Periodicity in representation theory of algebras

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Introduction

In this notes we discuss the periodicity problems in the representation theory of finite dimensional algebras over an algebraically closed field and exhibit their natural sources in the theory of finite groups, algebraic topology and commutative algebra.

One of the deepest and important results of the cohomological theory of finite groups is a complete classification of all periodic groups, that is, the finite groups with periodic cohomology groups. The class of periodic groups contains the cyclic groups, generalized quaternion groups as well as the binary dihedral, tetrahedral, octahedral and icosahedral groups. The classification problem of periodic groups is strongly related with the spherical space form problem concerning the classification of all finite groups G acting freely on spheres $\mathbb{S}^m$ and the homotopical type of the orbit (spherical) spaces $\mathbb{S}^m/G$. Namely, the finite groups acting freely on spheres are necessarily periodic. On the other hand, already in 1938–1939, P. A. Smith proved that free action of a finite group G on a sphere $\mathbb{S}^m$ forces that every abelian subgroup of G is cyclic. This was the topological motivation for the Zassenhaus problem concerning the classification of all finite groups with cyclic abelian subgroups. This problem was solved completely by M. Suzuki and H. Zassenhaus in 1949–1955. Moreover, it has been proved by E. Artin and J. Tate that the periodic groups are exactly the finite groups solving the Zassenhaus problem. In 1957 J. Milnor proved that there are periodic groups without free action on a sphere, and in 1960 G. Swan clarified the picture by showing that the periodic groups are all finite groups acting freely on finite CW-complexes homotopically equivalent to spheres.

In the representation theory of finite dimensional algebras a prominent role is played the syzygy operator which assigns to a module M over a finite dimensional algebra A the kernel $\Omega_A(M)$ of a projective cover of M. The main objective of the notes is to discuss the structure and homological invariants of finite dimensional algebras A over an algebraically closed field K for which all indecomposable nonprojective finite dimensional A-modules are periodic with respect to the action of the syzygy operator Ω_A. It turns out that all such algebras are selfinjective (projective modules are injective), and hence are Morita equivalent to the Frobenius algebras. Classical examples of selfinjective (Frobenius) algebras are provided by the group algebras of finite groups, or more generally the finite dimensional Hopf algebras. It follows from the classification of periodic groups and represenation theory of finite groups that a finite group G is periodic if and only if, for any algebraically closed field K, all indecomposable nonprojective finite dimensional modules over the group algebra KG are Ω_{KG}-periodic. Moreover, the group algebras KG of periodic groups G are symmetric algebras of tame representation type. One of the maing results presented in these notes is a complete classification (up to Morita equivalence) of all symmetric algebras of tame representation type with all indecomposable nonprojective finite dimensional modules periodic, established recently by K. Erdmann and A. Skowroński.

In the notes, the periodicity of a finite dimensional algebra A as an A-A-bimodule (equivalently, as a module over the enveloping algebra A^e) is also discussed. In particular, important recent results in this direction by E. L. Green, N. Snashall and Ø. Solberg, invoking the Hochschild cohomology algebras, are presented.

In the final part of the notes we exhibit natural examples of periodic selfinjective algebras coming from the commutative algebra. These are the stable Auslander algebras of the hypersurface singularities of finite Cohen-Macaulay type over an algebraically closed field. In particular, a large class of selfinjective algebras of wild representation type with all indecomposable nonprojective finite dimensional modules is presented.

We divide the notes into the following parts:
(1) Selfinjective algebras.
(2) Periodicity of modules and algebras.
(3) Periodicity of finite groups.
(4) Periodicity of tame symmetric algebras.
(5) Periodicity and hypersurface singularities.

1. Selfinjective algebras

In this section we introduce the classes of selfinjective algebras, Frobenius algebras and symmetric algebras as well as present their classical characterizations and examples.

Let A be a finite dimensional K-algebra and A^{op} its opposite algebra. We denote by $\operatorname{mod} A$ the category of finite dimensional (over K) right A-modules. Then $\operatorname{mod} A^{\text{op}}$ is the category of finite dimensional left A-modules. Moreover, the functor $D = \operatorname{Hom}_K(-, K) : \operatorname{mod} K \to \operatorname{mod} K$ induces a duality

$$\operatorname{mod} A \underset{D}{\overset{D}{\rightleftarrows}} \operatorname{mod} A^{\text{op}}$$

with $D \circ D \cong 1_{\operatorname{mod} A}$, $D \circ D \cong 1_{\operatorname{mod} A^{\text{op}}}$. Let

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

be a decomposition of the identity 1_A of A into a sum of pairwise orthogonal primitive idempotents e_{ij} such that

$$e_{ij}A \cong e_{ij'}A, \text{ for all } j, j' \in \{1, \dots, m_A(i)\},$$
$$e_{ij}A \ncong e_{i'j}A, \text{ for } i, i' \in \{1, \dots, n_A\} \text{ with } i \neq i'.$$

We will abbreviate $e_i = e_{i1}$ for $i \in \{1, \dots, n_A\}$. Hence,

$$P_i = e_iA, 1 \le i \le n_A,$$

is a complete set of pairwise nonisomorphic indecomposable projective right A-modules. Moreover,

$$I_i = D(Ae_i), 1 \leq i \leq n_A,$$

is a complete set of pairwise nonisomorphic indecomposable injective right A-modules. The algebra A is said to be **basic** if $m_A(i) = 1$ for all $i \in \{1, \dots, n_A\}$. In general, consider the **basic idempotent** of A

$$e = \sum_{i=1}^{n_A} e_{i1} = \sum_{i=1}^{n_A} e_i.$$

Then $A^{\mathrm{b}} = eAe$ is said to be **basic algebra** of A. By general theory, the pair of functors

$$\operatorname{mod} A \underset{-\otimes_{A^{\mathrm{b}}} A}{\overset{(-)e}{\rightleftarrows}} \operatorname{mod} A^{\mathrm{b}}$$

induces an equivalence of categories and A is said to be **Morita equivalent** to A^{b}.

We denote by $\operatorname{proj} A$ the category of projective modules in $\operatorname{mod} A$ and by $\operatorname{inj} A$ the category of injective modules in $\operatorname{mod} A$. Then we have the following dualities of categories

$$\operatorname{proj} A \underset{D}{\overset{D}{\rightleftarrows}} \operatorname{inj} A^{\mathrm{op}},$$

$$\operatorname{inj} A \underset{D}{\overset{D}{\rightleftarrows}} \operatorname{proj} A^{\mathrm{op}}.$$

Proposition 1.1. *Let A be an algebra. The following statements are equivalent:*

(1) A_A *is injective.*
(2) $\operatorname{proj} A = \operatorname{inj} A$.
(3) $\operatorname{proj} A^{\mathrm{op}} = \operatorname{inj} A^{\mathrm{op}}$.
(4) ${}_AA$ *is injective.*

An algebra A is said to be **selfinjective** if the modules A_A and ${}_AA$ are injective.

Therefore, for a selfinjective algebra A, $e_{11}A, e_{21}A, \dots, e_{n1}A$ is a complete set of pairwise nonisomorphic indecomposable injective right A-modules.

Hence, there exists a permutation ν of $\{1, \dots, n_A\}$, called the **Nakayama permutation**, such that

$$\operatorname{top} e_{i1}A \cong \operatorname{soc} e_{\nu(i)1}A \text{ for all } i \in \{1, \dots, n_A\}.$$

The following characterization of selfinjective algebras has been established by Nakayama in [**Na2**].

Theorem 1.2 (Nakayama, 1941)**.** *An algebra A is selfinjective if and only if there exists a permutation ν of $\{1, \dots, n_A\}$ such that $\operatorname{top} e_{i1}A \cong \operatorname{soc} e_{\nu(i)1}A$ for all $i \in \{1, \dots, n_A\}$.*

In the representation theory of selfinjective algebras an essential role is played by the A-A-bimodule $D(A) = \operatorname{Hom}_K(A, K)$, with the A-A-bimodule structure given by

$$(af)(b) = f(ba), \quad (fa)(b) = f(ab), \quad \text{for} \quad a, b \in A, f \in D(A).$$

Then $D(A)_A$ is an injective cogenerator in the category $\operatorname{mod} A$ and ${}_AD(A)$ is an injective cogenerator in the category $\operatorname{mod} A^{\mathrm{op}}$.

The following theorem is a combination of results proved by Brauer and Nesbitt [**BN**] and Nakayama [**Na1**].

Theorem 1.3 (Brauer, Nesbitt, Nakayama, 1937–1939)**.** *Let A be an algebra. The following statements are equivalent:*

(1) *There exists a nondegenerate K-bilinear form $(-,-) : A \times A \to K$ such that $(a, bc) = (ab, c)$ for all $a, b, c \in A$.*
(2) *There exists a K-linear form $\varphi : A \to K$ such that* $\ker\varphi$ *does not contain nonzero right ideal of A.*
(3) *There exists an isomorphism $\theta : A_A \to D(A)_A$ of right A-modules.*
(4) *There exists a K-linear form $\varphi' : A \to K$ such that* $\ker\varphi'$ *does not contain nonzero left ideal of A.*
(5) *There exists an isomorphism $\theta' : {}_AA \to {}_AD(A)$ of left A-modules.*

Proof. (1) $\Rightarrow$ (2). Let $(-,-) : A \times A \to K$ be a nondegenerate associative K-bilinear form. Define the K-linear map $\varphi : A \to K$ by

$$\varphi(a) = (a, 1) = (1, a) \text{ for } a \in A.$$

Let I be a right ideal of A such that $\varphi(I) = 0$. Take $a \in I$. Then $(a, A) = (aA, 1) = \varphi(aA) = 0$ implies $(a,-) = 0$, and so $a = 0$. Hence $I = 0$.

(2) $\Rightarrow$ (1), (3). Let $\varphi : A \to K$ be a K-linear map such that $\varphi(I) \neq 0$ for any nonzero right ideal I of A. Define the K-bilinear form $(-,-) : A \times A \to K$ by

$$(a, b) = \varphi(ab) \text{ for all } a, b \in A.$$

Observe that

$$(a, bc) = \varphi(a(bc)) = \varphi((ab)c) = (ab, c),$$

for $a, b, c \in A$. Let $a \in A$. If $(a,-) = 0$ then $\varphi(aA) = (a, A) = 0$ implies $a = 0$. Assume $(-, a) = 0$. Then $(a,-) = 0$, and hence $a = 0$. Indeed, consider a K-linear basis $a_1, \dots, a_m$ of A. Then $a = \sum_{i=1}^{m} \lambda_i a_i$ for some $\lambda_1, \dots, \lambda_m \in K$, and, for any $j \in \{1, \dots, m\}$, we have $0 = (a_j, a) = \sum_{i=1}^{n} \lambda_i (a_j, a_i)$, or equivalently

$$[(a_j, a_i)] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = 0.$$

Taking the transpose, we get

$$[\lambda_1, \dots, \lambda_m]\,[(a_i, a_j)] = 0,$$

or equivalently $0 = \sum_{i=1}^{n} \lambda_i (a_i, a_j) = (a, a_j)$ for any $j \in \{1, \dots, m\}$. Hence $(a,-) = 0$, as required. Therefore $(-,-)$ is a nondegenerate associative K-bilinear form, and (1) holds. For (3), define the K-linear map

$$\theta = \theta_\varphi : A \to D(A) = \operatorname{Hom}_K(A, K)$$

such that $\theta(a)(b) = \varphi(ab)$, for $a, b \in A$. Then θ is a homomorphism of right A-modules. Indeed, for $a, b, c \in A$, we have $\theta(ab)(c) = \varphi((ab)c) = \varphi(a(bc)) = \theta(a)(bc) = (\theta(a)b)(c)$, and hence $\theta(ab) = \theta(a)b$. Moreover, θ is a monomorphism, because, for $a \in A$, $\theta(a) = 0$ implies $\varphi(aA) = \theta(a)(A) = 0$, and hence $aA = 0$, and consequently $a = 0$, by the condition (2). Since $\dim_K A = \dim_K D(A)$, we conclude that θ is an isomorphism of right A-modules.

(3) $\Rightarrow$ (2) Assume that $\theta : A \to D(A)$ is an isomorphism of right A-modules. Define the K-linear map $\varphi = \varphi_\theta = \theta(1) \in D(A)$. Let I be a right ideal of A such that $\varphi(I) = 0$. Then, for any $a \in A$, we have $aA \subseteq I$, and hence we obtain $0 = \varphi(aA) = \theta(1)(aA) = (\theta(1)a)(A) = \theta(a)(A)$ and hence $a = 0$, because θ is and isomorphism of right A-modules. Hence $I = 0$, and (2) holds.

In a similar way, we prove the equivalences (1) $\iff$ (4) $\iff$ (5). □

An algebra A statisfying one of the equivalent conditions (1)–(5) is called a **Frobenius algebra**. Observe that every Frobenius algebra A is selfinjective, because an isomorphism $A_A \xrightarrow{\sim} D(A)_A$ implies that A_A is injective. Conversely, every basic, selfinjective algebra A is a Frobenius algebra.

In particular, we obtain that every selfinjective algebra A is Morita equivalent to a Frobenius algebra, namely its basic algebra A^{b}.

In general, we have the following result due to Nakayama [**Na1**].

Theorem 1.4 (Nakayama, 1939)**.** *Let A be a selfinjective algebra. Then A is a Frobenius algebra if and only if, for the Nakayama permutation $\nu = \nu_A$ of A, we have $m_A(i) = m_A(\nu(i))$ for all $i \in \{1, \dots, n_A\}$.*

The following example has been exhibited already by Nakayama [**Na1**].

Example 1.5. Let $\Lambda = KQ/I$ where Q is the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

$I = \langle \alpha\beta, \beta\alpha \rangle$. Then Λ is a basic, connected selfinjective algebra with $\operatorname{rad}^2 \Lambda = 0$. Moreover, $\Lambda = e_1 A \oplus e_2 \Lambda$. Take $P_{10} = e_1 A$, $P_{11} = e_1 A$, $P_2 = e_2 A$, and consider the endomorphism algebra $A = \Lambda(2,1) = \operatorname{End}_\Lambda(P_{10} \oplus P_{11} \oplus P_2)$. Let e_0, e_1, e_2 be the primitive idempotents of A corresponding to the direct summands P_{10}, P_{11}, P_2 of $P_{10} \oplus P_{11} \oplus P_2$. Further, denote by ${}_1u_0$ the identity map from P_{10} to P_{11}, and by ${}_0u_1$ the identity map from P_{11} to P_{10}. Finally, let ${}_2\alpha_0 : P_{10} \to P_2$, ${}_2\alpha_1 : P_{11} \to P_2$, and ${}_0\beta_2 : P_2 \to P_{10}$, ${}_1\beta_2 : P_2 \to P_{11}$ be the maps given by the left multiplications by α and β, respectively. Then we have in A the equalities

$$\begin{aligned} e_0 &= {}_0u_1 \cdot {}_1u_0, & e_1 &= {}_1u_0 \cdot {}_0u_1, \\ {}_2\alpha_0 &= e_2 \cdot {}_2\alpha_0 \cdot e_0, & {}_2\alpha_1 &= e_2 \cdot {}_2\alpha_1 \cdot e_1, \\ {}_0\beta_2 &= e_0 \cdot {}_0\beta_2 \cdot e_2, & {}_1\beta_2 &= e_1 \cdot {}_1\beta_2 \cdot e_2. \end{aligned}$$

Then A is a 9-dimensional **selfinjective non-Frobenius algebra**, isomorphic to the matrix algebra given by the matrices of the form

$$\left[\begin{array}{ccc|ccc} a_0 & {}_0b_1 & {}_0\mu_1 & & & \\ {}_1b_0 & a_1 & {}_1\mu_2 & & 0 & \\ 0 & 0 & a_2 & & & \\ \hline & & & a_2 & {}_2\lambda_0 & {}_2\lambda_1 \\ & 0 & & 0 & a_0 & {}_0b_1 \\ & & & 0 & {}_1b_0 & a_1 \end{array}\right]$$

where $a_0 \in Ke_0$, $a_1 \in Ke_1$, $a_2 \in Ke_2$, ${}_0b_1 \in K_0u_1$, ${}_1b_0 \in K_1u_0$, ${}_0\mu_2 \in K_0\beta_2$, ${}_1\mu_2 \in K_1\beta_2$, ${}_2\lambda_0 \in K_2\alpha_0$, ${}_2\lambda_1 \in K_2\alpha_1$, which is exactly the algebra presented by Nakayama in [**Na1**, p.624].

We refer to [**SY**] for the general form of non-Frobenius selfinjective algebras.

Hence, the class of Frobenius algebras is not closed under Morita equivalences. The class of selfinjective algebras is the smallest class of algebras containing the Frobenius algebras and closed under Morita equivalences.

An important class of Frobenius algebras is formed by the symmetric algebras. The following theorem is again a combination of results proved by Brauer and Nesbitt [**BN**] and Nakayama [**Na2**].

Theorem 1.6 (Brauer, Nesbitt, Nakayama, 1937–1941)**.** *Let A be an algebra. The following statements are equivalent:*

(1) *There exists a nondegenerate symmetric associative K-bilinear form $(-,-) : A \times A \to K$.*

(2) *There exists a K-linear form $\varphi : A \to K$ such that $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$, and $\ker\varphi$ does not contain nonzero one-sided ideal of A.*

(3) *There exists an isomorphism $\theta : {}_AA_A \to {}_AD(A)_A$ of A-A-bimodules.*

Proof. This follows from the proof of the characterizations of Frobenius algebras. □

An algebra A satisfying one of the equivalent conditions (1)–(3) is called a **symmetric algebra**.

Let A be a Frobenius K-algebra and $(-,-) : A \times A \to K$ a nondegenerate associative K-bilinear form. Then there exists a unique K-algebra isomorphism

$$\nu_A : A \to A$$

such that $(a, b) = (b, \nu_A(a))$ for all $a, b \in A$, called the **Nakayama automorphism** of A. We will see later that ν_A induces the Nakayama permutation of A. Moreover, $\nu_A = \mathrm{id}_A$, if A is symmetric.

The following fact has been observed by Nakayama [**Na1**].

Theorem 1.7 (Nakayama, 1939)**.** *Let A be a selfinjective algebra. Then* $\mathrm{soc}({}_AA) = \mathrm{soc}(A_A)$. *In particular,* $\mathrm{soc}(A) := \mathrm{soc}({}_AA) = \mathrm{soc}(A_A)$ *is an ideal of A.*

Two selfinjective algebras A and Λ are said to be **socle equivalent** if the factor algebras $A/\mathrm{soc}(A)$ and $\Lambda/\mathrm{soc}(\Lambda)$ are isomorphic.

Examples 1.8. (1) Let $A = K[X]/(X^n), n \geq 1$, be a truncated polynomial algebra. Then A is a commutative local K-algebra with $A \cong D(A)$ as A-A-bimodules, and hence A is a symmetric algebra. **More generally, every finite dimensional commutative selfinjective K-algebra is a symmetric algebra.**

(2) Let G be a finite group, and $A = KG$ the **group algebra** o G. Then

$$A = \left\{ \sum_{g \in G} \lambda_g g \ \mid \ \lambda_g \in K \right\}$$

and $\dim_K A = |G|$. Moreover, the map $(-,-) : A \times A \to K$ given by

$$\left(\sum_{g \in G} \lambda_g g, \sum_{h \in H} \mu_h h \right) = \sum_{g \in G} \lambda_g \mu_{g^{-1}}$$

is a symmetric, associative, nondegenerate K-bilinear form. Hence, $A = KG$ is a **symmetric algebra**.

(3) Let A be an arbitrary finite dimensional K-algebra. Consider the **trivial extension** $\mathrm{T}(A) = A \ltimes D(A)$ of A by the A-A-bimodule $D(A)$. That is, $\mathrm{T}(A) = A \oplus D(A)$ as K-vector space and the multiplication in $\mathrm{T}(A)$ is given by

$$(a, f)(a', f') = (aa', af' + fa'),$$

for $a, a' \in a,\ f, f' \in D(A)$. Obviously, $\dim_K \mathrm{T}(A) = 2 \dim_K A$.

Further, the map $(-,-) : \mathrm{T}(A) \times \mathrm{T}(A) \to K$ given by

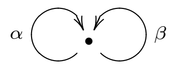
$$((a, f), (a', f')) = f(a') + f'(a), \text{ for } a, a' \in A, f, f' \in D(A),$$

is a symmetric, associative, nondegenerate, K-bilinear form. Therefore, $\mathrm{T}(A)$ is a **symmetric algebra**.

Observe that $D(A) = 0 \oplus D(A)$ is a two-sided ideal of $\mathrm{T}(A)$ and $A = \mathrm{T}(A)/D(A)$. Hence, every algebra A is a factor algebra of a symmetric algebra.

(4) For $\lambda \in K \setminus \{0\}$, let $A_\lambda = KQ/I_\lambda$, where

Q :



,

$I = \langle \alpha^2, \beta^2, \alpha\beta - \lambda\beta\alpha \rangle$. Then A_λ is a 4-dimensional local Frobenius algebra. But

$$A_\lambda \textbf{ is symmetric} \iff \lambda = 1.$$

Indeed, let $a = \alpha + I_\lambda$, $b = \beta + I_\lambda$. Then $1, a, b, ab = \lambda ba$ is a basis of A_λ over K. Define $\varphi_\lambda : A_\lambda \to K$ by

$$\varphi_\lambda(1) = \varphi_\lambda(a) = \varphi_\lambda(b) = 0, \quad \varphi_\lambda(ab) = 1.$$

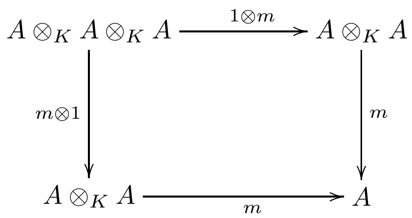
Then $\ker \varphi$ does not contain nonzero right (left) ideal of A_λ, and hence A_λ is a Frobenius algebra. For $\lambda = 1$, $\varphi = \varphi_1$ has the property $\varphi(xy) = \varphi(yx)$ for all $x, y \in A_1$, and hence A_1 is a symmetric algebra. For $\lambda \neq 1$, A_λ is not symmetric. Indeed, assume that $\psi : A \to K$ is a K-linear map such that $\psi(xy) = \psi(yx)$ for all $x, y \in A_\lambda$, and $\ker \psi$ does not contain nonzero one-sided ideal of A_λ. Then $Kab = Kba$ is a nonzero ideal of A_λ, and hence $0 \neq \psi(ba) = \psi(ab) = \psi(\lambda ba) = \lambda\psi(ba)$ implies $\lambda = 1$.

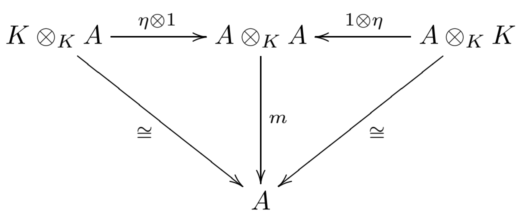
A distinguished class of Frobenius algebras is formed by the finite dimensional Hopf algebras.

A K-vector space A is a K**-algebra** if and only if there are K-linear maps

$$m : A \otimes_K A \longrightarrow A \quad \text{and} \quad \eta : K \longrightarrow A$$

called the **multiplication** and the **unit**, respectively, such that the following diagrams are commutative

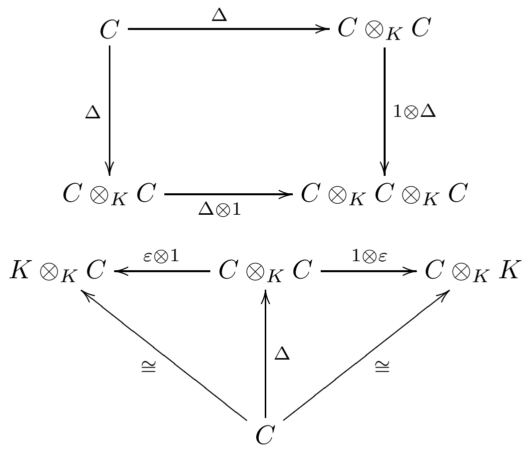




Dually, a K-vector space C is a K**-coalgebra** if there are K-linear maps

$$\Delta : C \longrightarrow C \otimes_K C \quad \text{and} \quad \varepsilon : C \longrightarrow K$$

called the **comultiplication** and the **counit**, respectively, such that the following diagrams are commutative



A K-vector space H is said to be a K**-bialgebra** if there are K-linear maps $m : H \otimes_K H \to H$, $\eta : K \to H$, $\Delta : H \to H \otimes_K H$ and $\varepsilon : H \to K$ such that the following conditions are satisfied:

(1) (H, m, η) is a K-algebra,

(2) (H, Δ, ε) is a K-coalgebra,

(3) Δ, ε are homomorphisms of K-algebras.

Let $H = (H, m, \eta, \Delta, \varepsilon)$ be a bialgebra over K. Consider the **convolution map**

$$* : \mathrm{Hom}_K(H, H) \times \mathrm{Hom}_K(H, H) \longrightarrow \mathrm{Hom}_K(H, H)$$

which assigns to $f, g \in \mathrm{Hom}_K(H, H)$ the composition

$$f * g : H \xrightarrow{\Delta} H \otimes_K H \xrightarrow{f \otimes g} H \otimes_K H \xrightarrow{m} H.$$

Then a bialgebra $H = (H, m, \eta, \Delta, \varepsilon)$ over K is said to be a **Hopf algebra** if there exists a K-linear map $s : H \to H$, called the **antipode**, such that $s * \mathrm{id}_H = \eta\varepsilon = \mathrm{id}_H * s$.

We provide now few examples of finite dimensional Hopf algebras.

Examples 1.9. (1) The group algebra KG of a finite group G is a Hopf algebra with the comultiplication Δ, the counit ε and the antipode s given by

$$\Delta(g) = g \otimes g, \ \varepsilon(g) = 1, \ s(g) = g^{-1}, \text{ for } g \in G.$$

(2) Let $H = (H, m, \eta, \Delta, \varepsilon, s)$ be a finite dimensional Hopf algebra over K. Then the dual space $H^* = \mathrm{Hom}_K(H, K)$ is again a Hopf algebra $H^* = (H^*, \Delta^*, \varepsilon^*, m^*, \eta^*, s^*)$ with

$$\Delta^* : H^* \otimes_K H^* \xrightarrow{\sim} (H \otimes_K H)^* \xrightarrow{\Delta^*} H^*,$$
$$\varepsilon^* : K = K^* \longrightarrow H^*,$$
$$m^* : H^* \xrightarrow{m^*} (H \otimes_K H)^* \xrightarrow{\sim} H^* \otimes_K H^*,$$
$$\eta^* : H^* \longrightarrow K^* = K,$$
$$s^* : H^* \longrightarrow H^*.$$

We note that, for an antipode s of a Hopf algebra H, we have $s(xy) = s(y)s(x)$ for $x, y \in H$ and $s(1) = 1$.

The following theorem is due to Radford [**Ra**].

Theorem 1.10 (Radford, 1976)**.** *An antipode s of a finite dimensional Hopf algebra H has a finite order. In particular, s is an antiisomorphism of the algebra H.*

Let $H = (H, m, \eta, \Delta, \varepsilon, s)$ be a Hopf algebra over K. Then the set

$$\int_H^r = \{x \in H \mid xh = \varepsilon(h)x \text{ for all } h \in H\}$$

is called the space of **right integrals of** H.

The following theorem proved by Larson and Sweedler in [**LaSw**] shows that every finite dimensional Hopf algebra is a Frobenius algebra.

Theorem 1.11 (Larson-Sweedler, 1969)**.** *Let H be a finite dimensional Hopf algebra over K. Then the following statements hold.*

(1) $\dim_K \int_H^r = 1$ *and* $\dim_K \int_{H^*}^r = 1$.

(2) *For $\varphi \in \int_{H^*}^r \setminus \{0\}$, the K-bilinear form*

$$(-,-) : H \times H \to K$$

such that $(a,b) = \varphi(ab)$ for $a, b \in H$, is nondegenerate and associative.

In particular, H is a ***Frobenius algebra****.*

Let H be a finite dimensional Hopf algebra over K. Then there exists a homomorphism of K-algebras $\xi : H \longrightarrow K$ called the **modular function on** H, such that $hx = \xi(h)x$ for all $h \in H, x \in \int_H^r$. Consider the associated convolution map

$$\xi * \mathrm{id}_H : H \xrightarrow{\Delta} H \otimes_K H \xrightarrow{\xi \otimes \mathrm{id}_H} K \otimes_K H \xrightarrow{\sim} H.$$

The following result proved by Fischman, Montgomery and Schneider in [**FMS**] shows that the finite dimensional Hopf algebras form a special class of Frobenius algebras.

Theorem 1.12 (Fischman-Montgomery-Schneider, 1997)**.** *Let H be a finite dimensional Hopf algebra over K. Then the following statements hold.*

(1) $\nu_H = (\xi * \mathrm{id}_H) \cdot s^{-2}$ *is the Nakayama automorphism of the Frobenius algebra H, that is, $(a,b) = (b, \nu_H(a))$ for all $a, b \in H$.*

(2) ν_H *has finite order dividing* $2 \dim_K H$.

Example 1.13. Let $H = KG$ be the group algebra of a finite group G. Then $\int_H^r = Kt$, where $t = \sum_{g \in G} g$. Moreover, $\xi = \varepsilon : H \to K$, $s^2 = \mathrm{id}_H$, $\xi * \mathrm{id}_H = \varepsilon * \mathrm{id}_H = \mathrm{id}_H$, and hence $\nu_H = (\xi * \mathrm{id}_H)s^{-2} = \mathrm{id}_H$. This is correct because KG is a symmetric algebra.

Example 1.14. Let $n \geq 2$ and λ be a primitive n-th root of unity (hence $\operatorname{char} K$ is not divisible by n). Let

$$H = H_{n^2}(\lambda) = K\langle g, x\rangle / (g^n - 1, x^n, xg - \lambda gx).$$

Then $H_{n^2}(\lambda)$ is an n^2-dimensional Hopf algebra, with K-basis $\{g^i x^j \mid 0 \leq i, j \leq n-1\}$, and the comultiplication Δ, counit ε and antipode s given by

$$\Delta(g) = g \otimes g, \ \Delta(x) = g \otimes x + x \otimes 1$$
$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0$$
$$s(g) = g^{-1}, \quad s(x) = -g^{-1}x$$

The algebra $H_{n^2}(\lambda)$ is called the **Taft algebra**.

Observe that the Taft algebra is neither commutative nor cocommutative. For $n = 2$, $H_4(\lambda)$ is called the 4-dimensional **Sweedler's algebra**. We compute now the order of the Nakayama automorphism ν_H of $H = H_{n^2}(\lambda)$.

Since $s^2(x) = \lambda x$, $s^2(g) = g$, the antipode s has order $2n$. Further, $\int_H^r = Kt$, where

$$t = (\sum_{m=0}^{n-1} \lambda^{-m} g^m) x^{n-1}.$$

Moreover, the modular function $\xi : H \to K$ is given by

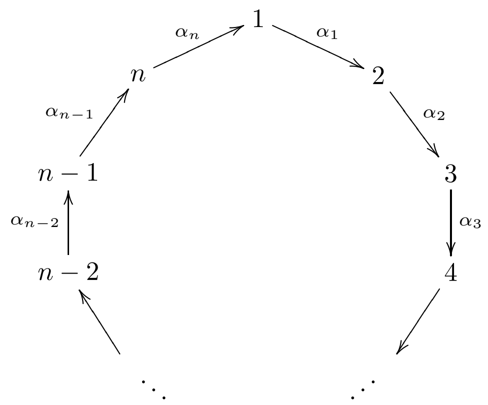
$$\xi(g) = \lambda, \quad \xi(x) = 0.$$

Then the convolution $\xi * \mathrm{id}_H : H \to H$ is given by $\lambda\, \mathrm{id}_H$, and hence the Nakayama automorphism $\nu_H = (\xi * \mathrm{id}_H)s^{-2}$ is given by

$$\nu_H(g) = \lambda g, \quad \nu_H(x) = x.$$

Therefore, ν_H has order n.

Observe also that, as an algebra, $H = H_{n^2}(\lambda)$ is isomorphic to the skew group algebra $A[G]$, where $A = K[x]/(x^n)$, $G = (g)$ is cyclic of order n, and G acts on A by $g(\bar{x}) = \lambda^{-1}\bar{x}$, where $\bar{x}$ is the residue class of x. Note that $g\bar{x}g = g(\bar{x})gg = \lambda^{-1}\bar{x}gg$ implies $\bar{x}g = \lambda g\bar{x}$. Moreover, the algebra $H = H_{n^2}(\lambda)$ is isomorphic to the bound quiver algebra KQ_n/I_n, where Q_n is the cyclic quiver of the form



and the ideal I_n is generated by the paths $\alpha_i\alpha_{i+1}\dots\alpha_{i+n-1}$, $1 \le i \le n$. Hence, as an algebra, $H_{n^2}(\lambda)$ is a selfinjective Nakayama algebra.

In the representation theory of selfinjective algebras a prominent role is played by the Galois coverings and the selfinjective orbit algebras.

A connected K-category R is said to be **locally bounded** if the following conditions are satisfied:

(1) distinct objects of R are nonisomorphic,

(2) $\forall_{x \in obR}\ R(x,x)$ is a local algebra,

(3) $\forall_{x \in obR} \sum_{y \in obR} (\dim_K R(x,y) + \dim_K R(y,x)) < \infty$.

It is known (see [**BG**]) that every locally bounded category R is of the form $R \cong KQ/I$, where Q is a locally finite connected quiver, and I is an admissible ideal of the path category KQ.

Denote by $\operatorname{mod} R$ the category of finitely generated contravariant functors $R \to \operatorname{mod} K$. If $R = KQ/I$, then $\operatorname{mod} R$ is equivalent to the category $\operatorname{rep}_K(Q, I)$ of K-linear representations of the bound quiver (Q, I).

A locally bounded category R with finitely many objects is said to be **bounded**. We may associate to a bounded category R the finite dimensional basic connected K-algebra $\bigoplus R = \bigoplus_{x,y \in obR} R(x, y)$.

We will identify a bounded K-category R with the associated finite dimensional algebra $\oplus R$.

Let R be locally bounded K-category and G a group of K-linear automorphisms of R. Then the group G is said to be **admissible** if G acts freely on the objects of R and has finitely many orbits. We may then consider the **orbit (bounded) category** R/G defined as follows (see [**Ga**]).

The objects of R/G are the G-orbits of objects of R, and the morphism spaces are given by

$$(R/G)(a, b) = \left\{ (f_{yx}) \in \prod_{(x,y) \in a \times b} R(x, y) \,\middle|\, g \cdot f_{yx} = f_{g(y),g(x)} \underset{g \in G, x \in a, y \in b}{\forall} \right\},$$

for all objects a, b of R/G. Then we have the canonical **Galois covering** $F : R \to R/G$ defined on the objects as follows

$$ob(R) \ni x \mapsto Fx = G \cdot x \in ob(R/G).$$

For each $x \in obR$ and $a \in ob(R/G)$, the functor F induces K-linear isomorphisms

$$\bigoplus_{Fy=a} R(x, y) \xrightarrow{\sim} (R/G)(Fx, a),$$

$$\bigoplus_{Fy=a} R(y, x) \xrightarrow{\sim} (R/G)(a, Fx).$$

The group G acts also on the category $\operatorname{mod} R$ by

$$\operatorname{mod} R \ni M \mapsto gM = Mg^{-1} \in \operatorname{mod} R$$

We have also the **push-down functor** (see Bongartz-Gabriel [**BG**])

$$F_\lambda : \operatorname{mod} R \longrightarrow \operatorname{mod} R/G$$

such that $(F_\lambda M)(a) = \bigoplus_{x \in a} M(x)$ for $M \in \operatorname{mod} R, a \in ob(R/G)$.

Assume G is torsion-free. Then F_λ induces an injection (see Gabriel [**Ga**])

$$\left\{ \begin{array}{c} G\text{-orbits of} \\ \text{isoclasses of} \\ \text{indecomposable} \\ \text{modules in } \operatorname{mod} R \end{array} \right\} \overset{F_\lambda}{\rightarrowtail} \left\{ \begin{array}{c} \text{isoclasses of} \\ \text{indecomposable} \\ \text{modules in} \\ \operatorname{mod} R/G \end{array} \right\}.$$

Following Dowbor and Skowroński [**DS1**] a locally bounded K-category R is said to be **locally support-finite** if, for any $x \in obR$,

$$\bigcup_{\substack{M \in \operatorname{ind} R \\ M(x) \neq 0}} \operatorname{supp}(M)$$

is a bounded category.

Then we have the **density theorem of Dowbor and Skowroński** (see [**DS1**], [**DS2**]): if R is a locally support-finite locally bounded K-category and G is an admissible torsion-free group of automorphisms of R then the push-down functor F_λ is dense. Moreover, then the Auslander-Reiten quiver $\Gamma_{R/G}$ of R/G is isomorphic to the orbit quiver Γ_R/G of the Auslander-Reiten quiver Γ_R of R with respect to the indicated action of G.

In particular, if R is a selfinjective locally bounded K-category and G is an admissible group of automorphisms of R then R/G is a basic connected finite dimensional selfinjective K-algebra.

Let B be a basic, connected, finite dimensional K-algebra and $1_B = e_1+\cdots+e_n$ a decomposition of the identity 1_B of B into sum of pairwise orthogonal primitive idempotents. We associate to B a selfinjective locally bounded K-category $\widehat{B}$, called the **repetitive category** of B (see Hughes-Waschbüsch [**HW**]). The objects of $\widehat{B}$ are $e_{m,i}$, $m \in \mathbb{Z}$, $1 \le i \le n$, and the morphism spaces are defined as follows

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_jBe_i & , r = m \\ D(e_iBe_j) & , r = m+1 \\ 0 & , \text{otherwise} \end{cases} .$$

Observe that $e_jBe_i = \operatorname{Hom}_B(e_iB, e_jB)$, $D(e_iBe_j) = e_jD(B)e_i$ and

$$\bigoplus_{(m,i)\in\mathbb{Z}\times\{1,\ldots,n\}} \widehat{B}(-, e_{r,j})(e_{m,i}) = e_jB \oplus D(Be_j).$$

Therefore, for any admissible group G of automorphisms of $\widehat{B}$, we obtain a basic, connected, finite dimensional selfinjective K-algebra $\widehat{B}/G$. We denote by $\nu_{\widehat{B}}$ the **Nakayama automorphism** of $\widehat{B}$ defined as follows

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}, \text{ for all } m, i \in \mathbb{Z} \times \{1, \ldots, n\}.$$

Then, for each positive integer r, the infinite cyclic group $(\nu_{\widehat{B}}^r)$ is an admissible group of automorphisms of $\widehat{B}$, and we have the selfinjective algebra

$$\mathrm{T}(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r) \cong \left\{ \begin{bmatrix} b_1 & 0 & 0 & & & & \\ f_2 & b_2 & 0 & & & 0 & \\ 0 & f_3 & b_3 & & & & \\ & & \ddots & \ddots & & & \\ & 0 & & f_{r-1} & b_{r-1} & 0 \\ & & & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \ldots, b_{r-1} \in B, f_1, \ldots, f_{r-1} \in D(B) \right\},$$

called the ***r*-fold trivial extension algebra** of B. We note that the Nakayama automorphism of $\mathrm{T}(B)^{(r)}$ has order r. Observe also that $\mathrm{T}(B)^{(1)} \cong \mathrm{T}(B) = B \ltimes D(B)$.

We illustrate the above construction by the following example.

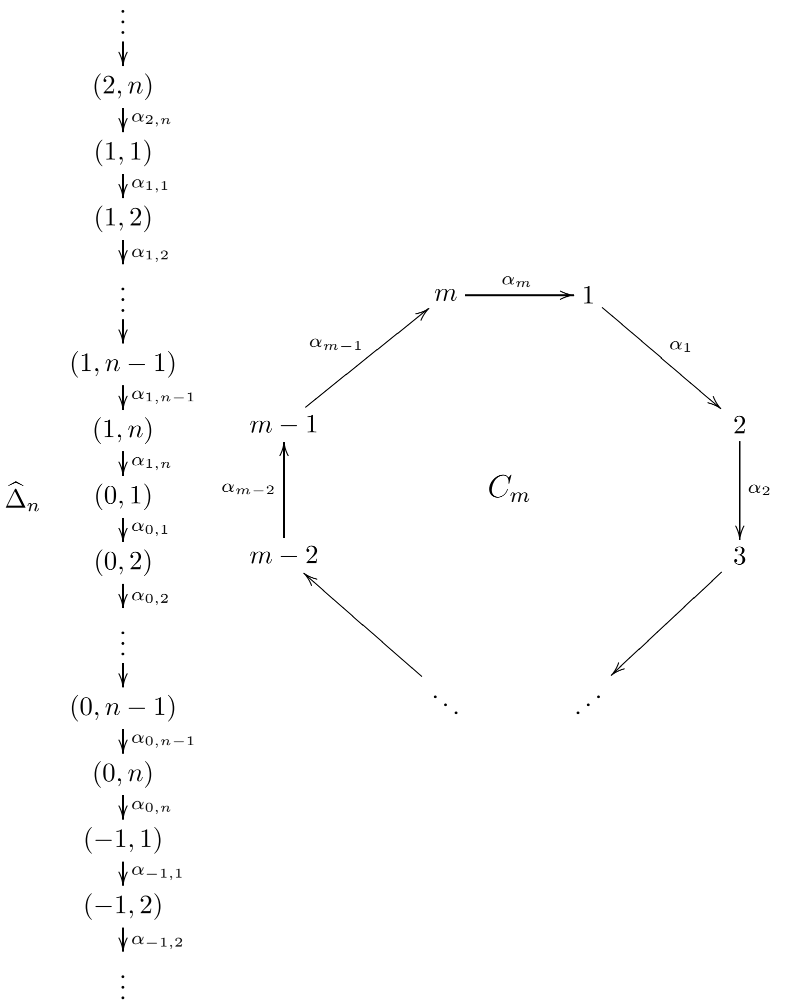
Example 1.15. Let B be the path algebra $K\Delta_n$ of the quiver

$$\Delta_n \qquad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \longrightarrow \cdots \longrightarrow n-1 \xrightarrow{\alpha_{n-1}} n.$$

Then $\widehat{B}$ is the bound quiver category $K\widehat{\Delta}_n/\widehat{I}_n$, where $\widehat{\Delta}_n$ is the left quiver bellow and $\widehat{I}_n$ is generated by all compositions of $n+1$ consecutive arrows in $\widehat{\Delta}_n$. Observe that the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ is given by

$$\nu_{\widehat{B}}(r,i) = (r+1,i), \quad \text{for } (r,i) \in \mathbb{Z} \times \{1,\dots,n\}.$$

Let φ be a unique positive automorphism of $\widehat{B}$ with $\varphi^n = \nu_{\widehat{B}}$. For each positive integer m, consider the orbit algebra $N_m^n = \widehat{B}/(\varphi^m)$. Then N_m^n is the bound quiver algebra $KC_m/J_{m,n}$, where C_m is the right quiver bellow and $J_{m,n}$ is the ideal in the path algebra KC_m generated by all compositions of $n+1$ consecutive arrows in C_m.



It is known that the algebras N_m^n, $m,n \geq 1$, exhaust (up to isomorphism) all non-simple basic connected selfinjective Nakayama algebras. Moreover, the Nakayama algebra N_m^n is symmetric if and only if $m \mid n$. Observe also that $N_n^n = \mathrm{T}(B)$, for $B = K\Delta_n$.

Recall that a finite dimensional selfinjective K-algebra A is called a **Nakayama algebra** if the indecomposable projective A-modules are **uniserial** (the sets of submodules are linearly ordered by inclusion). Then we have the following theorem.

Theorem 1.16. *Let A be an indecomposable finite dimensional selfinjective K-algebra. The following statements are equivalent:*

(1) *A is a Nakayama algebra.*
(2) *The indecomposable finite dimensional A-modules are uniserial.*
(3) *A is Morita equivalent to N_m^n for some $m, n \geq 1$.*

Assume now that B is **triangular** (the Gabriel quiver Q_B of B has no oriented cycles). Then the locally bounded K-category $\widehat{B}$ is also triangular. Moreover, B is the full bounded subcategory of $\widehat{B}$ given by the objects $e_{0,i}$, $1 \leq i \leq n$.

Let i be a sink of Q_B. The **reflection** of B at i is the full subcategory $S_i^+ B$ of $\widehat{B}$ given by the objects $e_{0,j}$, $1 \leq j \leq n$, $j \neq i$, and $e_{1,i} = \nu_{\widehat{B}}(e_{0,i})$. The associated quiver $\sigma_i^+ Q_B = Q_{S_i^+ B}$ is also called the reflection of Q_B at i. Observe that $\widehat{B} \cong \widehat{S_i^+ B}$, and hence

$$\mathrm{T}(B)^{(r)} \cong \mathrm{T}(S_i^+ B)^{(r)}, \text{ for any } r \geq 1.$$

A **reflection sequence of sinks** of Q_B is a sequence $i_1, \dots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \dots \sigma_{i_1}^+ Q_B$, for any $1 \leq s \leq t$.

Two triangular basic, connected algebras B and C are defined to be **reflection equivalent** if $C \cong S_{i_t}^+ \dots S_{i_1}^+ B$ for a reflection sequence of sinks $i_1, \dots, i_t$ of Q_B. Observe that, if B and C are reflection equivalent triangular algebra, then $\widehat{B} \cong \widehat{C}$, $\mathrm{T}(B)^{(r)} \cong \mathrm{T}(C)^r$ for all $r \geq 1$.

2. Periodicity of modules and algebras

In this section we introduce the periodic modules and the periodic algebras, and describe their properties and characterizations.

Let A be a finite dimensional selfinjective K-algebra. Then A^{op} is also selfinjective and we have the duality between $\operatorname{mod} A$ and A^{op}

$$\operatorname{mod} A \underset{\operatorname{Hom}_{A^{\mathrm{op}}}(-, {}_A A)}{\overset{\operatorname{Hom}_A(-, A_A)}{\rightleftarrows}} \operatorname{mod} A^{\mathrm{op}} .$$

Then we have the selfequivalence functor

$$\mathcal{N}_A = D\operatorname{Hom}_A(-, A_A) : \operatorname{mod} A \to \operatorname{mod} A,$$

called the **Nakayama functor**. Moreover,

$$\mathcal{N}_A^{-1} = \operatorname{Hom}_{A^{\mathrm{op}}}(-, {}_A A)D$$

is the inverse of $\mathcal{N}_A$.

Proposition 2.1. *The functors*

$$\mathcal{N}_A, - \otimes_A D(A) : \operatorname{mod} A \to \operatorname{mod} A$$

are equivalent.

Proof. For any module M in $\operatorname{mod} A$, we have a natural isomorphism of right A-modules

$$\phi_M : M \otimes_A D(A) \to D\operatorname{Hom}_A(M, A) = \mathcal{N}_A(M)$$

such that $\phi_M(m \otimes f)(g) = f(g(m))$ for $m \in M$, $f \in D(A) = \mathrm{Hom}_K(A, K)$ and $g \in \mathrm{Hom}_A(M, A)$. This induces an equivalence of functors

$$\phi : - \otimes_A D(A) \to \mathcal{N}_A.$$

□

For a K-algebra automorphism σ of A, we denote by

$$(-)_\sigma : \mathrm{mod}\, A \to \mathrm{mod}\, A$$

the induced functor such that, for any module M in $\mathrm{mod}\, A$, M_σ is the module with the twisted right A-module structure

$$m * a = m\sigma(a), \text{ for } m \in M \text{ and } a \in A.$$

Proposition 2.2. *Let A be a Frobenius algebra and ν_A its Nakayama automorphism. Then the functors*

$$\mathcal{N}_A, (-)_{\nu_A^{-1}} : \mathrm{mod}\, A \to \mathrm{mod}\, A$$

are equivalent.

Proof. Let $(-,-)$ be a nondegenerate associative K-bilinear form defining the Nakayama automorphism ν_A. Then a required equivalence

$$\psi : (-)_{\nu_A^{-1}} \longrightarrow \mathcal{N}_A$$

is given by the family of isomorphisms of right A-modules

$$\psi_M : M_{\nu_A^{-1}} \longrightarrow \mathcal{N}_A(M) = D\,\mathrm{Hom}_A(M, A),$$

where M are modules in $\mathrm{mod}\, A$, such that

$$\psi_M(m)(g) = (g(m), 1) = (1, g(m)),$$

for all $m \in M$, $g \in \mathrm{Hom}_A(M, A)$. □

Therefore, if A is a Frobenius algebra, and

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

is the standard decomposition of 1_A into the sum of pairwise orthogonal primitive idempotents, then we have isomorphisms of right A-modules

$$\mathcal{N}_A(e_{ij}A) \cong (e_{ij}A)_{\nu_A^{-1}} \xrightarrow{\sim} \nu_A(e_{ij})A = \nu_A(e_{ij}A)$$

such that

$$(e_{ij}a) * b = (e_{ij}a)\nu_A^{-1}(b) \mapsto \nu_A(e_{ij}a)b$$

for $a, b \in A$. Moreover, $\mathcal{N}_A(e_{ij}A) = D(Ae_{ij})$. Hence we obtain that

$$\mathrm{top}(e_{ij}A) \cong \mathrm{soc}\, \nu_A(e_{ij})A.$$

In particular, the Nakayama automorphism ν_A induces a Nakayama permutation $\nu = \nu_A$ of $\{1, \dots, n_A\}$.

For a symmetric algebra A, we have $\nu_A = \mathrm{id}_A$ and $\mathcal{N}_A \cong 1_{\mathrm{mod}\, A}$. In particular, for a symmetric algebra A, we have $\mathrm{top}\, P \cong \mathrm{soc}\, P$ for any indecomposable projective A-module P, that is, A is a **weakly symmetric algebra** (the trivial permutation of $\{1, \dots, n_A\}$ is a Nakayama permutation of A).

Let A be a finite dimensional selfinjective K-algebra. We denote by $\underline{\mathrm{mod}}A$ the **stable category** of $\mathrm{mod}\,A$. The objects of $\underline{\mathrm{mod}}A$ are the modules in $\mathrm{mod}\,A$ without nonzero projective direct summands, and, for any two objects M and N in $\underline{\mathrm{mod}}A$, the K-space $\underline{\mathrm{Hom}}_A(M,N)$ of morphisms from M to N is the quotient $\mathrm{Hom}_A(M,N)/P(M,N)$, where $P(M,N)$ is the subspace of $\mathrm{Hom}_A(M,N)$ consisting of all homomorphisms which factorize through a projective A-module. Then the Nakayama functors

$$\mathcal{N}_A, \mathcal{N}_A^{-1} : \mathrm{mod}\,A \to \mathrm{mod}\,A$$

induce the **Nakayama functors**

$$\mathcal{N}_A, \mathcal{N}_A^{-1} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A,$$

because $\mathcal{N}_A(\mathrm{proj}\,A) = \mathrm{inj}\,A = \mathrm{proj}\,A$.

We have also the **Auslander-Reiten translation functors**

$$\tau_A = D\,\mathrm{Tr}, \tau_A^{-1} = \mathrm{Tr}\,D : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A.$$

Consider also the **(Heller's) syzygy functors** [**He**]

$$\Omega_A, \Omega_A^{-1} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$$

defined by the exact sequences below. For a module M in $\mathrm{mod}\,A$ without projective direct summands, we have exact sequences

$$0 \to \Omega_A(M) \to P_A(M) \to M \to 0,$$
$$0 \to M \to I_A(M) \to \Omega_A^{-1}(M) \to 0,$$

where $P_A(M)$ is the projective cover of M and $I_A(M)$ is the injective envelope of M in $\mathrm{mod}\,A$.

Proposition 2.3. *Let A be a selfinjective algebra.*

(1) *The functors*

$$D\,\mathrm{Tr}, \Omega_A^2\mathcal{N}_A, \mathcal{N}_A\Omega_A^2 : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$$

are isomorphic.

(2) *The functors*

$$\mathrm{Tr}\,D, \Omega_A^{-2}\mathcal{N}_A^{-1}, \mathcal{N}_A^{-1}\Omega_A^{-2} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$$

are isomorphic.

Proof. (1) For a module M in $\mathrm{mod}\,A$ without projective direct summands, we have a minimal projective presentation of M in $\mathrm{mod}\,A$

$$0 \to \Omega_A^2(M) \to P_1(M) \to P_0(M) \to M \to 0.$$

Applying the exact functor $\mathrm{Hom}_A(-, A_A)$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_A(M, A_A) \to \mathrm{Hom}_A(P_0, A_A) \to \mathrm{Hom}_A(P_1(M), A_A) \to \mathrm{Tr}\,M \to 0$$

in $\mathrm{mod}\,A^{\mathrm{op}}$. Further, applying the duality functor $D : \mathrm{mod}\,A^{\mathrm{op}} \to \mathrm{mod}\,A$, we obtain the exact sequence

$$\begin{array}{ccccccccc} 0 \to D\,\mathrm{Tr}\,M \to & D\,\mathrm{Hom}_A(P_1, A_A) & \to & D\,\mathrm{Hom}_A(P_0(M), A_A) & \to & D\,\mathrm{Hom}_A(M, A_A) & \to 0\,, \\ & \| & & \| & & \| & \\ & \mathcal{N}_A(P_1(M)) & & \mathcal{N}_A(P_0(M)) & & \mathcal{N}_A(M) & \end{array}$$

which is a minimal projective presentation of $\mathcal{N}_A(M)$ in $\mathrm{mod}\,A$. Hence, we obtain isomorphisms $\Omega_A^2\mathcal{N}_A(M) \cong D\,\mathrm{Tr}\,M \cong \mathcal{N}_A\Omega_A^2(M)$. □

As a direct consequence we obtain the following facts.

Corollary 2.4. *Let A be a symmetric algebra. Then*

(1) *The functors $D\,\mathrm{Tr}, \Omega_A^2 : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$ are isomorphic.*
(2) *The functors $\mathrm{Tr}\,D, \Omega_A^{-2} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$ are isomorphic.*

By general theory, if P is an indecomposable projective-injective A-module, then we have in $\mathrm{mod}\,A$ an Auslander-Reiten sequence of the form

$$0 \to \mathrm{rad}\,P \to \mathrm{rad}\,P/\mathrm{soc}\,P \oplus P \to P/\mathrm{soc}\,P \to 0.$$

For A selfinjective, we denote by Γ_A^s the **stable Auslander-Reiten quiver** of A, obtained from the Auslander-Reiten quiver Γ_A of A by removing the projective-injective vertices and the arrows attached to them. Observe that we may recover Γ_A from Γ_A^s if we know the positions of $\mathrm{rad}\,P$ (equivalently, $P/\mathrm{soc}\,P$), for all indecomposable projective modules P, in Γ_A^s.

Two selfinjective algebras A and Λ are said to be **stably equivalent** if the stable module categories $\underline{\mathrm{mod}}A$ and $\underline{\mathrm{mod}}\Lambda$ are equivalent. Clearly, if A and Λ are stably equivalent selfinjective algebras then the stable Auslander-Reiten quivers Γ_A^s and Γ_Λ^s are isomorphic translation quivers.

Let A be a finite dimensional K-algebra. A module M in $\mathrm{mod}\,A$ is said to be **Ω_A-periodic** (shortly, **periodic**) if $\Omega_A^n(M) \cong M$ for some $n \geq 1$. The following problem occurs naturally.

PROBLEM 1. Determine the finite dimensional K-algebras A whose all indecomposable nonprojective finite dimensional right A-modules are periodic.

We will see later that all such algebras are selfinjective.

Similarly, a module M in $\mathrm{mod}\,A$ is called **$D\,\mathrm{Tr}$-periodic** if $(D\,\mathrm{Tr})^n(M) \cong M$ for some $n \geq 1$. Then we have the related natural problem.

PROBLEM 2. Determine the finite dimensional K-algebras A for which all indecomposable nonprojective finite dimensional right A-modules are $D\,\mathrm{Tr}$-periodic.

It is clear that all such algebras are selfinjective, because the $D\,\mathrm{Tr}$-orbit of an indecomposable injective A-module is not a finite periodic orbit, and hence consists of one module, which is then an indecomposable projective A-module.

Let A be a selfinjective algebra. Then $D\,\mathrm{Tr} \cong \Omega_A^2 \mathcal{N}_A$ as functors on $\underline{\mathrm{mod}}A$. Hence, the Ω_A-periodicity in $\underline{\mathrm{mod}}A$ coincides with the $D\,\mathrm{Tr}$-periodicity in $\underline{\mathrm{mod}}A$ if the Nakayama functor $\mathcal{N}_A$ on $\underline{\mathrm{mod}}A$ has finite order. For example, it is the case for all finite dimensional Hopf algebras H, because they are Frobenius algebras with the Nakayama automorphism ν_H of finite order, and $\mathcal{N}_H \cong (-)_{\nu_H^{-1}}$ on $\underline{\mathrm{mod}}H$. Obviously, it is also the case for all symmetric algebras. Moreover, we have the following fact.

Proposition 2.5. *Let A be a finite dimensional selfinjective K-algebra of finite representation type. Then all indecomposable nonprojective finite dimensional A-modules are Ω_A-periodic and $D\,\mathrm{Tr}$-periodic.*

Proof. Let M be an indecomposable nonprojective right A-module. If M is not Ω_A-periodic (respectively, $D\,\mathrm{Tr}$-periodic) then $\Omega_A^n(M)$, $n \geq 0$ (respectively, $(D\,\mathrm{Tr})^n(M)$, $n \geq 0$) is an infinite family of pairwise nonisomorphic indecomposable modules in mod A, and hence A is of infinite representation type, a contradiction. □

We will now discuss the Ω_A-periodicity of A-modules.

Lemma 2.6. *Let A be a basic, indecomposable, finite dimensional selfinjective K-algebra. The following statements are equivalent:*
(1) *$\Omega_A(S)$ is simple for any simple A-module S.*
(2) *$A \cong N_m^1$ for some $m \geq 1$.*

Proof. (1) ⇒ (2). For any simple A-module S, we have an exact sequence

$$0 \longrightarrow \Omega_A(S) \longrightarrow P(S) \longrightarrow S \longrightarrow 0.$$

Hence, $\Omega_A(S) \cong \operatorname{soc} P_A(S) = \operatorname{rad} P_A(S)$, and consequently $J(A)^2 = 0$, where $J(A)$ is the Jacobson radical of A. Then $A \cong N_m^1$ for some $m \geq 1$. For (2) ⇒ (1), note that $J(N_m^1)^2 = 0$. □

Corollary 2.7. *Let A be a basic, indecomposable, finite dimensional selfinjective K-algebra. The following statements are equivalent:*
(1) *$\Omega_A(S) \cong S$ for any simple A-module S.*
(2) *$\Omega_A(M) \cong M$ for any indecomposable nonprojective A-module M.*
(3) *$A \cong N_1^1$.*

We note that $N_1^1 \cong K[x]/(x^2) \cong \mathrm{T}(K) = K \ltimes D(K)$.

Theorem 2.8. *Let A be a basic, indecomposable, finite dimensional selfinjective K-algebra. The following statements are equivalent:*
(1) *$\Omega^2(S)$ is simple for any simple A-module S.*
(2) *$D\operatorname{Tr}(S)$ is simple for any simple A-module S.*
(3) *$A \cong N_m^n$ for some $m, n \geq 1$.*
(characterization of the Nakayama algebras)

Proof. For (1) ⇔ (2), observe that, for any simple A-module S, $\mathcal{N}_A(S)$ is simple, because the Nakayama functor $\mathcal{N}_A = D\operatorname{Hom}_A(-, A)$ is exact, and $D\operatorname{Tr}(S) \cong \Omega_A^2 \mathcal{N}_A(S)$.

(1) ⇒ (3). Since A is basic and indecomposable, $A \cong KQ/I$ for a connected quiver Q and an admissible ideal I of KQ. For a simple A-module S, we have an exact sequence

$$0 \to \Omega_A^2(S) \to P_A(\operatorname{rad} P_A(S)) \to P_A(S) \to S \to 0.$$

Then the assumption that $\Omega_A^2(S)$ is simple implies that $P_A(\operatorname{rad} P_A(S))$ is indecomposable, and hence $\operatorname{top}(\operatorname{rad} P_A(S))$ is simple. Therefore, every vertex of Q is the starting (respectively, ending) vertex of exactly one arrow. Then $A \cong N_m^n$ for some $m, n \geq 1$.

The implication (3) ⇒ (1) follows from the above exact sequences and the bound quiver presentation of N_m^n. □

We obtain the following immediate consequence of the above theorem.

Corollary 2.9. *Let A be a basic, indecomposable, finite dimensional selfinjective K-algebra. The following statements are equivalent:*
(1) *$\Omega_A^2(S) \cong S$ for any simple A-module S.*
(2) *$\Omega_A^2(M) \cong M$ for any indecomposable nonprojective finite dimensional A-module M.*

(3) $A \cong N_m^n$, *where* $n+1$ *is divisible by* m.

Corollary 2.10. *Let* A *be a basic, indecomposable, finite dimensional selfinjective* K*-algebra. The following statements are equivalent:*

(1) A *is symmetric and* $\Omega_A^2(S) \cong S$ *for any simple* A*-module* S.
(2) $D\,\mathrm{Tr}(S) \cong S$ *for any simple* A*-module* S.
(3) $D\,\mathrm{Tr}(M) \cong M$ *for any indecomposable nonprojective finite dimensional* A*-module* M.
(4) $A \cong N_1^n \cong K[x]/(x^{n+1})$ *for some* $n \geq 1$.

Proof. Observe that (1) $\Leftrightarrow A \cong N_m^n$ with $m \mid n$, $m \mid n+1 \Leftrightarrow A \cong N_1^n$. Hence (1) $\Leftrightarrow$ (4).

The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious. Finally, the implication (2) $\Rightarrow$ (4) also follows, because, by Theorem 2.8, (2) implies $A \cong N_m^n$. □

Example 2.11. Let $H = H_{n^2}(\lambda)$, $n \geq 2$, be the Taft (Hopf) algebra. Then $H \cong N_n^{n-1}$. Hence, for any indecomposable nonprojective finite dimensional H-module M, we have

$$\Omega_H^2(M) \cong M \quad \text{and} \quad \Omega_H(M) \ncong M.$$

On the other hand, we have

$$(D\,\mathrm{Tr})^n(M) \cong M \text{ and } (\mathrm{Tr}\,D)^r(M) \ncong M,$$

for $1 \leq r < n$, because

$$D\,\mathrm{Tr}(M) \cong \Omega_H^2 \mathcal{N}_H(M) \cong \mathcal{N}_H(M) \cong M_{\nu_H^{-1}},$$

and the Nakayama automorphism ν_H has order n.

Proposition 2.12. *Let* H *be a finite dimensional Hopf algebra over* K. *The following statements are equivalent:*

(1) *The trivial* H*-module* K *is* Ω_H*-periodic.*
(2) *All indecomposable nonprojective finite dimensional* H*-modules are* Ω_H*-periodic.*

Proof. Let $H = (H, m, \eta, \Delta, \varepsilon, s)$. Then the counit $\varepsilon : H \to K$ induces on K the structure of trivial right H-module by

$$\lambda * h = \lambda\varepsilon(h), \text{ for } \lambda \in K, h \in H.$$

Clearly, K is an indecomposable H-module. Moreover, K is projective if and only if H is semisimple. Hence (2) $\Rightarrow$ (1) holds.

For (1) $\Rightarrow$ (2), we first observe that for any projective module P in $\mathrm{mod}\,H$ and any module M in $\mathrm{mod}\,H$, $P \otimes_K M$ is a projective-injective module in $\mathrm{mod}\,H$. Indeed, the structure of right H-module on $P \otimes_K M$ is given by

$$\begin{array}{c} (P \otimes_K M) \otimes H \xrightarrow{1\otimes 1\otimes\Delta} P \otimes_K M \otimes_K H \otimes_K H \xrightarrow{1\otimes\tau\otimes 1} (P \otimes_K H) \otimes_K (M \otimes_K H) \\ \downarrow{\scriptstyle \alpha\otimes\beta} \\ P \otimes_K M \end{array}$$

where $\tau : M \otimes_K H \to H \otimes_K M$ is the exchanging map, and $\alpha : P \otimes_K H \to P$, $\beta : M \otimes_K H \to M$ are the right H-module structure maps. Moreover, the following well-known isomorphism of functors on $\mathrm{mod}\,K$

$$\mathrm{Hom}_K(P \otimes_K M, -) \xrightarrow{\sim} \mathrm{Hom}_K(P, \mathrm{Hom}_K(M, -))$$

induces an isomorphism of functors on $\operatorname{mod} H$

$$\operatorname{Hom}_H(P \otimes_K M, -) \xrightarrow{\sim} \operatorname{Hom}_H(P, \operatorname{Hom}_K(M, -)).$$

Hence the functor

$$\operatorname{Hom}_H(P \otimes_K M, -) : \operatorname{mod} H \longrightarrow \operatorname{mod} H$$

is exact, and consequently $P \otimes_K M$ is a projective right H-module. Since H is a Frobenius algebra, $P \otimes_K M$ is also injective.

Assume now that $\Omega_H^n(K) \cong K$ for some $n \geq 1$. Then there exists a long exact sequence in $\operatorname{mod} H$ of the form

$$0 \to \Omega_H^n(K) \to P_{n-1} \to \cdots \to P_1 \to P_0 \to K \to 0$$

with $P_0, P_1, \dots, P_{n-1}$ projective modules. Let M be an indecomposable nonprojective module in $\operatorname{mod} H$. Then we obtain a long exact sequence in $\operatorname{mod} H$

$$0 \to \Omega_H^n(K) \otimes_K M \to P_{n-1} \otimes_K M \to \cdots \to P_1 \otimes_K M \to P_0 \otimes_K M \to K \otimes_K M \to 0$$

with $P_0 \otimes_K M, P_1 \otimes_K M, \dots, P_{n-1} \otimes_K M$ projective H-modules.

We know that $\Omega_H^n(M)$ is an indecomposable nonprojective H-module. Hence

$$\Omega_H^n(K) \otimes_K M \cong \Omega_H^n(M) \oplus P$$

for some projective H-module P. On the other hand, we have

$$\Omega_H^n(K) \otimes_K M \cong K \otimes_K M \cong M.$$

Hence $\Omega_H^n(M) \cong M$, and M is Ω_H-periodic. Therefore, (1) $\Rightarrow$ (2) holds. □

Proposition 2.13. *Let A be a finite dimensional selfinjective K-algebra, M a module in* $\operatorname{mod} A$*, and r a positive integer. Then*

(1) *The functors* $\operatorname{Ext}_A^r(M, -), \underline{\operatorname{Hom}}_A(\Omega_A^r(M), -) : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ *are equivalent.*

(2) *The functors* $\operatorname{Ext}_A^r(-, M), \underline{\operatorname{Hom}}_A(-, \Omega_A^{-r}(M)) : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ *are equivalent.*

For a finite dimensional selfinjective K-algebra A and a module M in $\operatorname{mod} A$, consider the vector space

$$\begin{aligned}\operatorname{Ext}_A^*(M, M) &= \bigoplus_{r=0}^{\infty} \operatorname{Ext}_A^r(M, M)\\ &\cong \bigoplus_{r=0}^{\infty} \underline{\operatorname{Hom}}_A(\Omega_A^r(M), M).\end{aligned}$$

Then $\operatorname{Ext}_A^*(M, M)$ is a graded K-algebra, called the **Ext-algebra** of M, and the multiplication of

$$\underline{f} \in \underline{\operatorname{Hom}}_A(\Omega_A^r(M), M), \ \ \underline{g} \in \underline{\operatorname{Hom}}_A(\Omega_A^s(M), M)$$

is given by

$$\underline{f} * \underline{g} = \underline{f} \circ \Omega_A^r(\underline{g}), \quad \Omega_A^{r+s}(M) \to \Omega_A^r(M) \to M.$$

Observe that, if M is Ω_A-periodic of period d, then

$$\operatorname{Ext}_A^{i+d}(M, N) \cong \operatorname{Ext}_A^i(M, N),$$

for all $i \geq 1$ and modules N in $\operatorname{mod} A$. Indeed, we have isomorphisms of vector spaces

$$\begin{aligned}\operatorname{Ext}_A^{i+d}(M,N) &\cong \underline{\operatorname{Hom}}_A(\Omega_A^{i+d}(M),N)\\ &\cong \underline{\operatorname{Hom}}_A(\Omega_A^{i}(\Omega_A^{d}(M)),N)\\ &\cong \underline{\operatorname{Hom}}_A(\Omega_A^{i}(M),N)\\ &\cong \operatorname{Ext}_A^{i}(M,N).\end{aligned}$$

The following theorem was proved by Carlson [**Car**].

Theorem 2.14 (Carlson, 1977)**.** *Let A be a finite dimensional selfinjective K-algebra and M be an indecomposable Ω_A-periodic A-module of period d. Moreover, let $\mathcal{N}(M)$ be the ideal of the algebra $\operatorname{Ext}_A^*(M,M)$ generated by all nilpotent homogeneous elements. Then*

$$\operatorname{Ext}_A^*(M,M)/\mathcal{N}(M) \cong K[x]$$

as graded K-algebras, where x is of degree d.

Proof. We identify

$$\underline{\operatorname{Hom}}_A(\Omega_A^i(M),M) = \operatorname{Ext}_A^i(M,M) = \underline{\operatorname{Hom}}_A(M,\Omega_A^{-i}(M)),$$

for any $i \geq 1$. Let $\underline{f} \in \underline{\operatorname{Hom}}_A(\Omega_A^s(M),M)$ be a homogeneous nilpotent element of $\operatorname{Ext}_A^*(M,M)$ and $\underline{g} \in \underline{\operatorname{Hom}}_A(\Omega_A^m(M),M)$ an arbitrary homogeneous element of $\operatorname{Ext}_A^*(M,M)$. We claim that

$$\underline{f} * \underline{g} = \underline{f}\Omega_A^s(\underline{g}) \in \underline{\operatorname{Hom}}_A(\Omega_A^{m+s}(M),M)$$

is again a nilpotent element of $\operatorname{Ext}_A^*(M,M)$.

Choose r such that $r(m+s) = qd$, for some $q \geq 1$, and consider the element $\underline{h} = (\underline{f}\Omega_A^s(\underline{g}))^r$ in $\operatorname{Ext}_A^*(M,M)$. Then

$$\underline{h} \in \underline{\operatorname{Hom}}_A(\Omega_A^{qd}(M),M) \cong \underline{\operatorname{Hom}}_A(M,M),$$

because $\Omega_A^{qd}(M) \cong M$. Suppose $\underline{h}$ is an isomorphism. Then $f : \Omega_A^s(M) \to M$ is a split epimorphism, and hence an isomorphism, since M and $\Omega_A^s(M)$ are indecomposable. But then $\underline{f}$ is not nilpotent in $\operatorname{Ext}_A^*(M,M)$, a contradiction. Therefore, $\underline{h}$ belongs to the radical of the local algebra $\underline{\operatorname{End}}_A(M)$, and hence $\underline{h}$ is nilpotent. Then $\Omega_A^{id}(\underline{h}) \in \underline{\operatorname{End}}_A(M)$ are nilpotent elements for all $i \geq 0$, and hence belong to the radical of $\underline{\operatorname{End}}_A(M)$. Since $(\operatorname{rad}\underline{\operatorname{End}}_A(M))^l = 0$ for some $l \geq 1$, we get $\underline{h}^l = 0$. But then $\underline{f} * \underline{g} = \underline{f}\Omega_A^s(\underline{g})$ is a nilpotent element in $\operatorname{Ext}_A^*(M,M)$. Similarly, using

$$\operatorname{Ext}_A^i(M,M) = \underline{\operatorname{Hom}}_A(M,\Omega_A^{-i}(M)), i \geq 1,$$

we prove that $g * f$ is nilpotent in $\operatorname{Ext}_A^*(M,M)$.

Let $s \neq pd$, for all $p \geq 1$. We show that any element $\underline{f} \in \underline{\operatorname{Hom}}_A(\Omega_A^s(M),M)$ is a nilpotent element of $\operatorname{Ext}_A^*(M,M)$. Choose $r \geq 1$ such that $rs = qd$, for some $q \geq 1$, and take $\underline{h} = f^r$ in $\operatorname{Ext}_A^*(M,M)$. Since d is the period of M and s is not divisible by d, we conclude that f is not an isomorphism. Then $\underline{h}$ is not an isomorphism, hence $\underline{h} \in \underline{\operatorname{End}}_A(M)$ is nilpotent. Therefore, $\underline{h}$ is a nilpotent element in $\operatorname{Ext}_A^*(M,M)$, and so $\underline{f}$ is nilpotent in $\operatorname{Ext}_A^*(M,M)$.

Let $x \in \underline{\operatorname{Hom}}(\Omega_A^d(M),M) \cong \underline{\operatorname{Hom}}_A(M,M)$ corresponds to the residue class of the identity map from M to M. Observe that x is not nilpotent in $\operatorname{Ext}_A^*(M,M)$. We claim that $x^n \notin \mathcal{N}(M)$ for any $n \geq 1$. Suppose that $x^t \in \mathcal{N}(M)$ for some $t \geq 1$. Then $x^t = \sum g_i * f_i * h_i$, where f_i are homogeneous nilpotent elements of $\operatorname{Ext}_A^*(M,M)$ and g_i, h_i are elements of $\operatorname{Ext}_A^*(M,M)$. We may assume that the elements g_i, h_i are also homogeneous. It follows from the first part of the proof that $g_i * f_i * h_i = (g_i * f_i) * h_i$ are nilpotent elements in $\mathrm{Ext}_A^*(M,M)$, and hence are nilpotent in $\underline{\mathrm{End}}_A(M)$. But then $\sum g_i * f_i * h_i$ are nilpotent in $\underline{\mathrm{End}}_A(M)$, and hence in $\mathrm{Ext}_A^*(M,M)$. This implies that x^t, and hence x, is nilpotent in $\mathrm{Ext}_A^*(M,M)$, a contradiction. Since $\underline{\mathrm{End}}_A(M)/\operatorname{rad}\underline{\mathrm{End}}_A(M) \cong K$, we conclude that $\mathrm{Ext}_A^*(M,M)/\mathcal{N}(M) \cong K[x]$ as graded K-algebras, with x of degree d. □

Let A be a finite dimensional K-algebra and $1_A = e_1 + e_2 + \cdots + e_m$, where $e_1, e_2, \dots, e_m$ are pairwise orthogonal primitive idempotents of A. Then

$$A^e = A^{\mathrm{op}} \otimes_K A$$

is called the **enveloping algebra** of A. The identity of A^e has the decomposition

$$1_{A^e} = \sum_{1 \le i,j \le m} e'_i \otimes e_j,$$

where $e'_1 = e_1, e'_2 = e_2, \dots, e'_m = e_m$ are primitive idempotents of A^{op}. Moreover, the category $\operatorname{mod} A^e$ of finite dimensional right A^e-modules is the category of finite dimensional A-A-bimodules. In particular, the algebra A is a right A^e-module by $a(x \otimes y) = xay$, for $a \in A$, $x \in A^{\mathrm{op}}$, $y \in A$.

The indecomposable projective right A^e-module (projective A-A-bimodule) associated to the idempotent $e'_i \otimes e_j$ is of the form

$$P(i', j) = (e'_i \otimes e_j)A^e = e_i A^{\mathrm{op}} \otimes_K e_j A = Ae_i \otimes_K e_j A.$$

Moreover, we have a decomposition

$$A^e = \bigoplus_{1 \le i,j \le m} P(i', j)$$

of A^e into a direct sum of indecomposable projective right A^e-modules (projective A-A-bimodules) such that ${}_A P(i', j) \cong (Ae_i)^{\dim_K e_j A}$ is a projective left A-module and $P(i', j)_A \cong (e_j A)^{\dim_K Ae_i}$ is a projective right A-module. Hence every projective right A^e-module is a projective left A-module and a projective right A-module.

Lemma 2.15. *Let A be a finite dimensional K-algebra. For each $i \ge 0$, $\Omega^i_{A^e}(A)$ is a projective left A-module and a projective right A-module.*

Proof. Consider a minimal projective resolution of A in $\operatorname{mod} A^e$

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to A \to 0.$$

For each $i \ge 0$, we have an exact sequence in $\operatorname{mod} A^e$

$$0 \to \Omega^{i+1}_{A^e}(A) \to P_i \to \Omega^i_{A^e}(A) \to 0,$$

which is an exact sequence in $\operatorname{mod} A^{\mathrm{op}}$ and in $\operatorname{mod} A$. Since the projective right A^e-modules are projective left A-modules and projective right A-modules, by induction on i, we conclude that these sequences split in $\operatorname{mod} A^{\mathrm{op}}$ and in $\operatorname{mod} A$, and hence $\Omega^i_{A^e}(A)$ are projective left A-modules and projective right A-modules. □

Lemma 2.16. *Let A be a finite dimensional selfinjective algebra and M be a module in $\operatorname{mod} A$ without projective direct summands. Then, for each $i \ge 0$, we have $\Omega^i_A(M) \cong M \otimes_A \Omega^i_{A^e}(A)$ in $\underline{\operatorname{mod}} A$.*

Proof. We may assume that M is indecomposable. The splitting exact sequences (as in the above lemma)

$$0 \to \Omega^{i+1}_{A^e}(A) \to P_i \to \Omega^i_{A^e}(A) \to 0,$$

for $i \geq 0$, induce the exact sequences

$$0 \to M \otimes_A \Omega^{i+1}_{A^e}(A) \to M \otimes_A P_i \to M \otimes_A \Omega^i_{A^e}(A) \to 0$$

in $\operatorname{mod} A$, and hence

$$\cdots \to M \otimes_A P_{i+1} \to M \otimes_A P_i \to \cdots \to M \otimes_A P_0 \to M \otimes_A A \to 0$$

is a projective resolution of $M \cong M \otimes_A A$ in $\operatorname{mod} A$. Since, for each $i \geq 0$, $\Omega^i_A(M)$ is an indecomposable nonprojective A-module, we conclude that

$$M \otimes_A \Omega^i_{A^e}(A) \cong \Omega^i_A(M) \oplus P(i),$$

for some projective module $P(i)$ in $\operatorname{mod} A$. Therefore, we obtain a required isomorphism

$$\Omega^i_A(M) \cong M \otimes_A \Omega^i_{A^e}(A) \text{ in } \underline{\operatorname{mod}} A.$$

□

The following lemma proved by Green, Snashall and Solberg [**GSS**] will be essential for our considerations.

Lemma 2.17 (Green-Snashall-Solberg, 2003)**.** *Let A be a finite dimensional K-algebra. Assume there exists a positive integer d and an algebra automorphism σ of A such that $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\operatorname{mod} A^e$. Then A is selfinjective.*

Proof. We have an isomorphism of A-A-bimodules

$$\alpha : D(A) \otimes_A {}_1A_\sigma \longrightarrow D(A)_\sigma$$

such that $\alpha(f \otimes a) = fa$, for $f \in D(A)$ and $a \in {}_1A_\sigma$. Consider a minimal projective resolution

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to A \to 0$$

of A in $\operatorname{mod} A^e$. Hence we obtain an exact sequence

$$0 \to D(A) \otimes_A \Omega^d_{A^e}(A) \to D(A) \otimes_A P_{d-1} \to D(A) \otimes_A \Omega^{d-1}_{A^e}(A) \to 0$$

in $\operatorname{mod} A$. Moreover, $D(A) \otimes_A P_{d-1}$ is a projective right A-module. On the other hand, $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\operatorname{mod} A^e$ implies that there is a monomorphism $D(A)_\sigma \to D(A) \otimes_A P_{d-1}$ in $\operatorname{mod} A$. Further, the automorphism σ induces an isomorphism ${}_1A_{\sigma^{-1}} \xrightarrow{\sim} {}_\sigma A_1$ of A-A-bimodules, and then the right A-modules $D(A)_\sigma = D({}_\sigma A_1)$ and ${}_{\sigma^{-1}}D(A) = D({}_1A_{\sigma^{-1}})$ are isomorphic. Therefore, the injective cogenerator $D(A)$ in $\operatorname{mod} A$ is a direct summand of the projective module $D(A) \otimes_A P_{d-1}$, and so is projective. Clearly, then A is selfinjective. □

A finite dimensional K-algebra A is said to be **periodic** if A is a periodic module in $\operatorname{mod} A^e$, that is, $\Omega^d_{A^e}(A) \cong A$ in $\operatorname{mod} A^e$, for some $d \geq 1$. It follows from the above lemma that then A is selfinjective. Moreover, we have the following fact.

Corollary 2.18. *Let A be a finite dimensional periodic K-algebra. Then all indecomposable nonprojective modules in* $\operatorname{mod} A$ *are periodic.*

PROOF. Assume $\Omega^d_{A^e}(A) \cong A$ in $\operatorname{mod} A$, for some $d \geq 1$. Let M be an indecomposable nonprojective module in $\operatorname{mod} A$. Since A is selfinjective, invoking Lemma 2.16, we have in $\underline{\operatorname{mod}} A$ isomorphisms

$$\Omega^d_A(M) \cong M \otimes_A \Omega^d_{A^e}(A) \cong M \otimes_A A \cong M.$$

Then $\Omega^d_A(M) \cong M$ in $\operatorname{mod} A$, because $\Omega^d_A(M)$ and M are indecomposable nonprojective A-modules. □

The following problem occurs naturally.

PROBLEM 3. Determine the finite dimensional periodic algebras.

We also note the following fact.

Lemma 2.19. *Let A be a finite dimensional K-algebra. Then A is selfinjective if and only if A^e is selfinjective.*

PROOF. Since $(A^e)^{\mathrm{b}} \cong (A^{\mathrm{b}})^e$ and the class of selfinjective algebras is closed under Morita equivalences, we may assume that A is basic. Then A^e is basic. Assume A is selfinjective. Then A is a Frobenius algebra and we obtain isomorphisms

$$A^e \cong A^{\mathrm{op}} \otimes_K A \cong D(A^{\mathrm{op}}) \otimes_K D(A) \cong D(A^{\mathrm{op}} \otimes_A A) \cong D(A^e)$$

in $\operatorname{mod} A^e$, and hence A^e is selfinjective.

Conversely, if A^e is selfinjective then

$$A^{\mathrm{op}} \otimes_K A \cong D(A^{\mathrm{op}}) \otimes_K D(A)$$

in $\operatorname{mod} A^e$, and hence

$$A^{\dim_K (A^{\mathrm{op}})} \cong D(A)^{\dim_K D(A^{\mathrm{op}})}$$

in $\operatorname{mod} A$. Then A_A is injective, and hence A is selfinjective. □

Theorem 2.20 (Green-Snashall-Solberg, 2003)**.** *Let A be a finite dimensional indecomposable K-algebra. The following statements are equivalent:*

(1) *All simple right A-modules are Ω_A-periodic.*
(2) *There exists a natural number d and an algebra automorphism σ of A such that $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\operatorname{mod} A^e$, and $\sigma(e)A \cong eA$ for any primitive idempotent e of A.*

PROOF. (1) ⇒ (2). Let d be a minimal natural number such that $\Omega^d_A(S) \cong S$ for any simple right A-module S.

Let $B = \Omega^d_{A^e}(A)$. We know that $\Omega^d_{A^e}(A)$ is a projective left A-module. Hence we have the exact functor $- \otimes_A B : \operatorname{mod} A \to \operatorname{mod} A$. Moreover, for any simple right A-module S, we have $S \otimes_A B = S \otimes_A \Omega^d_A(A) \cong \Omega^d_A(S) \cong S$. Then by induction on the length of a module, we conclude that $\ell(M \otimes_A B) \cong \ell(M)$ for any module M in $\operatorname{mod} A$.

We prove now that $P \otimes_A B \cong P$ for any projective module P in $\operatorname{mod} A$. Let P be an indecomposable projective right A-module. Then the exact sequence

$$0 \to PJ(A) \to P \to P/PJ(A) \to 0,$$

where $J(A)$ is the Jacobson radical of A, induces the exact sequence of right A-modules

$$0 \to PJ(A) \otimes_A B \to P \otimes_A B \to (P/PJ(A)) \otimes_A B \to 0.$$

The module $P \otimes_A B$ is a projective right A-module, as a direct summand of the projective right A-module $A \otimes_A B \cong \Omega^d_{A^e}(A)$, and $\ell(P \otimes_A B) = \ell(P)$. Further, $(P/PJ(A)) \otimes_A B \cong P/PJ(A)$, and hence $P/PJ(A)$ is a direct summand of the top $P \otimes_A B/(P \otimes_A B)J(A)$ of $P \otimes_A B$. Then P is a direct summand of $P \otimes_A B$, and consequently $P \otimes_A B \cong P$, because $\ell(P \otimes_A B) = \ell(P)$. Therefore, there exists an isomorphism $A \otimes_A B \to A$ of right A-modules, and hence B as a right A-module is isomorphic to A_A.

We claim now that B as a left A-module is isomorphic to ${}_AA$. Let T be a simple left A-module. Since B is isomorphic to A_A in $\operatorname{mod} A$, we have $B \otimes_A T \cong A \otimes_A T \cong T$ as K-vector spaces. Further, for any simple right A-module S, we have $S \otimes_A B \otimes_A T \cong S \otimes_A T$ (from the first part of the proof) and $S \otimes_A T \neq 0$ if and only if $S = D(T) = \operatorname{Hom}_K(T, K)$. Then $(A/J(A)) \otimes_A B \otimes_A T \cong T$. On the other hand, we have in $\operatorname{mod} A^{\text{op}}$ the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & J(A) \otimes_A (B \otimes_A T) & \to & A \otimes_A (B \otimes_A T) & \to & (A/J(A)) \otimes_A (B \otimes_A T) & \to 0 \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & J(A)(B \otimes_A T) & \longrightarrow & B \otimes_A T & \longrightarrow & B \otimes_A T/J(A)(B \otimes_A T) & \to 0
\end{array}$$

and hence $(B \otimes_A T)/J(A)(B \otimes_A T) \cong T$ in $\operatorname{mod} A^{\text{op}}$. Since $\dim_K B \otimes_A T = \dim_K T$ we obtain that $B \otimes_A T \cong T$ as left A-modules. Therefore, $B \otimes_A T \cong T$ in $\operatorname{mod} A^{\text{op}}$ for all simple left A-modules T. Applying now arguments from the first part of the proof we conclude that B as a left A-module is isomorphic to ${}_AA$.

Let $\psi : A \to B$ be an isomorphism of left A-modules, and $b = \psi(1)$. Then $\psi(a) = ab$, for $a \in A$, and $Ab = B$. Define the map $\sigma : A \to A$ by $\sigma(a) = \psi^{-1}(ba)$, for $a \in A$. Then, for $a \in A$, we have

$$ba = \psi(\psi^{-1}(ba)) = \psi(\sigma(a)) = \psi(\sigma(a)1) = \sigma(a)\psi(1) = \sigma(a)b.$$

Next we show that σ is a homomorphism of K-algebras. Obviously, σ is K-linear and $\sigma(1) = \psi^{-1}(b) = 1$. Moreover, for $a, a' \in A$, we have

$$\begin{aligned}
\sigma(aa')b &= b(aa') = (ba)a' = (\sigma(a)b)a' = \sigma(a)(ba') \\
&= \sigma(a)(\sigma(a')b) = (\sigma(a)\sigma(a'))b.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\psi(\sigma(aa')) &= \psi(\sigma(aa')1) = \sigma(aa')\psi(1) = \sigma(aa')b = (\sigma(a)\sigma(a'))b \\
&= (\sigma(a)\sigma(a'))\psi(1) = \psi(\sigma(a)\sigma(a')),
\end{aligned}$$

and so $\sigma(aa') = \sigma(a)\sigma(a')$. Therefore, σ is a homomorphism of K-algebras.

We claim that σ is in fact an automorphism. It is enough to show that $\ker \sigma = 0$. Let $a \in \ker \sigma$. Then $0 = \sigma(a)b = ba$ and hence $Ba = (Ab)a = A(ba) = 0$. Since B is isomorphic to A as a right A-module, we obtain $Aa = 0$, and hence $a = 0$. Therefore, indeed $\ker \sigma = 0$.

Finally, observe that the isomorphism $\psi : A \to B$ of left A-modules is an isomorphism $\psi : {}_1A_\sigma \to B$ of A-A-bimodules. Indeed, for $x, a \in A$, we have

$$\psi(x\sigma(a)) = (x\sigma(a))b = x(\sigma(a)b) = x(ba) = (xb)a = \psi(x)a.$$

Therefore, we obtain $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\operatorname{mod} A^e$.

Let e be a primitive idempotent of A. Then we have isomorphisms of right A-modules

$$\begin{aligned}\sigma(e)A/\sigma(e)J(A) &\xrightarrow{\sim} \Omega_A^d(\sigma(e)A/\sigma(e)J(A))\\ &\xrightarrow{\sim} (\sigma(e)A/\sigma(e)J(A)) \otimes_A {}_1A_\sigma\\ &\xrightarrow{\sim} (\sigma(e)A/\sigma(e)J(A))_\sigma\\ &\xrightarrow{\sim} eA/eJ(A).\end{aligned}$$

Hence, $\sigma(e)A \xrightarrow{\sim} eA$ in $\operatorname{mod} A$. Therefore, the implication (1) ⇒ (2) holds.

(2) ⇒ (1). Let $\Omega_{A^e}^d(A) \cong {}_1A_\sigma$ for some $d \geq 1$ and an automorphism σ of A such that $\sigma(e)A \cong eA$ for any primitive idempotent e of A. We know that then A and A^e are selfinjective. Then for any simple right A-module S, the right A-modules $\Omega_A^d(S)$ and $S \otimes \Omega_A^d(A) \cong S \otimes_A A_\sigma \cong S_\sigma$ are isomorphic. Every simple right A-module S is isomorphic to a module of the form $eA/eJ(A)$ for some primitive idempotent e of A. Since $eA \cong \sigma(e)A$ in $\operatorname{mod} A$, the automorphism σ induces isomorphisms of right A-modules $eA \xrightarrow{\sim} (eA)_\sigma$, $eJ(A) \xrightarrow{\sim} (eJ(A))_\sigma$, and hence $S \xrightarrow{\sim} S_\sigma$ in $\operatorname{mod} A$. Therefore, $\Omega_A^d(S) \cong S$ for any simple right A-module S. □

As a direct consequence of Lemma 2.17 and Theorem 2.20 we obtain the following interesting fact.

Corollary 2.21. *Let A be a finite dimensional K-algebra whose all simple right A-modules are periodic. Then A is a selfinjective algebra.*

Let A be a finite dimensional K-algebra. Then the vector space

$$HH^*(A) = \operatorname{Ext}^*_{A^e}(A, A) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_{A^e}(A, A)$$

is a graded commutative K-algebra (with the Yoneda product), called the **Hochschild cohomology algebra** of A (see [**CE**], [**Ha2**], [**Hoc**] for more details). We note that $HH^0(A) \cong \mathcal{Z}(A)$ is the center of A, and $HH^1(A) \cong \operatorname{Der}_K(A, A)/\operatorname{Der}^0_K(A, A)$, where

$$\operatorname{Der}_K(A, A) = \left\{ \delta \in \operatorname{Hom}_K(A, A) \,\middle|\, \begin{array}{c}\delta(ab) = a\delta(b) + \delta(a)b \\ \text{for all } a, b \in A\end{array} \right\}$$

is the space of **derivations** of A, and

$$\operatorname{Der}^0_K(A, A) = \left\{ \delta_x \in \operatorname{Hom}_K(A, A) \,\middle|\, \begin{array}{c}\delta_x(a) = ax - xa \\ x, a \in A\end{array} \right\}$$

is the space of **inner derivations** of A. Hence $HH^1(A)$ is the space of **outer derivations** of A. We also mention that the vector spaces $HH^n(A)$, $n \geq 2$, control deformations of the algebra A (see [**GePe**], [**Ger**], for more details).

Two finite dimensional algebras A and B are said to be **derived equivalent** if the derived categories $D^{\mathrm{b}}(\operatorname{mod} A)$ and $D^{\mathrm{b}}(\operatorname{mod} B)$ are equivalent as triangulated categories. In [**Ric1**] Rickard proved his celebrated criterion: two algebras A and B are derived equivalent if and only if B is the endomorphism algebra of a tilting complex over A.

For selfinjective algebras we have the following implications

$$\text{Morita equivalence} \Longrightarrow \text{derived equivalence} \xRightarrow[\textbf{[Ric2]}]{\textbf{Rickard}} \text{stable equivalence.}$$

The following theorem proved by Happel [**Ha2**] and Rickard [**Ric3**] shows invariance of the Hochschild cohomologies on the derived equivalences.

Theorem 2.22 (Happel, Rickard, 1989–1991)**.** *Let A and B be two derived equivalent K-algebras. Then $HH^*(A) \cong HH^*(B)$ as graded K-algebras.*

We will prove now the following important theorem by Green, Snashall and Solberg [**GSS**].

Theorem 2.23 (Green-Snashall-Solberg, 2003)**.** *Let A be an indecomposable finite dimensional K-algebra. Assume that $\Omega^n_{A^e}(A) \cong {}_1A_\sigma$ for a positive integer n and an algebra automorphism σ of A. Then*

$$HH^*(A)/\mathcal{N}(A) \cong \begin{cases} K, \text{ or} \\ K[x] \end{cases}$$

where $\mathcal{N}(A)$ is the ideal of $HH^*(A)$ generated by all nilpotent homogeneous elements. Moreover, $HH^*(A)/\mathcal{N}(A) \cong K$, if $\Omega^m_{A^e}(A) \not\cong A$ for all $m \geq 1$.

Proof. Since $\Omega^n_{A^e}(A) \cong {}_1A_\sigma$, it follows from Lemma 2.17 that A is selfinjective. Then A^e is selfinjective, by Lemma 2.19, and we may identify

$$HH^i(A) = \operatorname{Ext}^i_{A^e}(A, A) = \underline{\operatorname{Hom}}_{A^e}(\Omega^i_{A^e}(A), A).$$

If $\Omega^m_{A^e}(A) \cong A$ for some $m \geq 1$, then, by the Carlson's Theorem 2.14, we have $HH^*(A)/\mathcal{N}(A) \cong K[x]$, where x is of degree d (= period of A in $\operatorname{mod} A^e$). In particular, it is the case if σ has finite order.

Assume now that $\Omega^m_{A^e}(A) \not\cong A$ in $\operatorname{mod} A^e$ for any $m \geq 1$. Then σ has infinite order. Let $s \geq 1$ and $\underline{\eta} \in \underline{\operatorname{Hom}}(\Omega^s_{A^e}(A), A) = HH^s(A)$. We claim that $\underline{\eta}$ is nilpotent in $HH^*(A)$. Assume first that $s = np$, for some $p \geq 1$. Then, for any $i \geq 1$, we have that $\Omega^{inp}_{A^e}(A) \cong {}_1A_{\sigma^{ip}}$ is an indecomposable right A^e-module and the homomorphism of A^e-modules

$$\Omega^{(i-1)np}_{A^e}(\underline{\eta}) : \Omega^{inp}_{A^e}(A) \longrightarrow \Omega^{(i-1)np}_{A^e}(A)$$

is not an isomorphism. Further, our assumption $\Omega^n_{A^e}(A) \cong {}_1A_\sigma$ implies that the A^e-modules $\Omega^{inp}_{A^e}(A)$, $i \geq 1$, have bounded length (dimension). Then, applying the Harada-Sai lemma (see [**ASS**, (IV.5.2)]), we conclude that there exists a natural number t such that

$$\underline{\eta}^t = \Omega^{tnp}_{A^e}(\underline{\eta}) \dots \Omega^{2np}_{A^e}(\underline{\eta})\Omega^{np}_{A^e}(\underline{\eta}) = 0$$

in the algebra $HH^*(A)$. Hence, $\underline{\eta}$ is nilpotent. Assume now that n is not divisible by s. Then there are positive integers r and q such that $rs = nq$. Then $\underline{\eta}^r \in HH^{nq}(A)$, and hence (by the above argument) $\underline{\eta}^r$ is nilpotent, and consequently $\underline{\eta}$ is nilpotent.

We proved that every homogeneous element of $HH^*(A)$ of positive degree is nilpotent. Moreover, A is indecomposable, and then $HH^0(A) \cong \mathcal{Z}(A)$ is a commutative local algebra, $J(\mathcal{Z}(A))$ is nilpotent, and $\mathcal{Z}(A)/J(\mathcal{Z}(A)) \cong K$. Therefore, we conclude that $HH^*(A)/\mathcal{N}(A) \cong K$. □

Corollary 2.24. *Let A be a finite dimensional indecomposable selfinjective K-algebra of finite representation type. Then*

$$HH^*(A)/\mathcal{N}(A) \cong \begin{cases} K, \text{ or} \\ K[x] \end{cases}.$$

Proof. Since all indecomposable nonprojective (hence simple) modules in $\operatorname{mod} A$ are periodic, applying Theorems 2.20 and 2.23, we get the claim. □

The next result of this section shows invariance of the periodicity of algebras under the derived equivalences.

Theorem 2.25. *Let A and B be two derived equivalent indecomposable finite dimensional selfinjective K-algebras. Then A is periodic if and only if B is periodic.*

Proof. We may assume that A and B are of Loevy length at least 3 (see [**ARS2**, X.1.8]). By Theorem 2.22, we conclude that $HH^*(A)$ and $HH^*(B)$ are isomorphic graded K-algebras. Assume that A is periodic in $\operatorname{mod} A^e$, say of period d. Then, by Carlson's Theorem 2.14, we have an isomorphism $HH^*(A)/\mathcal{N}(A) \cong K[x]$ of graded K-algebras, where x is of degree d. Hence $HH^*(B)/\mathcal{N}(B) \cong K[x]$. On the other hand, $\Omega^d_{A^e}(A) \cong A$, and hence $\Omega^d_A(M) \cong M$ for any indecomposable nonprojective module M in $\operatorname{mod} A$, by Corollary 2.18. Because of Rickard's theorem [**Ric2**], two derived equivalent selfinjective algebras are stably equivalent, there is a stable equivalence $F : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} B$. Further, because A and B are of Loevy length at least 3, it follows from [**ARS2**, X.1.12] that $F\Omega_A \cong \Omega_B F$ as functors from $\underline{\operatorname{mod}} A$ to $\underline{\operatorname{mod}} B$. Therefore, we obtain that $\Omega^d_B(N) \cong N$ for any indecomposable nonprojective module N in $\operatorname{mod} B$. In particular, we conclude that $\Omega^d_B(S) \cong S$ for any simple right B-module S. Applying now Theorem 2.20, we infer that there exists an isomorphism $\Omega^d_{B^e}(B) \cong {}_1B_\sigma$ in $\operatorname{mod} B^e$ for an algebra automorphism σ of B. Applying Theorem 2.23, we then infer that B is periodic in $\operatorname{mod} B^e$, because $HH^*(B)/\mathcal{N}(B) \cong K[x]$. In fact, we have $\Omega^d_{B^e}(B) \cong B$ in $\operatorname{mod} B^e$. □

We also note the following direct consequence of Theorems 2.20 and 2.23.

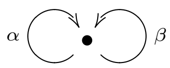
Corollary 2.26. *Let A be a finite dimensional indecomposable selfinjective K-algebra such that all simple A-modules are Ω_A-periodic. The following statements are equivalent:*

(1) *A is periodic.*
(2) *$HH^*(A)/\mathcal{N}(A) \cong K[x]$.*

In the final part of this section we discuss the relation between the boundedness and periodicity of modules over selfinjective algebras.

Let A be a selfinjective algebra. A module M in $\operatorname{mod} A$ is defined to be **(homologically) bounded** if there is a common bound on the dimensions of all syzygy modules, $\Omega^i_A(M)$, $i \geq 0$, of M. Clearly, every periodic A-module is bounded. In [**Al1**] Alperin proved that, if M is a bounded indecomposable nonprojective finite dimensional module over the group algebra KG of a finite group G, then M is periodic. The following examples show that it is not the case for arbitrary selfinjective algebras.

Examples 2.27. (1) Let λ be a nonzero element of K which is not a root of unity. Consider the 4-dimensional local Frobenius algebra $A_\lambda = KQ/I_\lambda$, where

$Q :$ 

and $I_\lambda = \langle \alpha^2, \beta^2, \alpha\beta - \lambda\beta\alpha \rangle$ (see Example 1.8(4)). Then, for the cosets $a = \alpha + I_\lambda$ and $b = \beta + I_\lambda$, 1, a, b, $ab = \lambda ba$ is a basis of A_λ over K. For each $i \in \mathbb{Z}$, take the element

$$x_i = (-1)^i \lambda^i a + b \in A_\lambda$$

and the cyclic right A_λ-module

$$M_i = x_i A_\lambda.$$

Then M_i, $i \in \mathbb{Z}$, is a family of pairwise nonisomorphic indecomposable A_λ-modules of dimension 2. Moreover, for each $i \in \mathbb{Z}$, we have a canonical exact sequence in $\operatorname{mod} A_\lambda$

$$0 \longrightarrow M_{i+1} \longrightarrow A_\lambda \xrightarrow{\pi_i} M_i \longrightarrow 0$$

where π_i is the canonical projection with $\pi_i(1) = x_i$. Hence, we obtain that $M_{i+1} \cong \Omega_A M_i$, for all $i \in \mathbb{Z}$. Therefore, M_i, $i \in \mathbb{Z}$, are bounded but nonperiodic indecomposable nonprojective modules in $\operatorname{mod} A_\lambda$.

(2) Following Liu and Schulz [**LiSc**] consider the trivial extension algebra $R_\lambda = A_\lambda \ltimes D(A_\lambda)$ of the algebra A_λ considered in (1). Then R_λ is a local symmetric algebra of dimension 8. Moreover, the Jacobson radical of R_λ is generated by the elements $a = (a, 0)$ and $b = (b, 0)$, with $a = \alpha + I_\lambda$ and $b = \beta + I_\lambda$ as above. For each $i \in \mathbb{Z}$, consider the element

$$y_i = (-1)^i \lambda^i a + b \in R_\lambda$$

and the cyclic right R_λ-module

$$N_i = y_i R_\lambda.$$

Then N_i, $i \in \mathbb{Z}$, is a family of pairwise nonisomorphic indecomposable right R_λ-modules of dimension 4. Moreover, similarly as in (1), we conclude that $N_{i+1} \cong \Omega_A N_i$, for all $i \in \mathbb{Z}$. Therefore, N_i, $i \in \mathbb{Z}$, are bounded but nonperiodic indecomposable nonprojective modules in $\operatorname{mod} R_\lambda$.

The following general criterion for the bounded modules to be periodic has been established by Schulz [**Schu**].

Theorem 2.28 (Schulz, 1986)**.** *Let A be a selfinjective algebra and M be a bounded indecomposable nonprojective A-module. The following statements are equivalent.*

(i) *M is periodic.*
(ii) *The algebra $\operatorname{Ext}^*_A(M, M)$ is right noetherian, and the right $\operatorname{Ext}^*_A(M, M)$-module $\operatorname{Ext}^*_A(M, S)$ is noetherian, for any simple right A-module S.*
(iii) *For any module X in $\operatorname{mod} A$, the right $\operatorname{Ext}^*_A(M, M)$-module $\operatorname{Ext}^*_A(M, X)$ is noetherian.*

We refer to [**Schu**] for examples showing that the noetherianness of the algebra $\operatorname{Ext}^*_A(M, M)$ is not sufficient for a bounded module M to be periodic.

We obtain also the following consequences of the above theorem.

Corollary 2.29. *Let A be a selfinjective algebra and M a periodic indecomposable finite dimensional A-module. Then the graded algebra $\operatorname{Ext}^*_A(M, M)$ is noetherian.*

Corollary 2.30. *Let A be a periodic algebra. Then the graded algebra $HH^*(A)$ is noetherian.*

3. Periodicity of finite groups

The aim of this section is to present characterizations of periodic finite groups and exhibit their topological sources.

Let G be a finite group, $\mathbb{Z}$ the ring of integers and $\mathbb{Z}G$ the group algebra of G over $\mathbb{Z}$. We may consider the group $\mathbb{Z}$ as the trivial $\mathbb{Z}G$-module by the action $m * g = m$, for any $m \in \mathbb{Z}$ and $g \in G$. For $n \geq 0$ and a $\mathbb{Z}G$-module M, let

$$H^n(G, M) = \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

be the n**-th cohomology group of** G **with coefficients in** M. In particular, we may consider the cohomology groups of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$H^i(G,\mathbb{Z}) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}),\ i \geq 0,$$

called the **cohomology groups of** G.

Definition 3.1. *A group G is defined to be **(globally) periodic** if there exists a positive integer d such that*

$$H^i(G,\mathbb{Z}) \cong H^{i+d}(G,\mathbb{Z}),\ \textit{for all } i \geq 1.$$

*The minimal such d is called **the (cohomological) period of** G.*

Example 3.2. Let $m \geq 2$, and $G = \mathbb{Z}_m$ be the cyclic group of order m, say generated by an element g. Then we have the following periodic free $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon(g) = 1$ for $g \in G$, $g-1$ is the left multiplication by $g-1$, and N is the left multiplication by $N = 1 + g + \cdots + g^{m-1}$. Applying $\mathrm{Hom}_{\mathbb{Z}G}(-,\mathbb{Z})$ we obtain the periodic complex whose i-th cohomology is the group $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) = H^i(G,\mathbb{Z})$. Then one obtains $H^0(G,\mathbb{Z}) \cong \mathbb{Z}$, $H^{2i}(G,\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ and $H^{2i-1}(G,\mathbb{Z}) = 0$ for $i \geq 1$. In particular, $G = \mathbb{Z}_m$ is a periodic group of period 2.

In fact, the following is true (see [**AM**], [**Br**], [**Sw2**]).

Theorem 3.3. *Let G be a finite group. Then G is periodic of period 2 if and only if G is cyclic.*

Moreover, we have also the following theorem (see [**AM**], [**Br**]).

Theorem 3.4. *Let G be a periodic finite group. Then $H^{2i-1}(G,\mathbb{Z}) = 0$ for any $i \geq 1$. Hence the period of G is **even**.*

Zassenhaus considered the following problem, motivated by some topological problems (free group actions on spheres).

PROBLEM 4 (Zassenhaus)**. Describe all finite groups** G **whose all commutative subgroups are cyclic.**

Zassenhaus solved this problem in the solvable case [**Za**]. This was completed by Suzuki [**Su**] to the general case.

Theorem 3.5 (Suzuki-Zassenhaus, 1955)**.** *A complete list of finite groups with all commutative subgroups cyclic is given by the following table*

Family	*Definition*	*Conditions*
I	$\mathbb{Z}/a \times_\alpha \mathbb{Z}/b$	$(a,b) = 1$
II	$\mathbb{Z}/a \times_\beta (\mathbb{Z}/b \times \mathcal{Q}_{2^i})$	$(a,b) = (ab,2) = 1$
III	$\mathbb{Z}/a \times_\gamma (\mathbb{Z}/b \times T_i)$	$(a,b) = (ab,6) = 1$
IV	$\mathbb{Z}/a \times_\tau (\mathbb{Z}/b \times O_i^*)$	$(a,b) = (ab,6) = 1$
V	$(\mathbb{Z}/a \times_\alpha \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$	$(a,b) = (ab,p(p^2-1)) = 1$
VI	$\mathbb{Z}/a \times_\mu (\mathbb{Z}/b \times TL_2(\mathbb{F}_p))$	$(a,b) = (ab,p(p^2-1)) = 1$

These 6 families of groups are given as semidirect products of certain finite groups (we refer to [**AM**, Chapter IV] *for more details on these groups).*

We will exhibit (now and later) only some natural examples of such groups.

Examples 3.6. (1) For $m \geq 1$, consider the **dihedral group**

$$D_{2m} = \{x, y \mid x^2 = 1 = y^m, yx = xy^{m-1}\}$$

of order $2m$.

For $m = 2r$, $\{1, x, y^r, xy^r = y^r x\}$ is a noncyclic commutative subgroup of D_{4r}.

For m odd, all commutative subgroups of D_{2m} are cyclic.

Hence, D_{2m} is periodic if and only if m is odd.

(2) For $m \geq 1$, consider the **generalized quaternion 2-group**

$$Q_{2^{m+2}} = \left\{x, y \mid x^{2^m} = y^2, xyx = x\right\}$$

of order 2^{m+2}. Then every commutative subgroup of $Q_{2^{m+2}}$ is cyclic and $Q_{2^{m+2}}$ is periodic.

(3) Let p be a prime and $\mathbb{F}_p$ the field with p elements, and

$$\mathrm{SL}_2(\mathbb{F}_p) = \{M \in M_{2\times 2}(\mathbb{F}_p) \mid \det M = 1\}$$

(2×2 **special linear group of** $\mathbb{F}_p$). Then

$$|\mathrm{SL}_2(\mathbb{F}_p)| = p(p-1)(p+1).$$

Moreover, all commutative subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ are cyclic, and hence $\mathrm{SL}_2(\mathbb{F}_p)$ is periodic. We also note that for p odd the groups $\mathrm{SL}_2(\mathbb{F}_p)$ are not solvable.

For a prime number p, the abelian group $\mathbb{Z}_p^r = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r}$ is said to be the **elementary p-group of rank** r.

For a finite group G and a prime p with $p \,\big|\, |G|$, denote by $r_p(G)$ the maximal rank of elementary p-subgroup of G, called the **p-rank** of G.

The following characterizations of periodic groups show that in fact the Suzuki-Zassenhaus theorem provides a complete classification of all periodic finite groups (see [**CE**]).

Theorem 3.7 (Artin-Tate, Cartan-Eilenberg, 1956)**.** *Let G be a finite group. The following statements are equivalent:*

(1) *G is periodic.*
(2) *$H^d(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$, for some $d \geq 1$.*
(3) *$H^{i+d}(G, M) \cong H^i(G, M)$, for some $d \geq 1$, all $i \geq 1$ and an arbitrary finitely generated $\mathbb{Z}G$-module M.*
(4) *$H^{i+d}(G, \mathbb{Z}_p) \cong H^i(G, \mathbb{Z}_p)$, for some $d \geq 1$, all $i \geq 1$ and any prime p dividing $|G|$.*
(5) *$r_p(G) \leq 1$, for any prime p dividing $|G|$.*
(6) *For any prime p dividing $|G|$, the p-Sylow subgroups of G are cyclic or generalized quaternion 2-groups.*
(7) *Every commutative subgroup of G is cyclic.*

Therefore, the subgroups of periodic groups are periodic.

Example 3.8. For a prime p, we have

$$\dim_{\mathbb{Z}_p} H^n(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) = n + 1, \text{ for any } n \geq 0,$$

and hence the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is not periodic (application of the **Künneth formula**).

Let p be a prime number. Consider the localization of $\mathbb{Z}$ at p

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q}, m, n \in \mathbb{Z}, p \not| n \right\}.$$

Let G be a finite group such that $p \bigm| |G|$. For each $i \geq 1$, let

$$H^i(G, \mathbb{Z})_{(p)} = H^i(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

Definition 3.9. *Let p be a prime number. A group G with $p \bigm| |G|$ is defined to be p-**periodic** if there exists a positive integer d such that*

$$H^i(G, \mathbb{Z})_{(p)} \cong H^{i+d}(G, \mathbb{Z})_{(p)}, \text{ for all } i \geq 1.$$

*The minimal such $d = d_p$ is called **the (cohomological) p-period of** G.*

The following theorem provides a characterization of p-periodic groups (see [**Br**]).

Theorem 3.10. *Let G be a finite group, p a prime number, and $p \bigm| |G|$. The following statements are equivalent:*

(1) *G is p-periodic.*
(2) *$H^{i+d}(G, \mathbb{Z}_p) \cong H^i(G, \mathbb{Z}_p)$, for some $d \geq 1$ and any $i \geq 1$.*
(3) *$\mathrm{Ext}^{i+d}_{\mathbb{Z}_pG}(\mathbb{Z}_p, M) \cong \mathrm{Ext}^{i}_{\mathbb{Z}_pG}(\mathbb{Z}_p, M)$, for some $d \geq 1$, any $i \geq 1$, and an arbitrary finite dimensional $\mathbb{Z}_pG$-module M.*
(4) *$\Omega^d_{\mathbb{Z}_pG}(\mathbb{Z}_p) \cong \mathbb{Z}_p$, for some $d \geq 1$.*
(5) *$r_p(G) \leq 1$.*
(6) *Every p-Sylow subgroup of G is either cyclic or generalized quaternion 2-group.*
(7) *Every commutative p-subgroup of G is cyclic.*
(8) *For any algebraically closed field K of characteristic p, $\Omega^d_{KG}(K) \cong K$, for some $d \geq 1$.*
(9) *For any algebraically closed field K of characteristic p, there exists $d \geq 1$ such that $\Omega^d_{KG}(M) \cong M$ for any indecomposable nonprojective finite dimensional KG-module M.*

Observe that a finite group G is periodic if and only if G is p-periodic for any prime p dividing $|G|$.

Example 3.11. Let p be an odd prime number, $q = p^n$, $n \geq 2$, $\mathbb{F}_q$ the field with q elements, and $G = SL_2(\mathbb{F}_q)$. Then $|G| = q(q^2 - 1)$. Moreover, we have

- the 2-Sylow subgroups of G are generalized quaternion 2-groups,
- for any odd prime $l \neq p$, the l-Sylow subgroups of G are cyclic,
- the p-Sylow subgroups of G are **not** cyclic,

Then G is not p-periodic, and hence is not periodic. Moreover, G is l-periodic for any prime such that $l \bigm| |G|$ and $l \neq p$.

We note that there is no chance for a classification of all finite p-periodic groups, for any fixed prime p.

Let G be a finite group, p a prime number, and $p \mid |G|$. Let

$$H^{ev}(G, \mathbb{Z}_p) \cong \bigoplus_{n \geq 0} H^{2n}(G, \mathbb{Z}_p)$$

be the **even cohomology algebra** of G at p. Then $H^{ev}(G, \mathbb{Z}_p)$ is a graded commutative ring and we have the following theorem proved independently by Evans [**Ev**] and Venkov [**Ve**].

Theorem 3.12 (Evans-Venkov, 1959-1961)**.** *Let G be a finite group, p a prime number, and $p \mid |G|$. Then $H^{ev}(G, \mathbb{Z}_p)$ is a noetherian ring.*

Denote by $\dim H^{ev}(G, \mathbb{Z}_p)$ the Krull dimension of $H^{ev}(G, \mathbb{Z}_p)$, that is, the length d of the maximal chain of distinct graded prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$$

of $H^{ev}(G, \mathbb{Z}_p)$. Then we have the following deep result proved by Quillen [**Q1**], [**Q2**].

Theorem 3.13 (Quillen, 1971)**.** *Let G be a finite group and p a prime number dividing $|G|$. Then*

$$\dim H^{ev}(G, \mathbb{Z}_p) = r_p(G).$$

Hence the Krull dimensions of the rings $H^{ev}(G, \mathbb{Z}_p)$, $p \mid |G|$, p prime, measure the complexity of the group G.

As a direct consequence of Theorems 3.10 and 3.13, we obtain the following characterization of p-periodic finite groups.

Corollary 3.14. *Let G be a finite group and p a prime number dividing $|G|$. Then G is p-periodic if and only if $\dim H^{ev}(G, \mathbb{Z}_p) = 1$.*

We describe now the representation type of the group algebras of p-periodic group.

Let K be an algebraically closed field of characteristic p. By the well-known **Maschke's theorem** the group algebra KG of a finite group G is semisimple if and only if $p \nmid |G|$.

The following classical theorem proved by Higman [**Hi**] describes the group algebras of finite representation type.

Theorem 3.15 (Higman, 1954)**.** *Let G be a finite group and $p \mid |G|$. Then the group algebra KG of G is of finite representation type if and only if the p-Sylow subgroups of G are cyclic.*

The following theorem proved by Bondarenko and Drozd [**BD**] gives a characterization of the tame group algebras of infinite type.

Theorem 3.16 (Bondarenko-Drozd, 1975)**.** *Let G be a finite group and $p \mid |G|$. Then KG is tame of infinite representation type if and only if $p = 2$ and the 2-Sylow subgroups of G are of one of the following types: dihedral, semidihedral, or generalized quaternion groups.*

Recall that the **semidihedral 2-groups** are the group of the form

$$S_m = \left\{ x, y \mid x^2 = y^{2^m} = 1, yx = xy^{2^{m-1}-1} \right\},\ m \geq 3.$$

The following consequence of Theorem 3.13 describes the representation type of the group algebras of p-periodic groups.

Corollary 3.17. *Let G be a finite group, $p \,\big|\, |G|$, and assume that G is p-periodic. Then*

(1) *KG is of tame representation type.*
(2) *If p is odd, then KG is of finite representation type.*

The following combination of results proved by Erdmann and Holm [**EH**] and Erdmann and Skowroński [**ES1**] shows that the group algebras of p-periodic groups are periodic as bimodules.

Theorem 3.18 (Erdmann-Holm (1999), Erdmann-Skowroński (2006))**.** *Let G be a finite group, $p \,\big|\, |G|$, and $A = KG$. If G is p-periodic then A is periodic in* $\operatorname{mod} A^e$.

We will indicate now topological sources of periodic groups.

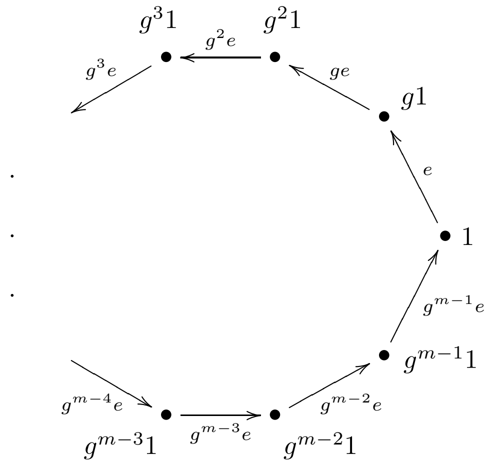
Let G be a finite group. We may consider G as a topological group with the discrete topology. We say that G **acts on a topological space** X if there is a group homomorphism of G into the group $\operatorname{Homeo}(X)$ homeomorphisms of X to X.

Assume X is a CW-complex (admits a cell decomposition) and G is a finite group of homeomorphisms of X. We say that G **acts freely** on X if G acts freely on a cell decomposition of X, that is,

$$g(\sigma) \subseteq \bigcup_{\tau \neq \sigma} \tau$$

for all $g \in G \setminus \{1\}$ and all cells σ of X.

Example 3.19. For any $m \geq 2$, the cyclic group $G = (g)$ of order m acts freely on the one-dimensional sphere $\mathbb{S}^1$, as the following cell decomposition and the action of G on $\mathbb{S}^1$ show



The following is one of the classical problems of algebraic topology (see [**AD**], [**MTW1**], [**MTW2**], [**TW**]).

PROBLEM 5 (Spherical space form problem)**. Describe the finite groups G acting freely on spheres $\mathbb{S}^m$ and the orbit spaces $\mathbb{S}^m/G$ (spherical spaces).**

The following theorem proved by Smith in [**Sm1**], [**Sm2**] was the topological motivation for the Zassenhaus problem.

Theorem 3.20 (Smith, 1938-1939)**.** *Let G be a finite group acting freely on a sphere $\mathbb{S}^m$. Then every abelian subgroup of G is cyclic.*

Moreover, we have the following theorem describing the periods of finite groups acting freely on spheres (see [**AD**]).

Theorem 3.21. *Let G be a finite group acting freely on a sphere $\mathbb{S}^m$. Then*
(1) *For m even, we have $|G| \leq 2$.*
(2) *For m odd, we have $H^{m+1}(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$. In particular, G is periodic with even period dividing $m+1$.*

Proof. (1) An application of Lefschetz fix point theorem.
(2) Application of cohomological methods (spectral sequence of the fibration $\mathbb{S}^m \to \mathbb{S}^m/G \to BG$). □

Example 3.22. Consider the (division) algebra of quaternions

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

with $ij = -ji = k$, $ki = -ik = j$, $jk = -kj = i$, $i^2 = j^2 = k^2 = -1$. Then

$$\mathbb{S}^3 = \left\{ a + bi + cj + dk \in \mathbb{H} \,\middle|\, a^2 + b^2 + c^2 + d^2 = 1 \right\}.$$

Hence $\mathbb{S}^3$ is the 3-dimensional sphere in $\mathbb{R}^4 = \mathbb{H}$. There is a canonical group epimorphism $\mathbb{S}^3 \to \mathrm{SO}(3, \mathbb{R})$ (group of rotations of $\mathbb{R}^3$) with the kernel $\{\pm 1\}$. Moreover, it is known that every noncyclic finite subgroup of $\mathbb{S}^3$ is conjugate in $\mathbb{S}^3$ (hence isomorphic) to one of the groups:

- $\mathbb{D}^*_{2n}$, $n \geq 2$, **binary dihedral group**,
- $\mathcal{T}^*$ **binary tetrahedral group**,
- $\mathcal{O}^*$ **binary octahedral group**,
- $\mathcal{I}^*$ **binary icosahedral group**.

We also note that the groups $\mathbb{D}^*_{2n}$, $\mathcal{T}^*$, $\mathcal{O}^*$, $\mathcal{I}^*$ admit a unique normal subgroup $\mathbb{Z}_2 = \{\pm 1\}$ of order 2 such that

- $\mathbb{D}^*_{2n}/\mathbb{Z}_2 = \mathbb{D}_{2n}$, the dihedral group,
- $\mathcal{T}^*/\mathbb{Z}_2 = \mathcal{T}$, the tetrahedral group of rotations of tetrahedron,
- $\mathcal{O}^*/\mathbb{Z}_2 = \mathcal{O}$, the octahedral group of rotations of octahedron (equivalently, cube),
- $\mathcal{I}^*/\mathbb{Z}_2 = \mathcal{I}$, the icosahedral group of rotations of icosahedron (equivalently, dodecahedron).

Then we get $|\mathbb{D}^*_{2n}| = 4n$, $|\mathcal{T}^*| = 24$, $|\mathcal{O}^*| = 48$, $|\mathcal{I}^*| = 120$.

Therefore, we conclude that the groups $\mathbb{D}^*_{2n}$, $\mathcal{T}^*$, $\mathcal{O}^*$, $\mathcal{I}^*$ act freely on the sphere $\mathbb{S}^3$, and hence are periodic groups of period 4, because only the cyclic groups may have period 2 (see Theorem 3.3 and [**Sw2**]). We also note that

$$Q_{4n} = \mathbb{D}^*_{2n} = \left\langle x, y \;\middle|\; x^n = y^2, xyx = y \right\rangle, \; n \geq 2.$$

The group Q_{4n} is called a **generalized quaternion group**. Hence, for $n = 2^m$, we get the generalized quaternion 2-group $Q_{2^{m+2}}$ considered before. Observe also that we have the following embedding of groups

$$Q_{4n} \longrightarrow \mathbb{S}^3 \subseteq \mathbb{H} = \mathbb{R}^4$$

by $x \longrightarrow e^{\pi i/n}$ and $y \longrightarrow j$. In particular, we have $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

Example 3.23 (Linear actions on spheres)**.** Let $V = \mathbb{R}^{2n}$, $n \geq 1$, $(-,-)$ be the Euclidean $\mathbb{R}$-bilinear form and $e_1, e_2, \ldots, e_{2n}$ the standard basis of $\mathbb{R}^{2n}$.

Let G be a finite group of $\mathbb{R}$-linear automorphisms of V. Assume G acts freely on $V \setminus \{0\}$: the eigenvalues of all $g \in G \setminus \{1\}$ are different from 1. Consider the G-invariant $\mathbb{R}$-bilinear form $(-,-)_G$ on $\mathbb{R}^{2n}$ induced by $(-,-)$, given by

$$(x,y)_G = \frac{1}{|G|}\sum_{g\in G}(g(x),g(y)), \text{ for } x,y \in V.$$

Then $\mathbb{S} = \{x \in V | (x,x)_G = 1\}$ is an $(2n-1)$-dimensional sphere and G acts freely on $\mathbb{S}$. In fact, G acts freely on a cell decomposition of $\mathbb{S}$ (see [**Il**]). Indeed, let C be the convex hull of the finite set $\{\pm g(e_i) \mid g \in G, 1 \le i \le 2n\}$ in $\mathbb{R}^{2n}$. Then $\mathbb{S}$ is the border of C and admits the induced cell decomposition. Since G acts freely on $V \setminus \{0\}$, we conclude that G acts freely on this cell decomposition of $\mathbb{S}$. In particular, we obtain that G is periodic of (even) period dividing $2n$. In fact, one can construct such groups of arbitrary even period $2n$.

The following question arises naturally.

Does every periodic group act freely on a sphere?

The following theorem proved by Milnor [**Mi**] gives negative answer.

Theorem 3.24 (Milnor, 1957)**.** *Let G be a finite group acting freely on a sphere $\mathbb{S}^m$. Then G admits at most one element of order 2, and such an element is central.*

Hence, for example, if m odd, then the dihedral group $\mathbb{D}_{2m}$ is periodic but does not act freely on a sphere. In particular, this is the case for the symmetric group $S_3 \cong \mathbb{D}_{2\cdot 3} = \mathbb{D}_6$.

On the other hand, the following theorem proved by Swan [**Sw1**] shows that the periodic groups are finite groups acting freely on CW-complexes homotopically equivalent to spheres.

Theorem 3.25 (Swan, 1960)**.** *Let G be a finite group. The following statements are equivalent:*

(1) *G is periodic.*
(2) *There exists an odd natural number m, an m-dimensional CW-complex X (Swan complex) homotopically equivalent to $\mathbb{S}^m$ such that G acts freely on X.*

The following theorem proved by Madsen, Thomas and Wall [**MTW2**] gives a complete characterization of finite groups acting freely on spheres.

Theorem 3.26 (Madsen-Thomas-Wall, 1976)**.** *Let G be a finite group. The following statements are equivalent:*

(1) *G acts freely on a sphere.*
(2) *G admits at most one element of order 2, and such an element is central.*
(3) *For each prime number p, every subgroup G of order p^2 or $2p$ is cyclic.*
(4) *G is periodic and has no dihedral subgroups.*

Example 3.27. For each odd prime p, the group $\mathrm{SL}_2(\mathbb{F}_p)$ acts freely on a sphere. Indeed,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the unique element of order 2 in $\mathrm{SL}_2(\mathbb{F}_p)$, and is central.

We note that $\mathrm{SL}_2(\mathbb{F}_2) \cong S_3 \cong \mathbb{D}_6$, $\mathrm{SL}_2(\mathbb{F}_3) \cong \mathcal{T}^*$, and $\mathrm{SL}_2(\mathbb{F}_5) \cong \mathcal{I}^*$. On the other hand, it is known that the groups $\mathrm{SL}_2(\mathbb{F}_p)$, $p > 5$, do not admit linear free actions on spheres.

We end this section with a theorem proved by Wolf [**Wo**] describing all finite groups having linear free actions on spheres.

Theorem 3.28 (Wolf, 1967). *A finite group G acts freely and linearly on some sphere if and only if the following conditions are satisfied:*

(1) *For all primes p and q, the subgroups of G of orders pq are cyclic.*

(2) *G has no subgroup isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$, for a prime $p > 5$.*

4. Periodicity of tame symmetric algebras

The aim of this section is to present a complete classification (up to Morita equivalence) of all symmetric algebras over an algebraically closed field for which the indecomposable nonprojective finite dimensional modules are periodic. Therefore, we may restrict to the symmetric algebras which are nonsimple, basic and indecomposable. The main classification theorem below proved by Erdmann and Skowroński in [**ES2**] relies on results of several authors which are described below in a new invariant algebra form.

Theorem 4.1 (Erdmann-Skowroński, 2006). *Let Λ be a nonsimple, basic, indecomposable, finite dimensional algebra over an algebraically closed field K. Then Λ is symmetric, tame, with all indecomposable nonprojective finite dimensional modules periodic if and only if Λ is isomorphic to an algebra of one of the forms:*

- *a symmetric algebra of Dynkin type;*
- *a symmetric algebra of tubular type;*
- *an algebra of quaternion type.*

The aim of the remaining part of this section is to describe the symmetric algebras of Dynkin type, the symmetric algebras of tubular type, the algebras of quaternion type, as well as properties of their module categories. In our description of the symmetric algebras of Dynkin and tubular type, a prominent role will be played by certain invariant algebras of the trivial extensions of algebras with respect to free actions of finite groups, as described bellow.

Let B be a basic connected K-algebra and $\mathrm{T}(B) = B \ltimes D(B)$ be the (symmetric) trivial extension algebra of B by its minimal injective cogenerator $D(B) = \mathrm{Hom}_K(B, K)$. Let G be a finite group of K-algebra automorphisms of $\mathrm{T}(B)$. Then we may consider the **invariant algebra**

$$\mathrm{T}(B)^G = \left\{ x \in \mathrm{T}(B) \;\middle|\; g(x) = x \text{ for all } g \in G \right\}.$$

Moreover, we say that the group G **acts freely** on $\mathrm{T}(B)$ if there is a decomposition of the identity of $\mathrm{T}(B)$

$$1_{\mathrm{T}(B)} = e_1 + e_2 + \cdots + e_n,$$

where $e_1, e_2, \ldots, e_n$ are orthogonal primitive idempotents of $\mathrm{T}(B)$ such that

(1) $g(e_i) \in \{e_1, \ldots, e_n\}$, for all $g \in G$ and $i \in \{1, \ldots, n\}$,

(2) if $g(e_i) = e_i$, for some $i \in \{1, \ldots, n\}$, then $g = 1$.

It is known that G acts freely on $\mathrm{T}(B)$ if and only if G acts freely on the isoclasses of simple $\mathrm{T}(B)$-modules, for the induced action of G on $\mathrm{mod}\,\mathrm{T}(B)$ (see [**ARS1**]).

We have the following general fact.

Proposition 4.2. *Assume G acts freely on $\mathrm{T}(B)$. Then $\mathrm{T}(B)^G$ is a weakly symmetric (hence selfinjective) algebra.*

Proof. The invariant algebra $\mathrm{T}(B)^G$ is isomorphic to the orbit algebra $\mathrm{T}(B)/G$ (in the sense of Gabriel [**Ga**]). Since $\mathrm{T}(B)$ is symmetric, $\mathrm{T}(B)$ is weakly symmetric, and hence $\mathrm{T}(B)^G \cong \mathrm{T}(B)/G$ is weakly symmetric. □

We note that in general $\mathrm{T}(B)^G$ is not necessarily a symmetric algebra.

We also note that by general theory the class of tame basic indecomposable algebras splits into two subclasses: the standard algebras, which admit simply connected Galois coverings, and the remaining nonstandard algebras (see [**Sk2**] for details).

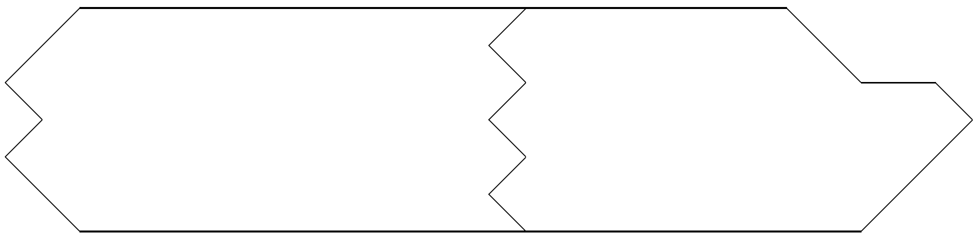
Symmetric algebras of Dynkin type

Let $\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ be a **Dynkin graph**, $\vec{\Delta}$ a Dynkin quiver with underlying graph Δ and $H = K\vec{\Delta}$ the path algebra of $\vec{\Delta}$.

Then a module T in $\operatorname{mod} H$ is said to be a **tilting** H-module if $\operatorname{Ext}^1_H(T,T) = 0$ and $T = T_1 \oplus \cdots \oplus T_n$, where $n = |\Delta_0|$ and $T_1, \dots, T_n$ are indecomposable pairwise nonisomorphic H-modules [**HR1**].

Then $B = \operatorname{End}_H(T)$ is called a **tilted algebra** of type $\vec{\Delta}$ and has the following properties

- $\operatorname{gl.dim} B \leq 2$;
- B is of finite type;
- The Auslander-Reiten quiver Γ_B of B is of the form

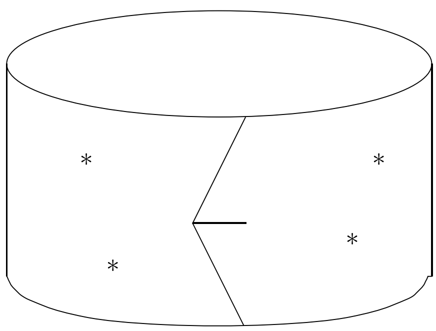


Dynkin section $\vec{\Delta}$

The following important result was proved by Hughes and Waschbüsch [**HW**].

Theorem 4.3 (Hughes-Waschbüsch,1983). *Let A be an algebra. Then $\mathrm{T}(A)$ is of finite type if and only if $\mathrm{T}(A) \cong \mathrm{T}(B)$ for a tilted algebra B of Dynkin type.*

Let B be a tilted of Dynkin type $\vec{\Delta}$. Then the Auslander-Reiten quiver of $\Gamma_{\mathrm{T}(B)}$ has the following shape



$\vec{\Delta}$

which is the stable finite cylinder $\mathbb{Z}\vec{\Delta}/(\tau^{m_\Delta})$ completed by $|\Delta_0|$-projective-injective modules. Moreover, if $m_\Delta = h_\Delta - 1$, where h_Δ is the **Coxeter number** of Δ, then the number of the isoclasses of indecomposable $\mathrm{T}(B)$-modules is the number $|\Delta_0|h_\Delta$ of roots of type Δ. Recall also that the Coxeter numbers are as follows $h_{\mathbb{A}_m} = m+1$, $h_{\mathbb{D}_m} = 2m-2$, $h_{\mathbb{E}_6} = 12$, $h_{\mathbb{E}_7} = 18$, $h_{\mathbb{E}_8} = 30$.

We also note that, if B, B' are tilted algebras of Dynkin type, then $\mathrm{T}(B) \cong \mathrm{T}(B') \iff B' = S^+_{i_t} \dots S^+_{i_1} B$ (finite number of reflections) (see [**HW**]).

The following problem occurs naturally.

PROBLEM 6. When a finite group G acts freely on the trivial extension $\mathrm{T}(B)$ of a tilted algebra B of Dynkin type?

By general theory such a group G is cyclic (see [**HW**]).

An additional information is given by the following theorem proved by Bretscher, Läser and Rietdmann [**BLR**].

Theorem 4.4 (Bretscher-Läser-Rietdmann, 1981)**.** *Let G be a finite group acting freely on the trivial extension $\mathrm{T}(B)$ of a tilted algebra B of Dynkin type $\vec{\Delta}$, with $\Delta \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$. Then $G = \{1\}$.*

There are respectively 22, 143, 598 isoclasses of the trivial extensions $\mathrm{T}(B)$ of tilted algebras B of types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (**Riedtmann**). These are all symmetric algebras of Dynkin types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

The tilted algebras B of Dynkin types for which $\mathrm{T}(B)$ admit a free action of a **nontrivial** finite group G are **very exceptional**.

In the representation theory of group algebras of finite groups a prominent role is played by the **Brauer tree algebras** (see [**Al2**]). Recall that a **Brauer tree** is a finite connected tree $T = T^m_S$ together with

- a circular ordering of the edges converging at each vertex,
- one exceptional vertex S with multiplicity $m \geq 1$.

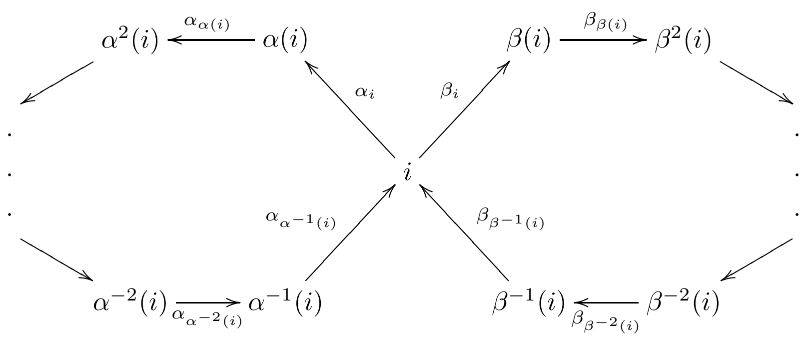
We associate to a Brauer tree T a **Brauer quiver** Q_T defined as follows:

- the vertices of Q_T are the edges of T;
- there is an arrow $i \to j$ in $Q_T \iff j$ is the consecutive edge of i in the circular ordering of the edges converging at a vertex of T.

Hence the quiver Q_T has the following structure:

- Q_T is a union of oriented cycles corresponding to the vertices of T;
- Every vertex of Q_T belongs to exactly two cycles.

The cycles of Q_T are divided into two camps: α**-camps** and β**-camps** such that two cycles of Q_T having nontrivial intersection belong to different camps. We assume that the cycle of Q_T corresponding to the exceptional vertex S of T is an α-cycle. Therefore, for each i vertex of Q_T, we have the arrow $i \xrightarrow{\alpha_i} \alpha(i)$ in α-camp of Q_T starting at i, the arrow $i \xrightarrow{\beta_i} \beta(i)$ in β-camp of Q_T starting at i, and the cycles $A_i = \alpha_i \alpha_{\alpha(i)} \dots \alpha_{\alpha^{-1}(i)}$, $B_i = \beta_i \beta_{\beta(i)} \dots \beta_{\beta^{-1}(i)}$ around i of the form



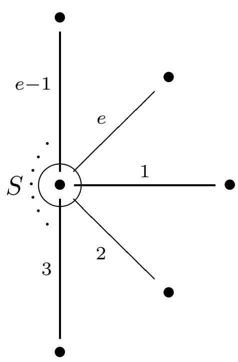
We associate to a Brauer tree $T = T_S^m$ the **Brauer tree algebra** $A(T) = A(T_S^m) = KQ_{T_S^m}/I_S^m$, where I_S^m is the ideal in the path algebra $KQ_{T_S^m}$ of $Q_{T_S^m}$ generated by the elements:

- $\beta_{\beta^{-1}(i)}\alpha_i$ and $\alpha_{\alpha^{-1}(i)}\beta_i$,
- $A_i^m - B_i$, if the α-cycle passing through i is exceptional,
- $A_i - B_i$, if the α-cycle passing through i is not exceptional.

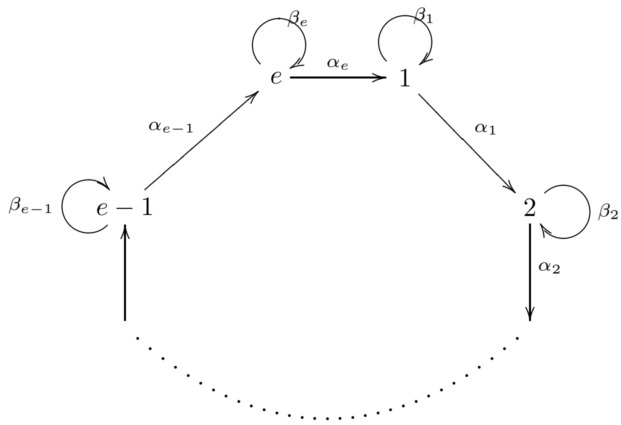
For the multiplicity $m = 1$, the Brauer tree algebras $A(T) = A(T_S^1)$ are exactly the trivial extension algebras $\mathrm{T}(B)$ of the tilted algebras of types $\mathbb{A}_n$.

For the multiplicity $m \geq 2$, we have $A(T_S^m) \cong \mathrm{T}(B)^{\mathbb{Z}_m}$ for an exceptional tilted algebra $B = B(T_S^m)$ of type $\mathbb{A}_n$ and the cyclic group $\mathbb{Z}_m$ acting freely on $\mathrm{T}(B)$. Here, $n = me$, where e is the number of edges of T_S^m.

Example 4.5. Let $T = T_S^m$ be the star



Then the associated Brauer quiver $Q_T = Q_{T_S^m}$ is of the form



and $A(T_S^m)$ is a **symmetric Nakayama algebra**. Moreover, $A(T_S^m) \cong A(T')^{\mathbb{Z}_m}$ for the star T' with me edges and the multiplicity 1, and $A(T') \cong T(B)$ for the path algebra $B = KQ$ of the equioriented quiver of type $\mathbb{A}_{me}$

$$1 \longrightarrow 2 \longrightarrow \dots \longrightarrow me.$$

Then we have the following classical result proved independently by Dade [**Da**], Janusz [**Ja**] and Kupisch [**Ku1**], [**Ku2**].

Theorem 4.6 (Dade-Janusz-Kupisch,1966-1969)**.** *Let B be a block of a group algebra KG with cyclic defect group D_B. Then B is Morita equivalent to a Brauer tree algebra $A(T_S^m)$.*

(Here $me + 1 = p^n$ if $| D_B |= p^n$ and B has e simple modules.)

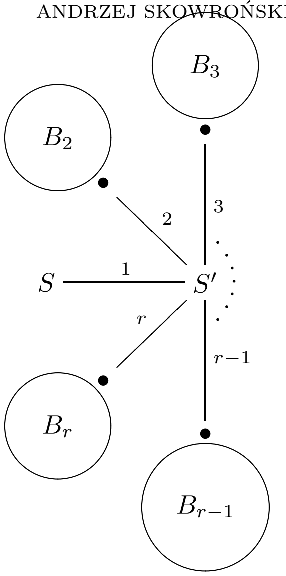
We refer to [**Fe**] for a description of the Brauer tree algebras $A(T_s^m)$ which are Morita equivalent to blocks of group algebras.

The following characterization of Brauer tree algebras was established by Gabriel and Riedtmann [**GR**] (equivalence (1) and (2)) and Rickard [**Ric2**] (equivalence of (1) and (3)).

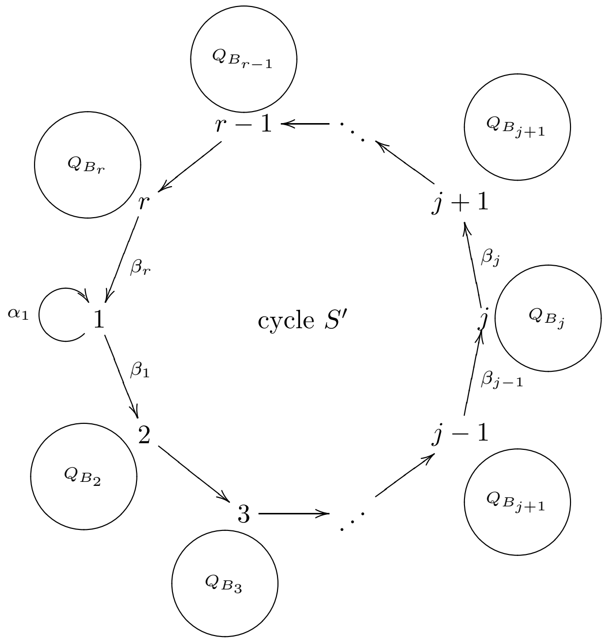
Theorem 4.7 (Gabriel-Riedtmann (1979), Rickard (1989))**.** *Let A be a selfinjective algebra. The following statements are equivalent:*

(1) *A is Morita equivalent to a Brauer tree algebra.*
(2) *A is stably equivalent to a symmetric Nakayama algebra.*
(3) *A is derived equivalent to a symmetric Nakayama algebra.*

Let $T = T_S$ be a Brauer tree with at least two edges and an **extreme** exceptional vertex S



Then the Brauer quiver Q_T is of the form



For each vertex i of Q_T, we have the cycles A_i and B_i around i. Define also the cycles $B'_j = \beta_j \dots \beta_r \alpha_1 \beta_1 \dots \beta_{j-1}$, around the vertices $j \in S'_0$, $j \neq 1$, of the cycle S'.

For each $\lambda \in K$, define the algebra

$$D(T_S, \lambda) = KQ_T / I(T_S, \lambda),$$

where $I(T_S, \lambda)$ is the ideal of KQ_T generated by the elements:

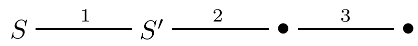
- $\beta_{\beta^{-1}(i)} \alpha_i$ and $\alpha_{\alpha^{-1}(i)} \beta_i$, $i \in (Q_T)_0 \setminus \{1\}$,
- $A_1^2 = B_1$,
- $A_i - B_i$, $i \in (Q_T)_0 \setminus S'_0$,
- $A_j - B'_j$, $j \in S'_0 \setminus \{1\}$,

- $\beta_r\beta_1 - \lambda\beta_r\alpha_1\beta_1$.

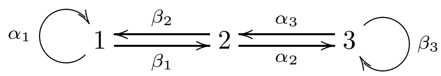
The following proposition describes the basic properties of the algebras $D(T_S, \lambda)$ (see Riedtmann [**Rd2**] and Waschbüsh [**W1**]).

Proposition 4.8. (1) *$D(T_S, \lambda)$, $\lambda \in K$, are symmetric algebras of finite type.*
(2) *For $\lambda, \mu \in K \setminus \{0\}$, $D(T_S, \lambda) \cong D(T_S, \mu)$.*
(3) $D(T_S, 0) \cong D(T_S, 1) \iff \operatorname{char} K \neq 2$.
(4) *$D(T_S, 0)$ and $D(T_S, 1)$ are socle equivalent.*
(5) *$D(T_S, 0) = \mathrm{T}(B)^{\mathbb{Z}_3}$, for an exceptional tilted algebra $B = B^*(T_S)$ of Dynkin type $\mathbb{D}_{3m}$ and $\mathbb{Z}_3$ acting freely on $\mathrm{T}(B)$.*
(6) *For $\operatorname{char} K = 2$, $D(T_S, 1)$ is nonstandard and degenerates to $D(T_S, 0)$.*

Example 4.9. Let $T = T_S^2$ be the Brauer tree of the form



Then the Brauer quiver $Q_T = Q_{T_S^2}$ is of the form



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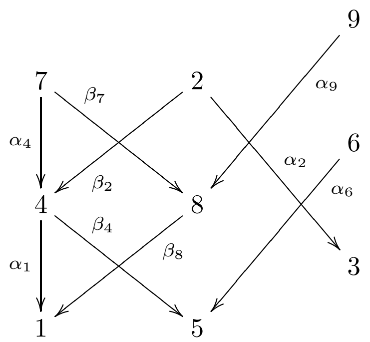
and the algebras $D(T_S, 0)$ and $D(T_S, 1)$ are of the form

$D(T_S, 0) = KQ_T/I(T_S, 0)$, with $I(T_S, 0)$ generated by	$D(T_S, 1) = KQ_T/I(T_S, 0)$, with $I(T_S, 1)$ generated by
$\beta_1\alpha_2$, $\alpha_3\beta_2$	$\beta_1\alpha_2$, $\alpha_3\beta_2$
$\beta_3\alpha_3$, $\alpha_2\beta_3$	$\beta_3\alpha_3$, $\alpha_2\beta_3$
$\alpha_1^2 - \beta_1\beta_2$	$\alpha_1^2 - \beta_1\beta_2$
$\alpha_2\alpha_3 - \beta_2\alpha_1\beta_1$	$\alpha_2\alpha_3 - \beta_2\alpha_1\beta_1$
$\alpha_3\alpha_2 - \beta_3$	$\alpha_3\alpha_2 - \beta_3$
$\beta_2\beta_1$	$\beta_2\beta_1 - \beta_2\alpha_1\beta_1$

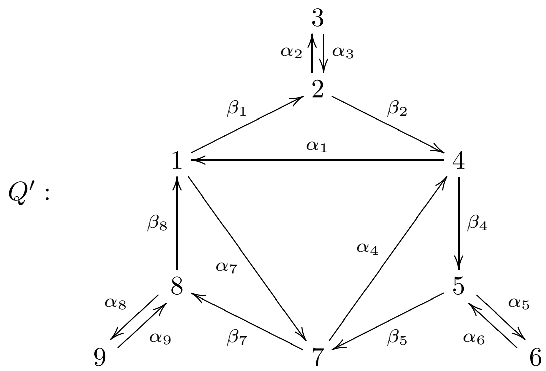
Let $B = KQ/I$, where Q is the quiver

Q :



and I is the ideal of KQ generated by the elements $\alpha_4\alpha_1 - \beta_7\beta_8$, $\beta_2\beta_4$, $\alpha_9\beta_8$. Then B is a tilted algebra of type $\mathbb{D}_9 = \mathbb{D}_{3\cdot 3}$.

Moreover, $\mathrm{T}(B) \cong KQ'/I'$, where Q' is the quiver



and I' is the ideal of KQ' generated by the elements

$$\alpha_4\alpha_1 - \beta_7\beta_8, \alpha_1\alpha_7 - \beta_4\beta_5, \alpha_7\alpha_4 - \beta_1\beta_2, \beta_2\beta_4, \beta_5\beta_7, \beta_8\beta_1, \beta_1\alpha_2, \alpha_3\beta_2,$$

$$\beta_4\alpha_5, \alpha_6\beta_5, \beta_7\alpha_8, \alpha_9\beta_8, \alpha_2\alpha_3 - \beta_2\alpha_1\beta_1, \alpha_5\alpha_6 - \beta_5\alpha_4\beta_4, \alpha_8\alpha_9 - \beta_8\alpha_7\beta_7.$$

Then the group $\mathbb{Z}_3$ acts freely on $\mathrm{T}(B)$ by the canonical rotation and we have $\mathrm{T}(B)^{\mathbb{Z}_3} \cong D(T_S, 0)$.

The following description of the nonsimple standard symmetric algebras of finite type follows from [**BLR**], [**Rd1**], [**Rd2**], [**HW**], [**W1**] and [**W2**].

Theorem 4.10 (Riedtmann, Waschbüsch, ...)**.** *Let Λ be a nonsimple standard selfinjective algebra. The following statements are equivalent:*

(1) Λ *is symmetric of finite type.*
(2) Λ *is isomorphic to* $\mathrm{T}(B)^G$*, for a tilted algebra B of Dynkin type and a finite group G acting freely on* $\mathrm{T}(B)$.
(3) Λ *is isomorphic to one of the algebras:*
   (a) $\mathrm{T}(B)$*, for a tilted algebra B of Dynkin type.*
   (b) $A(T_S^m)$*, for a Brauer tree T_S^m, with the exceptional vertex S of multiplicity $m \geq 2$.*
   (c) $D(T_S, 0)$*, for a Brauer tree T_S, and an extreme exceptional vertex S.*

The remaining (nonstandard) symmetric algebras of finite type are described by the following theorem proved independently by Riedtmann [**Rd2**] and Waschbüsch [**W1**].

Theorem 4.11 (Riedtmann (1983), Waschbüsch (1981))**.** *Let Λ be a selfinjective algebra over K. The following statements are equivalent:*

(1) Λ *is nonstandard of finite type.*
(2) Λ *is nonstandard symmetric of finite type.*
(3) $\Lambda \cong D(T_S, 1)$*, for a Brauer tree T_S, an extreme exceptional vertex S, and* $\operatorname{char} K = 2$.

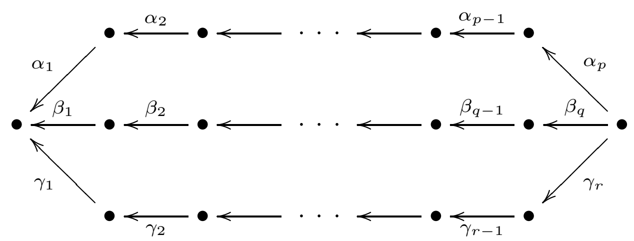
Definition 4.12. *A* ***symmetric algebra of Dynkin type*** *is defined to be a symmetric algebra A which is socle equivalent to an invariant symmetric algebra $\mathrm{T}(B)^G$, where B is a tilted algebra of Dynkin type and G is a finite group acting freely on $\mathrm{T}(B)$.*

Therefore, a symmetric algebra of Dynkin type is a symmetric algebra listed in the above Theorems 4.10 and 4.11.

Symmetric algebras of tubular type

In the description of the symmetric algebras of the symmetric algebras of tubular type a prominent role is played by the tubular algebras introduced by Ringel in [**Ri**].

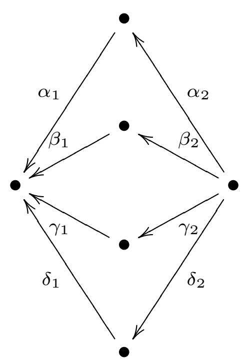
For a triple $(p, q, r) \in \{(3, 3, 3), (2, 4, 4), (2, 3, 6)\}$, we denote by $C(p, q, r)$ the **canonical tubular algebra of type** (p, q, r) given by the quiver



bound by the relation

$$\alpha_p \dots \alpha_2\alpha_1 + \beta_q \dots \beta_2\beta_1 + \gamma_r \dots \gamma_2\gamma_1 = 0.$$

Further, for $\lambda \in K \setminus \{0, 1\}$, denote by $C(2, 2, 2, 2, \lambda)$, the **canonical tubular algebra of type** $(2, 2, 2, 2)$ given by the quiver



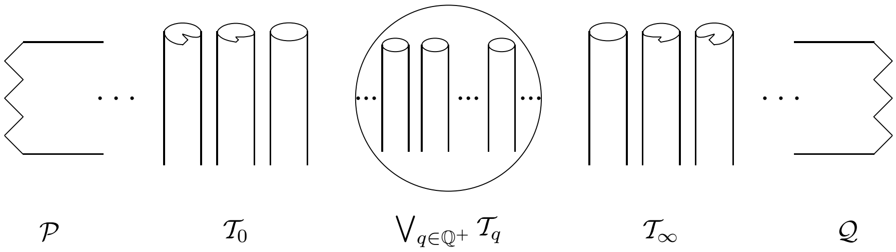
bound by the relations

$$\alpha_2\alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1 = 0, \qquad \alpha_2\alpha_1 + \lambda\beta_2\beta_1 + \delta_2\delta_1 = 0.$$

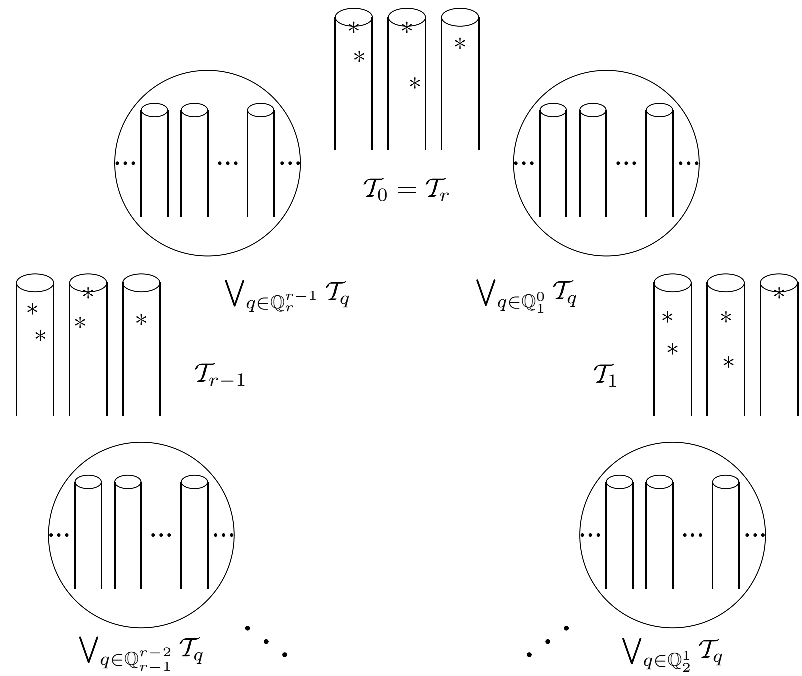
Then a **tubular algebra** is defined to be a tilted algebra $B = \operatorname{End}_C(T)$ of a canonical tubular algebra C of one of tubular types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$, and with T a tilting C-module of nonnegative rank.

A tubular algebra B has the following characteristic properties:

- $\operatorname{gl.dim} B = 2$;
- $\operatorname{rk} K_0(B) = 6$, 8, 9, or 10;
- B is tame of polynomial growth;
- The Auslander-Reiten quiver Γ_B of B is of the form



Let B be a tubular algebra. Then it follows from Nehring and Skowroński [**NS**] (see also Happel-Ringel [**HR2**]) that $\mathrm{T}(B)$ is a symmetric standard tame algebra of polynomial growth and the Auslander-Reiten quiver of $\mathrm{T}(B)$ is of the form



where $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_r$ are $\mathbb{P}_1(K)$-families of quasi-tubes (stable tubes with inserted projective-injective vertices $*$) and $\mathcal{T}_q, q \in \mathbb{Q}_i^{i-1} = \mathbb{Q} \cap (i-1, i), 1 \le i \le r$, are $\mathbb{P}_1(K)$-families of stable tubes.

The following theorem proved by Białkowski and Skowroński [**BiS1**] gives a characterization of the trivial extension algebras of tubular algebras.

Theorem 4.13 (Białkowski-Skowroński, 2003)**.** *Let Λ be a representation-infinite algebra. The following statements are equivalent:*

(i) *Λ is tame, standard, weakly symmetric, with all indecomposable nonprojective finite dimensional modules periodic, and singular Cartan matrix.*

(ii) *Λ is tame, standard, symmetric, with all indecomposable nonprojective finite dimensional modules periodic, and singular Cartan matrix.*

(iii) *$\Lambda \cong \mathrm{T}(B)$ for a tubular algebra B.*

We also note that by a result of [**NS**], for tubular algebras B and B', we have $\mathrm{T}(B) \cong \mathrm{T}(B') \iff B' = S_{i_t}^{+} \dots S_{i_1}^{+} B$ (finite number of reflections).

There are 4 families of nonisomorphic trivial extensions of tubular algebras of tubular type $(2,2,2,2)$, and 38, 85, 4953 isoclasses of the trivial extensions of tubular types $(3,3,3)$, $(2,4,4)$, $(2,3,6)$, respectively (**Białkowski**).

The following problem arises naturally.

PROBLEM 7. When a finite group G acts freely on the trivial extension $\mathrm{T}(B)$ of a tubular algebra B?

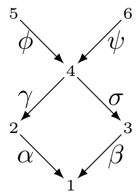
By general theory such a group G is cyclic (see [**Sk1**]).

An additional information is provided by the following theorem proved by Lenzing and Skowroński [**LeSk**].

Theorem 4.14 (Lenzing-Skowroński, 2000)**.** *Let G be a finite group acting freely on the trivial extension $\mathrm{T}(B)$ of a tubular algebra B of type $(2,3,6)$. Then $G = \{1\}$.*

A complete answer to the problem raised above is given by the following theorem proved by Białkowski and Skowroński [**BiS1**].

Theorem 4.15 (Białkowski-Skowroński, 2002)**.** *Let B be a tubular algebra such that a **nontrivial** finite group G acts freely on $\mathrm{T}(B)$. Then $\mathrm{T}(B) \cong \mathrm{T}(B')$ for a tubular algebra B' given by one of the following bound quivers:*

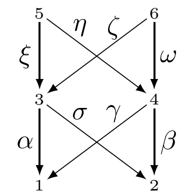


$\phi\gamma\alpha = \phi\sigma\beta$

$\psi\gamma\alpha = \lambda\psi\sigma\beta$

$B_1(\lambda)$

$\lambda \in K \setminus \{0,1\}$

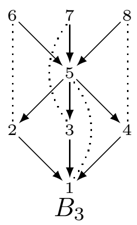


$\xi\alpha = \eta\gamma, \quad \zeta\alpha = \omega\gamma$

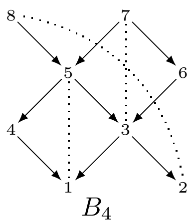
$\xi\sigma = \eta\beta, \quad \zeta\sigma = \lambda\omega\beta$

$B_2(\lambda)$

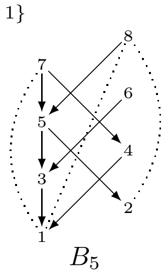
$\lambda \in K \setminus \{0,1\}$



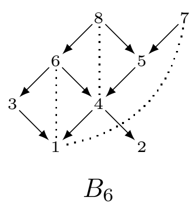
B_3



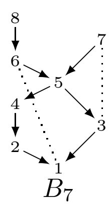
B_4



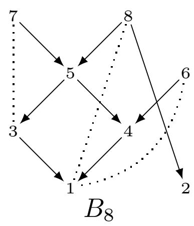
B_5



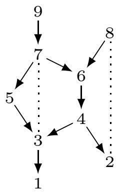
B_6



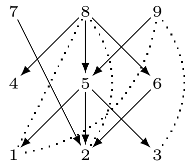
B_7



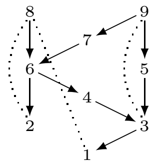
B_8



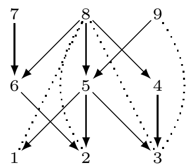
B_9



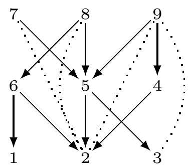
B_{10}



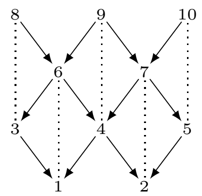
B_{11}



B_{12}



B_{13}



B_{14}

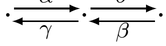
(where a dotted line means that the sum of paths indicated by this line is zero if it indicates exactly three parallel paths, the commutativity of paths if it indicates exactly two parallel paths, and the zero path if it indicates only one path).

Here, $B_1(\lambda)$, $B_2(\lambda)$ are tubular algebras of type $(2,2,2,2)$, B_3, B_4, B_5, B_6, B_7, B_8 are tubular algebras of type $(3,3,3)$, and B_9, B_{10}, B_{11}, B_{12}, B_{13} are tubular algebras of type $(2,4,4)$.

The following characterization of the nontrivial invariant algebras of the trivial extensions of tubular algebras of free actions of finite groups has been established by Białkowski and Skowroński [**BiS2**].

Theorem 4.16 (Białkowski-Skowroński, 2003). *Let* Λ *be a representation-infinite algebra. The following statements are equivalent:*

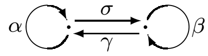
(i) Λ *is tame, standard, weakly symmetric, with all indecomposable nonprojective finite dimensional modules periodic and nonsingular Cartan matrix.*
(ii) $\Lambda \cong \mathrm{T}(B)^G$ *for a tubular algebra* B *and a* ***nontrivial*** *finite group* G *acting freely on* $\mathrm{T}(B)$.
(iii) Λ *is isomorphic to one of the bound quiver algebras.*



$\alpha\gamma\alpha = \alpha\sigma\beta$
$\beta\gamma\alpha = \lambda\beta\sigma\beta$
$\gamma\alpha\gamma = \sigma\beta\gamma$
$\gamma\alpha\sigma = \lambda\sigma\beta\sigma$

$A_1(\lambda) \cong \mathrm{T}(B_1(\lambda))^{\mathbb{Z}_2}$

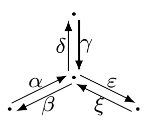
$\lambda \in K \setminus \{0,1\}$



$\alpha^2 = \sigma\gamma$
$\lambda\beta^2 = \gamma\sigma$
$\gamma\alpha = \beta\gamma$
$\sigma\beta = \alpha\sigma$

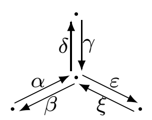
$A_2(\lambda) \cong \mathrm{T}(B_2(\lambda))^{\mathbb{Z}_3}$

$\lambda \in K \setminus \{0,1\}$



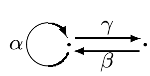
$\beta\alpha + \delta\gamma + \varepsilon\xi = 0$
$\alpha\beta = 0, \quad \xi\varepsilon = 0$
$\gamma\delta = 0$

$A_3 \cong \mathrm{T}(B_3)^{\mathbb{Z}_2}$



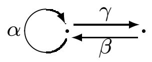
$\beta\alpha + \delta\gamma + \varepsilon\xi = 0$
$\alpha\beta = 0, \quad \gamma\varepsilon = 0$
$\xi\delta = 0$

$A_4 \cong \mathrm{T}(B_3)^{\mathbb{Z}_2}$



$\alpha^2 = \gamma\beta$
$\beta\alpha\gamma = 0$

$A_5 \cong \mathrm{T}(B_4)^{\mathbb{Z}_4}$



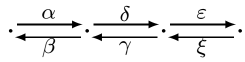
$\alpha^3 = \gamma\beta$

$\beta\gamma = 0$

$\beta\alpha^2 = 0$

$\alpha^2\gamma = 0$

$A_6 \cong \mathrm{T}(B_5)^{\mathbb{Z}_4}$



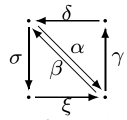
$\beta\alpha = \delta\gamma$

$\gamma\delta = \varepsilon\xi$

$\alpha\delta\varepsilon = 0$

$\xi\gamma\beta = 0$

$A_7 \cong \mathrm{T}(B_4)^{\mathbb{Z}_2}$



$\xi\beta\alpha = 0$

$\delta\alpha\beta = 0$

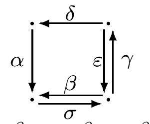
$\beta\alpha\gamma = 0$

$\alpha\beta\sigma = 0$

$\alpha\beta\alpha = \sigma\xi, \quad \xi\gamma = 0$

$\beta\alpha\beta = \gamma\delta, \quad \delta\sigma = 0$

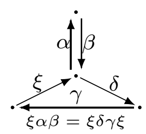
$A_8 \cong \mathrm{T}(B_5)^{\mathbb{Z}_2}$



$\delta\alpha = \varepsilon\beta, \gamma\varepsilon = \beta\sigma, \alpha\sigma\beta = 0$

$\varepsilon\gamma\delta = 0, \sigma\gamma\varepsilon\gamma = 0$

$A_9 \cong \mathrm{T}(B_6)^{\mathbb{Z}_2}$

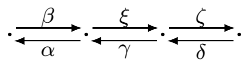


$\xi\alpha\beta = \xi\delta\gamma\xi$

$\alpha\beta\delta = \delta\gamma\xi\delta$

$\beta\alpha = 0, (\gamma\xi\delta)^2\gamma = 0$

$A_{10} \cong \mathrm{T}(B_7)^{\mathbb{Z}_2}$



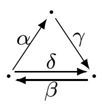
$\gamma\alpha\beta = \gamma\xi\gamma$

$\alpha\beta\xi = \xi\gamma\xi$

$\beta\alpha = 0, \quad \delta\gamma = 0$

$\xi\zeta = 0, \quad (\gamma\xi)^2 = \zeta\delta$

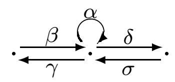
$A_{11} \cong \mathrm{T}(B_8)^{\mathbb{Z}_2}$



$\delta\beta\delta = \alpha\gamma$

$\gamma\beta\alpha = 0, \quad \beta(\delta\beta)^3 = 0$

$A_{12} \cong \mathrm{T}(B_9)^{\mathbb{Z}_3}$

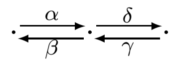


$\alpha^2 = \gamma\beta, \quad \beta\delta = 0, \quad \gamma\beta = 0$

$\sigma\gamma = 0, \quad \alpha\delta = 0, \quad \sigma\alpha = 0$

$\alpha^3 = \delta\sigma$

$A_{13} \cong \mathrm{T}(B_{10})^{\mathbb{Z}_3}$



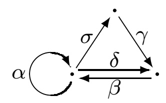
$\beta\alpha = \delta\gamma\delta\gamma$

$\alpha\delta\gamma\delta = 0$

$\gamma\delta\gamma\beta = 0$

$\alpha\beta = 0$

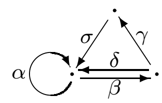
$A_{14} \cong \mathrm{T}(B_{11})^{\mathbb{Z}_3}$



$\gamma\beta\alpha = 0, \quad \alpha^2 = \delta\beta$

$\beta\delta = 0, \alpha\sigma = 0, \alpha\delta = \sigma\gamma$

$A_{15} \cong \mathrm{T}(B_{12})^{\mathbb{Z}_3}$



$\alpha\beta\gamma = 0, \quad \alpha^2 = \beta\delta$

$\delta\beta = 0, \sigma\alpha = 0, \delta\alpha = \gamma\sigma$

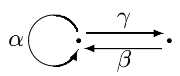
$A_{16} \cong \mathrm{T}(B_{13})^{\mathbb{Z}_3}$

We note that all algebras presented above, except A_4 for $\operatorname{char} K \neq 2$, are symmetric.

The following theorem proved by Białkowski and Skowroński in [**BiS3**] gives a complete description of the nonstandard symmetric algebras which are socle equivalent to the standard symmetric algebras described in Theorems 4.13 and 4.16.

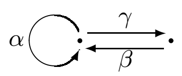
Theorem 4.17 (Białkowski-Skowroński, 2004). *Let Λ be a nonstandard symmetric algebra over an algebraically closed field K. Then Λ is socle equivalent to a standard representation-infinite tame symmetric algebra A with all indecomposable nonprojective modules periodic if and only if exactly one of the following cases holds:*

(i) *K is of characteristic 3 and Λ is isomorphic to one of the bound quiver algebras*



$\alpha^2 = \gamma\beta$
$\beta\alpha\gamma = \beta\alpha^2\gamma$
$\beta\alpha\gamma\beta = 0$
$\gamma\beta\alpha\gamma = 0$

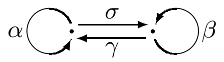
Λ_1



$\alpha^2\gamma = 0, \quad \beta\alpha^2 = 0$
$\gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0$
$\beta\gamma = \beta\alpha\gamma$
$\alpha^3 = \gamma\beta$

Λ_2

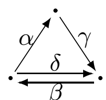
(ii) *K is of characteristic 2 and Λ is isomorphic to one of the bound quiver algebras*



$\alpha^4 = 0, \quad \gamma\alpha^2 = 0, \quad \alpha^2\sigma = 0$
$\alpha^2 = \sigma\gamma + \alpha^3, \quad \lambda\beta^2 = \gamma\sigma$
$\gamma\alpha = \beta\gamma, \quad \sigma\beta = \alpha\sigma$

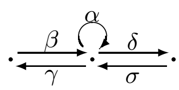
$\Lambda_3(\lambda)$

$\lambda \in K \setminus \{0, 1\}$



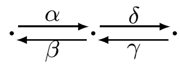
$\delta\beta\delta = \alpha\gamma, \ (\beta\delta)^3\beta = 0$
$\gamma\beta\alpha\gamma = 0, \ \alpha\gamma\beta\alpha = 0$
$\gamma\beta\alpha = \gamma\beta\delta\beta\alpha$

Λ_4



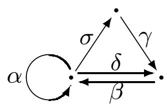
$\alpha^2 = \gamma\beta, \ \alpha^3 = \delta\sigma, \ \beta\delta = 0$
$\sigma\gamma = 0, \ \alpha\delta = 0, \ \sigma\alpha = 0$
$\gamma\beta\gamma = 0, \ \beta\gamma\beta = 0, \ \beta\gamma = \beta\alpha\gamma$

Λ_5



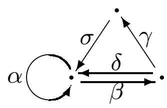
$\alpha\delta\gamma\delta = 0, \quad \gamma\delta\gamma\beta = 0$
$\alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0$
$\alpha\beta = \alpha\delta\gamma\beta$
$\beta\alpha = \delta\gamma\delta\gamma$

Λ_6



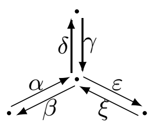
$\beta\delta = \beta\alpha\delta, \alpha\sigma = 0, \alpha\delta = \sigma\gamma$
$\gamma\beta\alpha = 0, \alpha^2 = \delta\beta, \gamma\beta\delta = 0$
$\beta\delta\beta = 0, \quad \delta\beta\delta = 0$

Λ_7



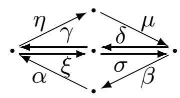
$\delta\beta = \delta\alpha\beta, \sigma\alpha = 0, \delta\alpha = \gamma\sigma$
$\alpha\beta\gamma = 0, \alpha^2 = \beta\delta, \delta\beta\gamma = 0$
$\beta\delta\beta = 0, \quad \delta\beta\delta = 0$

Λ_8



$\beta\alpha + \delta\gamma + \varepsilon\xi = 0$
$\gamma\delta = 0, \ \xi\varepsilon = 0, \ \alpha\beta\alpha = 0$
$\beta\alpha\beta = 0, \ \alpha\beta = \alpha\delta\gamma\beta$

Λ_9



$\mu\beta = 0, \ \alpha\eta = 0, \ \beta\alpha = \delta\gamma$
$\xi\sigma = \eta\mu, \ \sigma\delta = \gamma\xi + \sigma\delta\sigma\delta$
$\delta\sigma\delta\sigma = 0, \ \xi\gamma\xi\gamma = 0$

Λ_{10}

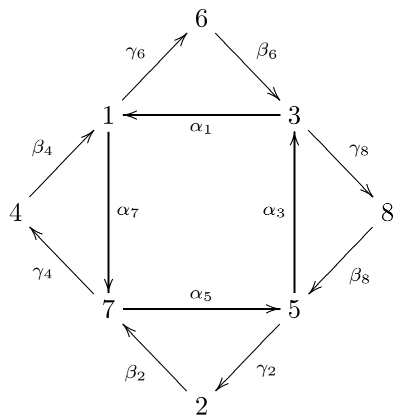
W also note that if Λ is a nonstandard algebra Λ_i, $i \in \{1, \dots, 10\}$, then Λ degenerates to a standard symmetric algebra $\Lambda' = \mathrm{T}(B)^G$, for an exceptional tubular algebra B and a nontrivial group G acting freely on $\mathrm{T}(B)$ (see Białkowski-Holm-Skowroński [**BHS**]).

Definition 4.18. *A **symmetric algebra of tubular type** is defined to be a symmetric algebra A which is socle equivalent to an invariant symmetric algebra $\mathrm{T}(B)^G$, where B is a tubular algebra and G is a finite group acting freely on $\mathrm{T}(B)$.*

Therefore, a symmetric algebra of tubular type is a symmetric algebra listed in the above Theorems 4.13, 4.16 and 4.17.

Example 4.19. The trivial extension $\mathrm{T}(B_5)$ of the tubular algebra B_5 of type $(3, 3, 3)$, presented in Theorem 4.15, is the bound quiver algebra $K\Omega/J$ given by the quiver

$\Omega :$



and the ideal J is generated by the elements

$$\alpha_1\alpha_7\alpha_5 - \gamma_8\beta_8, \alpha_3\alpha_1\alpha_7 - \gamma_2\beta_2, \alpha_5\alpha_3\alpha_1 - \gamma_4\beta_4, \alpha_7\alpha_5\alpha_3 - \gamma_6\beta_6, \beta_4\gamma_6, \beta_6\gamma_8, \beta_8\gamma_2,$$

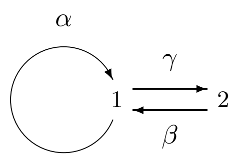
$$\beta_2\gamma_4, \beta_6\alpha_1\alpha_7, \beta_8\alpha_3\alpha_1, \beta_2\alpha_5\alpha_3, \beta_4\alpha_7\alpha_5, \alpha_1\alpha_7\gamma_4, \alpha_7\alpha_5\gamma_2, \alpha_5\alpha_3\gamma_8, \alpha_3\alpha_1\gamma_6.$$

Then the group $\mathbb{Z}_4$ acts on $\mathrm{T}(B_5)$ by the obvious rotation and

$$\mathrm{T}(B_5)^{\mathbb{Z}_4} \cong A_6 = KQ/I,$$

where

$Q :$



and the ideal I is generated by $\alpha^3 - \gamma\beta$, $\beta\gamma$, $\beta\alpha^2$, $\alpha^2\gamma$.

Consider the algebra

$$\Lambda_2 = KQ/I^{(1)}, \quad I^{(1)} = \left\langle \alpha^3 - \gamma\beta, \beta\gamma - \beta\alpha\gamma, \beta\alpha^2, \alpha^2\gamma \right\rangle.$$

Then A_6 and Λ_2 are selfinjective algebras of dimension 11, and we have

- $A_6 \cong \Lambda_2 \iff \operatorname{char} K \neq 3$,
- $\operatorname{char} K = 3 \;\Rightarrow\; \Lambda_2$ is nonstandard,
- $A_6/\operatorname{soc} A_6 \cong \Lambda_2/\operatorname{soc} \Lambda_2$.

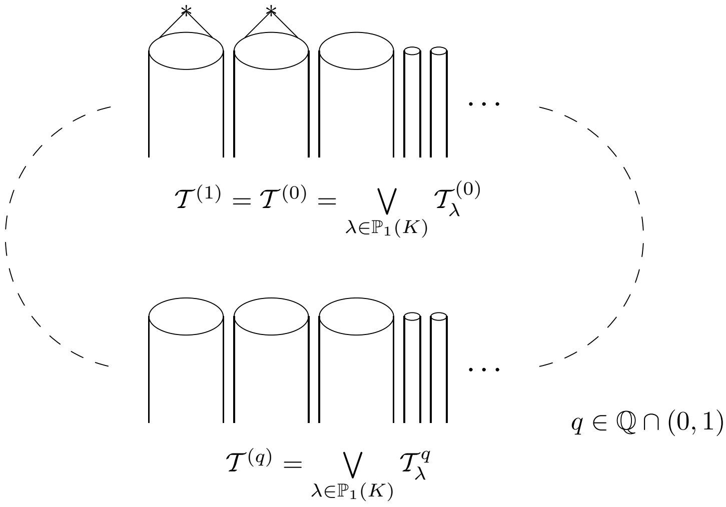
Consider also the family of algebras

$$\Lambda^{(t)} = KQ/I^{(t)}, \quad I^{(t)} = \left\langle \alpha^3 - \gamma\beta, \beta\gamma - t\beta\alpha\gamma, \beta\alpha^2, \alpha^2\gamma \right\rangle, \quad t \in K.$$

Then we have

- $\Lambda^{(t)} \cong \Lambda^{(1)} = \Lambda_2$, for $t \in K \setminus \{0\}$,
- $A_6 = \Lambda^{(0)} = \lim\limits_{t\to 0} \Lambda^{(t)}, \quad A_6 \in \overline{\mathrm{GL}_{11}(K)\Lambda_2}$.

Therefore A_6 is a **degeneration** of Λ_2 (Λ_2 is a **deformation** of A_6). Moreover, the Auslander-Reiten quivers Γ_{A_6} and Γ_{Λ_2} of A_6 and Λ_2 coincide and are of the form



Algebras of quaternion type

The following class of algebras of quaternion type has been introduced by Erdmann (see [**E1**], [**E2**], [**E3**]).

Definition 4.20. *An algebra A is said to be of* ***quaternion type*** *if the following conditions are satisfied:*

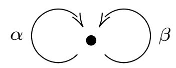
- *A is symmetric, indecomposable, tame of infinite type;*
- *The indecomposable nonprojective finite dimensional A-modules are Ω_A-periodic of period dividing 4;*
- *The Cartan matrix of A is nonsingular.*

This class of algebras includes all blocks of group algebras of finite groups with generalized quaternion defect groups. In [**E1**], [**E2**], [**E3**] Erdmann proved that any algebra of quaternion type is Morita equivalent to an algebra in 12 families of symmetric algebras defined by quivers and relations (presented in the theorem below). Later, Holm [**Hol**] has classified these algebras up to derived equivalence, and proved (applying the Geiss degeneration theorem [**Ge**] and the known results on selfinjective algebras of tubular type [**Sk1**]) that they are in fact tame. The problem whether all algebras listed by Erdmann are of quaternion type has been solved recently in the paper by Erdmann and Skowroński [**ES1**]. Therefore, we have the following theorem.

Theorem 4.21. *Let A be a selfinjective algebra. The following statements are equivalent:*

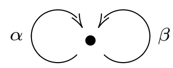
(i) *A is of quaternion type;*
(ii) *A is Morita equivalent to one of the bound quiver algebras*

$Q^k(c)$:



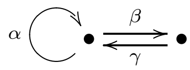
$\alpha^2 = (\beta\alpha)^{k-1}\beta + c(\alpha\beta)^k$
$\beta^2 = (\alpha\beta)^{k-1}\alpha$
$(\alpha\beta)^k = (\beta\alpha)^k,\ (\alpha\beta)^k\alpha = 0$
$k \geq 2$

$Q^k(c,d)$:



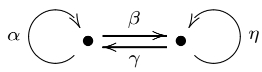
$\operatorname{char} K = 2$
$\alpha^2 = (\beta\alpha)^{k-1}\beta + c(\alpha\beta)^k$
$\beta^2 = (\alpha\beta)^{k-1}\alpha + d(\alpha\beta)^k$
$(\alpha\beta)^k = (\beta\alpha)^k,\ (\alpha\beta)^k\alpha = 0$
$(\beta\alpha)^k\beta = 0$
$k \geq 2,\ c, d \in K,\ (c,d) \neq (0,0)$

$Q(2\mathcal{A})^k(c)$:



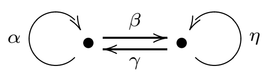
$\gamma\beta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$
$\beta\gamma\beta = (\alpha\beta\gamma)^{k-1}\alpha\beta$
$\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$
$\alpha^2\beta = 0$
$k \geq 2,\ c \in K$

$Q(2\mathcal{B})_1^{k,s}(a,c)$:



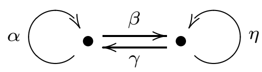
$\gamma\beta = \eta^{s-1},\ \beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta$
$\eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$
$\alpha^2 = a(\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$
$\alpha^2\beta = 0,\ \gamma\alpha^2 = 0$
$k \geq 1,\ s \geq 3,\ a \in K^*,\ c \in K$

$Q(2\mathcal{B})_2^s(a,c)$:



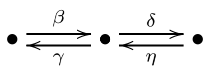
$\alpha\beta = \beta\eta,\ \eta\gamma = \gamma\alpha,\ \beta\gamma = \alpha^2$
$\gamma\beta = \eta^2 + a\eta^{s-1} + c\eta^s$
$\alpha^{s+1} = 0,\ \eta^{s+1} = 0$
$\gamma\alpha^{s-1} = 0,\ \alpha^{s-1}\beta = 0$
$s \geq 4,\ a \in K^*,\ c \in K$

$Q(2\mathcal{B})_3^t(a,c)$:



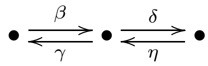
$\alpha\beta = \beta\eta,\ \eta\gamma = \gamma\alpha,\ \beta\gamma = \alpha^2$
$\gamma\beta = a\eta^{t-1} + c\eta^t$
$\alpha^4 = 0,\ \eta^{t+1} = 0,\ \gamma\alpha^2 = 0$
$\alpha^2\beta = 0$
$t \geq 3,\ a \in K^*,\ c \in K$
$(t = 3 \Rightarrow a \neq 1,\ t > 3 \Rightarrow a = 1)$

$Q(3\mathcal{A})_1^{k,s}(d)$:



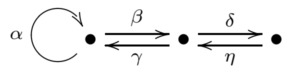
$\beta\delta\eta = (\beta\gamma)^{k-1}\beta$
$\delta\eta\gamma = (\gamma\beta)^{k-1}\gamma$
$\eta\gamma\beta = d(\eta\delta)^{s-1}\eta$
$\gamma\beta\delta = d(\delta\eta)^{s-1}\delta$
$\beta\delta\eta\delta = 0,\ \eta\gamma\beta\gamma = 0$
$k, s \geq 2,\ d \in K^*$
$(k = s = 2 \Rightarrow d \neq 1,\ \textit{else}\ d = 1)$

$Q(3\mathcal{A})_2^k$:



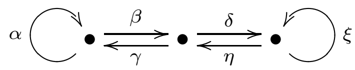
$\beta\gamma\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$
$\gamma\beta\gamma = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$
$\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$
$\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$
$\beta\gamma\beta\delta = 0,\ \eta\delta\eta\gamma = 0$
$k \geq 2$

$Q(3\mathcal{B})^{k,s}$:



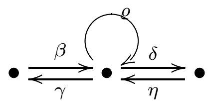
$\beta\gamma = \alpha^{s-1}$
$\alpha\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$
$\gamma\alpha = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$
$\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$
$\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$
$\alpha^2\beta = 0,\ \beta\delta\eta\delta = 0$
$k \geq 1,\ s \geq 3$

$Q(3\mathcal{D})^{k,s,t}$:



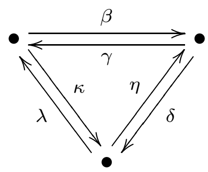
$\beta\gamma = \alpha^{s-1}$
$\gamma\alpha = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$
$\alpha\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$
$\eta\delta = \xi^{t-1}$
$\delta\xi = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$
$\xi\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$
$\alpha^2\beta = 0,\ \delta\eta\delta = 0$
$k \geq 1,\ s,t \geq 3$

$Q(3\mathcal{C})^{k,s}$:



$\beta\varrho = 0,\ \varrho\gamma = 0,\ \eta\varrho^2 = 0$
$\varrho^2\delta = 0$
$\delta\eta - \gamma\beta = \varrho^{s-1},\ \eta\varrho = (\eta\delta)^{k-1}\eta$
$\varrho\delta = (\delta\eta)^{k-1}\delta,\ (\beta\gamma)^{k-1}\beta\delta = 0$
$(\eta\delta)^{k-1}\eta\gamma = 0$
$k \geq 2,\ s \geq 3$

$Q(3\mathcal{K})^{a,b,c}$:



$\beta\delta = (\kappa\lambda)^{a-1}\kappa$
$\eta\gamma = (\lambda\kappa)^{a-1}\lambda$
$\delta\lambda = (\gamma\beta)^{b-1}\gamma$
$\kappa\eta = (\beta\gamma)^{b-1}\beta$
$\lambda\beta = (\eta\delta)^{c-1}\eta,\ \gamma\kappa = (\delta\eta)^{c-1}\delta$
$\gamma\beta\delta = 0,\ \delta\eta\gamma = 0,\ \lambda\kappa\eta = 0$
$a, b, c \geq 1$ *(at most one equal* 1*)*

We have also the following consequence of the classification of the tame symmetric algebras with all indecomposable nonprojective finite dimensional modules periodic [**ES2**].

Theorem 4.22 (Erdmann-Skowroński, 2006)**.** *Let* Λ *be a basic, indecomposable, finite dimensional symmetric, tame algebra over an algebraically closed field* K*, with all indecomposable nonprojective finite dimensional modules* Ω_Λ*-periodic. Then*

(1) *The Cartan matrix* C_Λ *of* Λ *is singular if and only if* Λ *is isomorphic to the trivial extension* $\mathrm{T}(B)$ *of a tubular algebra* B*.*
(2) *If* Λ *is representation-infinite with nonsingular Cartan matrix* C_Λ *then* Λ *has at most* 4 *isoclasses of simple modules.*
(3) *If* Λ *is representation-infinite then* Λ *has at most* 10 *isoclasses of simple modules.*

5. Periodicity and hypersurface singularities

The aim of this section is to present natural examples of periodic algebras arising in commutative algebra. For basic background on the commutative algebra considered here we refer to the books [**E**] and [**Yo**].

Let R be a commutative noetherian local ring and $\mathfrak{m}$ the maximal ideal of R. Denote by $\dim R$ the **Krull dimension** of R, that is, the length of maximal chain of prime ideals of R.

Let M be a right R-module. A sequence $x_1, \dots, x_n \in \mathfrak{m}$ is said to be a **regular sequence** on M if x_i is not a zero-divisor of $M/M(x_1, \dots, x_{i-1})$, for any $i \in$

$\{1, \ldots, n\}$. The maximal length of regular sequences on M is said to be **depth of** M and denoted by $\operatorname{depth}(M)$. Then M is said to be a **(maximal) Cohen-Macaulay** R-module if $\operatorname{depth}(M) = \dim R$. Further, R is said to be a **Cohen-Macaulay ring** if R_R is a Cohen-Macaulay R-module. Moreover, the ring R is said to be **regular (nonsingular)** if $\mathfrak{m}$ is generated by a regular sequence (equivalently, $\operatorname{gl.dim} R = \dim R$, by the **Auslander-Buchsbaum-Serre theorem**). Finally, R is said to be an **isolated singularity** if R is nonregular and the localization $R_{\mathfrak{p}}$ is regular (nonsingular) for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ of R.

Let K be an algebraically closed field and $S = K[[x_0, x_1, \ldots, x_n]]$ the formal power series K-algebra. Then S is a commutative, complete, noetherian, regular, local K-algebra with $\dim S = n+1$, and $\mathfrak{m} = (x_0, x_1, \ldots, x_n)$ is the unique maximal ideal of S. For $0 \neq f \in \mathfrak{m}^2$, the quotient algebra $R = S/(f)$ is called a **hypersurface singularity**. Then R is a commutative, complete, noetherian, local K-algebra with $\dim R = n$. The ideal

$$\mathcal{J}(f) = \left(f, \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

of $S = K[[x_0, x_1, \ldots, x_n]]$ is called the **Jacobian ideal** of f. Then it is known that $R = S/(f)$ is an **isolated hypersurface singularity** if and only if $\dim_k S/\mathcal{J}(f)$ is finite.

It has been observed by Greuel and Kröning [**GrKr**] that if $S = K[[x_0, x_1, \ldots, x_n]]$, $0 \neq f \in \mathfrak{m}^2$ and $R = S/(f)$ is an isolated hypersurface singularity then $R \cong S/(F)$ for a polynomial $F \in K[x_0, x_1, \ldots, x_n]$.

Let R be a hypersurface singularity. We denote by $CM(R)$ the category of finitely generated maximal Cohen-Macaulay R-modules. Then $CM(R)$ is a Krull-Schmidt category (unique decomposition of objects into direct sums of indecomposable objects). The hypersurface singularity R is called of **finite Cohen-Macaulay type** (shortly, **finite *CM*-type**) if $CM(R)$ has only a finite number of pairwise nonisomorphic indecomposable objects.

The following important fact was proved by Auslander in [**Au1**].

Theorem 5.1 (Auslander, 1986)**.** *Let R be a hypersurface singularity of finite CM-type. Then R is an isolated singularity.*

Let R be an isolated hypersurface singularity. Then the category $CM(R)$ has the following properties:

- $CM(R)$ is a Frobenius category (projective objects are injective), and R is a unique indecomposable projective object.
- $CM(R)$ admits Auslander-Reiten sequences (Auslander [**Au2**]).

Then $\Gamma_R = \Gamma_R(CM(R))$ is said to be the **Aulander-Reiten quiver** of R. We may also consider the stable category $\underline{CM(R)}$ of $CM(R)$, and the **stable Aulander-Reiten quiver** $\Gamma_R^s = \Gamma_R(\underline{CM(R)})$ of R (obtainded from Γ_R by deleting R and the arrows attached to R). Moreover, we have the following equivalences of functors from $\underline{CM(R)}$ to $\underline{CM(R)}$:

- $\Omega_R^2 \cong \mathrm{id}_{\underline{CM(R)}}$,
- $\tau_R \cong \mathrm{id}_{\underline{CM(R)}}$, if $\dim R$ is even,
- $\tau_R \cong \Omega_R$, if $\dim R$ is odd.

Let $R = S/(f)$ be a hypersurface singularity. Denote by $c(f)$ the set of all proper ideals I of $S = K[[x_0, x_1, \dots, x_n]]$ such that $f \in I^2$. Then R is called a **simple hypersurface singularity** if the set $c(f)$ is finite.

The following important theorem is due to Arnold [**Arn1**] (see also [**Arn2**]).

Theorem 5.2 (Arnold, 1972)**.** *Let R be a hypersurface singularity of dimension d over an algebraically closed field K of characteristic 0. Then the following statements are equivalent:*

(1) *R is a simple hypersurface singularity.*
(2) *R is of finite deformation type.*
(3) *$R \cong K[[x_0, x_1, \dots, x_d]]/(f_\Delta^{(d)})$, for a Dynkin graph Δ of type $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, where*

$$\begin{aligned}
f_{\mathbb{A}_n}^{(d)} &= x_0^2 + x_1^{n+1} + x_2^2 + \dots + x_d^2, \\
f_{\mathbb{D}_n}^{(d)} &= x_0^2 + x_1^{n-1} + x_2^2 + \dots + x_d^2, \\
f_{\mathbb{E}_6}^{(d)} &= x_0^3 + x_1^4 + x_2^2 + \dots + x_d^2, \\
f_{\mathbb{E}_7}^{(d)} &= x_0^3 + x_0 x_1^3 + x_2^2 + \dots + x_d^2, \\
f_{\mathbb{E}_8}^{(d)} &= x_0^3 + x_1^5 + x_2^2 + \dots + x_d^2.
\end{aligned}$$

Here, the finite deformation type means that R can be deformed only into finitely many other nonisomorphic singularities (see [**Arn1**] for more details).

The ring $K[[x_0, x_1, \dots, x_d]]/(f_\Delta^{(d)})$ is called the **Arnold's simple hypersurface singularity of dimension d and Dynkin type Δ.**

The following theorem proved by Buchweitz, Greuel, Schreyer [**BGS**] and Knörrer [**Kn2**] shows importance of Arnold's simple hypersurface singularities for Cohen-Macaulay modules.

Theorem 5.3 (Buchweitz-Greuel-Schreyer, Knörrer, 1985-1987)**.** *Let R be a hypersurface singularity of dimension d over an algebraically closed field K of characteristic 0. Then R is of finite Cohen-Macaulay type if and only if R is isomorphic to $K[[x_0, x_1, \dots, x_d]]/(f_\Delta^{(d)})$, for some Dynkin graph Δ.*

We will show now that the study of the categories of $CM(R)$ for hypersurface singularities R of finite Cohen-Macaulay type can be reduced to the dimensions 1 and 2. This is done by the Knörrer's and Solberg's periodicity theorems.

Let $S = K[[x_0, x_1, \dots, x_n]]$ and $R = S/(f)$ be an isolated hypersurface singularity. Consider the rings

$$S^\sharp = S[[u]] \text{ and } R^\sharp = S^\sharp/(f + u^2).$$

Then the Knörrer's periodicity theorem [**Kn2**] (see also [**Kn1**]) is as follows.

Theorem 5.4 (Knörrer, 1987)**.** *Let R be an isolated hypersurface singularity over an algebraically closed field K of characteristic $\neq 2$. Then R is of finite Cohen-Macaulay type if and only if $R^\sharp$ is of finite Cohen-Macaulay type. Moreover, if R is of finite Cohen-Macaulay type, then*

(1) *$\underline{CM(R^\sharp)} \cong \underline{CM(R)}[\mathbb{Z}_2]$ skew group category, and hence $\Gamma^s_{R^\sharp}$ is a twisted quiver of Γ^s_R.*

(2) $\underline{CM((R^\sharp)^\sharp)} \cong \underline{CM(R)}$, *and hence the translation quivers* $\Gamma^s_{(R^\sharp)^\sharp}$ *and* Γ^s_R *are isomorphic.*

Let $S = K[[x_0, x_1, \dots, x_n]]$ and $R = S/(f)$ be an isolated hypersurface singularity. Consider the ring

$$R^* = S[[u, v]]/(f + uv).$$

Then the Solberg's periodicity theorem [**So**] is as follows.

Theorem 5.5 (Solberg, 1989)**.** *Let $R = S/(f)$ be an isolated hypersurface singularity over an* ***arbitrary*** *algebraically closed field K. Then R is of finite Cohen-Macaulay type if and only if R^* is of finite Cohen-Macaulay type. Moreover, if R is of finite Cohen-Macaulay type, then there is an equivalence of categories $\underline{CM(R)} \xrightarrow{\sim} \underline{CM(R^*)}$, which induces an isomorphism of the stable Auslander-Reiten quivers $\Gamma^s_R \xrightarrow{\sim} \Gamma^s_{R^*}$.*

We note that, for K of characteristic $\neq 2$, the Solberg's periodicity is equivalent to the Knörrer's periodicity.

Let K be an algebraically closed field of characteristic 0. Consider the special linear group

$$\mathrm{SL}_2(K) = \{A \in M_{2\times 2}(K) \mid \det A = 1\}.$$

It is a classical result that every finite subgroup of $\mathrm{SL}_2(K)$ is conjugate in $\mathrm{SL}_2(K)$ to one of the following **Klein groups**

- $\mathcal{C}^*_n$, the cyclic group of order n, $n \geq 1$,
- $\mathcal{D}^*_{2n}$, the binary dihedral group of order $4n$, $n \geq 2$,
- $\mathcal{T}^*$, the binary tetrahedral group of order 24,
- $\mathcal{O}^*$, the binary octahedral group of order 48,
- $\mathcal{I}^*$, the binary icosahedral group of order 120.

Let G be a group of the above form. We associate to G a Dynkin graph $\Delta = \Delta(G)$ as follows:

- $\mathbb{A}_n = \Delta(\mathcal{C}^*_{n+1}), n \geq 1$,
- $\mathbb{D}_n = \Delta(\mathcal{D}^*_{2(n-1)}), n \geq 4$,
- $\mathbb{E}_6 = \Delta(\mathcal{T}^*)$,
- $\mathbb{E}_7 = \Delta(\mathcal{O}^*)$,
- $\mathbb{E}_8 = \Delta(\mathcal{I}^*)$.

Let G be a finite subgroup of $\mathrm{SL}_2(K)$. Then G acts on the algebra $K[[X,Y]]$ as follows: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K)$ and $f(X,Y) \in K[[X,Y]]$,

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(X,Y) &= f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}\right) \\ &= f(dX - bY, -cX + aY).\end{aligned}$$

Hence, we may consider the **invariant algebra**

$$K[[X,Y]]^G = \Big\{ f(X,Y) \in K[[X,Y]] \;\Big|\; gf(X,Y) = f(X,Y) \text{ for all } g \in \mathrm{SL}_2(K) \Big\}.$$

The following theorem is the classical result proved by Klein in his famous book on the icosahedron [**Kle**].

Theorem 5.6 (Klein, 1884). *Let K be an algebraically closed field of characteristic 0, and G a finite subgroup of $\mathrm{SL}_2(K)$. Then*

$$K[[X,Y]]^G \cong K[[x,y,z]]/(f_\Delta)$$

where $\Delta = \Delta(G)$ is the Dynkin graph of G, and

$$\begin{aligned} f_{\mathbb{A}_n} &= x^2 + y^{n+1} + z^2, \\ f_{\mathbb{D}_n} &= x^2 y + y^{n-1} + z^2, \\ f_{\mathbb{E}_6} &= x^3 + y^4 + z^2, \\ f_{\mathbb{E}_7} &= x^3 + xy^3 + z^2, \\ f_{\mathbb{E}_8} &= x^3 + y^5 + z^2. \end{aligned}$$

Hence, $f_\Delta = f_\Delta^{(2)}$ with $x = x_0$, $y = x_1$, $z = x_2$, and $K[[X,Y]]^G$ are the Arnold's simple hypersurface singularities of dimension 2. We note that, for $K = \mathbb{C}$, the orbit space $\mathbb{C}^2/G$ is a compact Riemann surface with at most 3 singular points, and the Dynkin graph $\Delta(G)$ describes the multiplicities of these singular points. We refer also to [**Len**] for a connection with the representation theory of tame hereditary algebras.

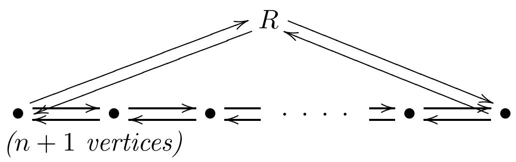
Therefore, we obtain the following result proved already by Artin-Verdier [**ArVe**], and Esnault-Knörrer [**EsKn**].

Theorem 5.7 (Artin-Verdier, Esnault-Knörrer, 1985). *Let R be a hypersurface singularity of dimension 2 over an algebraically closed field K of characteristic 0. Then R is of finite Cohen-Macaulay type if and only if $R \cong K[[X,Y]]^G$, for a finite subgroup G of $\mathrm{SL}_2(K)$.*

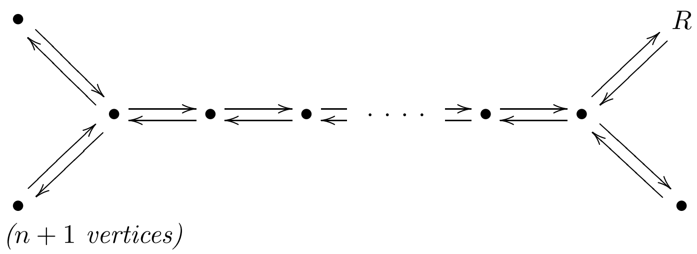
The following theorem proved by Auslander-Reiten [**AR1**], [**AR2**] describes the Auslander-Reiten quivers of the simple hypersurface singularities of dimension two in arbitrary characteristic.

Theorem 5.8 (Auslander-Reiten, 1986). *Let $R = K[[x,y,z]]/(f_\Delta)$ be an Arnold's simple hypersurface singularity of dimension 2 over an algebraically closed field K of arbitrary characteristic. Then the Auslander-Reiten quiver Γ_R is of the form*

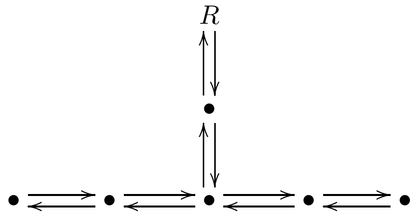
$\Delta = \mathbb{A}_n$
$n \geq 1$



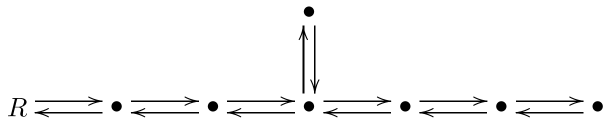
$\Delta = \mathbb{D}_n$
$n \geq 4$



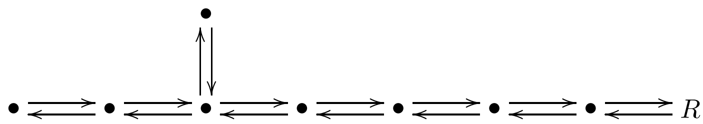
$\Delta = \mathbb{E}_6$



$\Delta = \mathbb{E}_7$



$\Delta = \mathbb{E}_8$



and, in all cases, the Auslander-Reiten translation τ_R is the identity.

Let $R = K[[x, y]]/(g_\Delta)$, where Δ is a Dynkin graph, and $g_\Delta = f_\Delta^{(1)}$, with $x = x_0$, $y = x_1$, is of the form

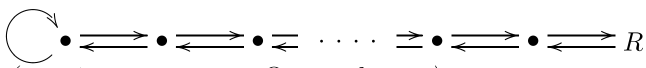
$$\begin{aligned} g_{\mathbb{A}_n} &= x^2 + y^{n+1}, \\ g_{\mathbb{D}_n} &= x^2 y + y^{n-1}, \\ g_{\mathbb{E}_6} &= x^3 + y^4, \\ g_{\mathbb{E}_7} &= x^3 + xy^3, \\ g_{\mathbb{E}_8} &= x^3 + y^5. \end{aligned}$$

Then R is called a **simple plane curve singularity**.

The following theorem has been proved by Dieterich and Wiedemann [**DW**] in characteristic $\neq 2$ and completed by Kiyek and Steineke [**KS**] in characteristic 2.

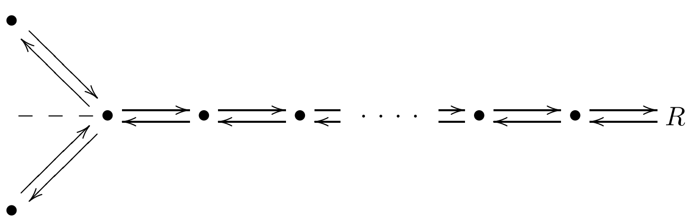
Theorem 5.9 (Dieterich-Wiedemann (1986), Kiyek-Steineke (1985))**.** *Let $R = K[[x, y]]/(g_\Delta)$ be a simple plane curve singularity over an algebraically closed field K of arbitrary characteristic. Then the Auslander-Reiten quiver Γ_R is of the form*

$\Delta = \mathbb{A}_{2m}$
$m \geq 1$



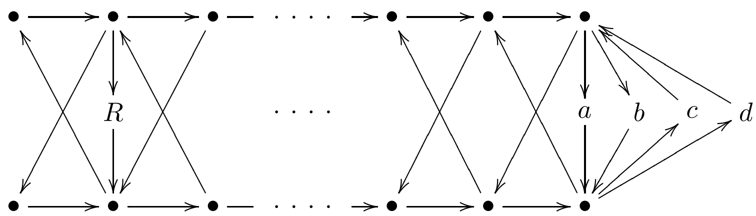
($m + 1$ vertices, $\tau_R = \Omega_R$ = identity)

$\Delta = \mathbb{A}_{2m-1}$
$m \geq 1$



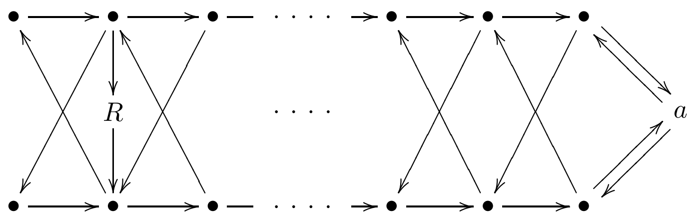
($m + 2$ vertices, $\tau_R = \Omega_R$ = reflection at the horizontal line)

$\Delta = \mathbb{D}_{2m}$
$m \geq 2$



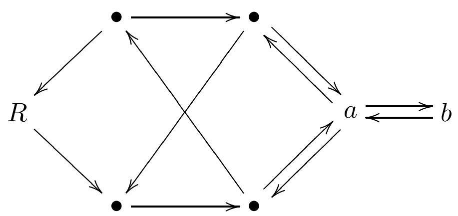
($4m+1$ *vertices,* $\tau_R = \Omega_R$ *= reflection at the horizontal line through* R, a, b, c, d*, with* $\tau_R a = d$*,* $\tau_R b = c$*,* $\tau_R^2 = \mathrm{id}$*)*

$\Delta = \mathbb{D}_{2m+1}$
$m \geq 2$



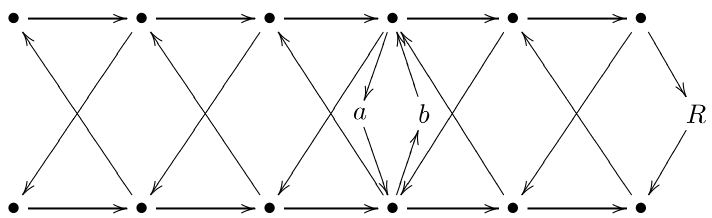
($4m$ *vertices,* $\tau_R = \Omega_R$ *= reflection at the horizontal line through* R *and* a*)*

$\Delta = \mathbb{E}_6$



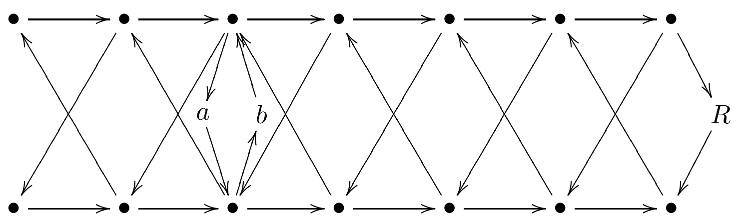
(7 vertices, $\tau_R = \Omega_R$ *= reflection at the horizontal line through* R*,* a *and* b*)*

$\Delta = \mathbb{E}_7$



(15 vertices, $\tau_R = \Omega_R$ *= reflection at the horizontal line through* R*,* a *and* b*, with* $\tau_R a = b$*,* $\tau_R b = a$*)*

$\Delta = \mathbb{E}_8$



(17 vertices, $\tau_R = \Omega_R$ *= reflection at the horizontal line through* R*,* a *and* b*, with* $\tau_R a = b$*,* $\tau_R b = a$*)*

Greuel and Kröning introduced in [**GrKr**] the concept of **finite deformation type** of hypersurface singularities for algebraically closed fields of positive characteristic and proved the theorem of the form.

Theorem 5.10 (Greuel-Kröning, 1990)**.** *Let R be a hypersurface singularity. The following statements are equivalent:*

(1) *R is a simple hypersurface singularity.*
(2) *R is of finite deformation type.*
(3) *R is of finite CM-type.*

We note that in characteristic $\neq 2, 3, 5$, the Arnold's simple hypersurface singularities are all simple hypersurface singularities.

The normal forms of simple hypersurface singularities of dimension 1 were classified by Kiyek and Steineke [**KS**].

The normal forms of simple hypersurface singularities of dimension 2 were classified by Artin [**Art1**], [**Art2**].

The normal forms of simple hypersurface singularities of dimensions ≥ 3 can be obtained from the normal forms of dimensions 1 and 2 by the Solberg's periodicity theorem (see [**So**] and [**GrKr**]).

The following theorem is a combination of results proved by Solberg in [**So**] and Greuel and Kröning in [**GrKr**].

Theorem 5.11 (Solberg (1989), Greuel-Kröning (1990))**.** *Let R be a hypersurface singularity of finite CM-type over an algebraically closed field K of arbitrary characteristic. Then the Auslander-Reiten quiver Γ_R of R is isomorphic to the Auslander-Reiten quiver of an Arnold's simple hypersurface singularity of dimension* 1 *or* 2 *(simple plane curve singularity or Kleinian singularity).*

Let R be a hypersurface singularity of finite CM-type over an algebraically closed field K of arbitrary characteristic. Then $CM(R)$ is a Frobenius category of finite type. Let $M_1, M_2, \dots, M_n$ be a complete set of pairwise nonisomorphic indecomposable nonprojective objects in $CM(R)$ and

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Then the endomorphism algebra

$$\underline{\mathcal{A}}(R) = \operatorname{End}_{\underline{CM(R)}}(\underline{M})$$

of $\underline{M} = M$ in the stable category $\underline{CM(R)}$, and called the **stable Auslander algebra** of R. For a Dynkin graph Δ, denote

$$P(\Delta) = \underline{\mathcal{A}}(K[[x, y, z]])/(f_\Delta)),$$

$$P(\Delta)^* = \underline{\mathcal{A}}(K[[x, y]])/(g_\Delta)).$$

The following theorem describes the basic properties of the algebras $P(\Delta)$ and $P(\Delta)^*$.

Theorem 5.12. *Let Δ be a Dynkin graph. The following statements hold:*

(1) *$P(\Delta)$ is a basic finite dimensional selfinjective K-algebra. Moreover, the Nakayama permutation ν of $P(\Delta)$ is the identity for $\Delta = \mathbb{A}_1$, $\mathbb{D}_n$ (n even), $\mathbb{E}_7$, $\mathbb{E}_8$, and of order 2 for $\Delta = \mathbb{A}_n$ ($n \geq 2$), $\mathbb{D}_n$ (n odd), $\mathbb{E}_6$.*
(2) *$P(\Delta)^*$ is a basic finite dimensional, symmetric K-algebra.*

It follows from the above remarks, that the stable Auslander algebra $\underline{\mathcal{A}}$ of any hypersurface singularity R of finite CM-type of even dimension (respectively, odd dimension) is isomorphic to $P(\Delta)$ (respectively, $P(\Delta)^*$), for some Dynkin graph Δ.

The algebra $P(\Delta)$ is called the **preprojective algebra of Dynkin type** Δ, and was introduced by Gelfand and Ponomarev in [**GePo**].

The algebra $P(\Delta)^*$ is called the **twisted preprojective algebra of Dynkin type** Δ.

For K of characteristic $\neq 2$, we have the Morita equivalences of

$$P(\Delta)^* \text{ and } P(\Delta)[\mathbb{Z}_2] \quad \text{(skew group algebra)},$$

$$P(\Delta) \text{ and } P(\Delta)^*[\mathbb{Z}_2] \quad \text{(skew group algebra)},$$

for the corresponding actions of $\mathbb{Z}_2$ on the algebras $P(\Delta)$ and $P(\Delta)^*$.

We also note that, with few exceptions, the algebras $P(\Delta)$ and $P(\Delta)^*$ are of **wild representation type** (see [**ES1**]).

We will show now that $P(\Delta)$ and $P(\Delta)^*$ are periodic algebras.

Let K be an algebraically closed field. Moreover, let $\mathcal{B}$ be a K-category $CM(R)$, for an isolated hypersurface singularity R over K. Then $\mathcal{B}$ is a Frobenius category with Auslander-Reiten sequences. Denote by $\mathscr{C} = \operatorname{mod} \underline{\mathcal{B}} = (\underline{\mathcal{B}}^{\mathrm{op}}, \mathbb{A}b)$ the category of finitely presented contravariant functors from the stable category $\underline{\mathcal{B}}$ of $\mathcal{B}$ to the category $\mathbb{A}b$ of abelian groups.

The following theorem due to Auslander and Reiten [**AR3**] describes the basic properties of the category $\mathscr{C}$.

Theorem 5.13 (Auslander-Reiten)**.** *The following statements hold:*

(1) *$\mathscr{C}$ is a Frobenius abelian K-category whose projective objects are the representable functors $\underline{\operatorname{Hom}}_{\mathcal{B}}(-, \underline{B})$, $\underline{B}$ objects of $\underline{\mathcal{B}}$.*

(2) *$\mathscr{C}$ admits Auslander-Reiten sequences.*

Moreover, denote by $\mathcal{N}_{\mathcal{B}}$, $\tau_{\mathcal{B}}$, $\Omega_{\mathcal{B}}$ (respectively, $\mathcal{N}_{\mathscr{C}}$, $\tau_{\mathscr{C}}$, $\Omega_{\mathscr{C}}$) the Nakayama, Auslander-Reiten and syzygy functors on $\underline{\mathcal{B}}$ (respectively, on $\underline{\mathscr{C}}$). Then we have the following two theorems proved by Auslander and Reiten in [**AR3**].

Theorem 5.14 (Auslander-Reiten, 1996)**.** *In the above notation, the following statements hold:*

(1) $\mathcal{N}_{\mathscr{C}}(\underline{\operatorname{Hom}}_{\mathcal{B}}(-, \underline{B})) = \underline{\operatorname{Hom}}_{\mathcal{B}}(-, \Omega_{\mathcal{B}}^{-1}\tau_{\mathcal{B}}(\underline{B}))$ *for any object $\underline{B}$ of $\underline{\mathcal{B}}$.*

(2) *The functors $\tau_{\mathscr{C}}, \Omega_{\mathscr{C}}^2\mathcal{N}_{\mathscr{C}}, \mathcal{N}_{\mathscr{C}}\Omega_{\mathscr{C}}^2 : \underline{\mathscr{C}} \to \underline{\mathscr{C}}$ are equivalent.*

(3) *If the functor $\Omega_{\mathcal{B}}^{-1}\tau_{\mathcal{B}} : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ has order s and the functor $\Omega_{\mathcal{B}}^2 : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ has order t, and $r = lcm(s, 3t)$, then $\tau_{\mathscr{C}}^r \xrightarrow{\sim} \mathrm{id}_{\underline{\mathscr{C}}}$.*

Theorem 5.15 (Auslander-Reiten, 1996)**.** *Let $\mathscr{C} = \operatorname{mod} \underline{CM(R)}$ for an isolated hypersurface singularity R over K. The following statements hold:*

(1) *If R has even dimension, then each indecomposable object of $\mathscr{C}$ is $\tau_{\mathscr{C}}$-periodic of period dividing* 6.

(2) *If R has odd dimension, then each indecomposable object of $\mathscr{C}$ is $\tau_{\mathscr{C}}$-periodic of period dividing* 3.

Proof. We have $\Omega_R^2 \xrightarrow{\sim} \mathrm{id}_{\underline{CM(R)}}$.

(1) If $\dim R$ is even then $\tau_R \xrightarrow{\sim} \mathrm{id}_{\underline{CM(R)}}$. Hence $\Omega_R^{-1}\tau_R = \Omega_R^{-1}$ has order 2, and so $r = lcm(2, 3 \cdot 1) = 6$.

(2) If $\dim R$ is odd then $\tau_R \xrightarrow{\sim} \Omega_R$. Hence $\Omega_R^{-1}\tau_R = \mathrm{id}_{\underline{CM(R)}}$, and so $r = lcm(1, 3 \cdot 1) = 3$.

□

Assume that R is a hypersurface singularity over K of finite CM-type. Then $CM(R)$ has only a finite number of indecomposable objects, and hence we have an equivalence

$$\operatorname{mod}\underline{CM(R)} \xrightarrow{\sim} \operatorname{mod}\underline{\mathcal{A}}(R)$$

which commutes with the Auslander-Reiten translations τ_R on $\operatorname{mod}\underline{CM(R)}$ and $\tau_{\underline{\mathcal{A}(R)}} = D\operatorname{Tr}$ on $\underline{\mathcal{A}}(R)$. Recall that $\tau_{\underline{\mathcal{A}(R)}} = \Omega^2_{\underline{\mathcal{A}(R)}}\mathcal{N}_{\underline{\mathcal{A}(R)}}$. We also note that the algebra $P(\Delta)$ (respectively, $P(\Delta)^*$) is semisimple if and only if $\Delta = \mathbb{A}_1$.

Therefore, we obtain we obtain the following periodicity theorem proved by Auslander and Reiten in [**AR3**].

Theorem 5.16 (Auslander-Reiten, 1996)**.** *Let Δ be a Dynkin graph $\neq \mathbb{A}_1$. The following statements hold:*

(1) $\tau^6_{P(\Delta)} \cong 1_{\underline{\operatorname{mod}}P(\Delta)}$, $\Omega^3_{P(\Delta)} \cong \mathcal{N}^{-1}_{P(\Delta)}$ *and* $\Omega^6_{P(\Delta)} \cong 1_{\underline{\operatorname{mod}}P(\Delta)}$ *as functors on* $\underline{\operatorname{mod}}P(\Delta)$.
(2) $\tau^3_{P(\Delta)^*} \cong 1_{\underline{\operatorname{mod}}P(\Delta)^*}$ *and* $\Omega^6_{P(\Delta)^*} \cong 1_{\underline{\operatorname{mod}}P(\Delta)^*}$ *as functors on* $\underline{\operatorname{mod}}P(\Delta)^*$.

In fact, we have the following theorem proved by Schofield [**Scho**] and Erdmann and Snashall [**ESn**].

Theorem 5.17 (Schofield (1990), Erdmann-Snashall (1998))**.** *Let Δ be a Dynkin graph $\neq \mathbb{A}_1$. Then $\Omega^6_{P(\Delta)^e}P(\Delta) \cong P(\Delta)$ in $\operatorname{mod}P(\Delta)^e$.*

Moreover, we have the following recent result proved by Białkowski, Erdmann and Skowroński [**BES1**], [**BES2**].

Theorem 5.18 (Białkowski-Erdmann-Skowroński (2006))**.** *Let Δ be a Dynkin graph $\neq \mathbb{A}_1$. Then $\Omega^6_{(P(\Delta)^*)^e}P(\Delta)^* \cong P(\Delta)^*$ in $\operatorname{mod}(P(\Delta)^*)^e$.*

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