, S_i \rangle_3 = \langle FC, S_i \rangle_3 + \langle \ker \alpha, S_i \rangle_3.$$

Note that $\langle FL, S_i \rangle_3 = \langle FL, S_i \rangle_\tau$, $\langle FM, S_i \rangle_3 = \langle FM, S_i \rangle_\tau$ and $\langle FC, S_i \rangle_3 = \langle FC, S_i \rangle_\tau$, which implies

$$\langle FL, S_i \rangle_\tau + \langle FM, S_i \rangle_\tau - \langle e + f, S_i \rangle_3 = \langle FC, S_i \rangle_\tau - \langle g, S_i \rangle_3.$$

□

Remark 2.4. *If B has finite global dimension, the Grothendieck group $K_0(\text{mod } B)$ has the Euler form $\langle \cdot, \cdot \rangle$. We can then define a Laurent polynomial X_M^f as follows*

$$X_M^f = \prod_{i=1}^n x_i^{\langle FM, S_i \rangle} \sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{i=1}^n x_i^{\langle S_i, e \rangle}.$$

One can show that in this case $X'_M = X_M^f$. In fact, if $\text{gldim } B < \infty$, then the perfect derived category $\text{per}(B)$ equals $D^b(\text{mod } B)$, and S_i belongs to $\text{per}(B)$ for all i . Thus, we have

$$\langle S_i, e \rangle = \sum_{i=0}^3 (-1)^i \dim \text{Ext}_B^i(S_i, e) = -\langle e, S_i \rangle_3$$

and $\langle FM, S_i \rangle_\tau = \langle FM, S_i \rangle$. The assumption that B is of finite global dimension holds for the examples constructed in [3] by Proposition I.2.5 b) of [loc. cit.] and for the examples constructed in [22] by Proposition 11.5 of [loc.cit.].

3. INDEX AND \mathbf{g} -VECTOR

We keep the assumptions of section 2. Let $\mathcal{D}(\text{Mod } B)$ be the derived category of B -modules, $\mathcal{D}^-(\text{mod } B)$ the right bounded derived category of $\text{mod } B$, $\mathcal{H}^-(\mathcal{P})$ the right bounded homotopy category of finitely generated projective B -modules. It is well known that there is an equivalence

$$\mathcal{H}^-(\mathcal{P}) \xrightarrow{\sim} \mathcal{D}^-(\text{mod } B).$$

Proposition 3.1. *For an arbitrary \underline{B} -module Z which is also a finitely presented B -module we have a canonical isomorphism*

$$D \text{Hom}_{\mathcal{D}^-(\text{mod } B)}(Z, ?) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}^-(\text{mod } B)}(?, Z[3]).$$

Proof. For arbitrary $X \in \mathcal{D}^-(\text{mod } B)$, by the equivalence, we have a $P_X \in \mathcal{H}^-(\mathcal{P})$ such that $X \cong P_X$ in $\mathcal{D}^-(\text{mod } B)$. Assume that P_X has the following form

$$\cdots \rightarrow P_m \rightarrow P_{m+1} \rightarrow \cdots \rightarrow P_{n-1} \rightarrow P_n \rightarrow 0 \rightarrow 0 \cdots .$$

Put

$$\begin{aligned} X_0 &= \cdots \rightarrow 0 \rightarrow 0 \rightarrow P_n \rightarrow 0 \cdots , \\ X_i &= \cdots \rightarrow 0 \rightarrow P_{n-i} \rightarrow \cdots \rightarrow P_n \rightarrow 0 \cdots , \text{ for } i > 0. \end{aligned}$$

We have $P_X \cong \text{hocolim } X_i$ in $\mathcal{D}(\text{Mod } B)$. Note that by Proposition 4 of [26], Z belongs to $\text{per } B$, i.e. Z is compact in $\mathcal{D}(\text{Mod } B)$. So we have

$$\begin{aligned} \text{Hom}_{\mathcal{D}(\text{Mod } B)}(Z, X) &\cong \text{Hom}_{\mathcal{D}(\text{Mod } B)}(Z, P_X) \\ &\cong \text{Hom}_{\mathcal{D}(\text{Mod } B)}(Z, \text{hocolim } X_i) \\ &\cong \text{colim } \text{Hom}_{\mathcal{D}(\text{Mod } B)}(Z, X_i). \end{aligned}$$

By the definition of X_i , we know that $X_i \in \text{per } B$. Since $\text{per } B$ is a full subcategory of $\mathcal{D}(\text{Mod } B)$, by the relative 3-Calabi-Yau property of $\text{per } B$, we have the following

$$\text{colim } \text{Hom}_{\mathcal{D}(\text{Mod } B)}(Z, X_i) \cong \text{colim } D \text{Hom}_{\mathcal{D}(\text{Mod } B)}(X_i, Z[3]).$$

It is easy to see that this colimit is a stationary system, i.e. $\exists N$ such that for $i > N$, we have

$$D \text{Hom}_{\mathcal{D}(\text{Mod } B)}(X_i, Z[3]) \cong D \text{Hom}_{\mathcal{D}(\text{Mod } B)}(X_{i+1}, Z[3]).$$

Thus, we have

$$\begin{aligned} \text{colim } D \text{Hom}_{\mathcal{D}(\text{Mod } B)}(X_i, Z[3]) &\cong D \lim \text{Hom}_{\mathcal{D}(\text{Mod } B)}(X_i, Z[3]) \\ &\cong D \text{Hom}_{\mathcal{D}(\text{Mod } B)}(\text{hocolim } X_i, Z[3]) \\ &\cong D \text{Hom}_{\mathcal{D}(\text{Mod } B)}(P_X, Z[3]). \end{aligned}$$

Note that since $\mathcal{D}^-(\text{mod } B)$ is a full subcategory of $\mathcal{D}(\text{Mod } B)$, we get the isomorphism

$$D \text{Hom}_{\mathcal{D}^-(\text{mod } B)}(Z, X) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}^-(\text{mod } B)}(X, Z[3]).$$

□

For each $X \in \mathcal{E}$, there is a unique minimal conflation (up to isomorphism)

$$0 \rightarrow T_X^1 \rightarrow T_X^0 \rightarrow X \rightarrow 0$$

with $T_X^0, T_X^1 \in \text{add } T$. As in [29], put

$$\text{ind}_T(X) = [T_X^0] - [T_X^1] \text{ in } K_0(\text{add } T).$$

By the proof of Theorem 2.2, we have

$$\text{ind}_T(X) = \sum_{i=1}^n \langle FX, S_i \rangle_{\tau} [T_i].$$

The following result is easily deduced from Theorem 2.3 of [13].

Lemma 3.2. *If X is a rigid object of \mathcal{E} , then X is determined up to isomorphism by $\text{ind}_T(X)$, i.e. if Y is rigid and $\text{ind}_T(X) = \text{ind}_T(Y)$, then X is isomorphic to Y .*

Proof. Since $\text{ind}_T(X) = \text{ind}_T(Y)$, we have $\text{ind}_{\underline{T}}(X) = \text{ind}_{\underline{T}}(Y)$ in the stable category $\underline{\mathcal{E}}$. By Theorem 2.3 of [13], we have $X \cong Y$ in $\underline{\mathcal{E}}$. Thus, there are \mathcal{E} -projectives P_X and P_Y such that $X \oplus P_X \cong Y \oplus P_Y$ in \mathcal{E} . Consider the minimal right T -approximation of $X \oplus P_X$

$$0 \rightarrow T^1 \rightarrow T^0 \rightarrow X \oplus P_X \rightarrow 0,$$

we have $\text{ind}_T(X \oplus P_X) = \text{ind}_T(Y \oplus P_Y) = [T^0] - [T^1]$. Note that

$$\text{ind}_T(X) = \text{ind}_T(X \oplus P_X) - [P_X] = \text{ind}_T(Y \oplus P_Y) - [P_Y] = \text{ind}_T(Y),$$

which implies $[P_X] = [P_Y]$ in $K_0(\text{add } T)$. Thus, we have $P_X \cong P_Y$ and $X \cong Y$ in \mathcal{E} . □

Let us recall the definition of \mathbf{g} -vector from section 7 of [20].

Let $1 < r \leq n$ be integers. Let $\tilde{B} = (\tilde{b}_{ij})$ be an $n \times r$ matrix with integer entries, whose principal part B (i.e. the submatrix formed by the first r rows) is antisymmetric. In [20], a cluster algebra with coefficients $\mathcal{A}(\tilde{B})$ is constructed from \tilde{B} . Let z be an element of $\mathcal{A}(\tilde{B})$. Suppose that we can write z as

$$z = R(\hat{y}_1, \dots, \hat{y}_r) \prod_{i=1}^n x_i^{g_i}, \text{ where } \hat{y}_j = \prod_{i=1}^n x_i^{\tilde{b}_{ij}},$$

where $R(\hat{y}_1, \dots, \hat{y}_r)$ is a primitive rational polynomial. If $\text{rank } \tilde{B} = r$, then the \mathbf{g} -vector of z is defined by

$$g(z) = (g_1, \dots, g_r).$$

Note that $\text{rank } \tilde{B} = r$ implies that the \mathbf{g} -vector is well-defined.

Recall that we have fixed a cluster-tilting object T in \mathcal{E} , that T_1, T_2, \dots, T_n are its pairwise non isomorphic indecomposable direct summands and that T_i is projective iff $r < i \leq n$. We define $B(T) = (b_{ij})_{n \times n}$ to be the antisymmetric matrix associated with the quiver of the endomorphism algebra of T . Let $B(T)^0$ be the submatrix formed by the first r columns of $B(T)$. We suppose that we have $\text{rank } B(T)^0 = r$. In analogy with the definition of \mathbf{g} -vectors in a cluster algebra, for $M \in \mathcal{E}$, if we can write X'_M as

$$X'_M = R(\hat{y}_1, \dots, \hat{y}_r) \prod_{i=1}^n x_i^{g_i}, \text{ where } \hat{y}_j = \prod_{i=1}^n x_i^{b_{ij}},$$

where $R(\hat{y}_1, \dots, \hat{y}_r)$ is a primitive rational polynomial, then we define the \mathbf{g} -vector $g_T(X'_M)$ of M with respect to T to be

$$g_T(X'_M) = (g_1, \dots, g_r).$$

As in the cluster algebra case, this is well-defined since $\text{rank } B(T)^0 = r$.

Proposition 3.3. *Assume that $\text{rank } B(T)^0 = r$. For arbitrary $M \in \mathcal{E}$, the \mathbf{g} -vector $g_T(X'_M)$ is well-defined and its i -th coordinate is given by*

$$g_T(X'_M)(i) = [\text{ind}_T(M) : T_i], \quad 1 \leq i \leq r.$$

Proof. By the relative 3-Calabi-Yau property of $\mathcal{D}^-(\text{mod } B)$, for $1 \leq i \leq n$, $1 \leq j \leq r$, we have

$$\begin{aligned} \langle S_i, S_j \rangle_3 &= [S_i, S_j] - {}^1[S_i, S_j] + {}^2[S_i, S_j] - {}^3[S_i, S_j] \\ &= [S_i, S_j] - {}^1[S_i, S_j] + {}^1[S_j, S_i] - [S_j, S_i] \\ &= {}^1[S_j, S_i] - {}^1[S_i, S_j] \\ &= b_{ij}, \end{aligned}$$

where the last equality follows from the definition of $B(T)$. Recall the definition of X'_M

$$X'_M = \prod_{i=1}^n x_i^{\langle FM, S_i \rangle_\tau} \sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{i=1}^n x_i^{-\langle e, S_i \rangle_3}.$$

Let e be the dimension vector of a B -submodule of $\text{Ext}_{\mathcal{E}}^1(T, M)$ and e_j its j -th coordinate in the basis of the S_i , $1 \leq i \leq n$. Then we have

$$-\langle e, S_i \rangle_3 = -\sum_{j=1}^r e_j \langle S_j, S_i \rangle_3 = \sum_{j=1}^r b_{ij} e_j.$$

Therefore, we get

$$\prod_{i=1}^n x_i^{-\langle e, S_i \rangle_3} = \prod_{i=1}^n x_i^{\sum_{j=1}^r b_{ij} e_j} = \prod_{j=1}^r \hat{y}_j^{e_j}.$$

Thus, we can write

$$X'_M = \prod_{i=1}^n x_i^{\langle FM, S_i \rangle_\tau} \left(\sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{j=1}^r \hat{y}_j^{e_j} \right).$$

The polynomial

$$R(\hat{y}_1, \dots, \hat{y}_r) = \sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{j=1}^r \hat{y}_j^{e_j}$$

is primitive since it has constant term 1. Thus, by definition we have $g_T(X'_M)(i) = \langle FM, S_i \rangle_\tau = [\text{ind}_T(M) : T_i]$. \square

Corollary 3.4. *Keep the assumptions above.*

- a) *The map $M \mapsto X'_M$ induces an injection from the set of isomorphism classes of non projective rigid indecomposables of \mathcal{E} into the set $\mathbb{Q}(x_1, \dots, x_n)$.*
- b) *Let I be a finite set and T^i , $i \in I$, cluster tilting objects of \mathcal{E} . Suppose that for each $i \in I$, we are given an object M_i which belongs to $\text{add } T^i$ and does not have non zero projective direct factors. If the M_i are pairwise non isomorphic, then the X'_{M_i} are linearly independent.*

Proof. a) clearly follows from b). Let us prove b). First, we will show that we can assign a degree to each x_i such that for every $1 \leq i \leq r$ the degree of \hat{y}_i is 1.

Indeed, it suffices to put $\deg(x_i) = k_i$, where the k_i are rationals such that we have

$$(k_1, k_2, \dots, k_n)B(T)^0 = (1, 1, \dots, 1).$$

Since $\text{rank } B(T)^0 = r$, this equation does admit a solution. Thus, the term of strictly minimal total degree in X'_{M_j} is

$$\prod_{i=1}^n x_i^{[\text{ind}_T(M_j):T_i]}.$$

Suppose that the X'_{M_i} are linearly dependent, *i.e.* there is a non-empty subset I' of I and rationals c_i , $i \in I'$, which are all non zero such that

$$\sum_{i \in I'} c_i X'_{M_i} = 0.$$

If we consider the terms of minimal total degree of the polynomial above, we find

$$\sum_{j \in I''} c_j \prod_{i=1}^n x_i^{[\text{ind}_T(M_j):T_i]} = 0$$

for some non-empty subset I'' of I . Since the M_j are all pairwise non isomorphic, Lemma 3.2 implies that the indices $\text{ind}_T(M_j)$ are all distinct. Thus, the monomials $\prod_{i=1}^n x_i^{[\text{ind}_T(M_j):T_i]}$ are linearly independent. Contradiction. \square

Remark 3.5. *If B has finite global dimension, then the condition $\text{rank } B(T)^0 = r$ is superfluous. Indeed, let C be the Cartan matrix. Then $B(T)^0$ is the submatrix formed by the first r columns of the invertible matrix C^{-t} .*

Next we will investigate the relation between the indices of an exchange pair.

Recall that F is the functor $\text{Hom}_{\mathcal{E}}(T, ?) : \mathcal{E} \rightarrow \text{mod } B$. A conflation of \mathcal{E}

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

is F -exact if

$$0 \rightarrow FX \rightarrow FY \rightarrow FZ \rightarrow 0$$

is exact in $\text{mod } B$. The F -exact sequences define a new exact structure on the additive category \mathcal{E} . For each X , we have an F -exact conflation

$$0 \rightarrow T_1 \rightarrow T_0 \rightarrow X \rightarrow 0.$$

This shows that \mathcal{E} endowed with the F -exact sequences has enough projectives and that its subcategory of projectives is $\text{add } T$. Moreover, if we denote by $\text{Ext}_F^i(X, Z)$ the i -th extension groups of the category \mathcal{E} endowed with the F -exact sequences, then $\text{Ext}_F^1(X, Z)$ is the cohomology at $\text{Hom}_{\mathcal{E}}(T_1, Z)$ of the complex

$$0 \rightarrow \text{Hom}_{\mathcal{E}}(X, Z) \rightarrow \text{Hom}_{\mathcal{E}}(T_0, Z) \rightarrow \text{Hom}_{\mathcal{E}}(T_1, Z) \rightarrow 0 \rightarrow \dots$$

Lemma 3.6. *For $X, Z \in \mathcal{E}$, there is a functorial isomorphism*

$$\text{Ext}_F^i(X, Z) \xrightarrow{\sim} \text{Ext}_B^i(FX, FZ).$$

Proof. Clearly, the derived functor

$$\mathbb{L}F : \mathcal{D}^b(\mathcal{E}) \rightarrow \mathcal{D}^b(\text{mod } B)$$

is fully faithful. Thus, $\text{Ext}_F^i(X, Z) \xrightarrow{\sim} \text{Ext}_B^i(FX, FZ)$. \square

Now Proposition 14.4 of [23] still holds in our general setting.

Proposition 3.7. *Let T and R be cluster tilting objects of \mathcal{E} . Let*

$$\eta' : 0 \rightarrow R_k \rightarrow R' \rightarrow R_k^* \rightarrow 0, \quad \eta'' : 0 \rightarrow R_k^* \rightarrow R'' \rightarrow R_k \rightarrow 0$$

be the two exchange sequences associated to an indecomposable direct summand R_k of R which is not \mathcal{E} -projective. Then exactly one of η' and η'' is F -exact. Moreover, we have

$$\underline{\dim} \text{Hom}_{\mathcal{E}}(T, R_k) + \underline{\dim} \text{Hom}_{\mathcal{E}}(T, R_k^*) = \max\{\underline{\dim} \text{Hom}_{\mathcal{E}}(T, R'), \underline{\dim} \text{Hom}_{\mathcal{E}}(T, R'')\}.$$

Proof. Using Lemma 3.6, the proof is the same as that of proposition 14.4 in [23]. \square

Corollary 3.8. *Under the assumptions of the above proposition, put*

$$I' = \text{ind}_T(R') - \text{ind}_T(R_k),$$

$$I'' = \text{ind}_T(R'') - \text{ind}_T(R_k).$$

Then we have

$$\text{ind}_T(R_k^*) = \begin{cases} I', & \text{if } \underline{\dim} FI' \geq \underline{\dim} FI'', \\ I'', & \text{if } \underline{\dim} FI' \leq \underline{\dim} FI''. \end{cases}$$

and exactly one of these cases occurs. Let $h(i) = [\text{ind}_T(R') - \text{ind}_T(R'') : T_i]$, for $1 \leq i \leq n$. Then h is a linear combination of the columns of $B(T)^0$.

Proof. The first part follows from Proposition 3.7 directly, because the index is additive on F -exact sequences.

Since (R_k, R_k^*) is an exchange pair, we have

$$X'_{R_k} X'_{R_k^*} = X'_{R'} + X'_{R''}.$$

For simplicity, we write

$$H_M = \sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}(T, M))) \prod_{i=1}^r \widehat{y}_i^{e_i}$$

for X'_M . By Proposition 3.3, we have

$$\prod_{i=1}^n x_i^{[\text{ind}_T(R_k) + \text{ind}_T(R_k^*):T_i]} H_{R_k} H_{R_k^*} = \prod_{i=1}^n x_i^{[\text{ind}_T(R'):T_i]} H_{R'} + \prod_{i=1}^n x_i^{[\text{ind}_T(R''):T_i]} H_{R''}.$$

Assume that $\text{ind}_T(R_k^*) = \text{ind}_T(R') - \text{ind}_T(R_k)$. We have

$$H_{R_k} H_{R_k^*} - H_{R'} = \prod_{i=1}^n x_i^{[\text{ind}_T(R'') - \text{ind}_T(R'):T_i]} H_{R''}.$$

By comparing the minimal total degree we get that $\prod_{i=1}^n x_i^{[\text{ind}_T(R'') - \text{ind}_T(R'):T_i]}$ is a monomial in \widehat{y}_i , $1 \leq i \leq r$, which implies the result. \square

4. FROBENIUS 2-CALABI-YAU REALIZATIONS

We define a bijection between antisymmetric integer $n \times n$ -matrices and finite quivers without loops nor 2-cycles with vertex set $\{1, 2, \dots, n\}$. Namely, the quiver Q corresponds to the matrix B iff $b_{ij} > 0$ exactly when there are arrows from i to j in Q and in this case their number is b_{ij} .

We call an $n \times n$ antisymmetric integer matrix B *acyclic* if the corresponding quiver Q does not have oriented cycles. Two matrices B and B' are called *mutation equivalent* if we can obtain B' from B by a series of matrix mutations followed by conjugation with a permutation matrix.

Let $0 \leq r < n$ be positive integers and Q a finite quiver without loops nor 2-cycles with vertex set $\{1, \dots, n\}$. We define \tilde{B} to be the $n \times r$ matrix formed by the first r columns of the skew-symmetric matrix associated with Q and we define $\mathcal{A}(Q, r) = \mathcal{A}(\tilde{B})$ to be the cluster algebra with coefficients associated with \tilde{B} , cf. [20]. A *Frobenius 2-Calabi-Yau realization* of the cluster algebra $\mathcal{A}(\tilde{B})$ is a Frobenius category \mathcal{E} with a cluster tilting object T as in section 2 such that

- \mathcal{E} has a cluster structure in the sense of [3];
- T has exactly n indecomposable pairwise non isomorphic summands T_1, T_2, \dots, T_n and among these, precisely T_{r+1}, \dots, T_n are projectives;
- The matrix \tilde{B} equals the matrix $B(T)^0$ associated with T in section 3.

Remark 4.1. *Suppose we have a Frobenius 2-CY realization of a cluster algebra $\mathcal{A}(Q, r)$ as above. Let $1 \leq k \leq r$. Then we have conflations*

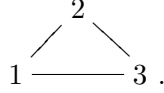
$$\begin{aligned} 0 \rightarrow T_k^* \rightarrow C \rightarrow T_k \rightarrow 0, \\ 0 \rightarrow T_k \rightarrow C' \rightarrow T_k^* \rightarrow 0. \end{aligned}$$

Here the middle terms are the sums

$$C = \bigoplus_{b_{ik} > 0} T_i^{b_{ik}}, C' = \bigoplus_{b_{ik} < 0} T_i^{-b_{ik}}.$$

Therefore, none of the first r vertices of Q can be a source or a sink.

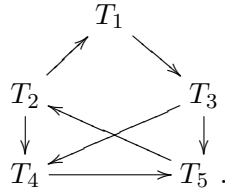
Example 4.2. All quivers obtained from Theorem 1.3 of [23] and, more generally, from Theorem II.4.1 of [3] admit Frobenius 2-Calabi-Yau realizations. We illustrate this on the following specific case taken from section II.4 of [loc. cit.]. Let Δ be the graph



Let Λ be the completion of the preprojective algebra of Δ and W the Weyl group associated with Δ . Let w be the element of W given by the reduced word $s_2s_1s_2s_3s_2$. Let $e_i, i = 1, 2, 3$, be the primitive idempotents corresponding to the vertices of Δ . Let $I_i = \Lambda(1 - e_i)\Lambda$. By Theorem II.2.8 of [3], the category $\text{Sub } \Lambda/I_w$ formed by all Λ -submodules of finite direct sums of copies of Λ/I_w is a Frobenius category whose associated stable category is 2-Calabi-Yau; moreover, it contains the cluster-tilting object

$$T = \Lambda/I_2 \oplus \Lambda/I_2I_1 \oplus \Lambda/I_2I_1I_2 \oplus \Lambda/I_2I_1I_2I_3 \oplus \Lambda/I_w.$$

According to Proposition II.1.11 of [loc. cit.], in this decomposition, each direct factor differs from the preceding one by one indecomposable direct summand $T_i, 1 \leq i \leq 5$, and among these, exactly T_3, T_4 and T_5 are projective-injective. Moreover, by Theorem II.4.1 of [loc. cit.], the quiver of the cluster-tilting object is



Using Theorem I.1.6 of [3], one can easily show that the category $\text{Sub } \Lambda/I_w$ is a Frobenius 2-Calabi-Yau realization of the cluster algebra $\mathcal{A}(\tilde{B})$ given by the matrix

$$\tilde{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} .$$

We return to the general setup. Following [13] we define a cluster-tilting object T' of \mathcal{E} to be *reachable from T* if it is obtained from T by a finite sequence of mutations. We define an indecomposable rigid objects M to be *reachable from T* if it occurs as a direct factor of a cluster-tilting object reachable from T .

Theorem 4.3. Let $\mathcal{A}(\tilde{B})$ be a cluster algebras with coefficients, where \tilde{B} is the initial $n \times r$ -matrix. We assume that $\text{rank } \tilde{B} = r$. Suppose that $\mathcal{A}(\tilde{B})$ admits a Frobenius 2-CY realization \mathcal{E} with cluster tilting object T .

- a) The map $M \mapsto X'_M$ induces a bijection from the set of isomorphism classes of indecomposable rigid non projective objects of \mathcal{E} reachable from T onto the set of cluster variables of $\mathcal{A}(\tilde{B})$. Under this bijection, the cluster-tilting objects reachable from T correspond to the clusters of $\mathcal{A}(\tilde{B})$.
- b) The map $M \mapsto \text{ind}_T(M)$ is a bijection from the set of isomorphism classes of indecomposable rigid non projective objects of \mathcal{E} reachable from T onto the set of \mathfrak{g} -vectors of cluster variables of $\mathcal{A}(\tilde{B})$.

Proof. a) It follows from Theorem 2.2 c) that X'_M is a cluster variable for each indecomposable rigid M reachable from T and from the existence of a cluster structure on \mathcal{E} that the map $M \mapsto X'_M$ is a surjection onto the set of cluster variables. The injectivity of the map $M \mapsto X'_M$ follows from Lemma 3.2 and Proposition 3.3. The second statement follows from the first one and the fact that \mathcal{E} has a cluster structure. b) follows from a), Lemma 3.2 and Proposition 3.3. \square

Proposition 4.4. *Suppose the assumptions of Theorem 4.3 hold for a cluster algebra $\mathcal{A} = \mathcal{A}(\tilde{B})$ with coefficients.*

- a) *Conjecture 7.2 of [20] holds for \mathcal{A} , i.e. cluster monomials are linearly independent.*
- b) *Conjecture 7.10 of [20] holds for \mathcal{A} , i.e.*
 - 1) *Different cluster monomials have different \mathbf{g} -vectors with respect to a given initial seed.*
 - 2) *The \mathbf{g} -vectors of the cluster variables in any given cluster form a \mathbb{Z} -basis of the lattice \mathbb{Z}^r .*
- c) *Conjecture 7.12 of [20] holds for \mathcal{A} , i.e. if (g_1, \dots, g_r) and (g'_1, \dots, g'_r) are the \mathbf{g} -vectors of one and the same cluster variable with respect to two clusters t and t' related by the mutation at l , then we have*

$$g'_j = \begin{cases} -g_l & \text{if } j = l \\ g_j + [b_{jl}]_+ g_l - b_{jl} \min(g_l, 0) & \text{if } j \neq l \end{cases}$$

where the b_{ij} are the entries of the $r \times r$ -matrix B associated with t and we write $[x]_+$ for $\max(x, 0)$ for any integer x .

Proof. a) follows from Corollary 3.4 and Theorem 4.3. b) follows from Lemma 3.2, Proposition 3.3, Theorem 4.3 and Theorem 2.6 of [13]. c) follows from Proposition 3.3, Theorem 4.3 and Theorem 3.1 of [13]. \square

Let \tilde{B} be a $2r \times r$ matrix whose principal (i.e. top $r \times r$) part B_0 is mutation equivalent to an acyclic matrix, and whose complementary (i.e. bottom) part is the $r \times r$ identity matrix. Let $\mathcal{A}(\tilde{B})$ be the cluster algebra with the initial seed (\mathbf{x}, \tilde{B}) .

Theorem 4.5. *With the above notation, the cluster algebra $\mathcal{A}(\tilde{B})$ does not admit a Frobenius 2-CY realization.*

Proof. Suppose that $\mathcal{A}(\tilde{B})$ has a Frobenius 2-CY realization \mathcal{E} . Then there is a cluster tilting object T of \mathcal{E} with $2r$ indecomposable direct summands. Then we have $B(T)^0 = \tilde{B}$. Since B_0 is mutation equivalent to an acyclic matrix B_c by a series of mutations, we have a cluster tilting object T' such that the quiver of the stable endomorphism algebra of T' corresponds to B_c . Let A be the stable endomorphism algebra of T' . By the main theorem of [27], we have a triangle equivalence $\underline{\mathcal{E}} \simeq \mathcal{C}_A$, where \mathcal{C}_A is the cluster category of A . Thus the cluster tilting graph of \mathcal{E} is connected and every rigid object of \mathcal{E} can be extended to a cluster tilting object of \mathcal{E} .

Let $F = \text{Hom}_{\mathcal{E}}(T, ?)$. Let S_i , $1 \leq i \leq 2r$, be the simple modules of $\text{End}_{\mathcal{E}}(T)$. For each object M of \mathcal{E} , we have the Laurent polynomial

$$X'_M = \prod_{i=1}^{2r} x_i^{\langle FM, S_i \rangle} \sum_e \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{i=1}^{2r} x_i^{-\langle e, S_i \rangle}.$$

Let

$$y_j = \prod_{i=1}^{2r} x_i^{b_{ij}}, 1 \leq j \leq r.$$

As in Proposition 3.3, we can rewrite X'_M as

$$X'_M = \prod_{i=1}^{2r} x_i^{\langle FM, S_i \rangle_\tau} \left(1 + \sum_{e \neq 0} \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{i=1}^r y_j^{e_j} \right),$$

where e_j is the j -th coordinate of e in the basis of the S_i , $1 \leq i \leq 2r$. If the indecomposable object M is rigid and not isomorphic to T_i for $r < i \leq 2r$, then X'_M is a cluster variable of $\mathcal{A}(\tilde{B})$. By the definition of the rational function $\mathcal{F}_{l,t}$ associated with the cluster variable $x_{l,t}$ in [20], we have

$$\begin{aligned} \mathcal{F}_M &= X'_M(x_1 = x_2 = \dots = x_r = 1) \\ &= \prod_{i=r+1}^{2r} x_i^{\langle FM, S_i \rangle_\tau} \left(1 + \sum_{e \neq 0} \chi(\text{Gr}_e(\text{Ext}_{\mathcal{E}}^1(T, M))) \prod_{j=r+1}^{2r} x_j^{e_j-r} \right). \end{aligned}$$

Put

$$G_M = 1 + \sum_{e \neq 0} \chi(\text{Gr}_e \text{Ext}_{\mathcal{E}}^1(T, M)) \prod_{j=r+1}^{2r} x_j^{e_j-r}.$$

Note that G_M is always a polynomial of x_i , $r + 1 \leq i \leq 2r$, with constant term 1. By Proposition 5.2 in [20], we know that the polynomial \mathcal{F}_M is not divisible by x_i , $r + 1 \leq i \leq 2r$. Now for $i > r$, we have $\langle FM, S_i \rangle_\tau \geq 0$ in general, which implies that $\langle FM, S_i \rangle_\tau = 0$. In particular, $\langle FM, S_i \rangle_\tau = [\text{ind}_T(M) : T_i] = 0$, for $r + 1 \leq i \leq 2r$. Consider $M = \Sigma T_1$, which is rigid and indecomposable, so X'_M is a cluster variable of the cluster algebra $\mathcal{A}(\tilde{B})$. But in the Frobenius category \mathcal{E} we have the conflation

$$0 \rightarrow T_1 \rightarrow P \rightarrow \Sigma T_1 \rightarrow 0,$$

where P is an injective hull of T_1 , which implies

$$\text{ind}_T(M) = [P] - [T_1].$$

Thus there is always some $r + 1 \leq i \leq 2r$ such that $[\text{ind}_T(M) : T_i] \neq 0$. Contradiction. \square

Remark 4.6. *In the above notation, if B_0 is acyclic, then it is easy to deduce that the cluster algebra $\mathcal{A}(\tilde{B})$ does not have a Frobenius 2-CY realization. Indeed in this case, one of the first r vertices of Q which corresponds to \tilde{B} is always a sink.*

5. TRIANGULATED 2-CALABI-YAU REALIZATIONS

5.1. Definitions. Let $B = (b_{ij})_{n \times n}$ be an antisymmetric integer matrix and $\mathcal{A}(B)$ the associated cluster algebra. A 2-Calabi-Yau triangulated category \mathcal{C} is called a *triangulated 2-Calabi-Yau realization* of the matrix B if \mathcal{C} admits a cluster tilting object T such that

- \mathcal{C} has a cluster structure in the sense [3];
- T has exactly n non isomorphic indecomposable direct summands T_1, \dots, T_n ;
- The matrix $B(T)$ as defined in section 3 equals B .

We denote a triangulated 2-CY realization of B by $\mathcal{C} \supset \text{add } T$.

Let n_1 and n_2 be positive integers. Let B_1 and B_2 be antisymmetric integer $n_1 \times n_1$ resp. $n_2 \times n_2$ -matrices. Let B_{21} be an integer $n_2 \times n_1$ -matrix with non negative entries. Let $\mathcal{C}_i \supset \mathcal{T}_i$ be a triangulated 2-CY realization of B_i , $i = 1, 2$. Let B be the matrix

$$\begin{pmatrix} B_1 & -B_{21}^t \\ B_{21} & B_2 \end{pmatrix}.$$

A *gluing* of $\mathcal{C}_1 \supset \mathcal{T}_1$ with $\mathcal{C}_2 \supset \mathcal{T}_2$ with respect to B is a triangulated 2-CY realization $\mathcal{C} \supset \mathcal{T}$ of B endowed with full additive subcategories \mathcal{T}'_1 and \mathcal{T}'_2 such that

- $\text{Hom}_{\mathcal{C}}(\mathcal{T}'_1, \mathcal{T}'_2) = 0$;
- The set $\text{indec}(\mathcal{T})$ is the disjoint union of $\text{indec}(\mathcal{T}'_1)$ with $\text{indec}(\mathcal{T}'_2)$;
- There is a triangle equivalence

$${}^{\perp}(\Sigma\mathcal{T}'_1)/(\mathcal{T}'_1) \xrightarrow{\sim} \mathcal{C}_2$$

inducing an equivalence $\mathcal{T}'_2 \xrightarrow{\sim} \mathcal{T}_2$;

- There is a triangle equivalence

$${}^{\perp}(\Sigma\mathcal{T}'_2)/(\mathcal{T}'_2) \xrightarrow{\sim} \mathcal{C}_1$$

inducing an equivalence $\mathcal{T}'_1 \xrightarrow{\sim} \mathcal{T}_1$.

A *principal gluing* of $\mathcal{C}_1 \supset \mathcal{T}_1$ is a gluing of $\mathcal{C}_1 \supset \mathcal{T}_1$ with $\mathcal{C}_2 \supset \mathcal{T}_2$ with respect to

$$\begin{pmatrix} B_1 & -I_{n_1} \\ I_{n_1} & 0 \end{pmatrix},$$

where \mathcal{C}_2 is the cluster category of $(A_1)^{n_1}$ and \mathcal{T}_2 the image of the subcategory of finitely generated projective modules.

It is well known that each acyclic matrix B admits a triangulated 2-CY realization \mathcal{C}_{Q_B} , where \mathcal{C}_{Q_B} is the cluster category of the quiver Q_B corresponding to B . In the last subsection, we will see that \mathcal{C}_{Q_B} does admit a principal gluing.

Conjecture 5.1. *If \mathcal{C}_1 and \mathcal{C}_2 are algebraic, a gluing exists for any matrix B_{21} with non negative entries.*

Ongoing work in [1] provides some evidence for this conjecture.

5.2. Cluster algebras with coefficients. Let B be an antisymmetric integer $n \times n$ -matrix. Suppose that the matrix B admits a triangulated 2-CY realization \mathcal{C} with the cluster tilting subcategory $\mathcal{T} = \text{add } T$. Let T_i , $1 \leq i \leq n$, be the non isomorphic indecomposable direct summands of T . By the definition, we have $B(T) = B$. The mutations of the matrix B correspond to the mutations of the cluster tilting object T . Fix an integer $0 < r < n$ and consider the submatrix B^0 of B formed by the first r columns of B . If $l \leq r$, then we have

$$\mu_l(B^0) = (\mu_l(B))^0,$$

where μ_l is the mutation in the direction l . Thus we can view the cluster algebra $\mathcal{A}(B^0)$ with coefficients as a sub-cluster algebra of $\mathcal{A}(B)$, cf. Ch. III of [3].

Denote by \mathcal{P} the full subcategory of \mathcal{C} whose objects are the finite direct sums of copies of T_{r+1}, \dots, T_n . We define a subcategory of \mathcal{C}

$$\mathcal{U} = {}^{\perp}\mathcal{P}[1] = \{X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^1(T_i, X) = 0, \text{ for } r < i \leq n\}.$$

By Theorem I.2.1 of [3], the quotient category \mathcal{U}/\mathcal{P} is a 2-Calabi-Yau triangulated category and there is a bijection between cluster tilting subcategories of \mathcal{C} containing \mathcal{P} and cluster tilting subcategories of \mathcal{U}/\mathcal{P} . Thus, a mutation of a cluster tilting object in \mathcal{U}/\mathcal{P} can be viewed as a mutation of a cluster tilting object in \mathcal{C} which does not affect the direct summands T_i , $r < i \leq n$. This exactly corresponds to a mutation of the matrix B in one of the first r directions. In particular, a mutation of the cluster algebra $\mathcal{A}(B^0)$ corresponds to a mutation of a cluster tilting object in \mathcal{U} .

Recall that there is a generalized Caldero-Chapoton map on \mathcal{C} defined by

$$X_M = \prod_{i=1}^n x_i^{-[\text{coind}_T M:T_i]} \sum_e \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, M))) \prod_{i=1}^n x_i^{\langle S_i, \epsilon \rangle_a}.$$

We consider the composition of this map with the shift:

$$X'_M = X_{\Sigma M} = \prod_{i=1}^n x_i^{[\text{ind}_T M : T_i]} \sum_e \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, \Sigma M))) \prod_{i=1}^n x_i^{\langle S_i, e \rangle_a}.$$

If we restrict the map $X'_?$ to the subcategory \mathcal{U} , then the multiplication theorem for $X'_?$ implies that if M is an indecomposable rigid object reachable from T in \mathcal{U} , then X'_M is a cluster variable of $\mathcal{A}(B^0)$.

If M belongs to \mathcal{U} , then $\text{Hom}_{\mathcal{C}}(T, \Sigma M)$ is an $\text{End}_{\mathcal{C}}(T)$ -module which vanishes at each vertex $r < i \leq n$. Let e be an element of the Grothendieck group of $\text{mod } \text{End}_{\mathcal{C}}(T)$. Let e_j be the j -th coordinate of e with respect to the basis S_i , $1 \leq i \leq n$. We have

$$\begin{aligned} \langle S_i, e \rangle_a &= \langle S_i, e \rangle_{\tau} - \langle e, S_i \rangle_{\tau} \\ &= \sum_{j=1}^r e_j (\langle S_i, S_j \rangle_{\tau} - \langle S_j, S_i \rangle_{\tau}) \\ &= \sum_{j=1}^r e_j (\text{Ext}_{\text{End}_{\mathcal{C}}(T)}^1(S_j, S_i) - \text{Ext}_{\text{End}_{\mathcal{C}}(T)}^1(S_i, S_j)) \\ &= \sum_{j=1}^r b_{ij} e_j. \end{aligned}$$

As in section 3, put

$$y_j = \prod_{i=1}^n x_i^{b_{ij}}, \text{ for } 1 \leq j \leq r.$$

Then X'_M can be rewritten as

$$X'_M = \prod_{i=1}^n x_i^{[\text{ind}_T(M) : T_i]} (1 + \sum_{e \neq 0} \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, \Sigma M))) \prod_{j=1}^r y_j^{e_j}).$$

As in section 3, when $\text{rank } B^0 = r$, we can define the \mathbf{g} -vector of $M \in \mathcal{U}$ with respect to a cluster tilting object T . Thus we have proved the following proposition.

Proposition 5.2. *Suppose that $\text{rank } B^0 = r$. The \mathbf{g} -vector of X'_M with respect to T is given by*

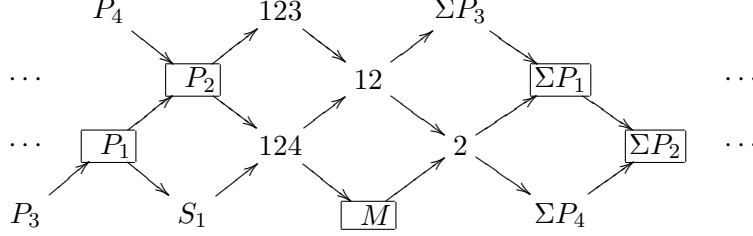
$$g_T(X'_M)(i) = [\text{ind}_T(M) : T_i], \text{ for } 1 \leq i \leq r.$$

It is easy to see that the $\sum_{i=1}^r g_T(X'_M)(i)[T_i]$ is just the index of M in the quotient category \mathcal{U}/\mathcal{P} . We use this in the proof of Theorem 5.3 below. In analogy with the definition in section 4, we define a cluster tilting object T' to be *reachable* from T if it is obtained from T by a sequence of mutations at indecomposable rigid objects not in \mathcal{P} . We define an indecomposable rigid object to be *reachable* from T if it is a direct factor of a cluster tilting object reachable from T .

Theorem 5.3. *If $\text{rank } B^0 = r$, then the statements of Theorem 4.3 and Proposition 4.4 hold for the cluster algebra $\mathcal{A}(B^0)$.*

Proof. We have already seen that the map $M \mapsto X'_M$ is well-defined and surjective onto the set of cluster variables. It is injective by Proposition 5.2 and because rigid objects are determined by their indices, which also implies part b) of Theorem 4.3. The same proof as for Corollary 3.4 implies a) of Proposition 4.4. Part b) of Proposition 4.4 follows from Theorem 2.6 of [13], Proposition 5.2 and the statement of Theorem 4.3. Part c) follows from Proposition 5.2 and the statement of Theorem 4.3 and Theorem 3.1 of [13]. \square

Example 5.4. Let A_4 be the quiver $3 \rightarrow 1 \rightarrow 2 \leftarrow 4$, \mathcal{C}_Q the corresponding cluster category. The following is the AR quiver of \mathcal{C}_Q , where P_i , $1 \leq i \leq 4$, are the indecomposable projective kQ -modules.



Let $T = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ be the canonical cluster tilting object in \mathcal{C}_Q , $\mathcal{P} = \text{add}(P_3 \oplus P_4)$. It is easy to see that the indecomposable objects in $\mathcal{U}/\mathcal{P} \cong \mathcal{C}_{A_2}$ are exactly $P_1, P_2, M, \Sigma P_1, \Sigma P_2$. In this case, the matrix $B(T)^0$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $\text{rank} B(T)^0 = 2$. Moreover, the cluster algebra $\mathcal{A}(B(T)^0)$ has principal coefficients.

5.3. Cluster algebras with principal coefficients. In this subsection, we suppose that $2r = n$ and that the complementary part of B^0 is the $r \times r$ identity matrix. Thus the cluster algebra $\mathcal{A}(B^0)$ has principal coefficients. Recall that for the matrix B , we have a triangulated 2-CY realization $\mathcal{C} \supset \text{add} T$ and we have fixed $\mathcal{P} = \text{add}(T_{r+1} \oplus \cdots \oplus T_{2r})$. Let $\mathcal{Q} = \text{add}(T_1 \oplus \cdots \oplus T_r)$. Let $\mathcal{C}_1 = \mathcal{U}/\mathcal{P}$ and $\mathcal{C}_2 = {}^\perp \mathcal{Q}[1]/\mathcal{Q}$ be the quotient categories, $\mathcal{T}_1 = \text{add}(\pi_1(T_1 \oplus \cdots \oplus T_r))$ and $\mathcal{T}_2 = \text{add}(\pi_2(T_{r+1} \oplus \cdots \oplus T_{2r}))$ the corresponding cluster tilting subcategories, where π_1 and π_2 are the respective projection functors. Then \mathcal{C} is a gluing of $\mathcal{C}_1 \supset \mathcal{T}_1$ with $\mathcal{C}_2 \supset \mathcal{T}_2$ with respect to the matrix B .

As in section 4, for a cluster variable $x_{l,t}$ of the cluster algebra $\mathcal{A}(B^0)$ which corresponds to an indecomposable rigid object $M \in \mathcal{U}$ and not in \mathcal{P} , we denote the rational function $\mathcal{F}_{l,t}$ defined in section 3 of [20] by \mathcal{F}_M . Since $x_{l,t} = X'_M$, we have

$$\begin{aligned} \mathcal{F}_M &= X'_M(x_1 = \cdots = x_r = 1) \\ &= \prod_{i=r+1}^{2r} x_i^{[\text{ind}_T(M):T_i]} (1 + \sum_{e \neq 0} \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, \Sigma M))) \prod_{j=r+1}^{2r} x_j^{e_{j-r}}). \end{aligned}$$

The following result is now a consequence of Proposition 3.6 and 5.2 in [20]. We give a proof based on representation theory. Note that conjecture 5.4 of [20] will be proved in full generality in [15].

Theorem 5.5. Conjecture 5.4 of [20] holds for $\mathcal{A}(B^0)$, i.e. the polynomial \mathcal{F}_M has constant term 1. Thus we have

$$\mathcal{F}_M = 1 + \sum_{e \neq 0} \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, \Sigma M))) \prod_{j=r+1}^{2r} x_j^{e_{j-r}}.$$

Proof. We need to show that for each $i > r$, $[\text{ind}_T(M) : T_i]$ is zero. Since X'_M is a cluster variable and M is indecomposable, we have the following two cases:

Case 1: $M \cong \Sigma T_j$ for some $j \leq r$. We have $\text{ind}_T(M) = -[T_j]$, which implies that $[\text{ind}_T(M) : T_i] = 0$.

Case 2: M is not isomorphic to ΣT_j for any $j \leq r$. Recall that by assumption, M is not isomorphic to T_j for any $j > r$. We have the following minimal triangle

$$T_M^1 \rightarrow T_M^0 \rightarrow M \rightarrow \Sigma T_M^1$$

with T_M^0, T_M^1 in $\text{add } T$ and $\text{ind}_T(M) = [T_M^0] - [T_M^1]$. Since M belongs to \mathcal{U} , for each $i > r$ we have $\text{Hom}_{\mathcal{C}}(M, \Sigma T_i) = 0$. If we had $[T_M^1 : T_i] \neq 0$ for some $i > r$, then the above minimal triangle would have a non zero direct factor

$$T_i \rightarrow T_i \rightarrow 0 \rightarrow \Sigma T_i.$$

Suppose that we have $[T_M^0 : T_i] \neq 0$ for some $i > r$. Applying the functor $F = \text{Hom}_{\mathcal{C}}(T, ?)$ to the triangle, we get a minimal projective resolution of FM as an $\text{End}_{\mathcal{C}}(T)$ -module. Note that for $i > r$, the projective module FT_i is also a simple module, which implies that FM is decomposable. Contradiction. \square

Let M be reachable from T and consider the polynomial \mathcal{F}_M of Theorem 5.5. We define the f -vector $f_T(M) = (f_1, \dots, f_r)$ of M with respect to T by

$$\mathcal{F}_M|_{\text{Trop}(u_1, \dots, u_r)}(u_1^{-1}, \dots, u_r^{-1}) = u_1^{-f_1} \dots u_r^{-f_r}.$$

Proposition 5.6. *Suppose that M is not isomorphic to T_i for $1 \leq i \leq 2r$, and let $\underline{\dim} \text{Hom}_{\mathcal{C}}(T, \Sigma M) = (d_1, \dots, d_r)$. Then we have*

$$d_i = f_i, \quad 1 \leq i \leq r.$$

Proof. By Theorem 5.5, we have

$$\mathcal{F}_M = 1 + \sum_{e \neq 0} \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, \Sigma M))) \prod_{j=r+1}^{2r} x_j^{e_j - r}.$$

Therefore, we obtain

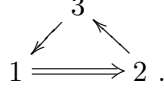
$$\begin{aligned} \mathcal{F}_M|_{\text{Trop}(u_1, \dots, u_r)}(u_1^{-1}, \dots, u_r^{-1}) &= 1 \oplus \bigoplus_{e \neq 0} \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T, \Sigma M))) \prod_{j=1}^r u_j^{-e_j} \\ &= u_1^{-d_1} \dots u_r^{-d_r}. \end{aligned}$$

\square

Under the assumptions above, we have proved that the dimension vector of $\text{Hom}_{\mathcal{C}}(T, \Sigma M)$ equals the f -vector $f_T(M)$. Conjecture 7.17 of [20] states that the f -vectors coincide with the denominator vectors in general. But by recent work of A. Buan, R. Marsh and I. Reiten [7], the dimension vectors do not always coincide with the denominator vectors. In fact, as shown in [7], for a quiver Q whose underlying graph is an affine Dynkin diagram, the vector $\dim \text{Hom}_{\mathcal{C}_Q}(T, M)$ is different from the denominator vector of X_M^T if $M = R$ and R is a direct factor of T , where R is a rigid regular indecomposable of maximal quasi-length. This leads to the following minimal counterexample to Conjecture 7.17 in [20]. Let us point out that the corresponding computations already appear in [12]. In subsection 5.5 below, we will show that in many cases, the f -vector is greater or equal to the denominator vector.

5.4. A counterexample.

Example 5.7. Let Q be the following quiver



Let $\mathcal{A}(Q)$ be the cluster algebra associated with the initial seed given by Q and $\mathbf{x} = (x_1, x_2, x_3)$. Consider the mutations at $3, 2, 1$. Let \mathbf{x}^{t_3} be the corresponding seed. We have

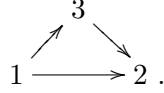
$$x_1^{t_3} = \frac{x_1^2 + 2x_1x_2 + x_2^2 + x_3}{x_1x_2x_3}$$

and the corresponding F -polynomial is

$$F_{x_1^{t_3}} = 1 + (1 + y_1 + y_1y_2)y_3 + y_1y_2y_3^2.$$

Then the f -vector of $x_1^{t_3}$ does not coincide with the denominator vector.

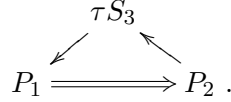
Let us interpret this counterexample in terms of representation theory. Let $A_{2,1}$ be the quiver



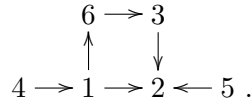
Consider the cluster category $\mathcal{C}_{A_{2,1}}$ of $kA_{2,1}$. Let P_i , $1 \leq i \leq 3$, be the indecomposable projective modules and S_i the corresponding simple modules. Then

$$T = P_1 \oplus P_2 \oplus \tau S_3,$$

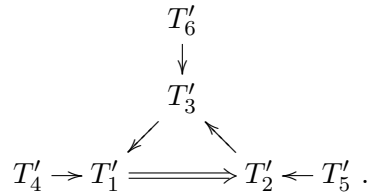
is a cluster tilting object of $\mathcal{C}_{A_{2,1}}$, where τ is the Auslander-Reiten translation functor. The quiver Q_T of T looks like



We will show that the cluster category $\mathcal{C}_{A_{2,1}} \supset \text{add } T$ admits a principal gluing. For this, consider the following quiver Q_1 :



It admits a cluster category \mathcal{C}_{Q_1} . Let $T_{Q_1} = kQ_1$ be the canonical cluster tilting object in \mathcal{C}_{Q_1} . Let $T' = \mu_3(\mu_6(T_{Q_1}))$ be the cluster tilting object obtained by mutations from T_{Q_1} . Denote the non isomorphic indecomposable direct summands of T' by T'_i , $1 \leq i \leq 6$. Then the quiver of $Q_{T'}$ is



Let $\mathcal{P} = \text{add}(T'_4 \oplus T'_5 \oplus T'_6)$. Then \mathcal{U}/\mathcal{P} is a 2-Calabi-Yau triangulated category and admits a cluster tilting object with the quiver Q_T . By the main theorem of [27], we know that there is a triangle equivalence $\mathcal{U}/\mathcal{P} \simeq \mathcal{C}_{A_{2,1}}$. Thus, we see that the matrix $B(T')$ admits a triangulated 2-CY realization \mathcal{C}_{Q_1} which is the required principal gluing of $\mathcal{C}_{A_{2,1}} \supset \text{add } T$.

We may assume that the images of T'_1, T'_2, T'_3 coincide with $P_1, P_2, \tau S_3$ in $\mathcal{C}_{A_{2,1}}$ respectively. Denote the shift functor in \mathcal{C}_{Q_1} (resp. $\mathcal{C}_{A_{2,1}}$) by Σ (resp. $[1]$).

Let N be the preimage of S_3 in \mathcal{C}_{Q_1} . Then one can easily compute

$$\underline{\dim} \operatorname{Hom}_{\mathcal{C}_{Q_1}}(T', \Sigma N) = \underline{\dim} \operatorname{Hom}_{\mathcal{C}_{A_{2,1}}}(T, \tau S_3) = (1, 1, 2).$$

Note that the denominator vector of X'_N equals the denominator vector of $X_{\tau S_3}^T$. Now the result follows from the Proposition above.

5.5. An inequality. Let \mathcal{T} be a 2-Calabi-Yau triangulated category with cluster-tilting object T . Recall that we have the generalized Caldero-Chapoton map

$$X_M^T = \prod_{i=1}^n x_i^{-[\operatorname{coind}_T(M):T_i]} \sum_e \chi(\operatorname{Gr}_e(GM)) \prod_{i=1}^n x_i^{\langle S_i, e \rangle_a},$$

where G is the functor $\operatorname{Hom}_{\mathcal{C}}(T, ?) : \mathcal{C} \rightarrow \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)$. The following proposition is proved in greater generality in [15].

Proposition 5.8. *For each M in \mathcal{T} , let $\underline{\dim} GM = (m_1, \dots, m_n)$ and let $1 \leq i \leq n$. We have*

$$-[\operatorname{coind}_T(M) : T_i] + \langle S_i, e \rangle_a \geq -m_i,$$

for each submodule N of GM with $\underline{\dim} N = e$. Thus the exponent of x_i in the denominator of X_M is less or equal to m_i .

Proof. This result holds for the case $M \cong \Sigma T'$, $T' \in \operatorname{add} T$ obviously. We assume that M is indecomposable and not isomorphic to any $\Sigma T'$. The case where M is decomposable is a consequence of the multiplication theorem for $X_?$. Now by Lemma 7 of [29], we have

$$-[\operatorname{coind}_T(M) : T_i] = -\langle S_i, GM \rangle_{\tau}.$$

Note that we have the short exact sequence of $\operatorname{End}_{\mathcal{C}}(T)$ -modules

$$0 \rightarrow N \rightarrow GM \rightarrow GM/N \rightarrow 0.$$

By applying the functor $\operatorname{Hom}(S_i, ?)$, we get

$$\langle S_i, N \rangle_{\tau} + \langle S_i, GM/N \rangle_{\tau} - \langle S_i, GM \rangle_{\tau} + \dim \operatorname{Ext}^2(S_i, N) \geq 0.$$

By the stable 3-Calabi-Yau property of $\operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)$ proved in [26], we have $\dim \operatorname{Ext}^2(S_i, N) \leq \dim \operatorname{Ext}^1(N, S_i)$. Therefore, we have

$$\begin{aligned} -[\operatorname{coind}_T(M) : T_i] + \langle S_i, e \rangle_a &\geq -\langle N, S_i \rangle_{\tau} - \langle S_i, GM/N \rangle_{\tau} - \dim \operatorname{Ext}^2(S_i, N) \\ &\geq -[N, S_i] - [S_i, GM/N] + {}^1[S_i, GM/N] \\ &\geq -m_i. \end{aligned}$$

□

5.6. Behaviour of the g-vectors under mutation. Let $B = (b_{ij})$ be an antisymmetric integer $r \times r$ -matrix. Let $\mathcal{C} \supset \operatorname{add} T$ be a triangulated 2-CY realization of B . Let T_1, \dots, T_r be the non isomorphic indecomposable factors of T . Let $1 \leq l \leq r$ be an integer and $T' = \mu_l(T)$ the mutation of T at T_l . Thus, the non isomorphic indecomposable factors of T' are $T_1, \dots, T_l^*, \dots, T_r$. Let \mathcal{C}_1 be a principal gluing of $\mathcal{C} \supset \operatorname{add} T$ and \mathcal{C}_2 a principal gluing of $\mathcal{C} \supset \operatorname{add} T'$. For each indecomposable object $M \in \mathcal{C}$ reachable from T , we denote by \mathcal{F}_M^T and $\mathcal{F}_M^{T'}$ the \mathcal{F} -polynomials of M with respect to \mathcal{C}_1 and \mathcal{C}_2 respectively. Following [20], we define the integers h_l and h'_l by

$$\begin{aligned} u^{h_l} &= \mathcal{F}_M^T|_{\operatorname{trop}(u)}(u^{[-b_{k1}]_+}, \dots, u^{-1}, \dots, u^{[-b_{kn}]_+}), \\ u^{h'_l} &= \mathcal{F}_M^{T'}|_{\operatorname{trop}(u)}(u^{[b_{k1}]_+}, \dots, u^{-1}, \dots, u^{[b_{kn}]_+}), \end{aligned}$$

where u^{-1} is in the l -th position.

The following proposition shows that if Conjecture 1 holds, then Conjecture 6.10 of [20] holds for the cluster algebra with principal coefficients associated with B .

Proposition 5.9. *In the above notation, we have*

$$h'_l = -[[\text{ind}_T(M) : T_l]_+]_+, \quad h_l = \min(0, [\text{ind}_T(M) : T_l]).$$

Proof. Let S_i , $1 \leq i \leq r$, be the top of the indecomposable right projective $\text{End}_{\mathcal{C}}(T')$ -module $\text{Hom}_{\mathcal{C}}(T', T'_i)$. First we will show that $g_l = [\text{ind}_T(M) : T_l] > 0$ iff S_l occurs as a submodule of the module $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$ and that the multiplicity of S_l in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$ equals $[\text{ind}_T(M) : T_l]$.

Suppose that $g_l > 0$. Then we have the following triangle

$$T_M^1 \rightarrow T_M^{0'} \oplus (T_l)^{g_l} \rightarrow M \rightarrow \Sigma T_M^1$$

with $T_M^1, T_M^{0'}$ in $\text{add } T$ and $[T_M^{0'} : T_l] = 0$, where $(T_l)^{g_l}$ is the sum of g_l copies of T_l . Applying the functor $\text{Hom}_{\mathcal{C}}(T', ?)$ to the shift of the above triangle, we get the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(T', \Sigma(T_l)^{g_l}) \rightarrow \text{Hom}_{\mathcal{C}}(T', \Sigma M) \rightarrow \text{Hom}_{\mathcal{C}}(T', \Sigma^2 T_M^1) \rightarrow \dots$$

Note that $\text{Hom}_{\mathcal{C}}(T', \Sigma(T_l)^{g_l}) \cong (S_l)^{g_l}$, *i.e.* S_l occurs with multiplicity $\geq g_l$ in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$. If the multiplicity of S_l in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$ was $> g_l$, then S_l would occur in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma^2 T_M^1)$. This is not the case since $\text{Hom}_{\mathcal{C}}(T', \Sigma^2 T_M^1)$ is the sum of injective indecomposables not isomorphic to the injective hull $\text{Hom}_{\mathcal{C}}(T', \Sigma^2 T_l)$ of S_l . Conversely, if S_l occurs in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$, thanks to the split idempotents property of \mathcal{C} , we have an irreducible morphism $\alpha : \Sigma T_l \rightarrow \Sigma M$ in \mathcal{C} . Thus, by the definition of the index, we get $g_l > 0$. Moreover, the multiplicity of S_l equals g_l by the same argument as before.

Assume that $g_l > 0$. For an arbitrary submodule U of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$, let $\dim U = (e_1, \dots, e_n)$. We will show that

$$e_l \leq g_l + \sum_i [b_{il}]_+ e_i.$$

Indeed, consider the projective resolution of the simple module S_l

$$\dots \rightarrow \bigoplus_i P_i^{b_{il}} \rightarrow P_l \rightarrow S_l \rightarrow 0.$$

Applying the functor $\text{Hom}_{\text{End}_{\mathcal{C}}(T')}(? , U)$, we get the exact sequence

$$0 \rightarrow \text{Hom}(S_l, U) \rightarrow \text{Hom}(P_l, U) \rightarrow \text{Hom}(\bigoplus_i P_i^{b_{il}}, U) \rightarrow \dots,$$

which implies the inequality because the dimension of $\text{Hom}(S_l, U)$ is less or equal to the multiplicity of S_l in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$, which equals g_l . By Theorem 5.5, we have

$$\begin{aligned} u^{h'_l} &= \mathcal{F}_M^{T'}|_{\text{Tr}_{\text{op}}(u)}(u^{[b_{k1}]_+}, \dots, u^{-1}, \dots, u^{[b_{kn}]_+}) \\ &= 1 \oplus \bigoplus_e \chi(\text{Gr}_e(\text{Hom}_{\mathcal{C}}(T', \Sigma M))) u^{-e_l} \prod_{i \neq l} (u^{[b_{ki}]_+})^{e_i}. \end{aligned}$$

We have just shown that for each e , we have

$$-e_l + \sum_i [b_{il}]_+ e_i \geq g_l,$$

and the equality occurs if e is the dimension vector of the submodule $(S_l)^{g_l}$. We conclude that we have $h'_l = -[\text{ind}_T(M) : T_l]$. If $g_l \leq 0$, then S_l does not occur in the socle of $\text{Hom}_{\mathcal{C}}(T', \Sigma M)$ and it is easy to see that $h'_l = 0$. Dually, we have the equality $h_l = \min(0, [\text{ind}_T(M) : T_l])$. \square

5.7. Acyclic cluster algebras with principal coefficients. Let B be an antisymmetric integer $r \times r$ -matrix. Assume that B is acyclic. Let Q be the corresponding quiver of B with set of vertices $Q_0 = \{1, \dots, r\}$ and with set of arrows Q_1 . Let \mathcal{C}_Q be the cluster category of Q , $T = kQ$ the canonical cluster tilting object of \mathcal{C}_Q . We claim that the cluster category $\mathcal{C}_Q \supset \text{add } T$ admits a principal gluing.

Indeed, we define a new quiver $\tilde{Q} = Q \overleftarrow{\coprod} Q_0$ associated with Q : Its set of vertices is $\{1, \dots, 2r\}$, and its arrows are those of Q and new arrows from $r+i$ to i for each vertex i of Q . Since Q is acyclic, so is \tilde{Q} , hence $k\tilde{Q}$ is finite-dimensional and hereditary. Thus, we have the cluster category $\mathcal{C}_{\tilde{Q}}$ which is a triangulated 2-CY realization of the matrix

$$\begin{pmatrix} B & -I_r \\ I_r & 0 \end{pmatrix}.$$

In particular, $\mathcal{C}_{\tilde{Q}} \supset \text{add } k\tilde{Q}$ is a principal gluing for $\mathcal{C}_Q \supset \text{add } T$. Thus, Proposition 5.2, Theorem 5.3, Theorem 5.5 and Proposition 5.6 hold for acyclic cluster algebras with principal coefficients.

Let P_i , $1 \leq i \leq 2r$, be the non isomorphic indecomposable projective right modules of $k\tilde{Q}$. Let $\mathcal{P} = \text{add}(P_{r+1} \oplus \dots \oplus P_{2r})$. We have a triangle equivalence

$${}^\perp \mathcal{P}[1]/\mathcal{P} \xrightarrow{\sim} \mathcal{C}_Q.$$

Recall that there is a partial order on \mathbb{Z}^r defined by

$$\alpha \leq \beta \text{ iff } \alpha(i) \leq \beta(i), \text{ for } 1 \leq i \leq r, \text{ where } \alpha, \beta \in \mathbb{Z}^r.$$

Proposition 5.10. *Let B be a $2r \times r$ integer matrix, whose principal part is antisymmetric and acyclic and whose complementary part is the identity matrix. Let σ be a sequence k_1, \dots, k_m with $1 \leq k_i \leq r$. Denote by B_σ the matrix*

$$\mu_{k_1} \circ \mu_{k_2} \cdots \circ \mu_{k_m}(B) = (b_{ij}^\sigma).$$

Let $E_\sigma = (e_1, e_2, \dots, e_r)$ be the complementary part of B_σ , where $e_i \in \mathbb{Z}^r$, $1 \leq i \leq r$. Then for each i , we have $e_i \leq 0$ or $e_i \geq 0$.

Proof. Suppose that there is some k such that $e_k \not\leq 0$ and $e_k \not\geq 0$. For simplicity, assume that $k = 1$, i.e. there are $r < i, j \leq 2r$ such that $b_{i1}^\sigma > 0$ and $b_{j1}^\sigma < 0$.

Let Q be the quiver corresponding to the principal part of B and \tilde{Q} as constructed above. By the argument above, there is a cluster tilting object T' of $\mathcal{C}_{k\tilde{Q}}$ such that $B(T')^0 = B_\sigma$. We have arrows $P_i \rightarrow T'_1$ and $T'_1 \rightarrow P_j$, where T'_1 is the indecomposable direct summand of T' corresponding to the first column of B_σ . Now if we consider the mutation in direction 1 of T' , we will have an arrow $P_i \rightarrow P_j$ in $Q_{\mu_1(T')}$. But this is impossible, since for $r < l \leq 2r$, the P_l are simple pairwise non isomorphic modules so we have

$$\text{Hom}_{\mathcal{C}_{k\tilde{Q}}}(P_i, P_j) = \text{Hom}_{k\tilde{Q}}(P_i, P_j) = 0.$$

□

REFERENCES

- [1] Claire Amiot, Ph.D. thesis in preparation.
- [2] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.
- [3] Aslak Bakke Buan, Osamu Iyama, Idun Reiten, Jeanne Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, arXiv:math/0701557.
- [4] Aslak Bakke Buan and Robert J. Marsh, *Cluster-tilting theory*, In: Trends in the Representation Theory of Algebras and Related Topics, Workshop, August 11–14, 2004, Querétaro, Mexico. Editors J. A. de la Pena and R. Bautista, Contemporary Mathematics **406** (2006), 1–30.

- [5] Aslak Bakke Buan, Robert J. Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204** (2006), 572–618.
- [6] Aslak Bakke Buan, Robert J. Marsh and Idun Reiten, *Cluster mutation via quiver representations*, arXiv:math/0412077, to appear in Commentarii Mathematici Helvetici.
- [7] ———, *Laurent polynomials in acyclic cluster algebras*, preprint in preparation.
- [8] Aslak Bakke Buan, Robert J. Marsh, Idun Reiten, and Gordana Todorov, *Clusters and seeds in acyclic cluster algebras*, with an appendix by Aslak Bakke Buan, Philippe Caldero, Bernhard Keller, Robert Marsh, Idun Reiten and Gordana Todorov, Proc. Amer. Math. Soc. **135** (2007), 3049–3060
- [9] Philippe Caldero and Bernhard Keller, *From triangulated categories to cluster algebras*, arXiv:math/0506018, to appear in Invent. Math.
- [10] Philippe Caldero and Bernhard Keller, *From triangulated categories to cluster algebras II*, Ann. Sci. École Norm. Sup. **39** (2006), 983–1009.
- [11] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616.
- [12] Giovanni Cerulli Irelli, *Structural theory of rank three cluster algebras of affine type*, Ph. D. thesis in preparation.
- [13] Raika Dehy and Bernhard Keller, *On the combinatorics of rigid objects in 2-Calabi-Yau categories*, arXiv:math/0709.0882.
- [14] Harm Derksen, Jerzy Weyman and Andrei Zelevinsky, *Quivers with potentials and their representations I: Mutations*, arXiv:math/0704.0649.
- [15] ———, *Quivers with potentials and their representations II*, in preparation.
- [16] Grégoire Dupont, *Caldero-Keller approach to the denominators of cluster variables*, arXiv:0711.4661v1 [math.RT].
- [17] V. V. Fock and A. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, arXiv:math/0311245.
- [18] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529 (electronic).
- [19] ———, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), 63–121.
- [20] ———, *Cluster algebras IV: Coefficients*, Compositio Mathematica **143** (2007), 112–164.
- [21] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), 193–253.
- [22] ———, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), 589–632.
- [23] ———, *Rigid modules over preprojective algebras II: The Kac-Moody case*, arXiv:math/0703039.
- [24] Michael Gekhtman, Michael Shapiro and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Moscow Math. J. **3** (2003), 899–934.
- [25] ———, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. **127** (2005), 291–311.
- [26] Bernhard Keller and Idun Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math., **211** (2007), 123–151.
- [27] ———, *Acyclic Calabi-Yau categories*, arXiv:math/0610594.
- [28] Robert Marsh, Markus Reineke, and Andrei Zelevinsky, *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. **355** (2003), 4171–4186 (electronic).
- [29] Yann Palu, *Cluster characters for 2-Calabi-Yau triangulated categories*, arXiv:math/0703540.
- [30] ———, *On algebraic Calabi-Yau categories*, Ph. D. thesis in preparation.
- [31] Idun Reiten, *Tilting theory and cluster algebras*, preprint available at www.institut.math.jussieu.fr/~keller/ictp2006/lecturenotes/reiten.pdf
- [32] C. M. Ringel, *Some remarks concerning tilting modules and tilted algebras. Origin. Relevance. Future.* (An appendix to the Handbook of Tilting Theory.) (ed. L. Angeleri-Hügel, D. Happel, H. Krause), London Math. Soc. Lecture Note Series vol. **332**. Cambridge University Press 2007, 413–472.
- [33] J. Scott, *Grassmannians and cluster algebras*, Proceedings of the London Mathematical Society **92** (2006), 345–380.

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