Reminders on cluster variables and their monoidal categorification

1. Preliminaries on cluster algebras

- Cluster algebras with coefficients ($1 \le n \le m$ integers)

ice quiver $\tilde{Q}$ of type (n,m): $\tilde{Q}_0 = \{1,\cdots,n\} \cup \{n+1,\cdots,m\}$ (frozen vertices)

$\tilde{Q}_1$: no arrows between any vertices $i,j \in \{n+1,\cdots,m\}$

The principal part of $\tilde{Q}$ is the full subquiver Q with $Q_0 = \{1,\cdots,n\}$

The cluster algebra $\mathcal{A}_{\tilde{Q}}$ ($\subset \mathbb{Q}(x_1,\cdots,x_n,f_{n+1},\cdots,f_m)$, if Q is a finite quiver without loops or 2-cycles "good")
are defined as before but only mutations w.r.t. vertices in Q_0 are allowed and arrows between frozen vertices are removed.

We call $\tilde{u} = (u_1,\cdots,u_n,f_{n+1},\cdots,f_m)$ an extended cluster and $u=(u_1,\cdots,u_n)$ a cluster
$u_1,\cdots,u_n$ cluster variables and $f_{n+1},\cdots,f_m$ (belonging to all extended clusters) frozen variables (or coefficients)

The cluster monomials are monomials in the variables of a single extended cluster.

Thm [FZ] Let $\tilde{Q}$ be an ice quiver of type (n,m) whose principal part Q is connected and good. Let $\mathcal{A}_{\tilde{Q}}$ be the associated cluster algebra.

1) Let $u=(u_1,\cdots,u_n)$ be any fixed single cluster. Then every cluster variable of $\mathcal{A}_{\tilde{Q}}$ is a Laurent polynomial in $\mathbb{Z}[u_1^{\pm},\cdots,u_n^{\pm},f_{n+1}^{\pm},\cdots,f_m^{\pm}]$ (i.e., its denominator is a monomial in $u_1,\cdots,u_n,f_{n+1},\cdots,f_m$)

2) The number of cluster variables in $\mathcal{A}_{\tilde{Q}}$ is finite $\Leftrightarrow$ Q is mutation equivalent to an orientation of a Dynkin diagram Δ of type A, D, E. In this case, Δ is unique (we say $\mathcal{A}_{\tilde{Q}}$ or Q is of cluster type Δ) and there is a bijection

$$\{\text{negative simple roots}\} \ni -\alpha_i \longleftrightarrow x[-\alpha_i] =: x_i \in \{\text{initial cluster variables}\}$$

$$\{\text{positive roots}\} \ni \alpha = \sum_{i=1}^{n} d_i\alpha_i \longleftrightarrow x[\alpha] = \frac{P_\alpha}{x_1^{d_1}\cdots x_n^{d_n}} \in \{\text{non initial cluster variables}\}, \quad P_\alpha \in \mathbb{N}[f_{n+1},\cdots,f_m][x_1,\cdots,x_n]$$

Examples 1)

Q:

X_3:
$$\begin{tikzcd} 1 \arrow[d] & 3 \arrow[l] \\ 2 \arrow[ur] & \end{tikzcd}$$

X_4:
$$\begin{tikzcd} 1 \arrow[d] & 3 \arrow[l] \arrow[d] \\ 2 \arrow[ur] & 4 \arrow[l] \end{tikzcd}$$

X_5:
$$\begin{tikzcd} 1 \arrow[d] & 3 \arrow[l] \arrow[d] & 5 \arrow[l] \\ 2 \arrow[ur] & 4 \arrow[l] & \end{tikzcd}$$

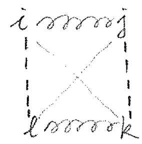
X_6 X_7 X_8 | X_9 $\cdots$

cluster type of $\mathcal{A}_{\tilde{Q}}$:	A_3	D_4	D_5	E_6	E_7	E_8	infinite type

3) $A = \mathbb{C}[Gr(2,n+3)]$ coordinate ring of the Grassmannian of 2-dim. subspaces in $\mathbb{C}^{n+3}$

algebra of polynomial functions on the cone over the Grassmannian of planes in $\mathbb{C}^{n+3}$

$= \mathbb{C}[x_{ij} \mid 1 \le i < j \le n+3] \big/ (\text{plücker relation})$ $\quad i<j<k<l$ $\quad x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}$

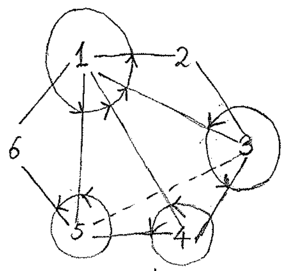


We parameterize the x_{ij} by the sides and diagonals $[ij]$ of an $(n+3)$-gon.

Proposition [FZ] The algebra A has a cluster structure (i.e. $\exists\, \mathcal{A}_{\tilde{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} A$) s.th.

- coefficients: $x_{ij} \mid [ij]$ a side
- cluster variables: $x_{ij} \mid [ij]$ a diagonal
- n-tuple of cluster variables is a cluster $\Leftrightarrow$ the associated diagonals form a triangulation of the $(n+3)$-gon.
- the exchange relations are the Plücker relations
- the cluster type is A_n.

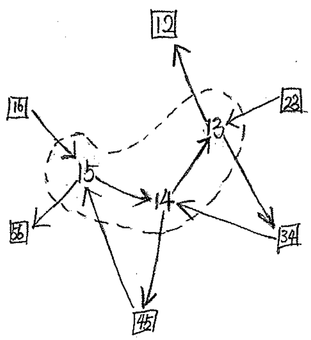
$n=3$



flip

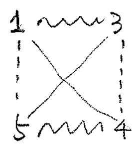
initial extended cluster: $(x_{13}, x_{14}, x_{15} \mid x_{12}, x_{23}, x_{34}, x_{45}, x_{56}, x_{16})$

$\tilde{Q}$:



(underlying graph: dual to the triangulation

orientation: induced by the orientation of the plane



$x_{14}\underbrace{x_{14}'}_{x_{35}} = x_{13}x_{45} + x_{15}x_{34}$

flip

$\tilde{Q}'$:

$\boxed{16} \longrightarrow 15 \longrightarrow 14 \longrightarrow 13$

$\Rightarrow \quad \mathcal{A}_{\tilde{Q}'} = \mathbb{C}[x_{ij} \mid 1 \le i < j \le n+3] \big/ \text{plücker relation} \ \& \ x_{12} = x_{23} = x_{34} = x_{45} = x_{56} = 1$

2. The cluster algebra $\mathcal{A}_l$ ($l \in \mathbb{N}$)

Let Q be a finite bipartite quiver. Set $Q_0 = \underset{\{\text{sinks}\}}{I_0} \sqcup \underset{\{\text{sources}\}}{I_1}$ and

$$\xi_i = \begin{cases} 0 & i \in I_0 \\ 1 & i \in I_1 \end{cases} \qquad \overset{1}{I_1} \longrightarrow \overset{0}{I_0}$$

Put $\hat{I}_0(\ell) = \{(i, \xi_i + 2k) \mid i \in Q_0,\ 0 \le k \le \ell\} \subset \hat{I}_0 = (I_0 \times 2\mathbb{Z}) \cup (I_1 \times (2\mathbb{Z}+1))$

Define a new quiver Γ_ℓ:

$(\Gamma_\ell)_0 = \hat{I}_0(\ell)$

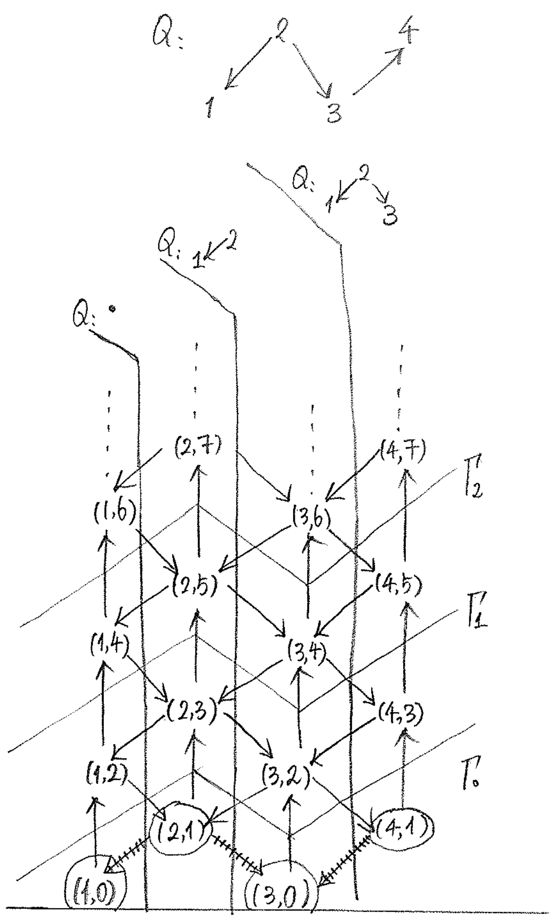
$(\Gamma_\ell)_1$: 1) for each arrow $i_1 \to i_0$ in Q,

drawing $\ell+1$ arrows $(i_1, 2k+1) \to (i_0, 2k)$, $0 \le k \le \ell$

ℓ arrows $(i_0, 2k) \to (i_1, 2k-1)$, $1 \le k \le \ell$

2) adding ℓ arrows $(i, r) \to (i, r+2)$,

where $i \in Q_0$, $r = \xi_i + 2k$ $(0 \le k \le \ell - 1)$



Let $Z = \{ z_{(i,r)} \mid (i,r) \in \hat{I}_0(\ell)\}$ be a set of indeterminates corresponding to $(\Gamma_\ell)_0$. Let (Γ_ℓ, Z) be the initial seed with frozen variables $z_{(i, \xi_i)}$ $(i \in Q_0)$. The associated cluster algebra will be denoted by $\mathcal{A}_\ell$.

Remarks: 1) The algebra $\mathcal{A}_1$ has the same cluster type as Q.

2) $Q: \bullet$, then $\mathcal{A}_\ell$ is of cluster type A_ℓ

$\bar{Q} = A_2$, then $\mathcal{A}_2$ is of cluster type D_4

$\mathcal{A}_3$ E_6

$\mathcal{A}_4$ E_8

$\bar{Q} = A_3$, then $\mathcal{A}_2$ is of cluster type E_6

$\bar{Q} = A_4$, then $\mathcal{A}_2$ is of cluster type E_8.

3) For other cases, $\mathcal{A}_\ell$ is not of finite type

3. Monoidal categorification of cluster algebras

A. In general sense B. Relation between $\mathcal{A}_\ell$ and $\mathcal{C}_\ell$.

A. Defn. A simple object S of a monoidal category $(\mathcal{M}, \otimes)$ is prime if there is no nontrivial factorization $S \simeq S_1 \otimes S_2$
real if $S \otimes S$ is simple.

observation : K field, H: K-bialgebra, $\mathcal{M} := \mathrm{mod}\,H$ is an abelian monoidal category and $K_0(\mathcal{M})$ is a free abelian group with a basis $\{[S] \mid S: \text{simple } H\text{-modules}\}$

ring multiplication: $[L][M] = [L \otimes M]$

where $L \otimes M = (L \otimes_K M,\ (\ell \otimes m)\cdot h = \sum \ell h_i' \otimes m h_i'')$ $\quad \Delta(h) = \sum h_i' \otimes h_i''$

Defn. Let $\mathcal{A}$ be a cluster algebra (with coefficients) and $\mathcal{M}$ an abelian monoidal category.
Then $\mathcal{M}$ is said to be a monoidal categorification of $\mathcal{A}$ if there is a ring isomorphism

$$K_0(\mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathcal{A}$$

s.th. [real simple object of $\mathcal{M}$] corresponds to 'cluster monomial of $\mathcal{A}$',

[real prime simple object of $\mathcal{M}$] corresponds to 'variable of $\mathcal{A}$'.
$$S(x) \longleftrightarrow x \qquad \text{variable: cluster / frozen}$$

Proposition [HL] Suppose that the cluster algebra $\mathcal{A}$ has a monoidal categorification $\mathcal{M}$. Then

(1) Cluster monomials of $\mathcal{A}$ are linearly independent.

(2) Every cluster variable of $\mathcal{A}$ has a Laurent expansion with positive coefficients w.r.t. any fixed single cluster. $(\mathbb{N}[u_1^{\pm}, \cdots, u_n^{\pm}, f_{n+1}^{\pm}, \cdots, f_m^{\pm}])$

Proof (1) $\{[S] \mid S: \text{simple object of } \mathcal{M}\}$ is a $\mathbb{Z}$-basis of $K_0(\mathcal{M})$

$\cup$ $\qquad$ Moreover, if all simple objects are real, then cluster monomials form a $\mathbb{Q}$-basis of $\mathcal{A}$.

$\{[S] \mid S: \text{real simple object of } \mathcal{M}\} \longleftrightarrow \{\text{cluster monomials of } \mathcal{A}\}$

$\Rightarrow$ linearly independent.

(2) Let $u = (u_1, \cdots, u_n)$ be a fixed cluster and z any cluster variable. Then

$$z = \frac{N(u_1, \cdots, u_n, f_{n+1}, \cdots, f_m)}{u_1^{d_1} \cdots u_n^{d_n} f_{n+1}^{d_{n+1}} \cdots f_m^{d_m}}, \qquad \mathcal{N} := N(u_1, \cdots, u_n, f_{n+1}, \cdots, f_m) \in \mathbb{Z}[u_1, \cdots, u_n, f_{n+1}, \cdots, f_m].$$

So we have $\mathcal{N} = z u_1^{d_1} \cdots u_n^{d_n} f_{n+1}^{d_{n+1}} \cdots f_m^{d_m}$ is a monomial in cluster variables (not a cluster monomial).
Hence, $\mathcal{N}$ corresponds to the isoclass of $N := S(z) \otimes S(u_1)^{\otimes d_1} \otimes \cdots \otimes S(u_n)^{\otimes d_n} \otimes S(f_{n+1})^{\otimes d_{n+1}} \otimes \cdots \otimes S(f_m)^{\otimes d_m}$.
Let $m = u_1^{r_1} \cdots u_n^{r_n} f_{n+1}^{r_{n+1}} \cdots f_m^{r_m}$ be a cluster monomial in $\mathcal{N}$. Then m corresponds to the isoclass of $M := S(u_1)^{\otimes r_1} \otimes \cdots \otimes S(u_n)^{\otimes r_n} \otimes S(f_{n+1})^{\otimes r_{n+1}} \otimes \cdots \otimes S(f_m)^{\otimes r_m}$ which is simple.
Therefore, the coefficient of m in $\mathcal{N}$ = # of M as a composition factor of $N \geq 0$.

Remark: A cluster algebra which has a monoidal categorification can also provide a combinatorial description about the tensor structure of its monoidal categorification.

B. Notation Let $\mathfrak{g}$ be a simple Lie algebra$/\mathbb{C}$ of type A, D, E and $q \in \mathbb{C}^*$ not a root of 1.

Denote by $I=[1,n]$ the set of vertices of the Dynkin diagram of $\mathfrak{g}$.

(it is a bipartite graph: $I = I_0 \sqcup I_1$ s.th. every edge connects a vertex of I_0 with a vertex of I_1. Set $\xi_i = \begin{cases} 0 & i\in I_0 \\ 1 & i \in I_1 \end{cases}$)

$\mathcal{C}$ = finite dimensional $U_q(\hat{\mathfrak{g}})$-module of type 1.

$\mathcal{M} = \{\text{Laurent monomials in variables } Y_{i,a}\}_{\substack{i\in I\\ a\in\mathbb{C}^*}}$

$\mathcal{M}_\ell = \{\text{monomials in } Y_{i,q^{\xi_i+2k}}\}_{\substack{i\in I\\ 0\le k\le \ell}} \subset \mathcal{M}_+ = \{\text{dominant monomials}\}$

[CP] $\forall\ m = \prod_{i\in I}\prod_{k=1}^{n_i} Y_{i,a_{i,k}} \in \mathcal{M}_+$,

$\exists!$ simple object $L(m)$ $\begin{cases} \text{with Drinfeld polynomials } P=(P_i)_{i\in I},\ P_i(z) = \prod_{k=1}^{n_i}(1-a_{i,k}z) \\ (\text{equ. saying: highest } \ell\text{-weight as } m.) \end{cases}$

Let $\mathcal{C}_\ell$ be the full subcategory of $\mathcal{C}$ whose objects have all their composition factors of the form $L(m)$ with $m\in\mathcal{M}_\ell$.

Proposition [HL] $\mathcal{C}_\ell$ is an abelien monoidal category with Grothendieck ring

$$K_0(\mathcal{C}_\ell) = \mathbb{Z}[[L(Y_{i,q^{\xi_i+2k}})] \mid i\in I,\ 0\le k\le \ell].$$

case $\ell=0$: every simple object of $\mathcal{C}_0$ is a tensor product of $L(Y_{i,q^{\xi_i}})_{i\in I}$,
[FM] and any tensor product of $L(Y_{i,q^{\xi_i}})_{i\in I}$ is simple in $\mathcal{C}_0$.

$$\mathcal{A}_0 = \mathbb{Q}[z_{(i,\xi_i)} \mid i\in I] \longrightarrow K_0(\mathcal{C}_0)\otimes_{\mathbb{Z}}\mathbb{Q}$$

$$z_{(i,\xi_i)} \longmapsto [L(Y_{i,q^{\xi_i}})]$$

frozen variable — real prime simple object

all simple objects are real.

$\mathcal{C}_0$ is a monoidal categorification of $\mathcal{A}_0$.

Conjecture [HL]. The assignment $z_{(i,\xi_i+2k)} \longrightarrow [W^{(i)}_{\ell+1-k,q^{\xi_i+2k}}]$,

where $W^{(i)}_{\ell+1-k,q^{\xi_i+2k}} = L\left(\prod_{j=0}^{\ell-k} Y_{i,q^{\xi_i+2(k+j)}}\right)$, extends to a ring isomorphism from $\mathcal{A}_\ell$ to $K_0(\mathcal{C}_\ell)\otimes_{\mathbb{Z}}\mathbb{Q}$.

Moreover, the map makes $\mathcal{C}_\ell$ to be a monoidal categorification of $\mathcal{A}_\ell$.

Remark: $\mathfrak{g}=\mathfrak{sl}_2$, then the conjecture holds for all $\ell\in\mathbb{N}$ [CP].

Theorem. [HL] [N] $\mathfrak{g} = A, D, E$ and $\ell=1$, the conjecture holds. Moreover, in this case, all simple modules are real.

• Illustration of the case $\mathfrak{g} = sl_2$ for any $\ell \in \mathbb{N}$.

Γ_ℓ: $\boxed{\cdot}_{z_0} \longrightarrow \cdot_{z_2} \longrightarrow \cdot_{z_4} \longrightarrow \cdot_{z_6} \cdots \longrightarrow \cdot_{z_{2\ell}}$ ($\ell=3$) (drop index i and let $\xi = 0$)

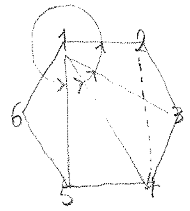
W_{4,q^0} W_{3,q^2} W_{2,q^4} W_{1,q^6}

$A_3 = \mathbb{Q}[x_{ij} \mid 1 \le i < j \le 6] \,/\,$ plücker relation & $x_{12} = x_{23} = x_{34} = x_{45} = x_{56} = 1$

9 cluster variables

$x_{16} = z_0, \quad x_{15} = z_2,$

$x_{14} = z_4, \quad x_{13} = z_6.$



[CP]: The kirillov-Reshetikhin modules $W_{k,a}$ ($a \in \mathbb{C}^*$, $k \in \mathbb{N}^*$) are the only (real) prime simple objects.

Then all real prime simple objects in $\mathcal{C}_3$ are as follows (Note that $K_0(\mathcal{C}_3) = \mathbb{Z}[[L(Y_{q^{2k}})] \mid 0 \le k \le \ell]$)

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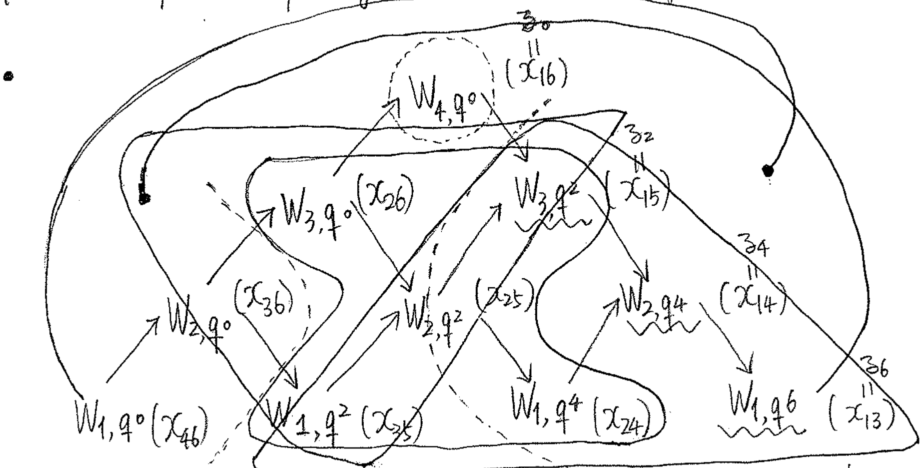


T-system for sl_2:

$[W_{k,a}][W_{k,aq^2}]$

$= [W_{k+1,a}][W_{k-1,aq^2}] + 1.$

Fact: the T-system equations satisfied by KR modules are of the same form as the exchange relations of the cluster algebra.

ex: $z_6' z_6 = x_{14}x_{23} + x_{12}x_{34} = z_4 + 1 \qquad (z_6' = x_{24})$

$[W_{1,q^4}][W_{1,q^6}] = [W_{2,q^4}] + 1$

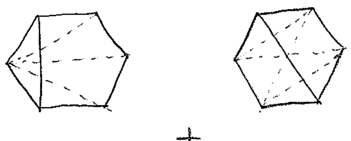
$z_0^{r_0} z_2^{r_2} z_4^{r_4} z_6^{r_6} \longleftrightarrow W_{4,q^0}^{\otimes r_0} \otimes W_{3,q^2}^{\otimes r_2} \otimes W_{2,q^4}^{\otimes r_4} \otimes W_{1,q^6}^{\otimes r_6}$ real simple.

roots: $\{q^{-6}, q^{-4}, q^{-2}, q^0\} \supset \{q^{-6}, q^{-4}, q^{-2}\} \supset \{q^{-6}, q^{-4}\} \supset \{q^{-6}\}$

$z_0^{r_0} z_2^{r_2} z_4^{r_4} z_6'^{s_6} \longleftrightarrow \cdots\cdots\cdots \otimes W_{1,q^4}^{s_6}$

$\cdots\cdots \supset \{q^{-4}\}$

• 14 clusters:



$6 \times 5 \times 2 + 3 \times 4 \times 2 \,/\, P_3^3 = 14$

general position:

• $\not\cup$ not a q-string

• $\subset$ or $\supset$

14 choices

• ring isomorphism:

$u = (x_{13}, x_{14}, x_{15})$

mut $\downarrow$ $\quad x_{24} \quad x_{35} \quad x_{46}$

[BFZ] The standard monomials in $x_{13}, x_{24}, x_{14}, x_{35}, x_{15}, x_{46}$ form a $\mathbb{Q}[x_{16}]$-basis of A_3.

Note: $x_{14} = x_{13}x_{24} - 1$, $x_{15} = x_{14}x_{35} - x_{13}$, $x_{16} = x_{15}x_{46} - x_{14}$

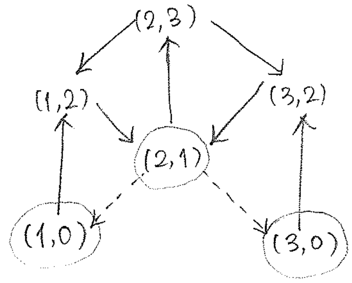
Therefore, $A_3 = \mathbb{Q}[x_{13}, x_{24}, x_{35}, x_{46}] \cong K_0(\mathcal{C}_3) \otimes_{\mathbb{Z}} \mathbb{Q}$.

• Illustration of the case $\mathfrak{g} = \mathfrak{sl}_4$ and $\ell = 1$

(« Cluster algebras and quantum affine algebras » Section 4)

Q: $\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 1 & & 3 \end{tikzcd}$

Γ_1:



frozen
$$\begin{cases} z_{(1,0)} =: f_1 \longmapsto W^{(1)}_{2,q^0} = L(Y_{1,q^0}Y_{1,q^2}) =: F_1 \\ z_{(2,1)} =: f_2 \longmapsto W^{(2)}_{2,q} = L(Y_{2,q}Y_{2,q^3}) =: F_2 \\ z_{(3,0)} =: f_3 \longmapsto W^{(3)}_{2,q^0} = L(Y_{3,q^0}Y_{3,q^2}) =: F_3 \end{cases}$$

initial
$$\begin{cases} x[-\alpha_1] = z_{(1,2)} \longmapsto W^{(1)}_{1,q^2} = L(Y_{1,q^2}) =: S(-\alpha_1) \\ x[-\alpha_2] = z_{(2,3)} \longmapsto W^{(2)}_{1,q^3} = L(Y_{2,q^3}) =: S(-\alpha_2) \\ x[-\alpha_3] = z_{(3,2)} \longmapsto W^{(3)}_{1,q^2} = L(Y_{3,q^2}) =: S(-\alpha_3) \end{cases}$$

$$K_0(\mathcal{C}_1) = \mathbb{Z}\big[\,[L(Y_{1,q^0})],\ [L(Y_{1,q^2})],\ [L(Y_{2,q})],\ [L(Y_{2,q^3})],\ [L(Y_{3,q^0})],\ [L(Y_{3,q^2})]\,\big]$$

Fact: the T-systems satisfied by KR modules:

$$[W^{(i)}_{k,a}][W^{(i)}_{k,aq^2}] = [W^{(i)}_{k+1,a}][W^{(i)}_{k-1,aq^2}] + \prod_{a_{ij}=-1}[W^{(j)}_{k,aq}]$$

are of the same form as the exchange relations of the cluster algebra.

$$x[\alpha_2] = \frac{z_{(1,2)}z_{(3,2)} + f_2}{z_{(2,3)}} \longmapsto [L(Y_{2,q})] =: S(\alpha_2)$$

$$x[\alpha_1+\alpha_2] = \frac{z_{(1,2)}z_{(3,2)} + f_2 + z_{(2,3)}f_1}{z_{(1,2)}z_{(2,3)}} \longmapsto [L(Y_{1,q^0})] =: S(\alpha_1+\alpha_2)$$

$$x[\alpha_2+\alpha_3] = \frac{z_{(1,2)}z_{(3,2)} + f_2 + z_{(2,3)}f_3}{z_{(3,2)}z_{(2,3)}} \longmapsto [L(Y_{3,q^0})] =: S(\alpha_2+\alpha_3)$$

not KR module
$$\begin{cases} x[\alpha_1] = \dfrac{z_{(2,3)}f_1 + f_2}{z_{(1,2)}} \rightsquigarrow [L(Y_{1,q^0}Y_{2,q^3})] =: S(\alpha_1) \\ x[\alpha_3] = \dfrac{z_{(2,3)}f_3 + f_2}{z_{(3,2)}} \rightsquigarrow [L(Y_{2,q^3}Y_{3,q^0})] =: S(\alpha_3) \\ x[\alpha_1+\alpha_2+\alpha_3] = \dfrac{z_{(1,2)}z_{(2,3)}f_2 + f_2^2 + z_{(2,3)}f_2f_3 + z_{(2,3)}f_1f_2 + z_{(2,3)}^2 f_1f_3}{z_{(1,2)}z_{(2,3)}z_{(3,2)}} \rightsquigarrow [L(Y_{1,q^0}Y_{2,q^3}Y_{3,q^0})] =: S(\alpha_1+\alpha_2+\alpha_3) \end{cases}$$

[this correspondence comes from the ring isom and equation (*), we have that $L(Y_{1,q^0}) \otimes L(Y_{2,q^3}) \twoheadrightarrow S(\alpha_1)$]

Then we have the following expressions

$$f_1 = x[-\alpha_1]\,x[\alpha_1+\alpha_2] - x[\alpha_2]$$

$$f_2 = x[-\alpha_2]\,x[\alpha_2] - x[-\alpha_1]\,x[-\alpha_3]$$

$$f_3 = x[-\alpha_3]\,x[\alpha_2+\alpha_3] - x[\alpha_2]$$

$$x[\alpha_1] = x[-\alpha_2]\,x[\alpha_1+\alpha_2] - x[-\alpha_3] \quad (*)$$

$$x[\alpha_3] = x[-\alpha_2]\,x[\alpha_2+\alpha_3] - x[-\alpha_1]$$

$$x[\alpha_1+\alpha_2+\alpha_3] = x[-\alpha_2]\,x[\alpha_1+\alpha_2]\,x[\alpha_2+\alpha_3] + x[\alpha_2] - x[-\alpha_1]\,x[\alpha_1+\alpha_2] - x[-\alpha_3]\,x[\alpha_2+\alpha_3]$$

[BFZ] $\quad \mathcal{U} = (x[-\alpha_1],\ x[\alpha_2],\ x[-\alpha_3])$

$\mu_1(x[-\alpha_1]) = x[\alpha_1+\alpha_2]$,
$\mu_2(x[\alpha_2]) = x[-\alpha_2]$,
$\mu_3(x[-\alpha_3]) = x[\alpha_2+\alpha_3]$.

Q':

$$\begin{tikzcd}
 & (2,3) \arrow[d] & \\
(1,2) \arrow[ur] & & (3,2) \arrow[ul] \\
 & (2,1) & \\
(1,0) \arrow[uu] & & (3,0) \arrow[uu]
\end{tikzcd}$$

$(Q', \mathcal{U})$ is an acyclic seed

(monomials containing no product of the form $x[-\alpha_1]x[\alpha_1+\alpha_2]$, $x[\alpha_2]x[-\alpha_2]$ or $x[-\alpha_3]x[\alpha_2+\alpha_3]$)

Then the standard monomials in $x[-\alpha_1], x[\alpha_1+\alpha_2], x[\alpha_2], x[-\alpha_2], x[-\alpha_3], x[\alpha_2+\alpha_3]$ form a $\mathbb{Q}[f_1, f_2, f_3]$-basis of A_1.

Since $f_1 = x[-\alpha_1]x[\alpha_1+\alpha_2] - x[\alpha_2]$, $f_2 = x[-\alpha_2]x[\alpha_2] - x[-\alpha_1]x[-\alpha_3]$ and $f_3 = x[-\alpha_3]x[\alpha_2+\alpha_3] - x[\alpha_2]$,

all monomials in $x[-\alpha_1], x[\alpha_1+\alpha_2], x[\alpha_2], x[-\alpha_2], x[-\alpha_3], x[\alpha_2+\alpha_3]$ is a $\mathbb{Q}$-basis of A_1.

Hence, $A_1 = \mathbb{Q}[x[\alpha_1+\alpha_2], x[-\alpha_1], x[\alpha_2], x[-\alpha_2], x[\alpha_2+\alpha_3], x[-\alpha_3]]$

$$\begin{array}{cccccccc}
A_1 & x[\alpha_1+\alpha_2] & x[-\alpha_1] & x[\alpha_2] & x[-\alpha_2] & x[\alpha_2+\alpha_3] & x[-\alpha_3] & \\
\downarrow S & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{ring isom.} \\
K_0(\mathcal{C}_1)\otimes_{\mathbb{Z}}\mathbb{Q} & L(Y_{1,q^0}) & L(Y_{1,q^2}) & L(Y_{2,q}) & L(Y_{2,q^3}) & L(Y_{3,q^0}) & L(Y_{3,q^2}) &
\end{array}$$

[Theorem-Conjecture]

$\mathcal{C}_1$: a monoidal categorification of A_1,

and every simple object of $\mathcal{C}_1$ is real.

$\Longrightarrow$ $\mathcal{C}_1$ has finitely many $(\# = 12)$ (real) prime simple objects: $S(\beta)_{\beta \in \Phi_{\geq -1}}$, $F_{i=1,2,3}$;

the simple objects of $\mathcal{C}_1$ are exactly all tensor products of the form

$$S(\beta_1)^{\otimes k_1} \otimes S(\beta_2)^{\otimes k_2} \otimes S(\beta_3)^{\otimes k_3} \otimes F_1^{\otimes l_1} \otimes F_2^{\otimes l_2} \otimes F_3^{\otimes l_3}, \quad k_i, l_i \in \mathbb{N}$$

where $\{\beta_1, \beta_2, \beta_3\}$ runs over the 14 clusters;

all the tensor powers of a simple object of $\mathcal{C}_1$ are simple;

all cluster monomials form a $\mathbb{Q}$-basis of A_1.