CLUSTER-TILTED ALGEBRAS ARE GORENSTEIN AND STABLY CALABI-YAU

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Abstract. We prove that in a 2-Calabi-Yau triangulated category, each cluster tilting subcategory is Gorenstein with all its finitely generated projectives of injective dimension at most one. We show that the stable category of its Cohen-Macaulay modules is 3-Calabi-Yau. We deduce in particular that cluster-tilted algebras are Gorenstein of dimension at most one, and hereditary if they are of finite global dimension. Our results also apply to the stable (!) endomorphism rings of maximal rigid modules of [27]. In addition, we prove a general result about relative 3-Calabi-Yau duality over non stable endomorphism rings. This strengthens and generalizes the Ext-group symmetries obtained in [27] for simple modules. Finally, we generalize the results on relative Calabi-Yau duality from 2-Calabi-Yau to d-Calabi-Yau categories. We show how to produce many examples of d-cluster tilted algebras.

1. Introduction

Let k be a field and H a finite-dimensional hereditary algebra. The associated cluster category $\mathcal{C}_H$ was introduced in [8] and, for algebras H of Dynkin type A_n, in [17]. It serves in the representation-theoretic approach to cluster algebras introduced and studied by Fomin-Zelevinsky in a series of articles [21] [22] [10]. We refer to [23] for more background on cluster algebras and to [1] [2] [12] [13] [14] [15] [16] [18] [19] [28] [33] [34] [46] [47] for some recent developments in the study of their links with representation theory.

The cluster category $\mathcal{C}_H$ is the orbit category of the bounded derived category of finite-dimensional right H-modules under the action of the unique autoequivalence F satisfying

$$D\,\mathsf{Hom}(X, SY) = \mathsf{Hom}(Y, SFX)\ ,$$

where D denotes the duality functor $\mathsf{Hom}_k(?, k)$ and S the suspension functor of the derived category. The functor F is τS^{-1}, where τ is the Auslander-Reiten translation in the bounded derived category, or alternatively, F is ΣS^{-2}, where Σ is the Serre functor of the bounded derived category. The cluster category is triangulated and Calabi-Yau of CY-dimension 2, *cf.* [38]. We still denote by S its suspension functor. A cluster-tilted algebra [9] is the endomorphism algebra of a cluster tilting object T of $\mathcal{C}_H$; this means that T has no self-extensions, *i.e.*

$$\mathsf{Hom}(T, ST) = 0\ ,$$

and any direct sum $T \oplus T'$ with an indecomposable T' not occuring as a direct summand of T does have selfextensions.

We will prove that each cluster-tilted algebra is Gorenstein in the sense of [5], *cf.* also [30], and that its projective objects are of injective dimension ≤ 1 (hence its injective objects are of projective dimension ≤ 1). We deduce that if it has finite global dimension, then it is hereditary. In particular, if it is given as the quotient of a directed quiver algebra by an

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admissible ideal, then this ideal has to vanish. We will also show that the stable category of Cohen-Macaulay modules over a cluster-tilted algebra is Calabi-Yau of CY-dimension 3.

We work in a more general framework which also covers the stable (!) endomorphism algebras of the maximal rigid modules of [27]. In our framework, the triangulated category $\mathcal{C}_H$ is replaced by an arbitrary triangulated category $\mathcal{C}$ with finite-dimensional Hom-spaces and which is Calabi-Yau of CY-dimension 2, *i.e.* we have

$$D\,\mathsf{Hom}(X,Y)=\mathsf{Hom}(Y,S^2X)\ ,\ X,Y\in\mathcal{C}.$$

This holds for cluster categories (including those coming from Ext-finite hereditary abelian categories with a tilting object, *cf.* [8] [38], and also those coming from the Ext-finite hereditary abelian k-categories in [44], [43], see [38]), for stable module categories of preprojective algebras of Dynkin graphs (*cf.* [7, 3.1, 1.2], [38, 8.5]) as investigated in [27] and also for stable categories $\underline{\mathsf{CM}}(R)$ of Cohen-Macaulay modules over commutative complete local Gorenstein isolated singularities of dimension 3, using [4]. The tilting object is replaced by a full k-linear subcategory $\mathcal{T}\subset\mathcal{C}$ such that, among other conditions, we have

$$\mathsf{Hom}(T,ST')=0$$

for all $T,T'\in\mathcal{T}$ and where $\mathcal{T}$ is maximal for this property. We then consider the category $\mathsf{mod}\,\mathcal{T}$ of finitely presented (right) modules over $\mathcal{T}$ and show that it is abelian and Gorenstein, and that the homological dimensions of the projectives and the injectives are bounded by 1. We also prove that $\mathsf{mod}\,\mathcal{T}$ is equivalent to the quotient of $\mathcal{C}$ by an ideal, thus extending a result of [9], *cf.* also [49], and we generalize Theorem 4.2 of [9] to give a relationship between $\mathsf{mod}\,\mathcal{T}$ and $\mathsf{mod}\,\mathcal{T}'$ for 'neighbouring' subcategories $\mathcal{T}$ and $\mathcal{T}'$ of the type we consider, where $\mathcal{C}$ is a Calabi-Yau category of CY-dimension 2 whose cluster tilting subcategories have 'no loops'. We give examples where $\mathcal{T}$ has an infinite number of isomorphism classes of indecomposables.

In section 3, we show that the stable category of Cohen-Macaulay modules over $\mathcal{T}$ is Calabi-Yau of CY-dimension 3. In the case where $\mathcal{T}$ is invariant under the Serre functor, this result was first proved in [25]. It closely resembles the Ext-group symmetries proved in Proposition 6.2 of [27] for simple modules over non stable endomorphism rings. We strengthen and generalize these Ext-group symmetries in section 4, where we prove a relative 3-Calabi-Yau duality over non stable endomorphism rings.

In section 5, we generalize the results obtained on relative Calabi-Yau duality to d-Calabi-Yau categories (*cf.* also [48], [47]). We also show how to produce many examples of d-cluster tilting subcategories in d-cluster categories. For further examples, we refer to [34].

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2. The Gorenstein property

2.1. **The main result.** Let k be a field and $\mathcal{C}$ a triangulated k-linear category with split idempotents and suspension functor S. We suppose that all Hom-spaces of $\mathcal{C}$ are finite-dimensional and that $\mathcal{C}$ admits a Serre functor Σ, *cf.* [44]. We suppose that $\mathcal{C}$ is Calabi-Yau of CY-dimension 2, *i.e.* there is an isomorphism of triangle functors

$$S^2 \xrightarrow{\sim} \Sigma.$$

Note that we do not exclude the possibility that there might already exist an isomorphism $S^d \xrightarrow{\sim} \Sigma$ for $d = 0$ or $d = 1$. We fix such an isomorphism once and for all. For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we put

$$\mathsf{Ext}^n(X, Y) = \mathcal{C}\,(X, S^n Y).$$

Assume that $\mathcal{T} \subset \mathcal{C}$ is a *cluster tilting subcategory*. By this, we mean that $\mathcal{T}$ is a maximal 1-orthogonal subcategory in the sense of Iyama [33], *i.e.*

a) $\mathcal{T}$ is a k-linear subcategory,
b) $\mathcal{T}$ is contravariantly finite in $\mathcal{C}$, *i.e.* the functor $\mathcal{C}(?, X)|\mathcal{T}$ is finitely generated for all $X \in \mathcal{C}$,
c) we have $\mathsf{Ext}^1(T, T') = 0$ for all $T, T' \in \mathcal{T}$ and
d) if $X \in \mathcal{C}$ satisfies $\mathsf{Ext}^1(T, X) = 0$ for all $T \in \mathcal{T}$, then X belongs to $\mathcal{T}$.

Note that condition b) means that for each $X \in \mathcal{C}$, there is a *right $\mathcal{T}$-approximation*, *i.e.* a morphism $T \to X$ with $T \in \mathcal{T}$ such that each morphism $T' \to X$ with $T' \in \mathcal{T}$ factors through T. Condition d) is self-dual (because of the Calabi-Yau property) and so are conditions a) and c). Part b) of the proposition below shows that the dual of condition b) also holds for $\mathcal{T}$.

We point out that there are examples of Calabi-Yau categories of CY-dimension 2 where we have cluster tilting subcategories with an infinite number of isomorphism classes of indecomposables. They arise from certain Ext-finite hereditary abelian categories with Serre duality from [44] [43]. For example, let

$$\cdots \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \cdots$$

be the A_∞^∞-quiver with linear orientation and $\mathcal{H}$ the category of its finite-dimensional representations over k. Let $\mathcal{C} = \mathcal{C}_\mathcal{H}$ be the corresponding cluster category. The Auslander-Reiten quiver of $\mathcal{H}$ is of the form $\mathbb{Z}A_\infty$. Corresponding to a section of $\mathbb{Z}A_\infty$ with zigzag orientation, there is a tilting class θ for $\mathcal{H}$ in the sense of [43], that is

a) $\mathsf{Ext}^1(T_1, T_2) = 0$ for T_1, T_2 in θ,
b) if, for $X \in \mathcal{H}$, we have $\mathsf{Hom}(T, X) = 0 = \mathsf{Ext}^1(T, X) = 0$ for all T in θ, then $X = 0$,
c) θ is locally bounded, *i.e.* for each indecomposable T_1 of θ, there are only finitely many indecomposables T_2 of θ such that $\mathsf{Hom}(T_1, T_2) \neq 0$ or $\mathsf{Hom}(T_2, T_1) \neq 0$.

It is easy to see that θ induces a cluster tilting subcategory $\mathcal{T}$ in $\mathcal{C}$.

By a *$\mathcal{T}$-module*, we mean a contravariant k-linear functor from $\mathcal{T}$ to the category of vector spaces. We denote by $\mathsf{mod}\,\mathcal{T}$ the category of finitely presented $\mathcal{T}$-modules and by $F : \mathcal{C} \to \mathsf{mod}\,\mathcal{T}$ the functor which sends $X \in \mathcal{C}$ to the module $T \mapsto \mathsf{Hom}(T, X)$. Note that, since idempotents split in $\mathcal{T}$, this functor induces an equivalence from $\mathcal{T}$ to the category of projectives of $\mathsf{mod}\,\mathcal{T}$. Following [5] and [30] we say that $\mathcal{T}$ is *Gorenstein* if the finitely presented projective $\mathcal{T}$-modules are of finite injective dimension and the finitely presented injective $\mathcal{T}$-modules of finite projective dimension. Statements b), b') and c) of the following theorem extend results of [9], *cf.* also [49].

Proposition. a) *The category* $\mathsf{mod}\,\mathcal{T}$ *is abelian.*

b) *For each object $X \in \mathcal{C}$, there is a triangle*

$$S^{-1}X \to T_1^X \to T_0^X \to X$$

of $\mathcal{C}$ with T_i^X in $\mathcal{T}$. In any such triangle, $T_0^X \to X$ is a right $\mathcal{T}$-approximation and $X \to ST_1^X$ a left $S\mathcal{T}$-approximation.

b') *For each object $X \in \mathcal{C}$, there is a triangle*

$$X \to \Sigma T_X^0 \to \Sigma T_X^1 \to SX$$

in $\mathcal{C}$ with T_X^i in $\mathcal{T}$. In any such triangle, $X \to \Sigma T_X^0$ is a left $\Sigma\mathcal{T}$-approximation and $\Sigma T_X^1 \to SX$ a right $\Sigma\mathcal{T}$-approximation.

c) *Denote by $(S\mathcal{T})$ the ideal of morphisms of $\mathcal{C}$ which factor through an object of the form ST, $T \in \mathcal{T}$. Then the functor F induces an equivalence $\mathcal{C}/(S\mathcal{T}) \xrightarrow{\sim} \mathsf{mod}\,\mathcal{T}$. Moreover, in addition to inducing an equivalence from $\mathcal{T}$ to the category of projectives of $\mathsf{mod}\,\mathcal{T}$, the functor F induces an equivalence from $\Sigma\mathcal{T}$ to the category of injectives of $\mathsf{mod}\,\mathcal{T}$.*

d) *The projectives of $\mathsf{mod}\,\mathcal{T}$ are of injective dimension at most 1 and the injectives of projective dimension at most 1. Thus $\mathcal{T}$ is Gorenstein of dimension at most 1.*

Proof. a) Indeed, $\mathcal{C}$ is a triangulated category. Thus $\mathcal{C}$ admits weak kernels. Now $\mathcal{T}$ is contravariantly finite in $\mathcal{C}$ and therefore $\mathcal{T}$ also admits weak kernels. This is equivalent to the fact that $\mathsf{mod}\,\mathcal{T}$ is abelian.

b) Since $\mathcal{T}$ is contravariantly finite in $\mathcal{C}$, there is a right $\mathcal{T}$-approximation $T_0 \to X$. Form a triangle

$$T_1 \to T_0 \to X \to ST_1.$$

For $T \in \mathcal{T}$, the long exact sequence obtained by applying $\mathcal{C}(T, ?)$ to this triangle shows that $\mathsf{Ext}^1(T, T_1)$ vanishes. Thus T_1 lies in $\mathcal{T}$. Conversely, if we are given such a triangle, then $T_0 \to X$ is a right $\mathcal{T}$-approximation because $\mathcal{C}(T, ST_1)$ vanishes for $T \in \mathcal{T}$, and $X \to ST_1$ is a left $S\mathcal{T}$-approximation because $\mathcal{C}(T_0, ST)$ vanishes for $T \in \mathcal{T}$.

b') We apply (b) to $Y = S^{-1}X$ and obtain a triangle

$$S^{-2}X \to T_1^Y \to T_0^Y \to S^{-1}X.$$

Now we apply $S^2 \xrightarrow{\sim} \Sigma$ to this triangle to get the required triangle

$$X \to \Sigma T_1^Y \to \Sigma T_0^Y \to SX.$$

c) Let M be a finitely presented $\mathcal{T}$-module. Choose a presentation

$$\mathcal{C}(?, T_1) \to \mathcal{C}(?, T_0) \to M \to 0.$$

Form a triangle

$$T_1 \to T_0 \to X \to ST_1.$$

Since $\mathcal{C}(T, ST_1) = 0$, we obtain an exact sequence

$$FT_1 \to FT_0 \to FX \to 0$$

and M is isomorphic to FX. One shows that F is full by lifting a morphism $FX \to FX'$ to a morphism between presentations and then to a morphism of triangles. Now let X, X' be objects of $\mathcal{C}$. Let $f : X \to X'$ be a morphism with $Ff = 0$. By b), we have a triangle

$$S^{-1}X \to T_1^X \to T_0^X \to X$$

with T_1^X and T_0^X in $\mathcal{T}$. Since $\mathcal{C}(T_0, f) = 0$, the morphism f factors through ST_1.

For $T \in \mathcal{T}$, we have

$$F\Sigma T = \mathcal{C}(?, \Sigma T) = D\mathcal{C}(T, ?)$$

so that $F\Sigma T$ is indeed injective. Since $\mathcal{C}(ST', S^2T) = 0$ for $T, T' \in \mathcal{T}$, the functor F induces a fully faithful functor from $\Sigma\mathcal{T}$ to the category of injectives of $\mathsf{mod}\,\mathcal{T}$ and since idempotents in $\mathcal{C}$ split, this functor is an equivalence.

d) Given $X = \Sigma T$, $T \in \mathcal{T}$, we form a triangle as in b). We have

$$FS^{-1}\Sigma T = FST = 0.$$

This shows that the image under F of the triangle of b) yields a projective resolution of length 1 of $F\Sigma T$.Similarly, to obtain an injective resolution of FT, $T \in \mathcal{T}$, we apply b') to $X = T$ and use the fact that $FST = 0$. □

Corollary. a) *Each $\mathcal{T}$-module is either of infinite projective (resp. injective) dimension or of projective (resp. injective) dimension at most* 1.
b) *The category* $\mathsf{mod}\,\mathcal{T}$ *is either hereditary or of infinite global dimension.*

Proof. a) Suppose that M is a $\mathcal{T}$-module of finite projective dimension. Denote by ΩM the kernel of an epimorphism $P \to M$ with P projective. Suppose that M is of projective dimension n. Then $\Omega^n M$ is projective. By induction on $p \geq 0$, we find that $\Omega^{n-p} M$ is of injective dimension at most 1. In particular, M is of injective dimension at most 1. Dually, one shows the statement on the injective dimensions. Part b) is immediate from a). □

2.2. **Comparing neighbours.** In this section, we briefly indicate how to generalize Theorem 4.2 of [9] beyond cluster categories, namely to Calabi-Yau categories of CY-dimension 2 whose cluster tilting subcategories have 'no loops'. This hypothesis holds not only for cluster categories but also for the stable categories of finite-dimensional preprojective algebras [27].

We recall some results from [33]. Let $\mathcal{C}$ be a Calabi-Yau category of CY-dimension 2. We assume that for each cluster tilting subcategory $\mathcal{T}$ of $\mathcal{C}$ and for each indecomposable T of $\mathcal{T}$, any non isomorphism $f : T \to T$ factors through an object T' of $\mathcal{T}$ which does not contain T as a direct summand. Now let $\mathcal{T}$ be a cluster tilting subcategory. Then, according to [33], the results on cluster categories from [8] generalize literally: For any indecomposable object T of $\mathcal{T}$, there is an indecomposable T^*, unique up to isomorphism, such that T^* is not isomorphic to T and the additive subcategory $\mathcal{T}'$ of $\mathcal{C}$ with indecomposables

$$\mathsf{ind}(\mathcal{T}') = \mathsf{ind}(\mathcal{T}) \setminus \{T\} \cup \{T^*\}$$

is a cluster tilting subcategory of $\mathcal{C}$. Moreover, there are approximation triangles

$$T^* \to B \to T \to ST^* \text{ and } T \to B' \to T^* \to ST$$

connecting T^* and T such that B and B' belong to both $\mathcal{T}$ and $\mathcal{T}'$. With these notations and assumptions, we get the following connection between $\mathsf{mod}\,\mathcal{T}$ and $\mathsf{mod}\,\mathcal{T}'$.

Proposition. *Let S_T and S_{T^*} be the simple tops of $\mathcal{C}(?, T)$ and $\mathcal{C}(?, T^*)$ respectively. Then there is an equivalence of categories*

$$\mathsf{mod}\,\mathcal{T} / \mathsf{add}\, S_T \xrightarrow{\sim} \mathsf{mod}\,\mathcal{T}' / \mathsf{add}\, S_{T^*}.$$

The proof, left to the reader, follows the proof of Theorem 4.2 of [9] and uses the previous proposition.

3. The Calabi-Yau property

3.1. **Reminder on exact categories.** An *exact category* in the sense of Quillen [42] is an additive category $\mathcal{A}$ endowed with a distinguished class of sequences

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0\ ,$$

where i is a kernel of p and p a cokernel of i. We will state the axioms these sequences have to satisfy using the terminology of [24]: The morphisms p are called deflations, the morphisms i inflations and the pairs (i, p) conflations. The given class of conflations has to satisfy the following natural axioms:

Ex0 The identity morphism of the zero object is a deflation.
Ex1 The composition of two deflations is a deflation.

Ex2 Deflations are stable under base change, *i.e.* if $p : Y \to Z$ is a deflation and $f : Z' \to Z$ a morphism, then there is a cartesian square

$$\begin{tikzcd} Y \arrow[r, "p"] & Z \\ Y' \arrow[u, "f'"] \arrow[r, "p'"'] & Z' \arrow[u, "f"'] \end{tikzcd}$$

where p' is a deflation.

Ex2' Inflations are stable under cobase change, *i.e.* if $i : X \to Y$ is an inflation and $g : X \to X'$ a morphism, there is a cocartesian square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "g"] & Y \arrow[d, "g'"] \\ X' \arrow[r, "i'"'] & Y' \end{tikzcd}$$

where i' is an inflation.

As shown in [35], these axioms are equivalent to Quillen's and they imply that if $\mathcal{A}$ is small, then there is a fully faithful functor from $\mathcal{A}$ into an abelian category $\mathcal{A}'$ whose image is an additive subcategory closed under extensions and such that a sequence of $\mathcal{A}$ is a conflation iff its image is a short exact sequence of $\mathcal{A}'$. Conversely, one easily checks that an extension closed full additive subcategory $\mathcal{A}$ of an abelian category $\mathcal{A}'$ endowed with all conflations which become exact sequences in $\mathcal{A}'$ is always exact. The fundamental notions and constructions of homological algebra, and in particular the construction of the derived category, naturally extend from abelian to exact categories, *cf.* [41] and [37].

A *Frobenius category* is an exact category $\mathcal{A}$ which has enough injectives and enough projectives and where the class of projectives coincides with the class of injectives. In this case, the *stable category* $\underline{\mathcal{A}}$ obtained by dividing $\mathcal{A}$ by the ideal of morphisms factoring through a projective-injective carries a canonical structure of triangulated category, *cf.* [31] [29] [39]. Its suspension functor $X \to SX$ is constructed by choosing, for each object X of $\mathcal{A}$, a conflation

$$X \to I \to SX ,$$

where I is injective.

3.2. **Reminder on Gorenstein categories.** We refer to [5] [6] [30] and the references given there for the theory of Gorenstein algebras and their modules in non commutative and commutative algebra. In this section, we restate some fundamental results of the theory in our setup.

Let $\mathcal{A}$ be a k-linear exact category with enough projectives and enough injectives. Assume that $\mathcal{A}$ is *Gorenstein*[1], *i.e.* the full subcategory $\mathcal{P}$ of the projectives is covariantly finite, the full subcategory $\mathcal{I}$ of injectives is contravariantly finite and there is an integer d such that all projectives are of injective dimension at most d and all injectives are of projective dimension at most d. Let $\mathcal{D}^b_f(\mathcal{A})$ denote the full triangulated subcategory of the bounded derived category $\mathcal{D}^b(\mathcal{A})$ generated by the projectives (equivalently: the injectives). Call an object X of $\mathcal{A}$ projectively (resp. injectively) Cohen-Macaulay if it satisfies

$$\mathsf{Ext}^i_{\mathcal{A}}(X, P) = 0 \text{ (resp. } \mathsf{Ext}^i_{\mathcal{A}}(I, X) = 0)$$

[1]We thank Xiao-Wu Chen for pointing out that it is not enough to assume that the projectives are of finite injective dimension and the injectives of finite projective dimension.

for all $i > 0$ and all projectives P (resp. injectives I). Let $\mathsf{CMP}(\mathcal{A})$ resp. $\mathsf{CMI}(\mathcal{A})$ denote the full subcategories of $\mathcal{A}$ formed by the projectively resp. injectively Cohen-Macaulay objects. Clearly, $\mathsf{CMP}(\mathcal{A})$ is an exact subcategory of $\mathcal{A}$. Endowed with this exact structure, $\mathsf{CMP}(\mathcal{A})$ is a Frobenius category and its subcategory of projective-injectives is $\mathcal{P}$. Dually, $\mathsf{CMI}(\mathcal{A})$ is a Frobenius category whose subcategory of projective-injectives is $\mathcal{I}$. We denote by $\underline{\mathsf{CMP}}(\mathcal{A})$ and $\underline{\mathsf{CMI}}(\mathcal{A})$ the stable categories associated with these Frobenius categories.

In analogy with a classical result on Frobenius categories (*cf.* [39], [45]), we have the following

Theorem ([30])**.** *The inclusions*

$$\mathsf{CMP}(\mathcal{A}) \longrightarrow \mathcal{A} \longleftarrow \mathsf{CMI}(\mathcal{A})$$

induce triangle equivalences

$$\underline{\mathsf{CMP}}(\mathcal{A}) \longrightarrow \mathcal{D}^b(\mathcal{A})/\mathcal{D}^b_f(\mathcal{A}) \longleftarrow \underline{\mathsf{CMI}}(\mathcal{A})$$

The *stable Cohen-Macaulay category* of $\mathcal{A}$ is by definition $\underline{\mathsf{CM}}(\mathcal{A}) = \mathcal{D}^b(\mathcal{A})/\mathcal{D}^b_f(\mathcal{A})$. It is a triangulated category and an instance of a stable derived category in the sense of [40].

We can make the statement of the theorem more precise: The canonical functor

$$\mathcal{A}/(\mathcal{P}) \to \underline{\mathsf{CM}}(\mathcal{A})$$

admits a fully faithful left adjoint whose image is $\underline{\mathsf{CMP}}(\mathcal{A})$ and the canonical functor

$$\mathcal{A}/(\mathcal{I}) \to \underline{\mathsf{CM}}(\mathcal{A})$$

admits a fully faithful right adjoint whose image is $\underline{\mathsf{CMI}}(\mathcal{A})$.

The stable Cohen-Macaulay category is also canonically equivalent to the triangulated category $\mathbb{Z}\underline{\mathcal{A}}$ obtained [31] from the suspended [39] category $\underline{\mathcal{A}} = \mathcal{A}/(\mathcal{I})$ by formally inverting its suspension functor S and to the triangulated category obtained from the cosuspended category $\overline{\mathcal{A}} = \mathcal{A}/(\mathcal{P})$ by formally inverting its loop functor Ω. Thus, we have

$$\underline{\mathsf{CM}}(\mathcal{A})(X, S^n Y) = \mathsf{colim}_p\, \underline{\mathcal{A}}(S^p X, S^{n+p} Y) = \mathsf{colim}_p\, \overline{\mathcal{A}}(\Omega^{p+n} X, \Omega^p Y)$$

for all $X, Y \in \mathcal{A}$.

We will need the following easily proved lemma (the statement of whose dual is left to the reader).

Lemma. a) *Suppose that all injectives of $\mathcal{A}$ are of projective dimension at most d. Then $S^d Y$ is injectively Cohen-Macaulay for all $Y \in \mathcal{A}$.*

b) *Let $n \geq 1$ and $X, Y \in \mathcal{A}$. If $S^n Y$ is injectively Cohen-Macaulay, we have*

$$\mathsf{Ext}^n_{\mathcal{A}}(X, Y)/\,\mathsf{Ext}^n_{\mathcal{A}}(I, Y) \xrightarrow{\sim} \overline{\mathcal{A}}(X, S^n Y) \xrightarrow{\sim} \underline{\mathsf{CM}}(\mathcal{A})(X, S^n Y)\,,$$

where $X \to I$ is an inflation into an injective. If $S^{n-1}Y$ is injectively Cohen-Macaulay, then we have

$$\mathsf{Ext}^n_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \overline{\mathcal{A}}(X, S^n Y) \xrightarrow{\sim} \underline{\mathsf{CM}}(\mathcal{A})(X, S^n Y).$$

3.3. **Modules over cluster tilting subcategories.** Let $\mathcal{C}$ and $\mathcal{T}$ be as in section 2. We have seen there that $\mathsf{mod}\,\mathcal{T}$ is a Gorenstein category in the sense of the preceding section.

Theorem. *The stable Cohen-Macaulay category of* $\mathsf{mod}\,\mathcal{T}$ *is Calabi-Yau of CY-dimension* 3.

Note that, by sections 3.2 and 2.1, the theorem implies that for all $\mathcal{T}$-modules X, Y, we have a canonical isomorphism

$$D\,\mathsf{Ext}^2_{\mathsf{mod}\,\mathcal{T}}(Y, X) \xrightarrow{\sim} \underline{\mathsf{Ext}}^1_{\mathsf{mod}\,\mathcal{T}}(X, Y)\,,$$

where $\underline{\mathsf{Ext}}^1$ denotes the cokernel of the map

$$\mathsf{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(I, Y) \to \mathsf{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(X, Y)$$

induced by an arbitrary monomorphism $X \to I$ into an injective.

The theorem will be proved below in section 3.5. If $\mathcal{T}$ is stable under Σ, then $\mathsf{mod}\,\mathcal{T}$ is a Frobenius category. In this case, the theorem is proved in [25] using the fact that then $\mathcal{T}$ is a 'quadrangulated category' which is Calabi-Yau of CY-dimension 1.

3.4. **Three simple examples.** Consider the algebra A given by the quiver with relations

$$\begin{tikzcd} & & 3 \arrow[dr, "\beta"] & \\ & 2 \arrow[ur, "\alpha"] & & 4 \arrow[ll, "\gamma"] \\ 1 \arrow[ur, "\delta"] & & & \end{tikzcd} \qquad \alpha\gamma = \beta\alpha = \gamma\beta = 0.$$

It is cluster-tilted of type A_4. For $1 \leq i \leq 4$, denote by P_i resp. I_i resp. S_i the indecomposable projective, resp. injective resp. simple (right) module associated with the vertex i. We have $I_1 = P_3$ and $I_3 = P_4$ and we have minimal projective resolutions

$$0 \to P_1 \to P_3 \to I_2 \to 0\,,\ 0 \to P_1 \to P_2 \to I_4 \to 0$$

and minimal injective resolutions

$$0 \to P_1 \to I_1 \to I_2 \to 0\,,\ 0 \to P_2 \to I_1 \oplus I_4 \to I_2 \to 0.$$

Thus the simple modules except S_1 are injectively Cohen-Macaulay and the simple modules except S_2 are projectively Cohen-Macaulay. Therefore, we have

$$\mathsf{Ext}^1_A(S_i, S_j) \xrightarrow{\sim} D\,\mathsf{Ext}^2_A(S_j, S_i)$$

unless $(i, j) = (1, 2)$. Now $\dim \mathsf{Ext}^1_A(S_i, S_j)$ equals the number of arrows from j to i and $\mathsf{Ext}^2_A(S_j, S_i)$ is isomorphic to the space of minimal relations from i to j, *cf.* [11]. Thus for example, the arrow α corresponds to the relation $\gamma\beta = 0$, and similarly for the arrows β and γ. Note that no relation corresponds to the arrow δ. The stable category $\underline{\mathsf{CM}}(\mathsf{mod}\,A)$ is equivalent to the stable category of the full subquiver with relations on $2, 3, 4$. It is indeed 3-Calabi-Yau.

Consider the algebra A given by the quiver with relations

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\delta"'] & & 3 \arrow[ll, "\varepsilon"] \\ & 4 \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \alpha\varepsilon = \varepsilon\beta = \delta\varepsilon = \varepsilon\gamma = 0, \beta\alpha = \gamma\delta.$$

It is cluster-tilted of type D_4. We have $P_1 = I_3$ and $P_3 = I_1$. There are minimal injective resolutions

$$0 \to P_2 \to I_1 \to I_4 \to 0\,,\ 0 \to P_4 \to I_1 \to I_2 \to 0$$

and minimal projective resolutions

$$0 \to P_4 \to P_3 \to I_2 \to 0\,,\ 0 \to P_2 \to P_3 \to I_4 \to 0.$$

Thus the simples S_1 and S_3 are both projectively and injectively Cohen-Macaulay and whenever i, j are connected by an arrow, we have

$$\mathsf{Ext}^1_A(S_i, S_j) \xrightarrow{\sim} D\,\mathsf{Ext}^2_A(S_j, S_i).$$

Since there are neither arrows nor relations between 2 and 4, we obtain a perfect correspondence between arrows and relations: the outer arrows correspond to zero relations and the inner arrow to the commutativity relation.

Consider the algebra A given by the quiver

$$\begin{tikzcd} & & 3 \arrow[dr] & & \\ & 2 \arrow[ur] \arrow[dr] & & 5 \arrow[ll] \arrow[dr] & \\ 1 \arrow[ur] & & 4 \arrow[ll] \arrow[ur] & & 6 \arrow[ll] \end{tikzcd}$$

subject to the following relations: each path of length two containing one of the outer arrows vanishes; each of the three rhombi containing one of the inner arrows is commutative. As shown in [26], this algebra is isomorphic to the stable endomorphism algebra of a maximal rigid module over the preprojective algebra of type A_4. It is selfinjective and thus Gorenstein. According to the theorem, it is stably 3-Calabi-Yau, a fact which was first proved in [25] using the fact that the projectives over A form a quadrangulated category. Note that the isomorphisms

$$\mathsf{Ext}^1_A(S_i, S_j) \xrightarrow{\sim} D\,\mathsf{Ext}^2_A(S_j, S_i)$$

translate into a perfect correspondence between arrows and relations: each outer arrow corresponds to a zero relation and each inner arrow to a commutativity relation.

3.5. **Proof of the theorem.** Denote by $\underline{\mathsf{mod}}\mathcal{T}$ the quotient of $\mathsf{mod}\,\mathcal{T}$ by the ideal of morphisms factoring through a projective and by $\overline{\mathsf{mod}}\mathcal{T}$ the quotient by the ideal of morphisms factoring through an injective. It follows from Proposition 2.1 that the functor F induces equivalences

$$\mathcal{C}/(\mathcal{T}, S\mathcal{T}) \xrightarrow{\sim} \underline{\mathsf{mod}}\mathcal{T} \text{ and } \mathcal{C}/(S\mathcal{T}, \Sigma\mathcal{T}) \xrightarrow{\sim} \overline{\mathsf{mod}}\mathcal{T}.$$

Thus the suspension functor $S : \mathcal{C} \to \mathcal{C}$ induces a well-defined equivalence $\tau : \underline{\mathsf{mod}}\mathcal{T} \to \overline{\mathsf{mod}}\mathcal{T}$. We first prove that Serre duality for $\mathcal{C}$ implies the Auslander-Reiten formula for τ in $\mathsf{mod}\,\mathcal{T}$. This shows in particular that τ is indeed isomorphic to the Auslander-Reiten translation of $\mathsf{mod}\,\mathcal{T}$. In the context of cluster categories, it was proved in [9] that the functor induced by S coincides with the Auslander-Reiten translation on objects of $\mathsf{mod}\,\mathcal{T}$. Note that if $\mathcal{C}$ and $\mathsf{mod}\,\mathcal{T}$ are Krull-Schmidt categories, then, as for cluster-tilted algebras, we obtain the Auslander-Reiten quiver of $\mathsf{mod}\,\mathcal{T}$ from that of $\mathcal{C}$ by removing the vertices corresponding to indecomposables in $S\mathcal{T}$.

Lemma. *Let $X, Y \in \mathsf{mod}\,\mathcal{T}$. Then we have canonical isomorphisms*

$$\overline{\mathsf{Hom}}(Y, \tau X) \xrightarrow{\sim} D\,\mathsf{Ext}^1(X, Y) \xrightarrow{\sim} \underline{\mathsf{Hom}}(\tau^{-1}Y, X).$$

Proof. Choose L, M in $\mathcal{C}$ such that $FM = X$ and $FL = Y$. Form triangles

$$S^{-1}M \to T_1^M \to T_0^M \to M \text{ and } S^{-1}\Sigma T_L^1 \to L \to \Sigma T_L^0 \to \Sigma T_L^1$$

in $\mathcal{C}$ as in Proposition 2.1. We obtain an exact sequence

$$FS^{-1}M \to FT_1^M \to FT_0^M \to FM \to 0$$

in $\mathsf{mod}\,\mathcal{T}$ and its middle two terms are projective. Using this sequence we see that $\mathsf{Ext}^1_{\mathcal{T}}(FM, FL)$ is isomorphic to the middle cohomology of the complex

$$\mathsf{Hom}(FT_0^M, FL) \to \mathsf{Hom}(FT_1^M, FL) \to \mathsf{Hom}(FS^{-1}M, FL). \tag{1}$$

Now if V is an object of $\mathcal{C}$ and

$$V \to \Sigma T_V^0 \to \Sigma T_V^1 \to SV$$

a triangle as in Proposition 2.1, then we have, for each U of $\mathcal{C}$, an exact sequence

$$\mathcal{C}(U, ST_V^1) \to \mathcal{C}(U, V) \to \mathsf{Hom}(FU, FV) \to 0.$$

By applying this to the three terms in the sequence (1) and using the vanishing of Ext^1 between objects of $\mathcal{T}$, we obtain that $E = \mathsf{Ext}^1(FM, FL)$ is the intersection of the images of the maps

$$\mathcal{C}(S^{-1}M, ST_L^1) \to \mathcal{C}(S^{-1}M, L) \text{ and } \mathcal{C}(T_1^M, L) \to \mathcal{C}(S^{-1}M, L).$$

By the exact sequences $\mathcal{C}(T_1^M, L) \to \mathcal{C}(S^{-1}M, L) \to \mathcal{C}(S^{-1}T_0^M, L)$ and

$$\mathcal{C}(S^{-1}M, ST_L^1) \to \mathcal{C}(S^{-1}M, L) \to \mathcal{C}(S^{-1}M, \Sigma T_L^0)\,,$$

the group E also appears as a kernel in the exact sequence

$$0 \to E \to \mathcal{C}(S^{-1}M, L) \to \mathcal{C}(S^{-1}M, \Sigma T_L^0) \oplus \mathcal{C}(S^{-1}T_0^M, L).$$

If we dualize this sequence and use the isomorphism

$$D\mathcal{C}(U, V) \xrightarrow{\sim} \mathcal{C}(V, \Sigma U) = \mathcal{C}(V, S^2U),$$

we obtain the exact sequence

$$\mathcal{C}(\Sigma T_L^0, SM) \oplus \mathcal{C}(L, ST_0^M) \to \mathcal{C}(L, SM) \to DE \to 0.$$

This proves the left isomorphism. The right one follows since τ is an equivalence. $\square$

Now let X, Y be $\mathcal{T}$-modules and $\mathcal{S}$ the stable Cohen-Macaulay category of $\mathsf{mod}\,\mathcal{T}$. We will construct a canonical isomorphism

$$D\mathcal{S}(Y, S^2X) \xrightarrow{\sim} \mathcal{S}(X, SY).\,,$$

Since the injectives of $\mathsf{mod}\,\mathcal{T}$ are of projective dimension at most 1 and the projectives of injective dimension at most 1, we have

$$\mathsf{Hom}_{\mathcal{S}}(Y, S^2X) = \mathsf{Ext}^2_{\mathsf{mod}\,\mathcal{T}}(Y, X)$$

and

$$\mathsf{Hom}_{\mathcal{S}}(X, SY) = \mathsf{cok}(\mathsf{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(I, Y) \to \mathsf{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(X, Y))\,,$$

by lemma 3.2 b), where $X \to I$ is a monomorphism into an injective. We will examine $\mathsf{Ext}^2_{\mathsf{mod}\,\mathcal{T}}(Y, X)$ and compare its dual with $\mathsf{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(X, Y)$ as described by the Auslander-Reiten formula. Choose M and L in $\mathcal{C}$ such that $X = FM$ and $Y = FL$. Form the triangle

$$M \to \Sigma T_M^0 \to \Sigma T_M^1 \to SM.$$

We obtain an exact sequence

$$0 \to FM \to F\Sigma T_M^0 \to F\Sigma T_M^1 \to FSM \to FS\Sigma T_M^0$$

whose second and third term are injective. This is sufficient to obtain that $\mathsf{Ext}^2(FL, FM)$ is isomorphic to the middle homology of the complex

$$\mathsf{Hom}(FL, F\Sigma T_M^1) \to \mathsf{Hom}(FL, FSM) \to \mathsf{Hom}(FL, FS\Sigma T_M^0).$$

Now $F\Sigma T_M^1 \to FSM$ is a right $\mathcal{I}$-approximation and $FM \to F\Sigma T_M^0$ is a monomorphism into an injective. Put $I = F\Sigma T_M^0$. Then we have

$$\mathsf{Ext}^2(FL, FM) = \mathsf{ker}(\overline{\mathsf{Hom}}(FL, \tau FM) \to \overline{\mathsf{Hom}}(FL, \tau I))\,,$$

Using the Auslander-Reiten formula, we obtain

$$D\,\mathsf{Ext}^2(FL, FM) \xrightarrow{\sim} \mathsf{cok}(\mathsf{Ext}^1(I, FL) \to \mathsf{Ext}^1(FL, FM))$$

and thus

$$D\,\mathsf{Ext}^2(Y, X) \xrightarrow{\sim} \mathsf{cok}(\mathsf{Ext}^1(I, Y) \to \mathsf{Ext}^1(X, Y))\,,$$

where $X \to I$ is a monomorphism into an injective. This yields

$$D\mathcal{S}(Y, S^2X) \xrightarrow{\sim} \mathcal{S}(X, SY)$$

as claimed.

4. Relative 3-Calabi-Yau duality over non stable endomorphism rings

Let $\mathcal{E}$ be a k-linear Frobenius category with split idempotents. Suppose that its stable category $\mathcal{C} = \underline{\mathcal{E}}$ has finite-dimensional Hom-spaces and is Calabi-Yau of CY-dimension 2. This situation occurs in the following examples:

(1) $\mathcal{E}$ is the category of finite-dimensional modules over the preprojective algebra of a Dynkin quiver as investigated in [27].
(2) $\mathcal{E}$ is the category of Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity of dimension 3.
(3) $\mathcal{E}$ is a Frobenius category whose stable category is triangle equivalent to the cluster category associated with an Ext-finite hereditary category. Such Frobenius categories always exist by [38], section 9.9.
(4) $\mathcal{E}$ is a Frobenius category whose stable category is triangle equivalent to the bounded derived category of coherent sheaves on a Calabi-Yau surface (*i.e.* a K3-surface). For example, one obtains such a Frobenius category by taking the full subcategory of the exact category of left bounded complexes of injective quasi-coherent sheaves whose homology is bounded and coherent.

Let $\mathcal{T} \subset \mathcal{C}$ be a cluster tilting subcategory and $\mathcal{M} \subset \mathcal{E}$ the preimage of $\mathcal{T}$ under the projection functor. In particular, $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of the projective-injective objects in $\mathcal{M}$. Note that $\mathcal{T}$ equals the quotient $\underline{\mathcal{M}}$ of $\mathcal{M}$ by the ideal of morphisms factoring through a projective-injective.

We know from section 2 that $\mathsf{mod}\,\underline{\mathcal{M}}$ is abelian. In general, we cannot expect the category $\mathsf{mod}\,\mathcal{M}$ of finitely presented $\mathcal{M}$-modules to be abelian (*i.e.* $\mathcal{M}$ to have weak kernels). However, the category $\mathsf{Mod}\,\mathcal{M}$ of all right $\mathcal{M}$-modules is of course abelian. Recall that the *perfect derived category* $\mathsf{per}(\mathcal{M})$ is the full triangulated subcategory of the derived category of $\mathsf{Mod}\,\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules. We identify $\mathsf{Mod}\,\underline{\mathcal{M}}$ with the full subcategory of $\mathsf{Mod}\,\mathcal{M}$ formed by the modules vanishing on $\mathcal{P}$. The following proposition is based on the methods of [3].

Proposition. a) *For each $X \in \mathcal{E}$, there is a conflation*

$$0 \to M_1 \to M_0 \to X \to 0$$

with M_i in $\mathcal{M}$. In any such conflation, the morphism $M_0 \to X$ is a right $\mathcal{M}$-approximation.

b) *Let Z be a finitely presented $\mathcal{M}$-module. Then Z vanishes on $\mathcal{P}$ iff there exists a conflation*

$$0 \to K \to M_1 \to M_0 \to 0$$

with M_0, M_1 in $\mathcal{M}$ such that $Z = \mathsf{cok}(M_1^\wedge \to M_0^\wedge)$, where $M^\wedge$ denotes the $\mathcal{M}$-module represented by $M \in \mathcal{M}$. In this case, we have

$$Z \xrightarrow{\sim} \mathsf{Ext}^1_{\mathcal{E}}(?, K)|\mathcal{M}.$$

c) *Let Z be a finitely presented $\underline{\mathcal{M}}$-module. Then Z considered as an $\mathcal{M}$-module lies in $\mathsf{per}(\mathcal{M})$ and we have a canonical isomorphism*

$$D\,\mathsf{per}(\mathcal{M})(Z, ?) \xrightarrow{\sim} \mathsf{per}(\mathcal{M})(?, Z[3]).$$

Proof. a) Indeed, we know that there is a triangle

$$M_1 \to M_0 \to X \to SM_1$$

in $\underline{\mathcal{E}}$ with M_1, M_0 in $\mathcal{T}$. We lift it to the required conflation.

b) Clearly, if Z is of the form given, it vanishes on $\mathcal{P}$. Conversely, suppose that Z is finitely presented with presentation

$$M_1^\wedge \xrightarrow{p^\wedge} M_0^\wedge \longrightarrow Z \longrightarrow 0$$

and vanishes on $\mathcal{P}$. Then each morphism $P \to M_0$ with $P \in \mathcal{P}$ lifts along p. Now let $q : P_0 \to M_0$ be a deflation with projective P_0. Then we have $q = pq'$ for some q'. Clearly, the morphism

$$M_1 \oplus P_0 \to M_0$$

with components p and q is a deflation and $M_1 \to M_0$ is a retract of this deflation: The retraction is given by $[\mathbf{1}, q']$ and the identity of M_0. Since $\mathcal{E}$ has split idempotents, it follows that $M_1 \to M_0$ is a deflation. The last assertion follows from the vanishing of $\mathsf{Ext}^1_{\mathcal{E}}(M, M_1)$ for all M in $\mathcal{M}$.

c) Clearly, each representable $\underline{\mathcal{M}}$-module is finitely presented as an $\mathcal{M}$-module. Since a cokernel of a morphism between finitely presented modules is finitely presented, the module Z considered as an $\mathcal{M}$-module is finitely presented. So we can choose a conflation

$$0 \to K \to M_1 \to M_0 \to 0$$

as in b). Now choose a conflation

$$0 \to M_3 \to M_2 \to K \to 0$$

with M_i in $\mathcal{M}$ as in a). Then the image of the spliced complex

$$0 \to M_3 \to M_2 \to M_1 \to M_0 \to 0$$

under the Yoneda functor $\mathcal{M} \to \mathsf{Mod}\,\mathcal{M}$ is a projective resolution P of Z. So Z belongs to $\mathsf{per}(\mathcal{M})$. We now prove the duality formula. We will exhibit a canonical linear form ϕ on $\mathsf{per}(\mathcal{M})(P, P[3])$ and show that it yields the required isomorphism. We have

$$\mathsf{per}(\mathcal{M})(P, P[3]) = \mathsf{per}(\mathcal{M})(P, Z[3]) = \mathsf{cok}(\mathsf{Hom}(M_2^\wedge, Z) \to \mathsf{Hom}(M_3^\wedge, Z))$$

and this is also isomorphic to

$$\mathsf{cok}(Z(M_2) \to Z(M_3)) = \mathsf{cok}(\mathsf{Ext}^1_{\mathcal{E}}(M_2, K) \to \mathsf{Ext}^1_{\mathcal{E}}(M_3, K)).$$

Now the conflation

$$0 \to M_3 \to M_2 \to K \to 0$$

yields an exact sequence

$$\mathsf{Ext}^1_{\mathcal{E}}(M_2, K) \to \mathsf{Ext}^1_{\mathcal{E}}(M_3, K) \to \mathsf{Ext}^2_{\mathcal{E}}(K, K).$$

Thus we obtain an injection

$$\mathsf{per}(\mathcal{M})(P, P[3]) \to \mathsf{Ext}^2_{\mathcal{E}}(K, K) = \underline{\mathcal{E}}(K, S^2K).$$

Now since $\mathcal{C} = \underline{\mathcal{E}}$ is Calabi-Yau of CY-dimension 2, we have a canonical linear form

$$\psi : \underline{\mathcal{E}}(K, S^2K) \to k$$

and we define ϕ to be the restriction of ψ to $\mathsf{per}(\mathcal{M})(P, P[3])$. Then ϕ yields a morphism

$$\mathsf{per}(\mathcal{M})(?, P[3]) \to D\,\mathsf{per}(\mathcal{M})(P, ?)$$

which sends f to the map $g \mapsto \phi(fg)$. Clearly this is a morphism of cohomological functors defined on the triangulated category $\mathsf{per}(\mathcal{M})$. To check that it is an isomorphism, it suffices to check that its evaluation at all shifts $S^i M^\wedge$, $i \in \mathbb{Z}$, of representable functors $M^\wedge$ is an isomorphism. Now indeed, we have

$$\mathsf{per}(\mathcal{M})(S^i M^\wedge, P[3]) = \mathsf{per}(\mathcal{M})(S^i M^\wedge, Z[3]) = \mathsf{Ext}^{3-i}(M^\wedge, Z).$$

This vanishes if $i \neq 3$ and is canonically isomorphic to

$$\mathsf{Ext}^1_{\mathcal{E}}(M, K)$$

for $i = 3$. On the other hand, to compute $\mathsf{per}(\mathcal{M})(P, S^i M^\wedge)$, we have to compute the homology of the complex

$$0 \to \mathcal{E}(M_0, M) \to \mathcal{E}(M_1, M) \to \mathcal{E}(M_2, M) \to \mathcal{E}(M_3, M) \to 0.$$

From the exact sequences

$$0 \to \mathcal{E}(M_0, M) \to \mathcal{E}(M_1, M) \to \mathcal{E}(K, M) \to \mathsf{Ext}^1_{\mathcal{E}}(M_0, M)$$

and

$$0 \to \mathcal{E}(K, M) \to \mathcal{E}(M_2, M) \to \mathcal{E}(M_3, M) \to \mathsf{Ext}^1_{\mathcal{E}}(K, M) \to \mathsf{Ext}^1_{\mathcal{E}}(M_2, M)\ ,$$

using that $\mathsf{Ext}^1_{\mathcal{E}}(M_0, M) = 0 = \mathsf{Ext}^1_{\mathcal{E}}(M_2, M)$, we get that the homology is 0 except at $\mathcal{E}(M_3, M)$, where it is $\mathsf{Ext}^1_{\mathcal{E}}(K, M_0)$. Now we know that the canonical linear form on $\mathsf{Ext}^2_{\mathcal{E}}(K, K)$ yields a canonical isomorphism

$$\mathsf{Ext}^1_{\mathcal{E}}(M, K) \xrightarrow{\sim} D\,\mathsf{Ext}^1_{\mathcal{E}}(K, M)$$

and one checks that it identifies with the given morphism

$$\mathsf{per}(\mathcal{M})(M^\wedge[3], P[3]) \to D\,\mathsf{per}(\mathcal{M})(P, M^\wedge[3]).$$

□

The following corollary generalizes Proposition 6.2 of [27]. The fact that $\mathsf{mod}\,\mathcal{M}$ is abelian and of global dimension 3 is a special case of the results of [32].

Corollary. *Suppose that $\mathcal{E}$ is abelian. Then $\mathsf{mod}\,\mathcal{M}$ is abelian of global dimension at most 3 and for each $X \in \mathsf{mod}\,\mathcal{M}$ and each $Y \in \mathsf{mod}\,\underline{\mathcal{M}}$, we have canonical isomorphisms*

$$\mathsf{Ext}^i_{\mathsf{mod}\,\mathcal{M}}(X, Y) \xrightarrow{\sim} D\,\mathsf{Ext}^{3-i}_{\mathsf{mod}\,\mathcal{M}}(Y, X)\ ,\ i \in \mathbb{Z}.$$

If $\underline{\mathcal{M}}$ contains a non zero object, then $\mathsf{mod}\,\mathcal{M}$ is of global dimension exactly 3.

Proof. Let X be a finitely presented $\mathcal{M}$-module and let

$$M_1^\wedge \to M_0^\wedge \to X \to 0$$

be a presentation. We form the exact sequence

$$0 \to K \to M_1 \to M_0$$

and then choose an exact sequence

$$0 \to M_3 \to M_2 \to K \to 0$$

using part a) of the proposition above. By splicing the two, we obtain a complex

$$0 \to M_3 \to M_2 \to M_1 \to M_0 \to 0\ ,$$

whose image under the Yoneda functor is a projective resolution of length at most 3 of X. This shows that $\mathcal{M}$ admits weak kernels (hence $\mathsf{mod}\,\mathcal{M}$ is abelian) and that $\mathsf{mod}\,\mathcal{M}$ is of global dimension at most 3. Thus the perfect derived category coincides with the bounded derived category of $\mathsf{mod}\,\mathcal{M}$. Now the claim about the extension groups is obvious from the proposition. For the last assertion, we choose X to be a non zero $\underline{\mathcal{M}}$-module. Then $\mathsf{Ext}^3_{\mathsf{mod}\,\mathcal{M}}(X, X)$ is non zero. □

5. Relative $(d+1)$-Calabi-Yau duality

5.1. **d-cluster tilting subcategories.** Let k be a field and $\mathcal{C}$ a triangulated k-linear category with split idempotents and suspension functor S. We suppose that all Hom-spaces of $\mathcal{C}$ are finite-dimensional and that $\mathcal{C}$ admits a Serre functor Σ, *cf.* [44]. Let $d \geq 1$ be an integer. We suppose that $\mathcal{C}$ is Calabi-Yau of CY-dimension d, *i.e.* there is an isomorphism of triangle functors

$$S^d \xrightarrow{\sim} \Sigma.$$

We fix such an isomorphism once and for all.

For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we put

$$\mathsf{Ext}^n(X,Y) = \mathcal{C}\,(X, S^n Y).$$

Assume that $\mathcal{T} \subset \mathcal{C}$ is a *d-cluster tilting subcategory.* By this, we mean that $\mathcal{T}$ is maximal $(d-1)$-orthogonal in the sense of Iyama [33], *i.e.*

a) $\mathcal{T}$ is a k-linear subcategory,
b) $\mathcal{T}$ is functorially finite in $\mathcal{C}$, *i.e.* the functors $\mathcal{C}(?,X)|\mathcal{T}$ and $\mathcal{C}(X,?)|\mathcal{T}$ are finitely generated for all $X \in \mathcal{C}$,
c) we have $\mathsf{Ext}^i(T,T') = 0$ for all $T, T' \in \mathcal{T}$ and all $0 < i < d$ and
d) if $X \in \mathcal{C}$ satisfies $\mathsf{Ext}^i(T,X) = 0$ for all $0 < i < d$ and all $T \in \mathcal{T}$, then T belongs to $\mathcal{T}$.

Note that a), b), c) are self-dual and so is d) (by the Calabi-Yau property). For $d = 1$, we have $\mathcal{T} = \mathcal{C}$ by condition d). This definition is slightly different from that in [47] but most probably equivalent in the context of [loc.cit.]. As in section 2, one proves that $\mathsf{mod}\,\mathcal{T}$, the category of finitely presented right $\mathcal{T}$-modules, is abelian. Let

$$F : \mathcal{C} \to \mathsf{mod}\,\mathcal{T}$$

be the functor which sends X to the restriction of $\mathcal{C}(?,X)$ to $\mathcal{T}$. For classes $\mathcal{U}$, $\mathcal{V}$ of objects of $\mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the full subcategory of all objects X of $\mathcal{C}$ appearing in a triangle

$$U \to X \to V \to SU.$$

Lemma. *Suppose that $d \geq 2$. For each finitely presented module $M \in \mathsf{mod}\,\mathcal{T}$, there is a triangle*

$$T_0 \to T_1 \to X \to ST_0$$

such that FX is isomorphic to M. The functor F induces an equivalence

$$\mathcal{U}/(S\mathcal{T}) \to \mathsf{mod}\,\mathcal{T}\ ,$$

where $\mathcal{U} = \mathcal{T} * S\mathcal{T}$ and $(S\mathcal{T})$ is the ideal of morphisms factoring through objects ST, $T \in \mathcal{T}$.

Proof. Since $\mathcal{T}$ has split idempotents, the functor F induces an equivalence from $\mathcal{T}$ to the category of projectives of $\mathsf{mod}\,\mathcal{T}$. Now let $P_1 \to P_0 \to M \to 0$ be a projective presentation of M. Choose a morphism $T_1 \to T_0$ of $\mathcal{T}$ whose image under F is $P_1 \to P_0$. Define X by the triangle

$$T_1 \to T_0 \to X \to ST_1.$$

Then, since $\mathsf{Ext}^1(T, T_1)$ vanishes for $T \in \mathcal{T}$, we get an exact sequence $FT_1 \to FT_0 \to FX \to 0$ and hence an isomorphism between FX and M. This also shows that F is essentially surjective since X clearly belongs to $\mathcal{U}$. One shows that F is full by lifting a morphism between modules to a morphism between projective presentations and then to a morphism between triangles. One shows that the kernel of $F|\mathcal{U}$ is $(S\mathcal{T})$ as in section 2. $\square$

5.2. **Examples of d-cluster tilting subcategories.** One can use proposition 5.6 below (*cf.* also [48] and [47] [20]) to produce the following examples: Consider the algebra A given by the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 3 \arrow[u] & 4 \arrow[l] \end{tikzcd}$$

with the relations given by all paths of length 2. Then A is 3-cluster tilted of type A_4. It is selfinjective hence Gorenstein. It is not hard to check that its stable category is 4-Calabi-Yau. More generally, a finite-dimensional algebra of radical square 0 whose quiver is an oriented cycle with $d+1$ vertices is d-cluster-tilted of type A_{d+1}, selfinjective and stably $(d+1)$-Calabi-Yau.

Let A be the algebra given by the quiver

$$\begin{tikzcd}[column sep=small] \circ \arrow[dr] & & \circ \arrow[dr] & & & & \\ & \circ \arrow[ur] & & \circ \arrow[ulll] \arrow[dr] & & \circ \arrow[dr] & \\ & & & & \circ \arrow[ur] & & \circ \arrow[ulll] \end{tikzcd}$$

subject to the relations $\alpha\beta = 0$ for all composable arrows α, β which point in different directions (the composition of the long skew arrows in non zero). Then A is 3-cluster tilted of type A_7. One can show that it is Gorenstein of dimension 1 and that its stable Cohen-Macaulay category is 3-Calabi-Yau.

5.3. **A counterexample.** The following example due to Osamu Iyama shows that part d) of proposition 2.1 does not generalize from the 2-dimensional to the d-dimensional case: It is an example of a 3-cluster tilting subcategory of a 3-Calabi Yau triangulated category which is not Gorenstein.

Let k be a field of characteristic $\neq 3$ and ω a primitive third root of 1. Let $S = k[[t, x, y, z]]$ and let the generator g of $G = \mathbb{Z}/3\mathbb{Z}$ act on S by $gt = \omega t$, $gx = \omega x$, $gy = \omega^2 y$ and $gz = \omega^2 z$. Then the algebra S^G is commutative, an isolated singularity and Gorenstein of dimension 4, its category of maximal Cohen-Macaulay modules $\mathcal{E} = \mathsf{CM}(S^G)$ is Frobenius and the stable category $\mathcal{C} = \underline{\mathsf{CM}}(S^G)$ is 3-Calabi-Yau. The S^G-module S is maximal Cohen-Macaulay and the subcategory $\mathcal{T}$ formed by all direct factors of finite direct sums of copies of S in $\mathcal{C}$ is a 3-cluster tilting subcategory. As an S^G-module, S decomposes into the direct sum of three indecomposable submodules S_i, $i = 0, 1, 2$, formed respectively by the $f \in S$ such that $gf = \omega^i f$. Note that $S_0 = S$ is projective. The full subcategory of $\mathsf{CM}(S^G)$ with the objects S_0, S_1, S_2 is isomorphic to the completed path category of the quiver

$$\begin{tikzcd} & 0 \arrow[dl, shift right=3] \arrow[dl, shift right=1] \arrow[dr, shift left=1] \arrow[dr, shift left=3] & \\ 1 \arrow[ur, shift right=1] \arrow[ur, shift right=3] \arrow[rr, shift right=1] \arrow[rr, shift right=3] & & 2 \arrow[ul, shift left=1] \arrow[ul, shift left=3] \arrow[ll, shift right=1] \arrow[ll, shift right=3] \end{tikzcd}$$

subject to all commutativity relations, where, between any two vertices, the arrows pointing counterclockwise are labelled t and x and the arrows pointing clockwise y and z. The vertex 0 corresponds to the indecomposable projective S_0, which vanishes in the stable category $\mathcal{C}$. Therefore the full subcategory $\mathsf{ind}(\mathcal{T})$ of $\mathcal{C}$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift right=1] \arrow[r, shift right=3] & 2 \arrow[l, shift right=1] \arrow[l, shift right=3] \end{tikzcd}$$

subject to the relations given by all paths of length ≥ 2. It is easy to see that the free modules $\mathcal{T}(?, S_i)$, $i = 1, 2$, are of infinite injective dimension. Thus, $\mathcal{T}$ is not Gorenstein.

5.4. **Relative $(d+1)$-Calabi-Yau duality.** Let d be an integer ≥ 1. Let $\mathcal{E}$ be a k-linear Frobenius category with split idempotents. Suppose that its stable category $\mathcal{C} = \underline{\mathcal{E}}$ has finite-dimensional Hom-spaces and is Calabi-Yau of CY-dimension d. Let $\mathcal{T} \subset \mathcal{C}$ be a d-cluster tilting subcategory and $\mathcal{M} \subset \mathcal{E}$ the preimage of $\mathcal{T}$ under the projection functor. In particular, $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of the projective-injective objects in $\mathcal{M}$. Note that $\mathcal{T}$ equals the quotient $\underline{\mathcal{M}}$ of $\mathcal{M}$ by the ideal of morphisms factoring through a projective-injective.

We call a complex $\mathcal{M}$-*acyclic* if it is acyclic as a complex over $\mathcal{E}$ (*i.e.* obtained by splicing conflations of $\mathcal{E}$) and its image under the functor $X \mapsto \mathcal{E}(?, X)|\mathcal{M}$ is exact.

Theorem. a) *For each $X \in \mathcal{E}$, there is an $\mathcal{M}$-acyclic complex*

$$0 \to M_{d-1} \to M_{d-2} \to \ldots \to M_0 \to X \to 0$$

with all M_i in $\mathcal{M}$.

b) *For each finitely presented $\underline{\mathcal{M}}$-module Z, there is an $\mathcal{M}$-acyclic complex*

$$0 \to M_{d+1} \to M_d \to \cdots \to M_1 \to M_0 \to 0$$

together with isomorphisms

$$Z \xrightarrow{\sim} \mathsf{cok}(\mathcal{E}(?, M_1) \to \mathcal{E}(?, M_0)) \xrightarrow{\sim} \mathsf{Ext}^1_{\mathcal{E}}(?, Z_1)\,,$$

where Z_1 is the kernel of $M_1 \to M_0$.

c) *Let Z be a finitely presented $\underline{\mathcal{M}}$-module. Then Z considered as an $\mathcal{M}$-module lies in $\mathsf{per}(\mathcal{M})$ and we have a canonical isomorphism*

$$\mathsf{per}(\mathcal{M})(?, Z[d+1]) \xrightarrow{\sim} D\,\mathsf{per}(\mathcal{M})(Z, ?).$$

Proof. a) We choose a right $\mathcal{T}$-approximation $M_0' \to X$ in $\mathcal{C}$ and complete it to a triangle

$$Z_0' \to M_0' \to X \to SZ_0'.$$

Now we lift the triangle to a conflation

$$0 \to Z_0 \to M_0 \to X \to 0.$$

By repeating this process we inductively construct conflations

$$0 \to Z_i \to M_i \to Z_{i-1} \to 0$$

for $1 \leq i \leq d-1$. We splice these to obtain the complex

$$0 \to Z_{d-1} \to M_{d-2} \to \cdots \to M_0 \to X \to 0.$$

If follows from 5.5 below that $M_{d-1} = Z_{d-1}$ belongs to $\mathcal{M}$.

b) As in section 4, one sees that there is a conflation

$$0 \to Z_1 \to M_1 \to M_0 \to 0$$

together with isomorphisms

$$Z \xrightarrow{\sim} \mathsf{cok}(\mathcal{E}(?, M_1) \to \mathcal{E}(?, M_0)) \xrightarrow{\sim} \mathsf{Ext}^1_{\mathcal{E}}(?, Z_1).$$

Now we apply a) to $X = Z_1$.

c) Let P be the complex constructed in b). We have

$$\mathsf{per}(\mathcal{M})(P, P[d+1]) = \mathsf{per}(\mathcal{M})(P, Z[d+1]) = \mathsf{cok}(Z(M_d) \to Z(M_{d+1})).$$

Since $Z \xrightarrow{\sim} \mathsf{Ext}^1_{\mathcal{E}}(?, Z_1)$, we have to compute

$$\mathsf{cok}(\mathsf{Ext}^1_{\mathcal{E}}(M_d, Z_1) \to \mathsf{Ext}^1_{\mathcal{E}}(M_{d+1}, Z_1)).$$

Put $X = SZ_1$. We have $\mathsf{Ext}^1_{\mathcal{E}}(?, Z_1) = \underline{\mathcal{E}}(?, SZ_1) = \underline{\mathcal{E}}(?, X)$ and so we have to compute

$$\mathsf{cok}(\underline{\mathcal{E}}(M_d, X) \to \underline{\mathcal{E}}(M_{d+1}, X)).$$

We now apply part c) of proposition 5.5 to the image in $\underline{\mathcal{E}}$ of the complex

$$0 \to M_{d+1} \to M_d \to \ldots \to M_d$$

and the object $Y = Z_1 = S^{-1}X$. In the notations of proposition 5.5, we obtain that

$$\mathsf{cok}(GM_d \to GM_{d+1}) = GS^{-(d-1)}Y = \underline{\mathcal{E}}(S^{-(d-1)}Y, ?) = \underline{\mathcal{E}}(S^{-d}X, ?).$$

This yields in particular that

$$\mathsf{cok}(\underline{\mathcal{E}}(M_d, X) \to \underline{\mathcal{E}}(M_{d+1}, X)) = \underline{\mathcal{E}}(S^{-d}X, X) = \mathcal{C}(X, S^d X).$$

Thus, the canonical linear form on $\mathcal{C}(X, S^d X)$ yields a morphism

$$\mathsf{per}(\mathcal{M})(?, P[d+1]) \to D\,\mathsf{per}(\mathcal{M})(P, ?).$$

We have to check that it is an isomorphism on all representable functors $M^\wedge[i]$, $i \in \mathbb{Z}$, $M \in \mathcal{M}$. Now the group

$$\mathsf{per}(\mathcal{M})(M^\wedge[i], P[d+1]) = \mathsf{per}(\mathcal{M})(M^\wedge, Z[d+1-i])$$

vanishes if $i \neq d+1$ and equals $\underline{\mathcal{E}}(M, X)$ for $i = d+1$. On the other hand, we have to compute the group

$$\mathsf{per}(\mathcal{M})(P, M^\wedge[i]).$$

For $i = d+1$, it is isomorphic to $\underline{\mathcal{E}}(S^{-d}X, M)$ by part c) of proposition 5.5. If we do not have $2 \leq i \leq d+1$, it clearly vanishes. For $2 \leq i \leq d$, it is the image of

$$\mathcal{E}(Z_{i-1}, M) \to \mathsf{Ext}^1(Z_{i-2}, M).$$

But $\mathsf{Ext}^1_{\mathcal{E}}(Z_{i-1}, M) = \underline{\mathcal{E}}(Z_{i-2}, SM)$ vanishes because Z_{i-2} is an iterate extension of objects in

$$S^{-(i-2)}\mathcal{T}, \ldots, S^{-1}\mathcal{T}, \mathcal{T}.$$

□

5.5. **Triangular resolutions.** We work with the notations and assumptions of section 5.1 and assume moreover that $d \geq 2$. Let Y be an object of $\mathcal{C}$. Let $T_0 \to Y$ be a right $\mathcal{T}$-approximation of Y. We define an object Z_0 by the triangle

$$Z_0 \to T_0 \to Y \to SZ_0.$$

Now we choose a right $\mathcal{T}$-approximation $T_1 \to Z_0$ and define Z_1 by the triangle

$$Z_1 \to T_1 \to Z_0 \to SZ_1.$$

We continue inductively constructing triangles

$$Z_i \to T_i \to Z_{i-1} \to SZ_i$$

for $1 < i \leq d-2$. By the proposition below, the object Z_{d-2} belongs to $\mathcal{T}$. We put $T_{d-1} = Z_{d-2}$. The triangles

$$Z_i \to T_i \to Z_{i-1} \to SZ_i$$

yield morphisms

$$S^{-(d-1)}Y \to S^{-(d-2)}Z_0 \to \cdots \to S^{-1}Z_{d-3} \to Z_{d-2}$$

so that we obtain a complex

$$S^{-(d-1)}Y \to T_{d-1} \to T_{d-2} \to \ldots \to T_1 \to T_0 \to Y.$$

Put $\mathcal{T}\text{-}\mathsf{mod} = \mathsf{mod}(\mathcal{T}^{op})$. Let $F : \mathcal{C} \to \mathsf{mod}\,\mathcal{T}$ and $G : \mathcal{C} \to \mathcal{T}\text{-}\mathsf{mod}$ be the functors which take an object X of $\mathcal{C}$ to $\mathcal{C}(?, X)|\mathcal{T}$ respectively $\mathcal{C}(X, ?)|\mathcal{T}$. The following proposition is related to Theorem 2.1 of [33].

Proposition. *In the above notations, we have:*

a) *The object $T_{d-1} = Z_{d-2}$ belongs to $\mathcal{T}$.*

b) *The image of the complex*

$$T_{d-1} \to T_{d-2} \to \ldots \to T_1 \to T_0 \to 0$$

under F has homology FY in degree 0.

c) *The image of the complex*

$$0 \to T_{d-1} \to T_{d-2} \to \ldots \to T_1 \to T_0$$

under G has homology $GS^{-(d-1)}Y$ in degree $d-1$. If Y belongs to $(S^{-1}\mathcal{T}) * \mathcal{T}$, then its homology in degree $d-2$ vanishes.

Proof. a) From the construction of the Z_i, we get that, for $T \in \mathcal{T}$, we have

$$\mathsf{Ext}^1(T, Z_i) = 0$$

for all i and that

$$\mathsf{Ext}^j(T, Z_i) \xrightarrow{\sim} \mathsf{Ext}^{j-1}(T, Z_{i-1}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathsf{Ext}^1(T, Z_{i-j+1}) = 0$$

for all $2 \leq j \leq i+1$. Thus we have $\mathsf{Ext}^j(T, Z_{d-2}) = 0$ for $1 \leq j \leq d-1$ and Z_{d-1} indeed belongs to $\mathcal{T}$.

b) This follows readily from the construction.

c) We consider the triangle

$$S^{-1}T_{d-2} \to S^{-1}Z_{d-3} \to T_{d-1} \to T_{d-2}.$$

For $T \in \mathcal{T}$, the long exact sequence obtained by applying $\mathcal{C}(?, T)$ combined with the vanishing of $\mathcal{C}(S^{-1}T_{d-2}, T)$ shows that we have the isomorphism

$$\mathsf{cok}(\mathcal{C}(T_{d-2}, T) \to \mathcal{C}(T_{d-1}, T)) \xrightarrow{\sim} \mathcal{C}(S^{-1}Z_{d-3}, T).$$

Now for $-1 \leq i \leq d-4$, the maps $S^{-1}Z_i \to Z_{i+1}$ induce isomorphisms

$$\mathcal{C}(S^{-(d-2)+i+1}Z_{i+1}, T) \xrightarrow{\sim} \mathcal{C}(S^{-(d-2)+i}Z_i, T)$$

because of the triangles

$$S^iT_{i+1} \to S^iZ_i \to S^{i+1}Z_{i+1} \to S^{i+1}T_{i+1}$$

and the fact that

$$\mathcal{C}(S^{-(d-2)+i}T_{i+1}, T) = 0 = \mathcal{C}(S^{-(d-2)+i+1}T_{i+1}, T).$$

Here, we have taken $Z_{-1} = Y$. Finally, we obtain the isomorphism

$$\mathsf{cok}(\mathcal{C}(T_{d-2}, T) \to \mathcal{C}(T_{d-1}, T)) \xrightarrow{\sim} \mathcal{C}(S^{-(d-1)}Y, T).$$

For the last assertion, let $T \in \mathcal{T}$. By the sequence

$$T_{d-1} \to T_{d-2} \to Z_{d-3} \to ST_{d-1}\,,$$

the sequence

$$\mathcal{C}(Z_{d-3}, T) \to \mathcal{C}(T_{d-2}, T) \to \mathcal{C}(T_{d-1}, T)$$

is exact so that it suffices to show that

$$\mathcal{C}(T_{d-3}, T) \to \mathcal{C}(Z_{d-3}, T)$$

is surjective. Thanks to the triangle

$$S^{-1}Z_{d-4} \to Z_{d-3} \to T_{d-3} \to Z_{d-4}\ ,$$

it suffices to show that $\mathcal{C}(S^{-1}Z_{d-4}, T)$ vanishes. This is clear because the object Z_{d-4} is an iterated extension of objects in

$$S^{-(d-2)}\mathcal{T}, S^{-(d-3)}\mathcal{T}, \dots, S^{-1}\mathcal{T}, \mathcal{T}.$$

□

5.6. **d-cluster tilting categories from tilting subcategories.** Let $\mathcal{H}$ be an Ext-finite hereditary k-linear Krull-Schmidt category whose bounded derived category $\mathcal{D} = \mathcal{D}^b(\mathcal{H})$ admits a Serre functor Σ. Let $d \geq 2$ be an integer. The *d-cluster category*

$$\mathcal{C} = \mathcal{D}/(S^{-d} \circ \Sigma)^{\mathbb{Z}}\ ,$$

has been considered in [38] [48] [47]. Let $\pi : \mathcal{D} \to \mathcal{C}$ be the projection functor. Let $\mathcal{T} \subset \mathcal{D}$ be a tilting subcategory, *i.e.* a full subcategory whose objects form a set of generators for the triangulated category $\mathcal{D}$ and such that

$$\mathcal{D}(T, S^i T') = 0$$

for all $T, T' \in \mathcal{T}$ and all $i \neq 0$. It follows that there is a triangle equivalence, *cf.* [36],

$$\mathcal{D}^b(\mathsf{mod}\,\mathcal{T}) \to \mathcal{D}^b(\mathcal{H})$$

which takes the projective module $T^\wedge = \mathcal{T}(?, T)$ in $\mathsf{mod}\,\mathcal{T}$ to T, for each $T \in \mathcal{T}$. We assume that $\mathcal{T}$ is locally bounded, *i.e.* for each indecomposable T, there are only finitely many indecomposables T' in $\mathcal{T}$ such that $\mathcal{T}(T, T') \neq 0$ or $\mathcal{T}(T', T) \neq 0$. We denote the *Nakayama functor* of $\mathsf{mod}\,\mathcal{T}$ by ν. Recall from section 3.7 of [24] that it is the unique functor endowed with isomorphisms

$$D\,\mathsf{Hom}(P, M) = \mathsf{Hom}(M, \nu P)$$

for all finitely generated projectives P and all finitely presented $\mathcal{T}$-modules M. It follows that if we view a projective P as an object of $\mathcal{D}^b(\mathsf{mod}\,\mathcal{T})$, we have $\Sigma P \xrightarrow{\sim} \nu P$. The category of finitely generated projectives is functorially finite in $\mathcal{D}^b(\mathsf{mod}\,\mathcal{T})$: Indeed, if M is an object of $\mathcal{D}^b(\mathsf{mod}\,\mathcal{T})$, we obtain a left approximation by taking the morphism $P \to M$ induced by an epimorphism $P \to H^0(M)$ with projective P, and we obtain a right approximation by taking the morphism $M \to P'$ induced by a monomorphism $H^0(\Sigma M) \to \nu P'$ into an injective $\nu P'$. Therefore, $\mathcal{T}$ is functorially finite in $\mathcal{D}$.

The first part of the following proposition is proved in section 3 of [8] for hereditary categories with a tilting object for $d = 2$, see also Proposition 2.6 of [49]. For arbitrary d, it was proved in [48] when $\mathcal{T}$ is given by a tilting module over a hereditary algebra. The second part has been proved in [1] for $d = 2$ and hereditary algebras.

Proposition. *Assume that the objects of $\mathcal{T}$ have their homology concentrated in degrees i with $-(d-2) \leq i \leq 0$. Then $\pi(\mathcal{T})$ is a d-cluster tilting subcategory in $\mathcal{C}$ and for all objects T, T' of $\mathcal{T}$, we have a functorial isomorphism compatible with compositions*

$$\pi(\mathcal{T})(\pi(T), \pi(T')) = \mathcal{T}(T, T') \oplus \mathsf{Ext}^d_{\mathcal{T}}(\nu T^\wedge, T'^\wedge).$$

Note that the proposition does not produce all d-cluster tilting subcategories of $\mathcal{C}$. For example, as one checks easily, the non connected algebra $k \times k$ is 3-cluster tilted of type A_2.

Proof. Put $F = \Sigma^{-1}S^d$. For $n \in \mathbb{Z}$, denote by $\mathcal{D}_{\leq n}$ the full subcategory of $\mathcal{D}$ formed by the objects X with $H^i(X) = 0$ for $i > n$ and by $\mathcal{D}_{\geq n}$ the full subcategory formed by the Y with $H^i(Y) = 0$ for $i < n$. Then $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ are the aisles of the natural t-structure on $\mathcal{D}$. Thus the subcategory $\mathcal{D}_{\geq 1}$ is the right perpendicular subcategory of $\mathcal{D}_{\leq 0}$ and we have in particular $\mathsf{Hom}(X,Y)=0$ if $X\in\mathcal{D}_{\leq 0}$ and $Y\in\mathcal{D}_{\geq 1}$. Since $\mathcal{H}$ is hereditary, we also have $\mathsf{Hom}(X,Y)=0$ if $X\in\mathcal{D}_{\geq 0}$ and $Y\in\mathcal{D}_{\leq -2}$. We have $F\mathcal{D}_{\leq 0}\subset\mathcal{D}_{\leq -(d-1)}$. Indeed, for $X\in\mathcal{D}_{\leq 0}$ and $Y\in\mathcal{D}_{\geq -(d-2)}$, we have

$$\mathcal{D}(FX,Y)=\mathcal{D}(\Sigma^{-1}S^dX,Y)=D\mathcal{D}(Y,S^dX)=0$$

because S^dX belongs to $\mathcal{D}_{\leq -d}$ and Y to $\mathcal{D}_{\geq -d+2}$.

Let $X\in\mathcal{D}$. Since we have $T\in\mathcal{D}_{\leq 0}\cap\mathcal{D}_{\geq -(d-2)}$, there are only finitely many integers n such that $\mathcal{D}(T,F^nX)\neq 0$ or $\mathcal{D}(F^nX,T)\neq 0$ for some $T\in\mathcal{T}$. Since $\mathcal{T}$ is functorially finite in $\mathcal{D}$, it follows that $\pi(\mathcal{T})$ is functorially finite in $\mathcal{C}$.

Let us show that for $T,T'\in\mathcal{T}$, the space $\mathcal{D}(T,S^iF^nT')$ vanishes for $0<i$ and $n\geq 0$. This is clear if $n=0$. For $n\geq 1$, we have

$$S^iF^nT'\in\mathcal{D}_{\leq -n(d-1)-i}\text{ and }T\in\mathcal{D}_{\geq -(d-2)}.$$

Since we have

$$-(d-2)+n(d-1)+i=(n-1)(d-1)+1+i\geq 2$$

it follows that $\mathcal{D}(T,S^iF^nT')$ vanishes because $\mathcal{H}$ is hereditary.

Let us show that $\mathcal{D}(F^nT,S^iT')$ vanishes for $i<d$ and $n>0$. Let us first consider the case $n=1$. Then we have

$$\mathcal{D}(FT,S^iT')=\mathcal{D}(\Sigma^{-1}S^dT,S^iT')=D\mathcal{D}(S^iT',S^dT)=D\mathcal{D}(T',S^{d-i}T)=0$$

since $i<d$. Now assume that $n>1$. Then we have

$$F^nT\in\mathcal{D}_{\leq -n(d-1)}\text{ and }S^iT'\in\mathcal{D}_{\geq -(d-2)-i}.$$

Since we have

$$-(d-2)-i+n(d-1)=(n-1)(d-1)+1-i\geq(d-1)+1-i=d-i>0\,,$$

it follows that $\mathsf{Hom}(F^nT,S^iT')$ vanishes.

We conclude that $\mathcal{D}(T,S^iF^nT')$ vanishes for $0<i<d$ and all $n\in\mathbb{Z}$ so that we have

$$\mathcal{C}(T,S^iT')=0$$

for $0<i<d$. By re-examining the above computation, we also find that $\mathcal{D}(T,F^nT')$ vanishes for each integer $n\neq 0,1$ and that we have

$$\mathcal{D}(T,FT')=\mathcal{D}(T,\Sigma^{-1}S^dT')=\mathcal{D}(\Sigma T,S^dT')=\mathsf{per}(\mathcal{T})(\Sigma T^\wedge,S^dT'^\wedge)=\mathsf{Ext}^d_{\mathcal{T}}(\nu T^\wedge,T'^\wedge).$$

Let $X\in\mathcal{D}$. Let us show that if

$$\mathsf{Ext}^i(\pi(T),\pi(X))=0$$

for all $0<i<d$ and all $T\in\mathcal{T}$, then $\pi(X)$ belongs to $\pi(\mathcal{T})$. We may assume that X is indecomposable and belongs to

$$\mathcal{D}_{\leq 0}\cap\mathcal{D}_{\geq -(d-1)}.$$

Let $\mathcal{U}_{\leq 0},\mathcal{U}_{\geq 0}$ be the aisles obtained as the images of the natural aisles in $\mathcal{D}^b(\mathsf{mod}\,\mathcal{T})$ under the triangle equivalence $\mathcal{D}^b(\mathsf{mod}\,\mathcal{T})\xrightarrow{\sim}\mathcal{D}$ associated with $\mathcal{T}$. We claim that we have

$$\mathcal{D}_{\leq 0}\subset\mathcal{U}_{\leq(d-1)}.$$

Indeed, for $Y\in\mathcal{D}_{\leq 0}$ and $i\geq d$, we have $S^{-i}T\in\mathcal{D}_{\geq 2}$ so that $\mathcal{D}(S^{-i}T,Y)=0$ and this implies the claim. Since X lies in $\mathcal{D}_{\leq 0}$ it follows that $X\in\mathcal{U}_{\leq(d-1)}$. Now the assumption on X yields that

$$\mathcal{D}(S^{-i}T,X)=0$$

for $0<i\leq d-1$. Thus X lies in $\mathcal{U}_{\leq 0}$. Now we claim that we have

$$\mathcal{D}_{\geq 0}\subset\mathcal{U}_{\geq 0}.$$

Indeed, for $Y \in \mathcal{D}_{\geq 0}$ and $i > 0$, we have $\mathcal{D}(S^iT, Y) = 0$ since $S^iT \in \mathcal{D}_{<0}$. This implies the claim. Since X lies in $\mathcal{D}_{\geq -(d-1)}$, it follows that X lies in $\mathcal{U}_{\geq -(d-1)}$. Finally, for $T \in \mathcal{T}$ and $0 < i < d$, we have

$$0 = \mathcal{D}(T, S^iFX) = \mathcal{D}(T, S^{i-d}\Sigma X) = D\mathcal{D}(S^{i-d}X, T) = D\mathcal{D}(X, S^{d-i}T).$$

Thus we have $\mathcal{D}(X, S^iT) = 0$ for $0 < i < d$. But since X lies in $\mathcal{D}_{\geq -(d-1)}$ and T in $\mathcal{D}_{\leq 0}$ we also have $\mathcal{D}(X, S^iT) = 0$ for all $i \geq d$. Thus X is left orthogonal to $\mathcal{U}_{\leq -1}$. Therefore, the object $Y \in \mathcal{D}^b(\operatorname{mod} \mathcal{T})$ corresponding to X via the triangle equivalence associated with $\mathcal{T}$ is in $\mathcal{D}^b_{\leq 0}(\operatorname{mod} \mathcal{T})$ and left orthogonal to $\mathcal{D}^b_{\leq -1}(\operatorname{mod} \mathcal{T})$. Since $\operatorname{mod} \mathcal{T}$ is of finite global dimension, this implies that Y is a projective $\mathcal{T}$-module. So X lies in $\mathcal{T}$. □

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