THE DIFFERENCE BETWEEN THE LITTLE AND THE BIG FINITISTIC DIMENSIONS

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ABSTRACT. This note is an expanded version of the article of Sverre O. Smalø "The supremum of the difference between the big and the little finitistic dimensions is infinite" [3].

1. Introduction

Our aim is to show that, in general, the little and the big (left) finitistic dimensions are different, and that their difference can be arbitrarily large. For an overview on the finitistic dimensions conjectures we refer to [6].

All modules, unless otherwise stated, are left modules, $\mathsf{Mod}\Lambda$ denotes the category of modules over Λ and $\mathsf{mod}\Lambda$ the category of modules of finite type. Recall that the left little finitistic dimension $\mathsf{fin.dim}\Lambda$ of an algebra Λ is defined as

$$\mathsf{fin.dim}\Lambda = \sup\{\mathsf{p.dim}M | M \in \mathsf{mod}\Lambda \text{ with } \mathsf{p.dim}\mathrm{M} < \infty\}$$

and the big little finitistic dimension $\mathsf{Fin.dim}\Lambda$ as

$$\mathsf{Fin.dim}\Lambda = \sup\{\mathsf{p.dim}M | M \in \mathsf{Mod}\Lambda \text{ with } \mathsf{p.dim}\mathrm{M} < \infty\}.$$

Put $\delta(A) = \mathsf{Fin.dim}\Lambda - \mathsf{fin.dim}\Lambda$. We will give two different proofs of the fact that this difference can be arbitrarily large:

- The first approach is based on work by Birge Huisgen-Zimmermann: In her paper [5], she showed that for monomial relation algebras Λ, we have $\delta(\Lambda) \leq 1$ and exhibited a monomial relation algebra Δ such that $\delta(\Delta) = 1$. This was the first counterexample to Bass' conjecture which claimed that $\delta(\Lambda)$ vanishes for all finite-dimensional algebras Λ.

  Rickard has shown that both finitistic dimensions are additive with respect to tensor products of algebras (if the ground field is perfect). Therefore δ is additive as well and we have

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$\delta(\Delta^{\otimes n}) = n$. So the difference between the two dimensions can indeed be arbitrarily large.

- The second approach is direct: We construct a sequence of algebras Λ_n such that $\delta(\Lambda_n) = n-1$. The computation of the big finitistic dimension will be based on the fact that for $n \geq 3$, Λ_n is a one point extension of Λ_{n-1} by an injective module.

The algebra Λ_n has a simpler structure than $\Delta^{\otimes n}$: Indeed the cube of the radical of Λ_n vanishes and its little finitistic dimension equals 1. The algebra $\Delta^{\otimes n}$ has little finitistic dimension equal to $2n$ and non vanishing radical cube even for $n = 1$. Indeed, it is known that for a monomial relation algebra with vanishing radical cube, the two dimensions coincide.

2. Additivity of the finitistic dimensions with respect to the tensor product of algebras

The algebras Δ_n will be constructed as tensor powers of the example of Huisgen-Zimmermann using the following theorem due to Jeremy Rickard.

Theorem 2.1. *Let Λ_1 and Λ_2 be two finite dimensional algebras over a an algebraically closed field k. We have the following equalities:*

(1) $\mathsf{fin.dim}\Lambda_1 \otimes \Lambda_2 = \mathsf{fin.dim}\Lambda_1 + \mathsf{fin.dim}\Lambda_2$
(2) $\mathsf{Fin.dim}\Lambda_1 \otimes \Lambda_2 = \mathsf{Fin.dim}\Lambda_1 + \mathsf{Fin.dim}\Lambda_2$

Corollary 2.2. *The supremum of the difference between the big and little finitistic dimensions is infinite.*

Proof of the corollary

The algebra $\Delta_n = \Delta^{\otimes n}$ has little finitistic dimension and big finitistic dimension differing by n.

Proof of the theorem

The facts from homological algebra used here can be found in [4].

Denote by $\mathsf{p}^{\Delta}U$ a projective resolution of a Δ-module U and by Λ the tensor product algebra $\Lambda_1 \otimes \Lambda_2$.

Recall the following facts :

(1) An arbitrary module M over a finite dimensional algebra A has projective dimension less or equal than l if and only if $\mathsf{Tor}^{A}_{l+1}(M, S)$ vanishes for all simple A-modules S.

(2) Let A and B be two finite dimensional algebras over an algebraically closed field. The simple modules over $A \otimes B$ are isomorphic to $S \otimes T$, where S are the simple A-modules and T the simple B-modules.

Suppose M is a Λ_1-module of projective dimension m and N is a Λ_2-module of projective dimension n.

For a Λ_1-module X_1 and a Λ_2-module X_2 , we have the isomorphism

$$\mathsf{Hom}_{\Lambda}(\mathsf{p}^{\Lambda_1} M \otimes \mathsf{p}^{\Lambda_2} N, X_1 \otimes X_2) \simeq \mathsf{Hom}_{\Lambda_1}(\mathsf{p}^{\Lambda_1} M, X_1) \otimes \mathsf{Hom}_{\Lambda_2}(\mathsf{p}^{\Lambda_2} N, X_2),$$

(since it is an isomorphism for $M = \Lambda_1$, $N = \Lambda_2$) hence an isomorphism

$$\mathsf{Ext}^p_{\Lambda}(M \otimes N, X_1 \otimes X_2) \simeq \oplus_{i+j=p} \mathsf{Ext}^i_{\Lambda_1}(M, X_1) \otimes \mathsf{Ext}^j_{\Lambda_2}(N, X_2).$$

This shows that the projective dimension of the Λ-module $M \otimes N$ is $m + n$, so we have the inequalities

(1) $\mathsf{fin.dim}\Lambda_1 \otimes \Lambda_2 \geq \mathsf{fin.dim}\Lambda_1 + \mathsf{fin.dim}\Lambda_2$,
(2) $\mathsf{Fin.dim}\Lambda_1 \otimes \Lambda_2 \geq \mathsf{Fin.dim}\Lambda_1 + \mathsf{Fin.dim}\Lambda_2$.

Let $\mathsf{fin.dim}\Lambda_1 = m$, $\mathsf{fin.dim}\Lambda_2 = n$ and X be a finite dimensional module over Λ. Since a projective Λ-resolution of $\mathsf{p}^{\Lambda} X \to X$ restricts to a projective Λ_1-resolution, the Λ_1-restriction of the m'th-syzygy $\Omega = \Omega_m X$ in the resolution $\mathsf{p}X$ is a projective Λ_1-module. Let us show that the Λ-module Ω has Λ-projective dimension n.

Let us compute $\mathsf{Tor}^{\Lambda}_*(\Omega, S \otimes T)$ where S is a simple Λ_1-module and T a simple Λ_2-module.

Let S be a simple Λ_1-module and T be a simple Λ_2-module. A complex to compute the functors $\mathsf{Tor}^{\Lambda}_*(\Omega, S \otimes_k T)$ is given by

$$\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1 \otimes \Lambda_2} (\mathsf{p}^{\Lambda_1} S \otimes_k \mathsf{p}^{\Lambda_2} T)$$

which is trivially isomorphic to $(\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1} \mathsf{p}^{\Lambda_1} S) \otimes_{\Lambda_2} \mathsf{p}^{\Lambda_2} T$. Let us show that the last one is quasi-isomorphic to

$$(\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1} S) \otimes_{\Lambda_2} \mathsf{p}^{\Lambda_2} T.$$

Let V be a Λ_2-module and $i \geq 0$. We deduce from the isomorphism

$$((\mathsf{p}^{\Lambda}\Omega)_i \otimes_{\Lambda_1} S) \otimes_{\Lambda_2} V \simeq (\mathsf{p}^{\Lambda}\Omega)_i \otimes_{\Lambda_1 \otimes \Lambda_2} (S \otimes_k V)$$

that the components $(\mathsf{p}^{\Lambda}\Omega)_i \otimes_{\Lambda_1} S$ of the Λ_2-complex $\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1} S$ are flat Λ_2-modules (of finite type) and hence projective. Thus $\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1} S$ is a projective Λ_2-resolution of $\Omega \otimes S$ and is then homotopically equivalent to the complex $\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1} \mathsf{p}^{\Lambda_1} S$ as Λ_2-modules. Tensorising over Λ_2 by $\mathsf{p}^{\Lambda_2} T$, this gives us the announced isomorphism above.

Since the complex $(\mathsf{p}^{\Lambda}\Omega \otimes_{\Lambda_1} S) \otimes_{\Lambda_2} \mathsf{p}^{\Lambda_2} T$ computes the derived functors $\mathsf{Tor}^{\Lambda_2}_*(\Omega \otimes_{\Lambda_1} S, T)$, this shows that $\mathsf{Tor_i}^{\Lambda_2}(\Omega \otimes_{\Lambda_1} S, T) \simeq \mathsf{Tor_i}^{\Lambda}(\Omega, S \otimes_k T)$ and hence we find that this functor vanishes for $i = n$. So the m'th syzygy of the Λ-module X has projective dimension at most n and hence $\mathsf{p.dim}^{\Lambda}(X) \leq m+n$. The assertion 1. of the theorem is proven.

The main ingredient for the above proof is that projective and flat finite dimensional modules coincide. For infinite dimensional modules, we know that, for artinian algebras, flatness is equivalent to projectivity, see for example [1]. Since finite dimensional algebras are artinian, we argue as above to prove the assertion 2.

3. The example

For each $n > 1$ we construct an algebra Λ_n whose little and big finitistic dimensions differ at least by $n-1$.

The algebra Λ_n will be given as the path algebra of a quiver modulo relations. We will obtain a lower bound for the big finitistic dimension using the fact that Λ_n is an iterated one-point-extension by injective modules.

Let k be any field and Γ_n be the quiver

$$\begin{tikzcd}
n \arrow[r, bend left, "\rho_n"] \arrow[r, "\sigma_n"] \arrow[r, bend right, "\tau_n"'] & n-1 \arrow[r, bend left, "\rho_{n-1}"] \arrow[r, "\sigma_{n-1}"] \arrow[r, bend right, "\tau_{n-1}"'] & n-2 & \cdots & 3 \arrow[r, bend left, "\rho_3"] \arrow[r, "\sigma_3"] \arrow[r, bend right, "\tau_3"'] & 2 \arrow[r, bend left, "\rho_2"] \arrow[r, "\sigma_2"] \arrow[r, bend right, "\tau_2"'] & 1 \arrow[r, bend left, "\rho_1"] \arrow[r, "\sigma_1"] \arrow[r, bend right, "\tau_1"'] & 0 \arrow[loop above, "\alpha"] \arrow[loop below, "\beta"]
\end{tikzcd}.$$

Let Λ_n be the path algebra of Γ_n over the field k modulo the ideal generated by the following relations : $\alpha^2, \beta^2, \alpha\beta, \beta\alpha, \alpha\rho_1, \alpha\sigma_1, \beta\tau_1, x_i y_{i+1}$ for $i = 1, \ldots, n-1$ and $x = y$ and $x_i x_{i+1} - y_i y_{i+1}$, where $x, y \in \{\sigma, \tau, \rho\}$ and $i = 1, \ldots, n-1$. Let $\mathcal{C}$ be the path category, i.e. the category obtained from the quiver Γ_n the following way: A path from i to j in Γ is a formal composition of arrows starting in i and ending in j. If $i = j$, we also admit the zero length path. The category $\tilde{\mathcal{C}}$ has the vertices of Γ as its objects; the set of morphisms from i to j is the *vector space* whose basis is the set of paths from i to j; the composition is induced by the composition of paths. The category $\mathcal{C}$ is the quotient of $\tilde{\mathcal{C}}$ by the ideal generated by the relations. The category of left modules over the path algebra Λ_n is equivalent to the category of functors $\mathcal{C} \to \mathsf{Vect}\,k$ where $\mathsf{Vect}k$ is the category of k-vector spaces.

Theorem 3.1. *For each natural number $n \geq 1$, the algebra Λ_n given above has*

1. *left little finitistic dimension equal to 1,*
2. *left big finitistic dimension equal to n.*

Reminder on projective modules and minimal resolutions(for details see [2])

Each vertex i of the quiver yields a minimal idempotent e_i in the path algebra. This idempotent corresponds to the identity of i in $\mathcal{C}$. By the Yoneda lemma, the functor $i \mapsto P_i = \Lambda_n e_i \simeq \mathsf{Hom}_{\mathcal{C}}(i, ?)$ is an equivalence from the category $\mathcal{C}^{op}$ to the category of indecomposable projective left modules. The simple left module $S_i = P_i/\mathfrak{r}P_i \simeq$ (where $\mathfrak{r}$ denotes the radical of the path algebra Λ_n whose generators are the arrows of the quiver Γ_n) has projective cover P_i. The *socle* $\mathsf{soc}M$ of a module M is the biggest semi-simple submodule of M. The *Loewy length* of a module M is the minimal number of extensions of semi-simple modules needed to obtain M. Since the biggest semi-simple quotient of $\mathfrak{r}^i M$ is $\mathfrak{r}^i M/\mathfrak{r}^{i+1}M$, the Loewy length can also be viewed as the first exponent n for which $\mathfrak{r}^n M$ equals zero. Note that $\mathfrak{r}^{n-1}M$ is contained in the socle of M. A *minimal projective resolution* of a module M is an exact sequence

$$0 \to Q_m \to Q_{m-1} \to \ldots \to Q_1 \to Q_0 \to M \to 0$$

where the Q_j are projective modules and such that any other projective resolution admits a epimorphism onto it. Hence minimal resolution are unique up to isomorphisms. By minimality, the module Q_{i-1} is a projective cover of $\mathsf{ker}d_{i-2}$ (i.e. the morphism $Q_{i-1} \twoheadrightarrow \mathsf{ker}d_{i-2}$ restricts to an isomorphism $Q_{i-1}/\mathfrak{r}Q_{i-1} \simeq \mathsf{ker}d_{i-2}/\mathfrak{r}\mathsf{ker}d_{i-2}$) and, thus, the composition $Q_i \to Q_{i-1} \to Q_{i-1}/\mathfrak{r}Q_{i-1}$ equals zero because the composition

$$Q_i \to Q_{i-1} \to \mathsf{ker}d_{i-2} \to \mathsf{ker}d_{i-2}/\mathfrak{r}\mathsf{ker}d_{i-2}$$

equals also zero. Now suppose that, in a resolution, each morphism $d_i : Q_i \to Q_{i-1}$ factors through $\mathfrak{r}Q_{i-1}$. The projective Q_{i-1} and the cokernel of d_i must then have isomorphic biggest semi-simple quotient, hence Q_{i-1} is the projective cover of $\mathsf{coker}d_i$. This gives us a minimality caracterisation : namely, each morphism $d_i : Q_i \to Q_{i-1}$ factors through $\mathfrak{r}Q_{i-1}$.

Computation of the left little finitistic dimension

All modules considered during this computation are of finite type. We will show in the next paragraph that the syzygy of a module can not have projective dimension one. Suppose that this has been proven and that we have a minimal resolution of a module M

$$0 \to Q_m \to Q_{m-1} \to \ldots \to Q_1 \to Q_0 \to M \to 0$$

with $m > 1$. For each $n > 1$, the cokernel of d_n is the syzygy of $\mathsf{coker}\, d_{n-1}$. Since the resolution above is minimal, the sequence $Q_m \to Q_{m-1} \to \mathsf{coker} d_m$ is a minimal resolution of $\mathsf{coker} d_m$. This contradicts the fact that it can not have projective dimension 1. Hence a minimal resolution of a non projective M of finite projective dimension should be of the form $Q_1 \to Q_0 \to M$. This proves the first statement of the theorem.

The Loewy length of the projective module P_0 equals 2 : the socle of P_0 is $S_0 \oplus S_0$ generated as k-vector space by α and β. The quotient of P_0 by its socle is the simple module $S_0 = P_0/\mathfrak{r}P_0$. The Loewy length of P_1 is 3 :

$$P_1 \simeq \begin{pmatrix} P_1/\mathfrak{r}P_1 \\ \mathfrak{r}P_1/\mathfrak{r}^2 P_1 \\ \mathfrak{r}^2 P_1 \end{pmatrix} \simeq \begin{pmatrix} S_1 \\ S_0 \oplus S_0 \oplus S_0 \\ S_0 \oplus S_0 \oplus S_0 \end{pmatrix} \simeq \begin{pmatrix} ke_1 \\ k\tau_1 \oplus k\sigma_1 \oplus k\rho_1 \\ k\alpha\tau_1 \oplus k\beta\rho_1 \oplus k\beta\sigma_1 \end{pmatrix}.$$

The Loewy length of P_i for $i > 1$ equals 3 :

$$P_i \simeq \begin{pmatrix} S_i \\ S_{i-1} \oplus S_{i-1} \oplus S_{i-1} \\ S_{i-2} \end{pmatrix} \simeq \begin{pmatrix} ke_i \\ k\tau_i \oplus k\sigma_i \oplus k\rho_i \\ k\sigma_{i-1}\sigma_i \end{pmatrix}.$$

This shows that the Loewy length of any projective is at most 3. Suppose we have an inclusion of a projective P into the radical of another projective Q. Since the Loewy length of $\mathfrak{r}Q$ is 2, and that if $M \subset N$ then the Loewy length of N is at least that of M, the projective P needs must have Loewy length 2 and is then the direct sum of copies of the projective P_0. However, the only non zero morphisms of source P_0 have targets P_0, P_1 and P_2. So the inclusion $P \to Q$ factors through an inclusion

$$f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} : P = P_0^r \hookrightarrow P_0^u \oplus P_1^v \oplus P_2^w$$

whose image is contained in the radical. Let us show that f_1 is a monomorphism. The morphism f_0 (resp. f_2) is a linear combination of the morphisms α^* and β^* (resp. of $(\sigma_1\sigma_2)^*$), so its image contained in $\mathfrak{r}P_0^u$ (resp. in $\mathfrak{r}^2 P_2^w$) is semi-simple. Hence f_0 and f_2 factor through the quotient $P \to P/\mathfrak{r}P$ and the morphism f through $P \to P/\mathfrak{r}P \oplus P_1^v$. Since it is an inclusion, the kernel of $f_1 : P_0^r \to P_1^v$ has a zero intersection with $\mathfrak{r}P$ which also happens to be the socle of P, so it is also zero and f_1 is an inclusion. Now notice that $\mathsf{dim}\alpha P_0^r$ is r and that the dimension of $\alpha\mathfrak{r}P_1^v$ (which contains $\alpha\mathsf{Im} f_1$) is v. So in order to have an inclusion f_1, we must have the inequality $v \geq r$. Now, the socle of P_1^v, whose dimension is $3v$, can not be contained in the image of $\mathsf{soc}P_0^r$ (of dimension $2r$). This implies that the Loewy length of the cokernel of f_1 (and hence of f) is 3. Now recall that the radical of any projective has Loewy length equal to 2. This implies that no submodule of the radical of a projective (e.g. a syzygy) can have projective dimension 1.

Computation of the left big finitistic dimension

We denote the set of strictly positive integers by $\mathbf{N}$. Let $e_{0,i}$ be the element of $P_0^{(\mathbf{N})}$ with e_0 in the i'th place and 0 elsewhere. These elements form a set of Λ_n-generators of $P_0^{(\mathbf{N})}$. In the same way, define the elements $e_{1,i}$ of $P_1^{(\mathbf{N})}$. Let $\phi_1 : P_0^{(\mathbf{N})} \to P_1^{(\mathbf{N})}$ be the morphism of Λ_n-modules defined by $\phi_1(e_{0,2i-1}) = \tau_1 e_{1,2i-1} + \sigma_1 e_{1,i}$ and $\phi_1(e_{0,2i}) = \tau_1 e_{1,2i} + \rho_1 e_{1,i}$. The map ϕ_1 sends the elements

$$\begin{gathered} \alpha e_{0,2i-1} \text{ to } \alpha\tau_1 e_{1,2i-1}, \\ \alpha e_{0,2i} \text{ to } \alpha\tau_1 e_{1,2i}, \\ \beta e_{0,2i-1} \text{ to } \beta\sigma_1 e_{1,i} \\ \text{and } \beta e_{0,2i} \text{ to } \beta\rho_1 e_{1,i}. \end{gathered}$$

The matrix of ϕ_1 with respect to the generators $e_{0,j}$ and $e_{1,i}$ is :

$$\begin{bmatrix} \tau_1+\sigma_1 & \rho_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \tau_1 & \sigma_1 & \rho_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \tau_1 & 0 & \sigma_1 & \rho_1 & 0 & \dots \\ 0 & 0 & 0 & \tau_1 & 0 & 0 & \sigma_1 & \dots \\ 0 & 0 & 0 & 0 & \tau_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \tau_1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This shows that the morphism ϕ_1 is injective. Its cokernel, denoted by X_1, is the module $P_1^{(\mathbf{N})}$ divided by the relations :

$$\alpha\tau_1 e_{1,i}, \beta\sigma_1 e_{1,i}, \beta\rho_1 e_{1,i}, \tau_1 e_{1,2i-1} + \sigma_1 e_{1,i}, \tau_1 e_{1,2i} + \rho_1 e_{1,i}.$$

Hence the residue classes of $e_{1,i}$ and $\tau_1 e_{1,i}$ form a k-basis of the cokernel X_1. Clearly it is annihilated by the residue classes of α and β and its socle is the submodule generated by the $\tau_1 e_{1,i}$ and is isomorphic to $S_0^{(\mathbf{N})}$. The quotient of X_1 by its socle is isomorphic to $S_1^{(\mathbf{N})}$.

Let us analyze symmetry and repetition occurring in the quiver Λ_n. One can show that for $n \geq 3$ the monomial relation algebra Λ_n is obtained from Λ_{n-1} as a one point extension by the injective envelope E of S_{n-2} considered as a Λ_{n-1}-module. This means that the algebra

Λ_n is isomorphic to the 2×2-matrix algebra

$$\tilde{\Lambda_n} = \begin{bmatrix} \Lambda_{n-1} & E \\ 0 & k \end{bmatrix}.$$

The injective envelope E is the dual of the projective cover $\mathsf{Hom}(?, n-2)$ of the right module S^*_{n-2}. We denote its k-generators by $\rho^o_{n-1}, \sigma^o_{n-1}, \tau^o_{n-1}$ and e^o_{n-1}. The isomorphism $\Lambda_n \to \tilde{\Lambda_n}$ associates the generator of the vector space k with the idempotent e_n and maps

$$\begin{gathered} \rho_n \text{ to } \rho^o_{n-1}, \\ \sigma_n \text{ to } \sigma^o_{n-1}, \\ \tau_n \text{ to } \tau^o_{n-1} \\ \text{and } \rho_n\rho_{n-1} \text{ to } e^o_{n-1}. \end{gathered}$$

This permits us to make the following construction : suppose we have a morphism of Λ_{n-1}-modules $P \to C$ where P is projective and where the socle of C is $S^{(N)}_{n-2}$. Since $E^{(N)}$ is also the injective envelope of $S^{(N)}_{n-2}$, the inclusion of $S^{(N)}_{n-2}$ into $E^{(N)}$ extends to another inclusion $C \hookrightarrow E^{(N)}$. In this way, one obtains morphisms of Λ_{n-1}-modules

$$P \to C \to E^{(N)}$$

and by composition

$$P \to E^{(N)}.$$

Now we can construct a morphism of projective Λ_n-modules by replacing the module P with the column module

$$\begin{bmatrix} P \\ 0 \end{bmatrix}$$

(which is again projective over Λ_n !) and by replacing E with the projective Λ_n-module (isomorphic to P_n)

$$\begin{bmatrix} E \\ k \end{bmatrix}.$$

One can apply the same construction to a morphism of Λ_1-modules $P \to C$ where P is projective and where the socle of C is $S^{(N)}_0$ noting that the algebra Λ_2 is the one point extension of Λ_1 by the injective $\Lambda_1/(\alpha, \beta)$-envelope of S_0 considered as a Λ_1-module (here α and β denote their residue classes in Λ_1).

Now we apply the above construction to the morphism $P_1 \to X_1$. We get a monomorphism $X_1 \to P_2$. To see what is happening precisely let us analyze P_2 in terms of extensions of simple modules. The socle of P_2 is the k-vector space with basis $\sigma_1\sigma_2$ and is isomorphic to S_0. The quotient of P_2 by its socle has socle $S_1 \oplus S_1 \oplus S_1$ generated as a vector space by the residue classes of σ_2, τ_2, ρ_2. The embedding of X_1 into $P_2^{(N)}$ sends the element

$$e_{1,i} \text{ to } \tau_2 e_{2,i} - \sigma_2 e_{2,2i-1} - \rho_2 e_{2,2i}.$$

Its matrix is

$$\begin{bmatrix} \tau_2 - \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\rho_2 & \tau_2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -\sigma_2 & \tau_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & -\rho_2 & 0 & \tau_2 & 0 & 0 & 0 & \dots \\ 0 & 0 & -\sigma_2 & 0 & \tau_2 & 0 & 0 & \dots \\ 0 & 0 & -\rho_2 & 0 & 0 & \tau_2 & 0 & \dots \\ 0 & 0 & 0 & -\sigma_2 & 0 & 0 & \tau_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By the same kind of arguments as for ϕ_1, one can show that the cokernel of ϕ_2 (denoted by X_2) has socle $S_1^{(\mathbf{N})}$. At the level of composition factors, one has the diagram

$$0 \to X_1 \simeq \left(\begin{array}{c} S_1^{(\mathbf{N})} \\ S_0^{(\mathbf{N})} \end{array} \right) \xrightarrow{\phi_2} P_2^{\mathbf{N})} \simeq \left(\begin{array}{c} S_2 \\ S_1^{(3)} \\ S_0 \end{array} \right)^{(\mathbf{N})} \twoheadrightarrow X_2 \simeq \left(\begin{array}{c} S_2^{(\mathbf{N})} \\ S_1^{(\mathbf{N})} \end{array} \right) \to 0.$$

Now we can embed X_2 into P_3 using the same arguments as we did for the embedding of X_1 into P_2. We get a morphism ϕ_3 whose cokernel X_3 has socle $S_1^{(\mathbf{N})}$ and the quotient of X_3 by its socle is isomorphic to $S_2^{(\mathbf{N})}$. This process can be repeated and then leads to an exact sequence of modules

$$0 \to P_0^{(N)} \to P_1^{(N)} \to \dots \to P_{n-1}^{(N)} \to P_n^{(N)} \to X_n \to 0.$$

This sequence gives rise to a minimal projective resolution of X_n since for each i we have isomorphisms $(P_i)^{(\mathbf{N})}/\mathfrak{r}(P_i)^{(\mathbf{N})} \simeq X_i/\mathfrak{r}X_i \simeq S_i^{(\mathbf{N})}$. This shows that the big finitistic dimension of Λ_n is at least n.

To prove the other inequality, let us look at the algebra $\tilde{\Lambda_n}$ corresponding to the paths algebra of the subquiver of Γ_n

$$\begin{tikzcd} n \arrow[r, bend left, "\rho_n"] \arrow[r, "\sigma_n"] \arrow[r, bend right, "\tau_n"'] & n-1 \arrow[r, bend left, "\rho_{n-1}"] \arrow[r, "\sigma_{n-1}"] \arrow[r, bend right, "\tau_{n-1}"'] & n-2 & \cdots & 3 \arrow[r, bend left, "\rho_3"] \arrow[r, "\sigma_3"] \arrow[r, bend right, "\tau_3"'] & 2 \arrow[r, bend left, "\rho_2"] \arrow[r, "\sigma_2"] \arrow[r, bend right, "\tau_2"'] & 1 \end{tikzcd}$$

divided by the same relations (when they make sense). The restriction $\tilde{}\ : \mathsf{Mod}\Lambda_n \to \mathsf{Mod}\tilde{\Lambda_n}$ is exact, sends projectives to projectives and radical morphisms to radical morphisms. The restriction of a minimal projective resolution of a Λ_n-module M

$$\mathsf{p}M_* = (\ldots \to Q_{n+1} \to Q_n \xrightarrow{\delta_n} Q_{n-1} \to \ldots Q_0 \to M \to 0)$$

is thus a minimal projective resolution of the restriction of M. Suppose we know that the big finitistic dimension of $\tilde{\Lambda_n}$ is less or equal than $n-1$. This means that the restriction of $\mathsf{p}M_*$ must be such that $\tilde{Q}_l = 0$ for $l \geq n$. Hence the projective module Q_l (for $l \geq n$) in a minimal Λ_n-resolution is a direct sum of copies of P_0. Now suppose the morphism δ_n is not an injection. This means that its kernel will contain a copy of S_0 in its biggest semi-simple quotient. Since S_0 is of infinite projective dimension, so is the module M.
We have shown that, for a module M of finite projective dimension, the differential δ_n must be injective, i.e. $\mathsf{p.dim} M \leq n$.

The proof of the theorem will be complete when we will have proven that $\mathsf{Fin.dim}\tilde{\Lambda_n} \leq n-1$. By definition, a minimal projective resolution of a $\tilde{\Lambda_n}$-module M has projective components and radical differential. Any radical morphism between projectives is given by a linear combination of paths of non-zero length. Thus, the coefficients of the matrix (with respect to the projective components of the Q_i's) of a differential $\delta_i : Q_i \to Q_{i-1}$ in a minimal resolution are of the form $P_j \to P_k$ with $j < k$. Hence the maximal length of a minimal projective resolution can not exceed $n-1$ and there no module of infinite projective dimension.

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