THE HALL ALGEBRA OF A SPHERICAL OBJECT

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Abstract. We determine the Hall algebra, in the sense of Toën, of the triangulated category generated by a spherical object of dimension greater or equal to 3.

1. Introduction

This note is motivated by recent developments in the categorification of cluster algebras and cluster varieties. Let us recall the context: To a finite quiver Q without loops and without 2-cycles, one can associate the cluster algebra $\mathcal{A}_Q$ and the cluster variety $\mathcal{X}_Q$ (endowed with a Poisson structure), *cf.* [4] and [3]. If Q does not have oriented cycles, we have at our disposal a very good categorical model for the combinatorics of the cluster algebra $\mathcal{A}_Q$, *cf.* the surveys [1] [11] [12] [9]. In contrast, for the moment, there is no corresponding theory for the cluster variety $\mathcal{X}_Q$. Ongoing work by Kontsevich-Soibelman [10], Bridgeland [2] and others shows that there is a close link between the quantized version [3] of $\mathcal{X}_Q$ and the Hall algebra [13] of a certain triangulated 3-Calabi-Yau category $\mathcal{T}_Q$ associated with Q. The category $\mathcal{T}_Q$ can be described as the triangulated category generated by the objects in a 'generic' collection of 3-spherical objects whose extension spaces have dimensions encoded by the quiver Q. Alternatively, it may be described as the derived category of dg modules with finite-dimensional total homology over the Ginzburg dg algebra [5] associated with Q and a generic potential. In this note, we consider the case where Q is reduced to a single vertex without any arrows. For this simplest non empty quiver, we classify the objects of $\mathcal{T}_Q$ (Theorem 3.1), compute its Hall algebra (Theorem 4.1) and establish the link with the cluster variety, which in this case is just a one-dimensional torus (section 6). The Hall algebra of the triangulated category generated by a spherical object of arbitrary dimension ≥ 3 can be determined similarly. We give the result in section 5. For the classification theorem, we establish more generally the classification of the indecomposable objects in a triangulated category admitting a generator whose graded endomorphism algebra is hereditary, a result which may be useful in other contexts as well.

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2. Classification

In this section, we present a general classification theorem for indecomposable objects in a triangulated category admitting a 'generator' G whose graded endomorphism algebra is hereditary. We first consider the case where G compactly generates a triangulated category with arbitrary direct sums. Then we consider the case where G is a classical generator. We apply it to the perfect and the finite dimensional derived categories of the Ginzburg algebra of type A_1.

2.1. **Compactly generated case.** Let k be a field, and $\mathcal{T}$ a k-linear Hom-finite triangulated category with suspension functor Σ. Assume $\mathcal{T}$ has arbitrary direct sums. Let G be a *compact generator* for $\mathcal{T}$, i.e. the functor $\mathrm{Hom}_{\mathcal{T}}(G, ?)$ commutes with arbitrary direct sums, and given an object X of $\mathcal{T}$, if $\mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)$ vanishes for all integers p, then X vanishes. Let

$$A = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p G)$$

be the graded endomorphism algebra of G. Then for any object X of $\mathcal{T}$, the graded vector space

$$\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)$$

has a natural graded (right) module structure over A. We define a functor

$$F : \mathcal{T} \to \mathrm{Grmod}(A), X \mapsto \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X).$$

Notice that since G is a compact generator, a morphism of $\mathcal{T}$ is invertible if and only if its image under F is invertible.

We say that A is *graded hereditary*, if the category $\mathrm{Grmod}(A)$ of graded A-modules is hereditary, or in other terms, each subobject of a projective object of $\mathrm{Grmod}(A)$ is projective.

Theorem 2.1. *With the notations above, suppose that A is graded hereditary. The functor $F : \mathcal{T} \to \mathrm{Grmod}(A)$ is full, essentially surjective, and its kernel has square zero. In particular, it induces a bijection from the set of isoclasses of objects (respectively, of indecomposable objects) of $\mathcal{T}$ to that of* $\mathrm{Grmod}(A)$.

Remarks. *a) Notice that we have an isomorphism of functors $F\circ\Sigma \simeq [1]\circ F$, where $[1]$ denotes the shift functor in* $\mathrm{Grmod}(A)$.

b) The functor F is obviously a homological functor. We will use this fact implicitly.

The theorem is a consequence of the following lemmas.

For a class $\mathcal{S}$ of objects of an additive category $\mathcal{A}$ with arbitrary direct sums, we denote by $\mathrm{Add}(\mathcal{S})$ the closure of $\mathcal{S}$ under taking all direct sums and direct summands.

Lemma 2.2. *a) The functor $F : \mathcal{T} \to \mathrm{Grmod}(A)$ induces an equivalence between $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ and $\mathrm{Add}(A[p] | p \in \mathbb{Z})$.*

b) An object X belongs to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ if and only if FX belongs to $\mathrm{Add}(A[p] | p \in \mathbb{Z})$.

Proof. Part a) and the necessity in b) are easy, using that G is compact. Let us prove the sufficiency in b). If we have an isomorphism $f : P \to FX$ in $\mathrm{Add}(A[p]|p \in \mathbb{Z})$, we can lift it to a morphism $\tilde{f} : M \to X$ in $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$. Then $\tilde{f}$ is an isomorphism since $f = F\tilde{f}$ is. □

We remark that the class of projective objects of $\mathrm{Grmod}(A)$ is exactly $\mathrm{Add}(A[p]|p \in \mathbb{Z})$.

Lemma 2.3. *The functor F is essentially surjective.*

Proof. Let M be an object in $\mathrm{Grmod}(A)$. We want to find X in $\mathcal{T}$ such that $FX \cong M$. Since $\mathrm{Grmod}(A)$ is hereditary, there exists a short exact sequence of graded A-modules

$$0 \to P_1 \xrightarrow{u} P_0 \to M \to 0$$

with $P_0, P_1 \in \mathrm{Add}(A[p]|p \in \mathbb{Z})$. By Lemma 2.2, there are two objects G_0, G_1 in $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$ and $v \in \mathrm{Hom}_{\mathcal{T}}(G_1, G_0)$ such that

$$F(G_1 \xrightarrow{v} G_0) \cong (P_1 \xrightarrow{u} P_0).$$

Let X be a cone of v, i.e. we have a triangle

$$G_1 \xrightarrow{v} G_0 \xrightarrow{w} X \to \Sigma G_1$$

in $\mathcal{T}$. We apply the homological functor F and obtain an exact sequence

$$F(\Sigma^{-1}X) \to FG_1 \xrightarrow{Fv} FG_0 \xrightarrow{Fw} FX \to F(\Sigma G_1) \xrightarrow{F(\Sigma v)} F(\Sigma G_0).$$

Recall that Fv is injective, and so is $F(\Sigma v) = (Fv)[1]$. Therefore Fw is surjective. Then we obtain a short exact sequence in $\mathrm{Grmod}(A)$

$$0 \to P_1 \xrightarrow{u} P_0 \to FX \to 0,$$

and hence $FX \cong M$. □

Lemma 2.4. *The functor F is full.*

Proof. We prove this in three steps.

Step 1: By definition, we have

$$\mathrm{Hom}_A(A, FX) \cong (FX)_0 = \mathrm{Hom}_{\mathcal{T}}(G, X)$$

for any object X in $\mathcal{T}$, and so the map $F(G, X)$ is bijective. Therefore the map $F(G_0, X)$ is an isomorphism for any G_0 in $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$ and any X in $\mathcal{T}$.

Step 2: Let X be an object of $\mathcal{T}$. We will show that there exists a triangle

$$G_1 \to G_0 \to X \to \Sigma G_1$$

in $\mathcal{T}$ such that G_0, G_1 belong to $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$. We choose $w : G_0 \to X$ such that Fw is surjective. We form the triangle

$$Y \to G_0 \xrightarrow{w} X \to \Sigma Y.$$

We apply F and obtain an exact sequence

$$F(\Sigma^{-1}G_0) \xrightarrow{F(\Sigma^{-1}w)} F(\Sigma^{-1}X) \to FY \to FG_0 \xrightarrow{Fw} FX \to F(\Sigma Y).$$

Both Fw and $F(\Sigma^{-1}w) = (Fw)[-1]$ are surjective, so we obtain a short exact sequence

$$0 \to FY \to FG_0 \to FX \to 0.$$

Thus FY belongs to $\mathrm{Add}(A[p]|p \in \mathbb{Z})$ since $\mathrm{Grmod}(A)$ is hereditary. By Lemma 2.2 b), the object Y belongs to $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$. Now it suffices to take $G_1 = Y$.

Step 3: Let X, Y be objects in $\mathcal{T}$. By Step 2, there is a triangle in $\mathcal{T}$

$$G_1 \to G_0 \to X \to \Sigma G_1,$$

where G_0, G_1 belong to $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$, whose image under F is a short exact sequence in $\mathrm{Grmod}(A)$

$$0 \to FG_1 \to FG_0 \to FX \to 0.$$

If we apply $\mathrm{Hom}_{\mathcal{T}}(?, Y)$ to the triangle and $\mathrm{Hom}_A(?, FY)$ to the short exact sequence, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathcal{T}(\Sigma G_1, Y) & \longrightarrow & \mathcal{T}(X, Y) & \longrightarrow & \mathcal{T}(G_0, Y) & \longrightarrow & \mathcal{T}(G_1, Y) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (FX, FY) & \longrightarrow & (FG_0, FY) & \longrightarrow & (FG_1, FY), \end{array}$$

where the parentheses $(,)$ in the second row denote the groups of A-linear maps. By Step 1, the rightmost two vertical maps are isomorphisms. Therefore, the leftmost vertical map is surjective. Since X and Y are arbitrary, we have proved that F is full. $\square$

Lemma 2.5. *Let $J = \{f \in \mathrm{Mor}(\mathcal{T}) | Ff = 0\}$. Then $J^2 = 0$.*

Proof. Let $f : X \to Y$ be a morphism in J, that is, for any $p \in \mathbb{Z}$ and for any morphism $u : G \to \Sigma^p X$, we have $\Sigma^p f \circ u = 0$.

Let $G_1 \xrightarrow{u} G_0 \xrightarrow{v} X \xrightarrow{w} \Sigma G_1$ be a triangle in $\mathcal{T}$ such that G_0, G_1 belong to $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$. Since f belongs to J, we have $f \circ v = 0$. Therefore, the morphism f factors through w, that is, there is $f' \in \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, Y)$ such that $f = f' \circ w$.

Let $G'_1 \xrightarrow{u'} G'_0 \xrightarrow{v'} Y \xrightarrow{w'} \Sigma G'_1$ be a triangle in $\mathcal{T}$ such that G'_0, G'_1belong to $\mathrm{Add}(\Sigma^p G|p \in \mathbb{Z})$ and Fv' is surjective. Then the induced homomorphism $\mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, G'_0) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, Y)$ is surjective. Therefore there is $h \in \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, G'_0)$ such that $f' = v' \circ h$.

Now let $g : Y \to Z$ be another morphism in J. By the arguments in the second paragraph there is $g' : \Sigma G'_1 \to Z$ such that $g = g' \circ w'$. Thus we have $g \circ f = g' \circ w' \circ v' \circ h \circ w = 0$, and we are done. $\square$

2.2. **Classically generated case.** Let $\mathcal{T}$ be a k-linear Hom-finite triangulated category with suspension functor Σ. Let G be a *classical generator* for $\mathcal{T}$, i.e. $\mathcal{T}$ is the closure of G under taking shifts, extensions and direct summands. Let

$$A = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p G)$$

be the graded endomorphism algebra of G. We assume that the category $\mathrm{grmod}(A)$ of finitely presented graded A-modules is abelian (i.e. A is graded right coherent) and hereditary.

Theorem 2.6. *The functor*

$$F : \mathcal{T} \to \mathrm{grmod}(A), X \mapsto \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)$$

is well-defined, full, essentially surjective, and its kernel has square zero. In particular, it induces a bijection from the set of isoclasses of objects (respectively, of indecomposable objects) of $\mathcal{T}$ to that of $\mathrm{grmod}(A)$*.*

Proof. Lemma 2.2, 2.3, 2.4 and 2.5 and their proofs are still valid, mutatis mutandis. For example, we need to replace Add by add in the statement of Lemma 2.2, where for a class $\mathcal{S}$ of objects of an additive category $\mathcal{A}$, we denote by $\mathrm{add}(\mathcal{S})$ the closure of $\mathcal{S}$ under taking direct summands and finite direct sums. It remains to prove that F is well-defined, that is, for any object X of $\mathcal{T}$, the graded A-module FX is indeed finitely presented.

Let $\mathcal{T}'$ be the full additive subcategory of $\mathcal{T}$ consisting of those objects X such that $F(X)$ is a finitely presented A-module. Evidently G belongs to $\mathcal{T}'$. Thus, in order to conclude that $\mathcal{T}'$ equals $\mathcal{T}$, it suffices to show that $\mathcal{T}'$ is stable under shifts, direct summands and extensions. The first two points are clear.

Suppose that we have a triangle

$$Y \xrightarrow{u} Z \xrightarrow{v} X \xrightarrow{w} \Sigma Y$$

in $\mathcal{T}$ such that FY and FX are finitely presented. Then the objects

$$F(\Sigma^{-1}X) = (FX)[-1] \text{ and } F(\Sigma Y) = (FY)[1]$$

are also finitely presented. We apply F to the above triangle to obtain an exact sequence

$$F(\Sigma^{-1}X) \xrightarrow{F(\Sigma^{-1}w)} F(Y) \xrightarrow{Fu} FZ \xrightarrow{Fv} FX \xrightarrow{Fw} F(\Sigma Y).$$

Note that all components except possibly FZ are finitely presented. Since the category $\mathrm{grmod}(A)$ of finitely presented graded A-modules is abelian, the kernel $\ker Fv = \mathrm{coker} F(\Sigma^{-1}w)$ of Fv and the image $\mathrm{im} Fv = \ker Fw$ of Fv are also finitely presented. Consequently FZ is finitely presented and $\mathcal{T}'$ is stable under extensions. Therefore, the functor F is well-defined. □

Examples 2.7. *a) Let B be a finite dimensional hereditary algebra over the field k. Let $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}B)$ be the bounded derived category of finite dimensional B-modules, and let G be the free B-module of rank* 1*. Then $A = B$ and the functor $F : \mathcal{D}^b(\mathrm{mod}B) \to \mathrm{grmod}(B)$ takes X to its total homology H^*X.*

b) Let $B = k[\varepsilon]/(\varepsilon^2)$ be the algebra of dual numbers. Let $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}B)$ be the bounded category of finite dimensional B-modules, and G be the simple module k. Then $A = \mathrm{Ext}^*_B(k,k) \cong k[u]$ with $\deg(u) = 1$.

c) Let $\tilde{A}$ be a differential graded algebra (dg algebra for short) such that the category of finitely presented graded modules over the graded algebra $A = H^*(\tilde{A})$ is abelian and hereditary. Let $\mathcal{T} = \mathrm{per}(\tilde{A})$ be the perfect derived category and let G be the free dg $\tilde{A}$-module of rank 1*. Then the functor F takes X to its total homology viewed as a graded A-module.*

3. Application of the classification

Let k be a field. Let Γ denote the Ginzburg dg algebra of type A_1 over k, i.e. Γ is the dg algebra $k[t]$ with $\deg(t) = -2$ and trivial differential.

Denote by $\mathrm{per}(\Gamma)$ the perfect derived category, i.e. the smallest thick subcategory of the derived category $\mathcal{D}(\Gamma)$ containing Γ and by $\mathcal{D}_{fd}(\Gamma)$ the finite dimensional derived category, i.e. the full triangulated subcategory consisting of the dg Γ-modules whose homology has finite total dimension (cf. [7]). The triangulated category $\mathcal{D}_{fd}(\Gamma)$ is Hom-finite and 3-Calabi-Yau (cf. [8]). Notice that a triangulated category classically generated by a 3-sperical object is triangle equivalent to $\mathcal{D}_{fd}(\Gamma)$.

Let [1] denote the shift functor of the category $\mathrm{grmod}(\Gamma)$ of finitely presented graded Γ-modules. For an integer p and a strictly positive integer n, the finite dimensional graded Γ-module $\Gamma/(t^n\Gamma)[p]$, viewed as an object in $\mathcal{D}_{fd}(\Gamma)$, is indecomposable.

Theorem 3.1. *a) Each indecomposable object in* $\mathrm{per}(\Gamma)$ *is isomorphic to either* $\Gamma/(t^n\Gamma)[p]$ *for some integer* p *and some strictly positive integer* n *or* $\Gamma[p]$ *for some integer* p*.*

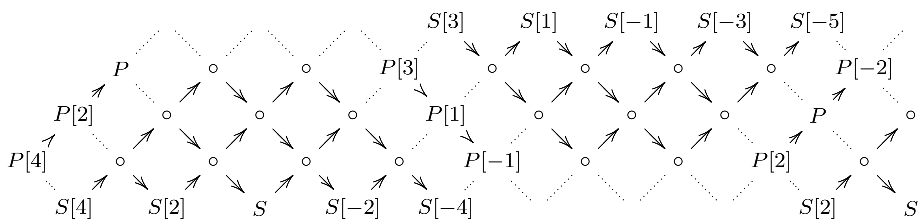
b) Each indecomposable object in $\mathcal{D}_{fd}(\Gamma)$ *is isomorphic to* $\Gamma/(t^n\Gamma)[p]$ *for some integer* p *and some strictly positive integer* n*.*

Proof. It is readily seen that the category $\mathrm{grmod}(A)$ for $A = H^*(\Gamma)(= \Gamma$ as graded algebras) is abelian and hereditary. We are therefore in a particular case of Example 2.7 c). The functor

$$F = H^* : \mathrm{per}(\Gamma) \to \mathrm{grmod}(\Gamma)$$

induces a bijection between the set of isoclasses of indecomposable objects of $\mathrm{per}(\Gamma)$ and that of $\mathrm{grmod}(A)$. Moreover, the full subcategory $\mathcal{D}_{fd}(\Gamma)$ of $\mathrm{per}(\Gamma)$ is sent by F to the full subcategory of $\mathrm{grmod}(\Gamma)$ consisting of finite dimensional graded Γ-modules. Now the theorem follows from the classification of indecomposable objects for the latter category, which is well-known. □

Remark. *It is not hard to check that the Auslander-Reiten quiver of the perfect derived category has the following shape*



where the picture is periodic as indicated by the labels. The Auslander-Reiten quiver of $\mathcal{D}_{fd}(\Gamma)$ *is the subquiver consisting of the components containing the simples* S *and* $S[1]$*.*

4. The Hall algebra

In this section, we prove the structure theorem (Theorem 4.1) for the (derived) Hall algebra of the the Ginzburg dg algebra of type A_1. We begin with some reminders on Hall algebras of triangulated categories.

4.1. **The Hall algebra.** We follow [13] and [14]. Let q be a prime power and let $\mathbb{F}_q$ be the finite field with q elements. Let $\mathcal{C}$ be a Hom-finite triangulated $\mathbb{F}_q$-category with suspension functor Σ, such that for all objects X and Y of $\mathcal{C}$, the space of morphisms from X to $\Sigma^{-i}Y$ vanishes for all but finitely many nonnegative integers i.

Let X, Y and Z be three objects of $\mathcal{C}$. We denote by $\mathrm{Aut}(Y)$ the group of automorphisms of Y and by $[Y,Z]_X$ the set of morphisms from Y to Z with cone isomorphic to X. Following [13], we define the Hall number by

$$F_{XY}^{Z} = \frac{|[Y,Z]_X|}{|\mathrm{Aut}(Y)|} \cdot \frac{\prod_{i>0} |\mathrm{Hom}(Y,\Sigma^{-i}Z)|^{(-1)^i}}{\prod_{i>0} |\mathrm{Hom}(Y,\Sigma^{-i}Y)|^{(-1)^i}},$$

where $|\cdot|$ denotes the cardinality. The *Hall algebra* of $\mathcal{C}$ over $\mathbb{Q}$, denoted by $\mathcal{H}(\mathcal{C})$, is the $\mathbb{Q}$-vector space with basis the isoclasses $[X]$ of objects X of $\mathcal{C}$ whose multiplication is given by

$$[X][Y] = \sum_{[Z]} F_{XY}^{Z}[Z].$$

It is shown in [13] [14] that it is an associative algebra with unit [0]. Notice however that the algebra we define here is opposite to that in [13] [14].

4.2. **The structure theorem.** Let $\mathbb{Q}$ be the field of rational numbers, and let q be prime power.

Let A be the $\mathbb{Q}$-algebra with generators x_i and y_i, $i \in \mathbb{Z}$, subject to the following relations:

$$x_i^2 x_{i-1} - (1+q^{-1}) x_i x_{i-1} x_i + q^{-1} x_{i-1} x_i^2 \tag{1}$$
$$x_i x_{i-1}^2 - (1+q^{-1}) x_{i-1} x_i x_{i-1} + q^{-1} x_{i-1}^2 x_i \tag{2}$$
$$x_i x_j - x_j x_i \qquad \text{if } |i-j|>1 \tag{3}$$
$$y_i x_i - q x_i y_i + \tfrac{q}{q-1} \tag{4}$$
$$y_i x_{i+1} - q^{-1} x_{i+1} y_i - \tfrac{1}{q-1} \tag{5}$$
$$y_i x_j - x_j y_i \qquad \text{if } j \neq i, i-1 \tag{6}$$
$$y_i^2 y_{i-1} - (1+q^{-1}) y_i y_{i-1} y_i + q^{-1} y_{i-1} y_i^2 \tag{7}$$
$$y_i y_{i-1}^2 - (1+q^{-1}) y_{i-1} y_i y_{i-1} + q^{-1} y_{i-1}^2 y_i \tag{8}$$
$$y_i y_j - y_j y_i \qquad \text{if } |i-j|>1. \tag{9}$$

Let Γ be the Ginzburg dg algebra of type A_1 over the finite field $\mathbb{F}_q$, and $\mathcal{D}_{fd}(\Gamma)$ the finite dimensional derived category with suspension functor Σ. Let $\mathcal{H} = \mathcal{H}(\mathcal{D}_{fd}(\Gamma))$ be the Hall algebra.

Theorem 4.1. *We have a $\mathbb{Q}$-algebra isomorphism*

$$\phi : A \longrightarrow \mathcal{H}, \qquad x_i \mapsto [\Sigma^{2i}S], y_i \mapsto [\Sigma^{2i+1}S],$$

where $S = \Gamma/(t\Gamma)$ is the simple dg Γ-module concentrated in degree 0.

One checks by a direct computation that ϕ is indeed an algebra homomorphism, i.e. the relations (1)–(9) are satisfied if we replace x_i and y_i by $[\Sigma^{2i}S]$ and $[\Sigma^{2i+1}S]$ respectively. It remains to prove the surjectivity and the injectivity.

4.3. **Surjectivity of ϕ.**

Proposition 4.2. *The $\mathbb{Q}$-algebra $\mathcal{H}$ is generated by the $[\Sigma^p S], p \in \mathbb{Z}$.*

Proof. Let M be an object of $\mathcal{D}_{fd}(\Gamma)$. Suppose $M \cong \bigoplus_{p \leq p_0} M_p$, where p_0 is an integer, M_p is a dg Γ-module which has trivial differential and which is generated in degree p, and M_{p_0} is nontrivial. Without loss of generality, we may assume $p_0 = 0$. Write $L = M_0$ and $N = \bigoplus_{p<0} M_p$.

We have a triangle

$$\tau_{\leq -1}(L) \oplus N \to M \to \tau_{\geq 0}(L) \to \Sigma(\tau_{\leq -1}(L) \oplus N),$$

which gives rise to a short exact sequence

$$0 \to \mathrm{rad}(H^*L) \oplus H^*N \to H^*M \to \mathrm{top}(H^*L) \to 0.$$

Now let E be an extension of $\tau_{\geq 0}(L)$ by $\tau_{\leq -1}(L) \oplus N$, i.e. we have a triangle

$$\tau_{\leq -1}(L) \oplus N \to E \to \tau_{\geq 0}(L) \xrightarrow{f} \Sigma(\tau_{\leq -1}(L) \oplus N),$$

which gives rise to a long exact sequence

$$\mathrm{top}(H^*L)[-1] \xrightarrow{(H^*f)[-1]} \mathrm{rad}(H^*L) \oplus H^*N \longrightarrow H^*E \longrightarrow \mathrm{top}(H^*L) \xrightarrow{H^*f} \mathrm{rad}(H^*L)[1] \oplus (H^*N)[1].$$

If $H^*f \neq 0$, then the dimensions of H^*E and H^*M are equal. If $H^*f = 0$, then the dimensions are equal but H^*E has more indecomposable direct summands than H^*M, and hence by Theorem 2.6, the object E has more indecomposable direct summands than M; moreover, the object E has precisely the same number of indecomposable direct summands as M if and only if $E \cong M$, since this holds if we replace E and M by H^*E and H^*M respectively. By induction the proof reduces to the case where H^*M is concentrated in one degree, namely, the case where M is isomorphic to the direct sum of m copies of S for some positive integer m. But we have

$$[S^{\oplus m}] = \frac{q-1}{q^m - 1}[S]^m,$$

which finishes the proof. □

As a consequence, we have

Corollary 4.3. *The algebra homomorphism $\phi : A \to \mathcal{H}$ is surjective.*

4.4. **Injectivity of ϕ.** Let R_x (respectively, R_y) be the subalgebra of A generated by $\{x_i | i \in \mathbb{Z}\}$ (respectively, by $\{y_i | i \in \mathbb{Z}\}$). The image of R_x under ϕ is the subalgebra of $\mathcal{H}$ generated by $\{[\Sigma^{2i} S] | i \in \mathbb{Z}\}$, denoted by $\mathcal{H}_x$, which has a $\mathbb{Q}$-basis $\{[M] | M \in \mathcal{D}_{fd}(\Gamma), H^{odd}M = 0\}$, where $H^{odd}M$ is the direct sum of homology spaces of M in odd degrees. Similarly, the image $\mathcal{H}_y$ of R_y under ϕ is the subalgebra of $\mathcal{H}$ generated by $\{[\Sigma^{2i+1} S] | i \in \mathbb{Z}\}$, and has a $\mathbb{Q}$-basis $\{[M] | M \in \mathcal{D}_{fd}(\Gamma), H^{even}M = 0\}$, where $H^{even}M$ is the direct sum of homology spaces of M in even degrees.

Thanks to (4)(5)(6), we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\psi : R_x \otimes R_y \to A, \qquad f(x) \otimes g(y) \mapsto f(x)g(y).$$

In particular, the product of a basis of R_x and a basis of R_y is a basis of A. Now the injectivity is implied by the following two lemmas.

Lemma 4.4. *The restriction* $\phi|_{R_x} : R_x \to \mathcal{H}_x$ *(respectively,* $\phi|_{R_y} : R_y \to \mathcal{H}_y$*) is an isomorphism.*

Lemma 4.5. *The set* $\{[M][N]|M,N \in \mathcal{D}_{fd}(\Gamma), H^{odd}M = H^{even}N = 0\}$ *is a* $\mathbb{Q}$*-basis of* $\mathcal{H}$.

Proof of Lemma 4.4: Let $\mathcal{P}$ be the path category of the quiver $\vec{A}_\infty$ whose vertices are the integers and which has one arrow from n to $n+1$ for each integer n. We have a fully faithful functor F from $\mathcal{P}$ to $\mathrm{per}\,\Gamma$ taking the object n to the dg module $P[-2n]$. Let F^* be the functor $\mathcal{D}\Gamma \to \mathrm{Mod}\,\mathcal{P}$ taking an object X to the module $n \mapsto \mathrm{Hom}\,(Fn, X)$. It induces an equivalence from the full subcategory whose objects are the dg modules M whose homology is finite-dimensional and vanishes in odd degrees to the category $\mathrm{mod}\,\mathcal{P}$ of finite-dimensional representations of the quiver $\vec{A}_\infty$. It is immediate to check that this functor induces an isomorphism from $\mathcal{H}_x$ to the Hall algebra of $\mathrm{mod}\,\mathcal{P}$, i.e. the Hall algebra of the quiver $\vec{A}_\infty$. The claim now follows from the (well-known) structure of this Hall algebra.

□

Proof of Lemma 4.5: By the surjectivity of ϕ, the set of products

$$\{[M][N]|M,N \in \mathcal{D}_{fd}(\Gamma), H^{odd}M = H^{even}N = 0\}$$

generates the $\mathbb{Q}$-vector space $\mathcal{H}$. It remains to prove that these products are linearly independent.

Following [6], we define a partial order $\leq_\Delta$ on the set of isoclasses of objects in $\mathcal{D}_{fd}(\Gamma)$ as follows: if X and Y are two objects of $\mathcal{D}_{fd}(\Gamma)$, then $[Y] \leq_\Delta [X]$ if there exists an object Z of $\mathcal{D}_{fd}(\Gamma)$ and a triangle in $\mathcal{D}_{fd}(\Gamma)$:

$$X \to Y \oplus Z \to Z \to \Sigma X.$$

We extend the partial order $\leq_\Delta$ to a total order $\preceq$.

Now suppose $(M_1, N_1), \ldots, (M_r, N_r)$ are pairwise distinct pairs of objects of $\mathcal{D}_{fd}(\Gamma)$ such that

$$H^{odd}M_1 = \ldots = H^{odd}M_r = H^{even}N_1 = \ldots = H^{even}N_r = 0.$$

Suppose that $\lambda_1, \ldots, \lambda_r$ are rational numbers such that

$$\lambda_1[M_1][N_1] + \ldots + \lambda_r[M_r][N_r] = 0.$$

By the assumption on the M_i's and N_i's, there is a unique maximal element among all $[M_i \oplus N_i]$'s, say $[M_1 \oplus N_1]$. Then we have

$$\lambda_1[M_1][N_1] + \ldots + \lambda_r[M_r][N_r] = \lambda_1 F_{M_1N_1}^{M_1 \oplus N_1}[M_1 \oplus N_1] + \text{smaller terms},$$

since a nontrivial extension of two objects is always smaller than the direct sum of them. The (derived) Hall number $F_{M_1N_1}^{M_1 \oplus N_1}$ is a nonzero rational number. Therefore λ_1 has to be zero. An induction on r shows that $\lambda_1 = \ldots = \lambda_r = 0$. □

5. General case

The general case can be treated similarly. Here we only give the final result and the key point of the proof.

Theorem 5.1. *Let d be an integer. Let $\mathcal{D}$ be a triangulated category classically generated by a $(d+1)$-spherical object, and let $\mathcal{H}(\mathcal{D})$ be the Hall algebra of $\mathcal{D}$ over $\mathbb{Q}$.*

(i) When $d \geq 2$, the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

$$z_i^2 z_{i-d} - (q+1)q^{(-1)^{d+1}} z_i z_{i-d} z_i + q^{2(-1)^{d+1}+1} z_{i-d} z_i^2 \tag{10}$$

$$z_i z_{i-d}^2 - (q+1)q^{(-1)^{d+1}} z_{i-d} z_i z_{i-d} + q^{2(-1)^{d+1}+1} z_{i-d}^2 z_i \tag{11}$$

$$z_i z_{i+1} - q^{-1} z_{i+1} z_i - \frac{1}{q-1} \tag{12}$$

$$z_i z_j - q^{(-1)^{j-i}(1+(-1)^{-d-1})} z_j z_i \qquad \text{if } i-j \leq -d-1 \tag{13}$$

$$z_i z_j - q^{(-1)^{j-i}} z_j z_i \qquad \text{if } -d-1 < i-j < 0, i-j \neq -1, -d. \tag{14}$$

(ii) When $d = 1$, the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

$$z_i^2 z_{i-1} - (q+1)q z_i z_{i-1} z_i + q^3 z_{i-1} z_i^2 - q(q+1) z_i \tag{15}$$

$$z_i z_{i-1}^2 - (q+1)q z_{i-1} z_i z_{i-1} + q^3 z_{i-1}^2 z_i - q(q+1) z_{i-1} \tag{16}$$

$$z_i z_j - q^{2(-1)^{j-i}} z_j z_i \qquad \text{if } i-j \leq -2. \tag{17}$$

(iii) When $d = 0$, the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

(iv) When $d = -1$, the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

(v) When $d \leq -2$, the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

$$z_i^2 z_{i-d} - (q+1)q^{-1-(-1)^{-d-1}} z_i z_{i-d} z_i + q^{-1-2(-1)^{-d-1}} z_{i-d} z_i^2 \tag{18}$$

$$z_i z_{i-d}^2 - (q+1)q^{-1-(-1)^{-d-1}} z_{i-d} z_i z_{i-d} + q^{-1-2(-1)^{-d-1}} z_{i-d}^2 z_i \tag{19}$$

$$z_i z_{i+1} - q^{-1} z_{i+1} z_i - \frac{1}{q^{(-1)^{-d-1}} q - 1} \tag{20}$$

$$z_i z_j - q^{-(-1)^{j-i}(1+(-1)^{-d-1})} z_j z_i \qquad \text{if } i-j < d+1, i-j \neq d \tag{21}$$

$$z_i z_j - q^{-(-1)^{j-i}} z_j z_i \qquad \text{if } -d-1 \leq i-j < 0, i-j \neq -1. \tag{22}$$

Proof. The proofs for (i) and (v) are similar to that for Theorem 4.1. For (ii), notice that both the Hall algebra $\mathcal{H}(\mathcal{D})$ and the desired algebra are filtered, and the algebra homomorphism from the desired algebra to the Hall algebra $\mathcal{H}(\mathcal{D})$ is a morphism of filtered algebras, and the associated graded algebra homomorphism is an isomorphism, which has a similar proof to that for Theorem 4.1. □

6. An algebra of Laurent polynomials

Let $v = \sqrt{q}$. We tensor A with $\mathbb{Q}(v)$ over $\mathbb{Q}(q)$, and still denote the resulting algebra by A. Let I be the ideal of A generated by the space $[A, A]$ of commutators of A.

Lemma 6.1. *The assignment* $\varphi : x_i \mapsto \frac{v}{v^2-1}x, y_i \mapsto \frac{v}{v^2-1}x^{-1}$ *defines an algebra homomorphism from* A *to* $\mathbb{Q}(v)[x, x^{-1}]$ *with kernel* I.

Proof. We have

$$A/I \cong \mathbb{Q}(v)[x_i, y_i]_{i\in\mathbb{Z}}/(x_i y_i = x_{i+1} y_i = \frac{q}{(q-1)^2}).$$

Now it is clear that $x_i \mapsto \frac{v}{v^2-1}x, y_i \mapsto \frac{v}{v^2-1}x^{-1}$ defines an algebra isomorphism from A/I to $\mathbb{Q}(v)[x, x^{-1}]$. □

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