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A-infinity algebras, modules and functor categories

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Abstract. In this survey, we first present basic facts on A-infinity algebras and modules including their use in describing triangulated categories. Then we describe the Quillen model approach to A-infinity structures following K. Lefèvre's thesis. Finally, starting from an idea of V. Lyubashenko's, which is being fully developped in [4], we give a conceptual construction of A-infinity functor categories using a suitable closed monoidal category of cocategories. In particular, this yields a natural construction of the bialgebra structure on the bar construction of the Hochschild complex of an associative algebra.

1. Introduction

A-infinity spaces and A-infinity algebras were invented at the beginning of the sixties by Stasheff [**39**]. In the seventies and the eighties, they were developped further by Smirnov [**38**], Kadeishvili [**19**], Prouté [**33**], Huebschmann [**16**], [**17**], ... especially with a view towards applications in topology. At the beginning of the nineties, the relevance of A-infinity structures in geometry and physics became apparent through the work of Getzler-Jones [**13**], Stasheff [**40**], Fukaya [**10**], Kontsevich [**22**], ..., and later Kontsevich-Soibelman [**25**], Seidel [**37**], ... We refer to [**20**], [**21**] for a more detailed introduction with numerous references. The present survey consists of three parts:

1. In sections 1–3, we will define A_∞-algebras and examine their basic properties. Then we will define A_∞-modules, the derived category and conclude with the description of triangulated categories via A_∞-algebras.
2. In section 4, we will present the interpretation of A_∞-algebras (resp. A_∞-modules) as the fibrant objects in the model category of certain differential graded coalgebras (resp. comodules) following Lefèvre [**26**].
3. In section 5, starting from an idea of V. Lyubashenko's [**28**] we will give a conceptual construction of A_∞-functor categories and use it to construct the canonical bialgebra structure on the cobar construction of the Hochschild complex of an associative algebra.

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2. A-infinity algebras

2.1. Notations. We will follow Fukaya's sign and degree conventions. For this, we need to introduce some notation:

Let k be a field. If V is a graded vector space, *i. e.*

$$V = \bigoplus_{p \in \mathbf{Z}} V^p ,$$

we denote by SV or $V[1]$ the graded space with $(SV)^p = V^{p+1}$ for all $p \in \mathbf{Z}$. We call SV the *suspension* or the *shift* of V.

If $f : V \to V'$ and $g : W \to W'$ are homogeneous maps between graded spaces, their *tensor product*

$$f \otimes g : V \otimes W \to V' \otimes W'$$

is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g|\,|v|} f(v) \otimes g(w)$$

for all homogeneous elements $v \in V$ and $w \in W$.

If V and V' are complexes, *i. e.* endowed with differentials d homogeneous of degree 1 and square 0, we put $d_{SV} = -d_V$ and, for a homogeneous map $f : V \to V'$,

$$d(f) = d_{V'} \circ f - (-1)^{|f|} f \circ d_V .$$

Thus, f is a morphism of complexes iff $d(f) = 0$ and two morphisms of complexes f and f' are homotopic iff there is a morphism of graded spaces h such that $f' = f + d(h)$. We will use that homotopic morphisms induce the same map in homology.

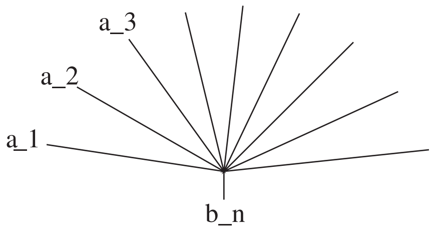
2.2. A-infinity algebras. An A_∞-*algebra* is a graded space A endowed with maps

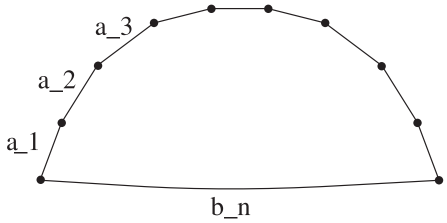
$$b_n : (SA)^{\otimes n} \to SA$$

defined for $n \geq 1$, homogeneous of degree 1 and such that, for all $n \geq 1$, we have

$$\sum_{i+j+l=n} b_{i+1+l} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes l}) = 0 \tag{2.1}$$

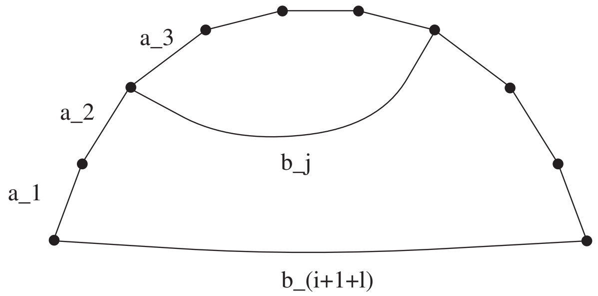
as maps from $(SA)^{\otimes p}$ to SA. Here, the symbol $\mathbf{1}$ denotes the identity map of SA. We visualize b_n either as a planar tree with n leaves and one root or as a halfdisk whose upper arc is divided into segments, each of which symbolizes an 'input', and whose base segment symbolizes the 'output' of the operation.





Using this last representation, the defining identity (2.1) is depicted as follows:

$\sum \pm$



$= 0.$

Let us analyze the defining identity for small values of n: For $n = 1$, it states that $b_1^2 = 0$, so that (SA, b_1) is a complex. We also make A into a complex by endowing it with the shifted differential

$$m_1 = -b_1.$$

For $n = 2$, the defining identity becomes

$$b_1 b_2 + b_2(b_1 \otimes \mathbf{1} + \mathbf{1} \otimes b_1) = 0.$$

Note that $b_1 \otimes \mathbf{1} + \mathbf{1} \otimes b_1$ is the differential of $SA \otimes SA$ so that we obtain $d(b_2) = 0$, which means that $b_2 : SA \otimes SA \to SA$ is a morphism of complexes.

For $n = 3$, the identity (2.1) becomes

$$b_2\,(b_2 \otimes \mathbf{1} + \mathbf{1} \otimes b_2) + b_1 b_3 + b_3(b_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes b_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes b_1) = 0.$$

Here the second summand is $d(b_3)$ whereas the first is, up to a sign, the associator for the binary operation b_2. If we define $m_2 : A \otimes A \to A$ by

$$m_2(x, y) = (-1)^{|x|} b_2(x, y),$$

then we obtain that m_2 is associative up to a homotopy given by b_3.

For each $n > 3$, the identity (2.1) states that the maps $b_2, \ldots, b_{n-1}$ satisfy a certain quadratic identity up to a homotopy given by b_n. In this sense, an A_∞-algebra is an algebra associative up to a given system of higher homotopies.

It is a direct consequence of the definition, that if b_n vanishes for each $n \geq 3$, then (A, m_1, m_2) is a differential graded (=dg) algebra, *i. e.* m_2 is associative and m_1 a differential compatible with m_2 through the (graded) Leibniz rule. Conversely, each dg algebra gives rise to an A_∞-algebra with vanishing b_n, $n \geq 3$.

In particular, each ordinary associative algebra can be viewed as an A_∞-algebra concentrated in degree 0, and conversely.

2.3. Examples via deformations. Following an idea of Penkava-Schwarz [**32**], let us exhibit a large class of easily constructed but non trivial examples of A_∞-algebras. Let B be an ordinary algebra and $N \geq 1$ an integer. Let ε be an indeterminate of degree $2-N$. We first endow the graded space $A = B[\varepsilon]/(\varepsilon^2)$ with the trivial A_∞-structure given by the map b_2 induced by the multiplication of B and the maps $b_n = 0$ for all $n \neq 2$. Now let

$$c : B^{\otimes N} \to B$$

be any linear map. Define deformed multiplications

$$b'_n = \begin{cases} b_n & n \neq N \\ b_N + \varepsilon c & n = N. \end{cases}$$

Then it is easy to see that A *endowed with the* b'_n *is an* A_∞*-algebra iff* c *is a Hochschild cocycle for* B.

2.4. Weak A-infinity algebras. A *weak* A_∞*-algebra* is a graded space A endowed with maps $b_0 : k \to SA$ and b_n, $n \geq 0$, such that the identity 2.1 holds for all $n \geq 0$. The preceding example then naturally extends to the case where $N = 0$, where we start from a Hochschild 0-cocycle, *i. e.* a central element c of B. In general, in a weak A_∞-algebra, we have

$$b_1^2 = -b_2(b_0 \otimes \mathbf{1} + \mathbf{1} \otimes b_0) \neq 0$$

so that the homology with respect to b_1 is no longer defined and the above remarks no longer apply. Little is known about weak A_∞-algebras in general, but they do appear in nature as deformations.

2.5. Morphisms and quasi-isomorphisms. A *morphism of* A_∞*-algebras* $f : A \to B$ is given by maps

$$f_n : (SA)^{\otimes n} \to SB\ ,\ n \geq 1\ ,$$

homogeneous of degree 0 such that, for all $n \geq 1$, we have

$$\sum_{i+j+l=n} f_{i+1+l} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes l}) = \sum_{i_1+\cdots i_s=n} b_s \circ (f_{i_1} \otimes \cdots \otimes f_{i_s}).$$

By looking at this equation for $n = 1$ and $n = 2$ we see that f_1 then induces a morphism of complexes from (A, m_1) to (B, m_1) which is compatible with m_2 up to an homotopy given by f_2. In particular, f_1 induces an *algebra morphism*

$$H^*A \to H^*B.$$

By definition, f is an A_∞-quasi-isomorphism if f_1 is a quasi-isomorphism (*i. e.* induces an isomorphism in homology).

The *composition* $f \circ g$ of two morphisms is given by

$$(f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_s \circ (g_{i_1} \otimes \cdots g_{i_s}).$$

The *identical morphism of* SA is given by $f_1 = \mathbf{1}$ and $f_n = 0$ for all $n \geq 2$.

It is easy to see that we do obtain a category. It contains the category of dg algebras and their morphisms as a non-full subcategory.

PROPOSITION 2.1. (*cf.* section 4.8) For each A_∞-algebra A, there is a universal A_∞-algebra morphism $\varphi : A \to U(A)$ to a dg algebra $U(A)$. Moreover, φ is an A_∞-quasi-isomorphism.

The universal property means that for each dg algebra B, each A_∞-morphism $f : A \to B$ factors as $f = g \circ \varphi$ for a unique morphism of dg algebras $g : U(A) \to B$. The proposition tells us that, up to A_∞-quasi-isomorphism, A_∞-algebras are quite similar to dg algebras. However, in other respects, they are radically different from dg algebras, as the following proposition shows.

Proposition 2.2. Let A be an A_∞-algebra, V a complex and $f_1 : A \to V$ a quasi-isomorphism of complexes. Then V admits a structure of A_∞-algebra such that f_1 extends to an A_∞-quasi-isomorphism $f : A \to V$.

The analogous statement for dg algebras and their morphisms is of course completely wrong. For our complex V, we can take in particular the graded space H^*V with the zero differential (since we work over a field, the canonical surjection from the cycles to the homology of V splits and we obtain f_1 by composing a right inverse with the inclusion of the cycles). Then we obtain the first part of the

Theorem 2.3. *If A is an A_∞-algebra, then H^*A admits an A_∞-algebra structure such that*

1. *$b_1 = 0$ and b_2 is induced from b_2^A, and*
2. *there is an A_∞-quasi-isomorphism $A \to H^*A$ inducing the identity in homology.*

Moreover, this structure is unique up to (non unique) A_∞-isomorphism.

Note that uniqueness up to A_∞-quasi-isomorphism is trivial. The point is that here we can omit 'quasi'. An A_∞-algebra is *minimal* if $b_1 = 0$. The *minimal model* of an A_∞-algebra A is the space H^*A endowed with 'the' structure provided by the theorem. This structure can be computed as follows [**24**]: Choose

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "p"] & H^*A \arrow[l, shift left, "i"] \end{tikzcd}$$

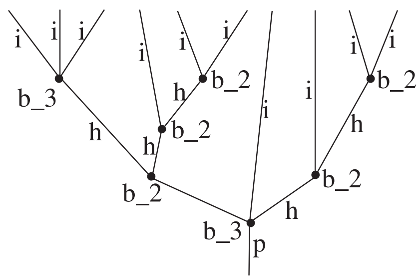
such that p and i are morphisms of complexes of degree 0 and h is a homogeneous map of degree -1 such that

$$pi = \mathbf{1}\ ,\ ip = \mathbf{1} + d(h)\ ,\ h^2 = 0.$$

Then the nth multiplication of the minimal model is constructed as

$$b_n^{min} = \sum_T b_n{}^T$$

where T ranges over the planar rooted trees T with n leaves and b_n^T is given by composing the tree-shaped diagram obtained by labelling each leaf by i, each branch point with m branches by b_m, each internal edge by h and the root by p.



2.6. Yoneda algebras. Let B be a unital associative algebra, M a (right) B-module and $P \to M$ a projective resolution. Let $A = \mathsf{Hom}_B(P,P)$ be the differential graded endomorphism algebra of P (its nth component consists of the morphisms of graded objects of degree n and its differential is the supercommutator with the differential of P). Then A is in particular an A_∞-algebra and thus has a minimal model. Now the homology H^*A is isomorphic, as an algebra for m_2, to the Yoneda algebra

$$\mathsf{Ext}^*_B(M,M).$$

Thus we obtain higher multiplications on the Yoneda algebra. The simplest case where these are non-trivial is that of the algebra B given by the quiver

$$1 \xrightarrow{\gamma} 2 \xrightarrow{\beta} 3 \xrightarrow{\alpha} 4$$

with the relation $\alpha\beta\gamma = 0$ and the module M equal to the sum of the four simple B-modules. Then the Yoneda algebra is given by the quiver

$$\begin{tikzcd} 1 & 2 \arrow[l, "c"'] & 3 \arrow[l, "b"'] & 4 \arrow[l, "a"'] \arrow[lll, bend right, "e"'] \end{tikzcd} ,$$

where the arrows a, b, c are of degree 1, the arrow e is of degree 2, we have $m_2(c,b) = 0$, $m_2(b,a) = 0$ and $b_3(c,b,a) = e$.

Another example is given by the algebras $B = k[x]/(x^n)$, where n is an integer ≥ 3, and the simple B-module $M = k$. In this case, the underlying graded algebra of the Yoneda algebra is isomorphic to $k[u,v]/(v^2)$, where u is of degree 2 and v of degree 1. Note that this is independent of n. In contrast, the A_∞-structure depends on n: it is given by the operations b_l which vanish for $l \neq 2, n$ such that b_2 is given by the Yoneda multiplication m_2 and b_n is the unique $k[u]$-multilinear map which vanishes if one of the arguments is in $k[u]$ and satisfies

$$b_n(v, \ldots, v) = u.$$

This example was treated by D. Madsen [**29**]. More generally, J. Chuang and A. King have computed the A_∞-structure on $Ext^*_B(M,M)$ for the sum M of the simple modules over an arbitrary monomial algebra B, *cf.* [**7**]. Other examples have been computed by Lu-Palmieri-Wu-Zhang [**27**].

2.7. On the meaning of the higher multiplications. This section should be skipped on a first reading. Suppose that A is a minimal A_∞-algebra. Endow $H^*(A)$ with the associative product obtained from b_2. Then b_3 yields a 3-cocycle homogeneous of degree -1 for the natural grading on H^*A and thus an element in Hochschild cohomology

$$\gamma \in HH^{3,-1}(H^*A, H^*A).$$

This class determines the Massey products in $H^*(A)$, *cf.* for example [**30**]. It also controls the realizability (up to direct factors) of graded H^*A-modules M as the homology $H^*\widetilde{M}$ of an A-module $\widetilde{M}$, *cf.* [**3**].

One can also interpret the 'higher' operations b_n. To simplify the exposition, let us change the setting slightly: Let $\mathcal{B}$ be a small k-linear abelian category and consider its bounded derived category $\mathsf{D}^b(\mathcal{B})$. Thanks to the shift functor, it becomes a graded category with morphism spaces

$$\mathsf{Hom}^\bullet(L,M) = \bigoplus_{n\in\mathbf{Z}} \mathsf{Hom}_{\mathsf{D}^b(\mathcal{B})}(L, M[n]).$$

This category can be completed into a minimal A_∞-category in much the same way that a Yoneda algebra can be completed into a minimal A_∞-algebra. In particular, the operation b_3 determines an element in the Hochschild cohomology

$$\gamma \in HH^{3,-1}(\mathsf{D}^b(\mathcal{B}), \mathsf{D}^b(\mathcal{B}))$$

of the graded category $\mathsf{D}^b(\mathcal{B})$ (defined as in [**31**]). One can show that this element determines the triangulated structure of the k-linear category $\mathsf{D}^b(\mathcal{B})$. But in fact, it determines more, namely it allows one to reconstruct the k-linear category

$$\mathsf{D}^b(\mathsf{fun}(\cdot \to \cdot, \mathcal{B}))$$

from $\mathsf{D}^b(\mathcal{B})$, where $\cdot \to \cdot$ denotes the k-category with 2 objects and one non trivial morphism and fun the functor category. For this, one notes that γ corresponds to $\mathsf{D}^b(\mathsf{fun}(\cdot \to \cdot, \mathcal{B}))$ considered as an extension of $\mathsf{fun}(\cdot \to \cdot, \mathsf{D}^b(\mathcal{B}))$ by the bimodule (in the sense of [**31**])

$$(L \to M) \mapsto \mathsf{Hom}_{\mathsf{D}^b(\mathcal{B})}(L, M[1]).$$

In a similar manner, for $n \geq 1$, the operations $b_2, \ldots, b_{n+2}$ allow one to reconstruct

$$\mathsf{D}^b(\mathsf{fun}((\cdot \to \cdot)^n, \mathcal{B}))$$

from $\mathsf{D}^b(\mathcal{B})$.

2.8. Units. A *strict unit* for an A_∞-algebra A is an element $1 \in A^0$ which is a unit for m_2 and such that, for $n \neq 2$, the map b_n takes the value 0 as soon as one of its arguments equals 1. Unfortunately, strict unitality is not preserved by A_∞-quasi-isomorphisms. A *homological unit* for A is a unit for the associative algebra H^*A with the multiplication induced by m_2. Homological unitality is clearly preserved under A_∞-quasi-isomorphism but is not easy to handle in practical computations. Fortunately, it turns out that the two notions are not very different:

PROPOSITION 2.4 ([**26**, 3.2.1]). *Each (resp. minimal) homologically unital A_∞-algebra is A_∞-quasi-isomorphic (resp. A_∞-isomorphic) to a strictly unital A_∞-algebra.*

3. A-infinity modules

Let A be a homologically unital A_∞-algebra. An *A_∞-module* is a graded space M with maps

$$b_n : SM \otimes (SA)^{\otimes n-1} \to SM\ ,\ n \geq 1\ ,$$

homogeneous of degree 1 such that the identity 2.1 holds for all $n \geq 1$ (where we have to interpret b_n as b_n^A or b_n^M according to the type of its arguments) and that the induced action

$$H^*M \otimes H^*A \to H^*M$$

is unital. For example, the A_∞-algebra A can be viewed as a module over itself: the free module of rank one. The notions of *morphism* and *quasi-isomorphism* of A_∞-modules are defined in the natural way. The *derived category* $\mathsf{D}_\infty A$ is defined as the localization of the category of A_∞-modules (with degree 0 morphisms) with respect to the class of quasi-isomorphisms. Thus, its objects are all A_∞-modules and its morphisms are obtained from morphisms of A_∞-modules by formally inverting all quasi-isomorphisms. It turns out that the derived category is naturally a triangulated category. The *perfect derived category* $\mathsf{per}\, A$ is defined as the closure of the free A-module of rank one under shifts in both directions, extensions and passage to direct factors.

When A is an ordinary unital associative algebra, we have a natural functor from the category of complexes over the category $\mathsf{Mod}\, A$ of (right) A-modules to the category of A_∞-modules. This functor is faithful but neither full nor essentially surjective. Nevertheless, it induces an equivalence

$$\mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}_\infty\, A.$$

Under this equivalence, the perfect derived category corresponds to the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective A-modules.

Let B be another homologically unital A_∞-algebra. An A_∞-*bimodule* is given by a graded space X with maps

$$b_{i,j} : (SA)^{\otimes i} \otimes SX \otimes (SB)^{\otimes j} \to SX \, , \; i+j \geq 0 \, ,$$

satisfying the identity 2.1 for all $i, j \geq 0$ and such that H^*X becomes a unital H^*A-H^*B-bimodule. For such a bimodule, one can define the *tensor product*

$$? \overset{\infty}{\otimes}_A X : \mathsf{D}_\infty\, A \to \mathsf{D}_\infty\, B \, , \; M \mapsto \bigoplus_{i=0}^{\infty} M \otimes (SA)^{\otimes i} \otimes X$$

(we only indicate the underlying graded space). The A_∞-algebras A and B are *derived equivalent* if there is an A_∞-bimodule X such that the associated tensor product is an equivalence.

This generalizes the now classical notion of derived equivalence for ordinary algebras: If A and B are ordinary algebras, then, by J. Rickard's theorem [**35**], they are derived equivalent iff B admits a tilting complex (*e. g.* a tilting module) with endomorphism ring A.

Let us call *algebraic* a triangulated category which is the homotopy category associated with a stable k-linear Quillen model category, *cf.* section 4.1 and [**15**] (recall that k is the ground field). The class of algebraic triangulated categories contains all homotopy categories of complexes over k-linear categories and is stable under passage to triangulated subcategories and Verdier localizations. Thus, it contains all triangulated categories 'of algebraic origin', *e. g.* derived categories of categories of coherent sheaves.

Recall that an additive category has *split idempotents* if every idempotent endomorphism admits a kernel (and thus gives rise to a direct sum decomposition). By a *generator* of a triangulated category $\mathcal{T}$, we mean an object G whose closure under shifts in both directions, extensions and passage to direct factors equals $\mathcal{T}$.

THEOREM 3.1 ([**26**, 7.6]). *Let $\mathcal{T}$ be a (k-linear) algebraic triangulated category with split idempotents and a generator G. Then there is a structure of A_∞-algebra on*

$$A = \bigoplus_{n \in \mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(G, G[n])$$

such that $m_1 = 0$, m_2 is given by composition and that the functor

$$\mathcal{T} \to \mathsf{Grmod}(A, m_2) \, , \; U \mapsto \bigoplus_{n \in \mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(G, U[n])$$

lifts to a triangle equivalence

$$\mathcal{T} \to \mathsf{per}(A).$$

Here Grmod denotes the category of graded right modules. For example, suppose that $\mathcal{T}$ is the bounded derived category of the category of coherent sheaves on projective n-space. Then, as Beilinson has shown [**5**] by 'resolving the diagonal', the object $G = \bigoplus_{i=0}^{n} O(-i)$ generates $\mathcal{T}$. Moreover, the algebra A is concentrated in degree 0 and of finite dimension and finite global dimension. Then we obtain equivalences

$$\mathsf{D}^b(\mathsf{coh}\,\mathbf{P}^n) \xrightarrow{\sim} \mathsf{per}(A) \xleftarrow{\sim} \mathsf{D}^b(\mathsf{mod}\,A)\,,$$

where $\mathsf{mod}\,A$ denotes the category of finite-dimensional A-modules.

Beautiful theorems on the existence of generators in triangulated categories of geometric origin are due to Bondal-Van den Bergh [**6**] and to Rouquier [**36**].

4. The bar-cobar equivalence

In this section, we present the interpretation of A_∞-algebras as the fibrant objects in the model category of certain differential graded coalgebras following [**26**]. For the reader's convenience we first recall the notion of model category.

4.1. Reminder on model categories. Model categories were invented by D. Quillen [**34**]. A good modern introduction can be found in [**8**]. A *model category* is a category $\mathcal{C}$ endowed with three classes of morphisms

- the class $\mathcal{W}$ whose elements are called *weak equivalences* and denoted by arrows with tilde $\xrightarrow{\sim}$,
- the class $\mathcal{F}ib$ whose elements are called *fibrations* and denoted by double-headed arrows $\twoheadrightarrow$.
- the class $\mathcal{C}of$ whose elements are called *cofibrations* and denoted by feathered arrows $\rightarrowtail$,

Each of these classes is supposed to be stable under compositions and to contain the identities. Moreover, the classes are supposed to satisfy the following axioms

MC1 The category $\mathcal{C}$ admits all finite limits and colimits (in particular, it admits an initial object $\emptyset$ and a terminal object $*$).

MC2 The class $\mathcal{W}$ is *saturated*, *i. e.* if two among f, g, and fg belong to $\mathcal{W}$ so does the third.

MC3 The three classes $\mathcal{W}$, $\mathcal{F}ib$ and $\mathcal{C}of$ are stable under retracts in the category of morphisms of $\mathcal{C}$.

MC4 In each commutative square of solid arrows of the following two types

$$\begin{tikzcd} {} \arrow[r] \arrow[d, tail, "i"', "\sim"] & {} \arrow[d, two heads, "p"] \\ {} \arrow[r] \arrow[ur, dotted] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r] \arrow[d, tail, "j"'] & {} \arrow[d, two heads, "q", "\sim"'] \\ {} \arrow[r] \arrow[ur, dotted] & {} \end{tikzcd}$$

the dotted arrow exists so as to make the two triangles commutative.

MC5 Each morphism f factors as $f = pi$, where p is a fibration and i a cofibration and a weak equivalence. It also factors as $f = qj$, where q is a fibration and a weak equivalence and j a cofibration.

Note that these axioms are selfdual. In the situation of MC4, one says that p has the *right lifting property* with respect to i and i the *left lifting propery* with respect to p. It is not hard to see that if $\mathcal{C}$ is a model category, then the class $\mathcal{F}ib$ is determined by the class of *trivial cofibrations* $\mathcal{W} \cap \mathcal{C}of$ and the class $\mathcal{C}of$ is determined by the class of *trivial fibrations* $\mathcal{W} \cap \mathcal{F}ib$. In fact, $\mathcal{F}ib$ consists of the morphisms with the right lifting property for all trivial cofibrations and $\mathcal{C}of$ of the morphisms with the left lifting property with respect to all trivial fibrations. A model category is *left proper* if any pushout of a weak equivalence along a cofibration is a weak equivalence.

As an example, consider the category $\mathcal{C} = \mathcal{C}^+(\mathsf{Mod}\, B)$ of left bounded complexes

$$\ldots \to 0 \to \ldots \to X^p \to X^{p+1} \to \ldots$$

of right modules over a ring B. We define $\mathcal{W}$ to be the class of quasi-isomorphisms, $\mathcal{C}of$ to be the class of morphisms $i : X \to Y$ such that each i^n, $n \in \mathbf{Z}$, is injective and $\mathcal{F}ib$ to be the class of morphisms $p : X \to Y$ such that each morphism p^n, $n \in \mathbf{Z}$, is surjective and has an injective kernel. It is not hard to show, *cf.* [**15**], that $\mathcal{C}$ is a model category. Note that in $\mathcal{C}$ the initial and the terminal object coincide with the complex of zero modules, that the morphism $0 \to X$ is always a cofibration and that the morphism $X \to 0$ is a fibration iff all components X^n, $n \in \mathbf{Z}$, are injective. Therefore, in this example, the first half of axiom MC5 corresponds to the existence of injective resolutions for complexes

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, tail, "\sim"'] & & 0. \\ & I \arrow[ur, two heads] & \end{tikzcd}$$

In an arbitrary model category $\mathcal{C}$, an object X is *fibrant* if the morphism $X \to *$ is a fibration; an object Y is *cofibrant* if the morphism $\emptyset \to Y$ is a cofibration.

Thus, in the above example, all complexes X are cofibrant and a complex Y is fibrant iff it has injective components.

For an arbitrary model category $\mathcal{C}$, the *homotopy category* $\mathsf{Ho}(\mathcal{C})$ is defined as the localization $\mathcal{C}[\mathcal{W}^{-1}]$ of the category $\mathcal{C}$ with respect to the class $\mathcal{W}$. Thus, it has the same objects as $\mathcal{C}$ and its morphisms are obtained from the morphisms of $\mathcal{C}$ by formally inverting all elements of $\mathcal{W}$. In the above example, the homotopy category equals the left bounded derived category $\mathsf{D}^+(\mathsf{Mod}\, B)$.

THEOREM 4.1. *Let $\mathcal{C}$ be a model category and $\mathcal{C}_{cf}$ the full subcategory formed by the objects X which are both fibrant and cofibrant. Then, for $X, Y \in \mathcal{C}_{cf}$, the canonical map*

$$\mathsf{Hom}_{\mathcal{C}}(X,Y) \to \mathsf{Hom}_{\mathsf{Ho}(\mathcal{C})}(X,Y)$$

is surjective.

The proof of the theorem can be found in [**8**], for example. With the notations of the theorem, define two morphisms f, g from X to Y to be *homotopic* if their images in the homotopy category coincide. Is is then clear that the canonical functor

$$\mathcal{C}_{cf}/\text{homotopy} \to \mathsf{Ho}(\mathcal{C})$$

is an equivalence.

In our example, a complex X belongs to $\mathcal{C}_{cf}$ iff it has injective components. Two morphisms f, g between such complexes are then homotopic in the sense of the model category iff they are homotopic in the classical sense, *i. e.* there is a B-linear morphism $r : X \to Y$ homogeneous of degree -1 such that

$$f - g = d \circ r + r \circ d.$$

Finally, the above equivalence specializes to the classical equivalence

$$\mathcal{C}^+(\mathsf{Inj}\, B)/\text{homotopy} \xrightarrow{\sim} \mathsf{D}^+(\mathsf{Mod}\, B)\ ,$$

where $\mathsf{Inj}\, B$ denotes the full subcategory of injective B-modules.

In a model category $\mathcal{C}$, an object I is *minimal* if it is fibrant and cofibrant and for all endomorphisms f of I in $\mathcal{C}$, we have

$$f \text{ is invertible in } \mathcal{C} \Leftrightarrow f \text{ becomes invertible in } \mathsf{Ho}(\mathcal{C}).$$

A *minimal model* for $X \in \mathcal{C}$ is an isomorphism

$$X \xrightarrow{\sim} I$$

of $\mathsf{Ho}(\mathcal{C})$ such that I is minimal.

In our example, assume that $X = M$ is a B-module, *i. e.* a complex concentrated in degree 0. Suppose that we have an injective resolution

$$0 \to M \to I^0 \to I^1 \to \dots.$$

Such a resolution gives rise to an isomorphism $M \xrightarrow{\sim} I$ in the derived category, *i. e.* $\mathsf{Ho}(\mathcal{C})$. This is a minimal model for M iff the above resolution is a minimal injective resolution.

4.2. Algebras. Let k be a field. Let Alg be the category of augmented dg algebras, *i. e.* unital associative dg algebras A endowed with a morphism

$$\varepsilon : A \to k$$

of unital dg algebras. In Alg, define $\mathcal{W}$ to be the class of quasi-isomorphisms and $\mathcal{F}ib$ to be the class of surjective morphisms. Let $\mathcal{C}of$ be the class of morphisms i such that in each commutative square of solid arrows in Alg

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, tail, "i"'] & \cdot \arrow[d, two heads, "\sim"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd}$$

there is a dotted arrow which makes the two triangles commutative. Note that the algebra k is both the initial and the terminal object of Alg.

THEOREM 4.2 (Hinich [**14**]). *The category* Alg *endowed with the above classes* $\mathcal{W}$, $\mathcal{F}ib$ *and* $\mathcal{C}of$ *is a model category. All its objects are fibrant.*

Below (in Theorem 4.3), we will give a description of the cofibrant objects of Alg. We will see that if an object of Alg is cofibrant, then its underlying graded algebra is free. However, one can show that this condition is not sufficient. One can also show that not all objects $A \in \mathsf{Alg}$ admit minimal models.

4.3. Coalgebras and the bar/cobar constructions. Let Cog' be the category of augmented dg coalgebras. Thus, an object of Cog' is a complex C endowed with a coassociative comultiplication

$$\Delta : C \to C \otimes C$$

for which d is a *coderivation*, *i. e.*

$$\Delta \circ d = (d \otimes \mathbf{1} + \mathbf{1} \otimes d) \circ \Delta\,,$$

such that Δ admits a counit $\eta : C \to k$ and a (co-)augmentation $\varepsilon : k \to C$, *i. e.* a counital coalgebra morphism such that $\eta\varepsilon = \mathbf{1}_k$. For example, if V is a complex, then we let $T^c(V)$ be the tensor coalgebra on V: Its underlying complex is

$$\bigoplus_{n \geq 0} V^{\otimes n}$$

and its comultiplication 'separates tensors':

$$\Delta(v_1, \ldots, v_n) = \sum_{i=0}^{n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n) \ ,$$

where $(v_1, \ldots, v_n)$ stands for $v_1 \otimes \ldots \otimes v_n$ and the empty parentheses () are to be interpreted as 1. Then $T^c(V)$ is endowed with a canonical projection

$$T^c(V) \to V$$

dual to the inclusion $V \to T(V)$ of V into the tensor algebra, but in the category Cog', the coalgebra $T^c(V)$ does not have the universal property dual to that of the tensor algebra: it is not cofree on V. To resolve this technical problem is one of the motivations for replacing Cog' by a smaller category, namely that of cocomplete augmented dg coalgebras. A coalgebra $C \in \mathsf{Cog}'$ is *cocomplete* if it is the union of the kernels of the maps

$$C \to C^{\otimes n} \to (C/k)^{\otimes n} \ , \ n \geq 2 \ ,$$

which are the compositions of the canonical projection with the iterated comultiplications. Define Cog to be the full subcategory of Cog' whose objects are the cocomplete coalgebras $C \in \mathsf{Cog}'$. Then it is not hard to show that $T^c(V)$ is cocomplete and that it is cofree on V in the category Cog.

For a dg coalgebra C and a dg algebra A, consider the complex

$$\mathsf{Hom}^{\bullet}{}_k(C, A)$$

whose nth component is the space of homogeneous k-linear maps $f : C \to A$ of degree n and whose differential maps f to

$$d \circ f - (-1)^n f \circ d.$$

This complex becomes a dg algebra for the *convolution* defined by

$$f * g = \mu \circ (f \otimes g) \circ \Delta.$$

Define a homogeneous k-linear map $\tau : C \to A$ to be a *twisting cochain* if it is homogeneous of degree 1 and satisfies

$$d(\tau) + \tau * \tau = 0 \text{ and } \varepsilon \circ \tau \circ \varepsilon = 0.$$

Let $\mathsf{Tw}(C, A)$ be the set of twisting cochains. Then it is not hard to show that for a given $A \in \mathsf{Alg}$, the functor

$$\mathsf{Cog} \to \mathsf{Sets} \ , \ C \mapsto \mathsf{Tw}(C, A)$$

is representable. The *bar construction* BA is by definition a representative. It is endowed with the *universal twisting cochain* τ_0 which corresponds to the identity of BA under the bijection

$$\mathsf{Tw}(BA, A) = \mathsf{Hom}_{\mathsf{Cog}}(BA, BA).$$

One checks that we have $BA = T^c(SA)$ endowed with a differential which takes into account the differential of A and its multiplication. The universal twisting cochain is the canonical projection $BA \to A$.

Dually, one shows that for a coalgebra C, the functor

$$\mathsf{Alg} \to \mathsf{Sets} \ , \ A \mapsto \mathsf{Tw}(C, A)$$

is corepresentable. The *cobar construction* $\Omega(C)$ is a corepresentative. One has $\Omega C = T(S^{-1}C)$ and the differential of ΩC takes into account the differential of C

and the comultiplication. The universal twisting cochain is the canonical inclusion $C \to \Omega C$.

It is clear from these constructions that for $A \in \mathsf{Alg}$ and $C \in \mathsf{Cog}$, we have

$$\mathsf{Hom}_{\mathsf{Alg}}(\Omega(C), A) = \mathsf{Tw}(C, A) = \mathsf{Hom}_{\mathsf{Cog}}(C, B(A))$$

so that Ω is left adjoint to B. If A is an ordinary augmented algebra (concentrated in degree 0) and C an ordinary augmented coalgebra, one checks that

$$H_*(BA) = \mathsf{Tor}^A_*(k,k) \text{ and } H^*(\Omega C) = \mathsf{Cotor}^*_C(k,k).$$

In Cog, we define $\mathcal{W}$ to be the class of morphisms $f : C \to D$ such that $\Omega(f)$ is a quasi-isomorphism; we define $\mathcal{C}of$ to be the class of injective morphisms and $\mathcal{F}ib$ to be the class of morphisms p such that in each commutative square of solid arrows in Cog

$$\begin{tikzcd} {} \arrow[r] \arrow[d, tail, "\sim"'] & {} \arrow[d, "p"] \\ {} \arrow[r] \arrow[ur, dotted] & {} \end{tikzcd}$$

there is a dotted arrow which makes the two triangles commutative.

Theorem 4.3 (Lefèvre [**26**, 1.3]). a) *The category* Cog *endowed with the above classes* $\mathcal{W}$*,* $\mathcal{C}of$ *and* $\mathcal{F}ib$ *is a model category. The functors* B *and* Ω *induce quasi-inverse equivalences*

$$\mathsf{Ho}(\mathsf{Alg}) \xrightarrow{\sim} \mathsf{Ho}(\mathsf{Cog}).$$

Moreover, B *preserves fibrations and trivial fibrations and* Ω *preserves cofibrations and trivial cofibrations.*

b) *All objects of* Alg *are fibrant. An object of* Alg *is cofibrant iff it is a retract of* ΩC *for some* C *in* Cog*. All objects of* Cog *are cofibrant. An object is fibrant iff its underlying graded coalgebra is a tensor coalgebra.*

c) *Two morphisms* $f, g : A \to A'$ *between fibrant-cofibrant objects of* Alg *are homotopic iff there is a* k*-linear map* $h : A \to A'$ *homogeneous of degree* -1 *such that*

$$\varepsilon \circ h = 0 \text{ and } h \circ m_A = m_{A'} \circ (f \otimes h + h \otimes g) \text{ and } f - g = d \circ h + h \circ d.$$

The dual statement holds for morphisms between fibrant-cofibrant objects of Cog.

Notice that Alg and Cog behave similarly in many respects but that the descriptions of their fibrant-cofibrant objects are quite different. Each cofibrant dg algebra has a free underlying graded algebra (since retracts of free algebras are free). Below, in section 4.5, we give an example of a non cofibrant object of Alg whose underlying graded algebra is free. Hinich [**14**] describes the cofibrations of Alg as 'transfinite cell attachments'.

4.4. The link with A_∞-algebras and minimal models. An *augmented* A_∞-algebra is a strictly unital A_∞-algebra endowed with a strict unit-preserving morphism $\varepsilon : A \to k$. Such an algebra decomposes as $A = k \oplus \overline{A}$ and the functor $A \mapsto \overline{A}$ yields an equivalence between the category of augmented A_∞-algebras and the category of non unital A_∞-algebras.

For a coalgebra $C \in \mathsf{Cog}$ and an augmented A_∞-algebra A, the complex $\mathsf{Hom}^\bullet{}_k(C, A)$ becomes an augmented A_∞-algebra for the convolution operations

$$b_n(f_1, \ldots, f_n) = b_n^A \circ (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{(n)} ,$$

where $\Delta^{(n)}$ is the iterate of Δ taking values in $C^{\otimes n}$. Define $\mathsf{Tw}_\infty(C, A)$ to be the set of solutions τ of the 'Maurer-Cartan equation'

$$\sum_{n \geq 1} b_n(\tau, \ldots, \tau) = 0 ,$$

cf. [**20**]. Note that the sum yields a well-defined map $C \to A$ thanks to the fact that C is cocomplete. Then the functor

$$\mathsf{Cog} \to \mathsf{Sets} \, , \; C \to \mathsf{Tw}_\infty(C, A)$$

is representable and we denote by $B_\infty A$ a representative. It is not hard to check that $B_\infty A$ is $T^c(SA)$ endowed with the unique coderivation whose composition with the projection $BA \to SA$ has the components

$$b_n : (SA)^{\otimes n} \to SA \, , \; n \geq 1.$$

In fact, if V is a graded vector space, then there is a bijection between A_∞-structures on V and coalgebra differentials on TSV. Moreover, if A and A' are A_∞-algebras, then the A_∞-morphisms $f : A \to A'$ are in canonical bijection with the morphisms $B_\infty A \to B_\infty A'$ of augmented dg coalgebras.

THEOREM 4.4 ([**26**, 1.3]). a) *An object of* Cog *is fibrant-cofibrant iff it is isomorphic to* $B_\infty A$ *for some* A_∞*-algebra* A*.*
b) *Each object* C *of* Cog *admits a minimal model and, if* A *is an* A_∞*-algebra and* A_{min} *its minimal model, then* $B_\infty(A_{min})$ *is the minimal model of* $B_\infty A$*.*

It is not hard to see that an A_∞-quasi-isomorphism $f : A \to A'$ between A_∞-algebras yields a weak equivalence $B_\infty A \to B_\infty A'$. Since both of these coalgebras are fibrant-cofibrant, such a weak equivalence has an inverse up to homotopy. This yields that f has an inverse up to A_∞-homotopy. The theorem also yields a good interpretation of the notion of minimality for A_∞-algebras. The following square summarizes the situation. We denote the category of augmented A_∞-algebras by Alg_∞.

$$\begin{tikzcd}
\mathsf{Alg} \arrow[r, hook, "\text{not full}"'] \arrow[d, shift left, "B"] & \mathsf{Alg}_\infty \arrow[d, "B_\infty", "\sim"'] \\
\mathsf{Cog} \arrow[u, shift left, "\Omega"] & \mathsf{Cog}_{cf} \arrow[l, "\text{fully faithful}"']
\end{tikzcd}$$

If we pass to the homotopy categories (where homotopy means localization for Alg and Cog and passage to quotient categories for Alg_∞ and Cog_{cf}), then all the arrows of the diagram become equivalences. Thus we obtain four descriptions of the localization of the category of dg algebras with respect to all quasi-isomorphisms. In particular, A_∞-algebras do not yield 'new homotopy types' of algebras but a new description of the existing types.

4.5. A non cofibrant dg algebra. Here is an example of a dg algebra $A \in \mathsf{Alg}$ which is free as a graded algebra but not cofibrant: Take $A = TV$, where $V = k$ is concentrated in degree 1. Endow A with the unique differential whose restriction to $V \subset TV$ is

$$V = k \xrightarrow{\sim} k \otimes k = V^{\otimes 2} \subset TV.$$

Then clearly A is quasi-isomorphic to its subalgebra k, which is fibrant-cofibrant. If A was fibrant-cofibrant as well, then the inclusion $k \to TV$ should admit a left inverse up to homotopy in the sense of c) in theorem 4.3. But there are no non-zero maps $h : A \to k$ of degree -1 such that $\varepsilon \circ h = 0$. Therefore, A cannot be fibrant-cofibrant. Note that A is the cobar construction on a non cocomplete dg coalgebra.

4.6. Modules and comodules. Let A be an augmented dg algebra and C a cocomplete augmented dg coalgebra. Let $\tau : C \to A$ be a twisting cochain. For a dg right A-module L, we endow the C-comodule $L \otimes C$ with the differential defined by

$$d = d_L \otimes \mathbf{1}_C + \mathbf{1}_L \otimes d_C + (\mu \otimes \mathbf{1}_C)(\mathbf{1}_L \otimes \tau \otimes \mathbf{1}_C)(\mathbf{1}_L \otimes \Delta)\,,$$

where $\Delta : C \to C \otimes C$ is the comultiplication of C and $\mu : L \otimes A \to L$ the multiplication of L. We write $L \otimes_\tau C$ for the resulting dg C-comodule. Similarly, if M is a dg right C-comodule, we define a differential on $M \otimes A$ by

$$d = d_M \otimes \mathbf{1}_A + \mathbf{1}_M \otimes d_A + (\mathbf{1}_M \otimes \mu_A)(\mathbf{1}_M \otimes \tau \otimes \mathbf{1}_A)(\delta_M \otimes \mathbf{1}_A).$$

We denote the resulting dg A-module by $M \otimes_\tau A$.

Let us denote by $\mathsf{Mod}\, A$ the category of right dg A-modules and by $\mathsf{Comc}\, C$ the category of dg C-comodules M which are *cocomplete*, *i. e.* M is the union of the kernels of the maps induced by iterated comultiplications

$$M \to M \otimes \overline{C}^{\otimes n}\ ,\ n \geq 2.$$

Then the pair

$$\begin{tikzcd} \mathsf{Mod}\, A \arrow[d, shift left, "?\otimes_\tau C"] \\ \mathsf{Comc}\, C \arrow[u, shift left, "?\otimes_\tau A"] \end{tikzcd}$$

is a pair of adjoint functors (*cf.* Lemme 2.2.1.2 of [**26**]). From now on, we suppose that τ is *acyclic*, *i. e.* that the following equivalent (*cf.* Proposition 2.2.4.1 of [**26**]) conditions hold

(i) The adjunction morphism $M \otimes_\tau C \otimes_\tau A \to M$ is a quasi-isomorphism for each dg A-module M.
(ii) The adjunction morphism $A \otimes_\tau C \otimes_\tau A \to A$ is a quasi-isomorphism.
(iii) The morphism $BC \to A$ induced by τ is a quasi-isomorphism.
(iv) The morphism $C \to \Omega A$ induced by τ is a weak equivalence.

If follows from [**14**] that the category $\mathsf{Mod}\, A$ admits a unique structure of model category whose weak equivalences are the quasi-isomorphisms and whose fibrations are the surjective morphisms. We write $\mathsf{D}(A)$ for the localization of $\mathsf{Mod}\, A$ with respect to the class of quasi-isomorphisms.

THEOREM 4.5 (Théorème 2.2.2.2 of [**26**]). a) *The category* $\mathsf{Comc}\, C$ *admits a unique structure of model category whose weak equivalences are the* *morphisms f such that $f \otimes_\tau A$ is a quasi-isomorphism and whose cofibrations are the injetive morphisms. We write $\mathsf{D}(C)$ for the localization of $\mathsf{Comc}\, C$ with respect to the class of weak equivalences.*

b) *The functors $? \otimes_\tau C$ and $? \otimes_\tau A$ induce quasi-inverse equivalences*

$$\mathsf{D}(A) \xrightarrow{\sim} \mathsf{D}(C).$$

4.7. Example: Koszul algebras. Let V be a vector space and $R \subset V \otimes V$ a subspace such that the algebra $A = TV/(R)$ is Koszul [**2**], *i. e.* the A-module k admits a minimal projective resolution $P \to k$ in the category of graded A-modules such that P_i is generated in degree i for all $i \geq 0$. Let C be the subspace of the tensor coalgebra $T^c(V)$ whose nth component is k for $n = 0$, V for $n = 1$ and

$$\bigcap_{p+2+q=n} V^{\otimes p} \otimes R \otimes V^{\otimes q}$$

for $n \geq 2$. Then clearly C is a subcoalgebra. We endow C with the grading such that V is in degree -1 and with the vanishing differential. We consider A as a dg algebra concentrated in degree 0. We let $\tau : A \to C$ be given by the composition

$$C \to V \to A$$

of the inclusion with the projection. Clearly, τ is a twisting cochain. Then the complex

$$A \otimes_\tau C \otimes_\tau A$$

is the Koszul resolution of the bimodule A. Thus, the twisting cochain τ is acyclic and we obtain an equivalence of categories

$$? \otimes_\tau C : \mathsf{D}(A) \to \mathsf{D}(C)$$

which takes A to k (weakly equivalent to $A \otimes_\tau C$) and k to C. An analogous equivalence can be constructed between suitable (Adams-) graded versions of the categories $\mathsf{D}(A)$ and $\mathsf{D}(C)$. We refer to [**9**] for related developments.

4.8. Link with A_∞-modules. Let A be an augmented A_∞-algebra. Let $C = B_\infty A$. The adjunction morphism

$$B_\infty A = C \to B\Omega C = B_\infty(\Omega C)$$

corresponds to a canonical A_∞-morphism $A \to \Omega B_\infty A$, which is a quasi-isomorphism and which is universal among the A_∞-morphisms from A to a dg algebra (Lemme 2.3.4.3 of [**26**]). Let us put $U(A) = \Omega B_\infty A$. Thus we have a canonical acyclic twisting cochain $\tau : B_\infty(A) \to U(A)$. By theorem 4.5, we have an equivalence

$$\mathsf{D}(U(A)) \xrightarrow{\sim} \mathsf{D}(B_\infty(A)).$$

Both of these categories can be regarded as the (unbounded) derived category of the A_∞-algebra A. The link with A_∞-modules is the following: Let $\mathsf{Mod}_\infty A$ be the category of A_∞-modules over A which are strictly unital (*i. e.* b_n^M, $n \geq 2$, vanishes as soon as one of the arguments is 1) and whose morphisms are strictly unital (*i. e.* f_n, $n \geq 2$, vanishes as soon as one of the arguments is 1). This category is isomorphic to the category of *all* A_∞-modules with *all* A_∞-morphisms over the reduction $\overline{A}$. By restriction along the A_∞-morphism $A \to U(A)$, each dg module over $U(A)$ yields an A_∞-module in $\mathsf{Mod}_\infty A$. However, the resulting functor

$$\mathsf{Mod}\, U(A) \to \mathsf{Mod}_\infty A$$

is not full (its image only contains strict morphisms).

Let M be in $\mathsf{Mod}_\infty A$. The datum of its strictly unital A_∞-module structure over A is equivalent to the datum of an arbitrary A_∞-module structure over $\overline{A}$. This in turn is equivalent to giving a comodule differential on the induced comodule $M \otimes B_\infty A$. We write $B_\infty M$ for the induced comodule endowed with the differential corresponding to a given A_∞-module structure on M. Thus we obtain a functor

$$\mathsf{Mod}_\infty A \to \mathsf{Comc}\, B_\infty(A)\ ,\ M \to B_\infty M.$$

It is not hard to see that this functor is in fact fully faithful. We define two morphisms of $\mathsf{Mod}_\infty A$ to be homotopic if they become homotopic in $\mathsf{Comc}\, B_\infty(A)$.

Proposition 4.6. a) The functor $M \mapsto B_\infty M$ induces an equivalence onto the full subcategory $(\mathsf{Comc}\, B_\infty A)_{cf}$ of fibrant-cofibrant objects of $\mathsf{Comc}\, B_\infty(A)$.

b) The functor $M \mapsto B_\infty M$ induces an equivalence

$$(\mathsf{Mod}_\infty A)/\text{homotopy} \xrightarrow{\sim} \mathsf{D}(C).$$

Proof. a) In Proposition 2.4.1.3 of [**26**], it is shown that $B_\infty M$ is fibrant (and cofibrant) in $\mathsf{Comc}\, B_\infty A$ and that each fibrant object is a direct factor of an object $M \otimes_\tau B_\infty A$ for some $U(A)$-module M. It remains to be shown that such a direct factor is of the form $B_\infty M'$ for some $M' \in \mathsf{Mod}_\infty A$. Put $C = B_\infty A$. We will use that a morphism $g : V \otimes C \to W \otimes C$ of induced C-comodules is invertible iff it induces an invertible morphism in the spaces of primitive elements

$$V = (V \otimes C)_{[1]} = \ker(\ V \otimes C \xrightarrow{\mathbf{1}_V \otimes \Delta - \mathbf{1}_V \otimes \mathbf{1}_V \otimes \eta} V \otimes C \otimes C\ ).$$

If f is an idempotent endomorphism of $M \otimes_\tau C$, it induces an idempotent endomorphism of graded vector spaces in $M = (M \otimes_\tau C)_{[1]}$. Let $M = M' \oplus M''$ be the corresponding decomposition in the category of graded vector spaces. Then we have $M \otimes C = (M' \otimes C) \oplus (M'' \otimes C)$ as C-comodules and in the corresponding decomposition

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

the diagonal entries are invertible. It follows that the image I of f is isomorphic to $M' \otimes C$ as a C-comodule. The differential of $M \otimes_\tau C$ yields a comodule differential on $I \xrightarrow{\sim} M' \otimes C$ and thus an A_∞-module structure on M'. We have $I \xrightarrow{\sim} B_\infty M'$. Point b) is proved in Lemme 2.4.2.3 of [**26**]. $\surd$

To sum up, we have the following diagram

$$\begin{tikzcd} \mathsf{Mod}\, U(A) \arrow[r, hook, "\text{not full}"] \arrow[d, shift left, "? \otimes_\tau B_\infty A"] & \mathsf{Mod}_\infty A \arrow[d, "B_\infty", "\sim"'] \\ \mathsf{Comc}\, B_\infty(A) \arrow[u, shift left, "? \otimes_\tau U(A)"] & (\mathsf{Comc}\, B_\infty A)_{cf} \arrow[l, "\text{fully faithful}"'] \end{tikzcd}$$

All functors induce equivalences in the localizations (for the two categories on the left) respectively the quotients by the homotopy relations (for the two categories on the right).

4.9. The derived category of a non augmented A_∞-algebra. Let A be a (non augmented) A_∞-algebra. Let $A^+ = A \oplus k$ be the augmented A_∞-algebra obtained by adjoining k. The augmentation $A^+ \to k$ yields a functor

$$\mathsf{Mod}_\infty A^+ \to \mathsf{Mod}_\infty k$$

which passes to the derived categories, *cf.* section 4.1.1 of [**26**]. By definition, the (unbounded) derived category $\mathsf{D}_\infty(A)$ is the kernel of the functor

$$\mathsf{D}_\infty(A^+) \to \mathsf{D}_\infty(k).$$

One can show that a strictly unital A_∞-module M over A^+ is in the kernel iff $B_\infty M$ is acyclic (Remarque 4.1.3.5 of [**26**]).

Now suppose that A is homologically unital, *i. e.* $H^*(A)$ endowed with the multiplication induced by m_2 is unital. Then M belongs to the kernel iff M is homologically unital, *i. e.* $H^*(M)$ is a unital $H^*(A)$-module, *cf.* Lemme 4.1.3.7 of [**26**]. Moreover, in this case $\mathsf{D}_\infty(A)$ is a compactly generated triangulated category and has the free A-module of rank one as a compact generator. Indeed, A^+ is quasi-isomorphic to $U(A^+)$ and one checks that if $B = k \oplus \overline{B}$ is an augmented dg algebra with homologically unital augmentation ideal $\overline{B}$, then $\overline{B}$ is a compact generator for the kernel of $\mathsf{D}(B) \to \mathsf{D}(k)$.

5. A conceptual construction of A_∞-functor categories

The notion of A_∞-algebra naturally generalizes to that of A_∞-category. It has been known for some time, *cf.* [**23**], that for two A_∞-categories $\mathcal{A}$, $\mathcal{B}$, there is a natural A_∞-category $\mathsf{Fun}_\infty(\mathcal{A},\mathcal{B})$ and for three A_∞-categories, a natural 'A_∞-bifunctor'

$$\mathsf{Fun}_\infty(\mathcal{B},\mathcal{C}) \times \mathsf{Fun}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Fun}_\infty(\mathcal{A},\mathcal{C}).$$

One can use this composition bifunctor to give a natural construction for the bialgebra structure of [**1**], [**12**] on the bar construction BC of the Hochschild complex $C = C(A,A)$ of an algebra A, *cf.* section 5.7.

The A_∞-category $\mathsf{Fun}_\infty(\mathcal{A},\mathcal{B})$ was constructed in [**26**] using twists of A_∞-structures. Here, we further develop an idea of V. Lyubashenko's [**28**] and interpret $\mathsf{Fun}_\infty(\mathcal{A},\mathcal{B})$ as an internal Hom-object in a tensor category, namely the category of cocomplete augmented dg cocategories. In their work [**4**], Yu. Bespalov, V. Lyubashenko and O. Manzyuk are developing a comprehensive theory based on this interpretation. The main difference with V. Lyubashenko's original approach in [**28**] is that he only considered the tensor subcategory generated by all cofree cocategories. The construction of Fun_∞ given in [**4**] agrees with the one below.

5.1. Augmented k-quivers. Let k be a commutative ring. A *k-quiver* V consists of a set of *objects* $\mathsf{obj}(V)$ and of k-modules $V(x,y)$, for all objects x,y of V. We call the elements of $V(x,y)$ the *morphisms* from x to y in V. A *morphism of k-quivers* $F : V \to W$ consists of a map $F : \mathsf{obj}(V) \to \mathsf{obj}(W)$ and of maps

$$F_{x,y} : V(x,y) \to W(Fx,Fy)\,,$$

for all objects x,y of V. For each set S, we have the *discrete k-quiver* kS with $\mathsf{obj}(kS) = S$, $(kS)(x,x) = k\mathbf{1}_x$ and $(kS)(x,y) = 0$ for all objects $x \neq y$ of kS. An *augmented k-quiver* is a k-quiver V endowed with morphisms

$$k\,\mathsf{obj}(V) \xrightarrow{\eta} V \xrightarrow{\varepsilon} k\,\mathsf{obj}(V)$$

whose composition is the identity of $k\,\mathsf{obj}(V)$. A *morphism of augmented k-quivers* is a morphism of the underlying quivers compatible with the morphisms η and ε. The *tensor product* of two augmented k-quivers V and W is the naturally augmented k-quiver $V \otimes W$ whose set of objects is $\mathsf{obj}(V) \times \mathsf{obj}(W)$ and such that

$$(V \otimes W)\left((x, x'), (y, y')\right) = V(x, y) \otimes W(x', y').$$

The *unit* augmented k-quiver E has one object $*$ and $E(*, *) = k$. It is endowed with the identity morphisms η and ε. Clearly, we have natural isomorphisms $E \otimes V \xrightarrow{\sim} V \xleftarrow{\sim} V \otimes E$ for each augmented k-quiver V. For two augmented k-quivers V, W, we define $\underline{Hom}_q(V, W)$ to be the k-quiver whose objects are the morphisms of augmented k-quivers $f : V \to W$ and such that for two morphisms of augmented k-quivers f and g, the k-module $\underline{Hom}_q(V, W)(f, g)$ is formed by the families $(\varphi_{x,y})$ of

$$\prod_{x,y \in \mathsf{obj}(V)} \mathsf{Hom}_k(V(x, y), W(fx, gy))$$

such that, for $f \neq g$, we have $\varepsilon_{fx,gy}\varphi_{x,y} = 0$ for all x, y of V, and for $f = g$, the map $\varphi_{x,y}$ takes the kernel of $\varepsilon_{x,y}$ to the kernel of $\varepsilon_{fx,fy}$ and the scalar

$$\varepsilon_{fx,fx} \circ \varphi_{x,x} \circ \eta_{x,x}(\mathbf{1}_x)$$

is independent of $x \in \mathsf{obj}(V)$. We endow $\underline{Hom}_q(V, W)$ with the morphism ε which takes a family $(\varphi_{x,y})$ to this scalar and with the morphism η which takes $\mathbf{1}_f$ to the family

$$\varphi_{x,y} : v \mapsto f(v).$$

Clearly we obtain an augmented k-quiver $\underline{Hom}_q(V, W)$. For example, we have $\underline{Hom}_q(E, W) \xrightarrow{\sim} W$.

A *graded k-quiver* (resp. a *differential graded k-quiver*) is a k-quiver V such that $V(x, y)$ is a $\mathbf{Z}$-graded k-module (resp. a cochain complex of k-modules) for all objects x, y of of V. One defines the tensor product and the functor $\underline{Hom}_q$ for graded and differential graded augmented k-quivers analogously.

Lemma 5.1. *The category of augmented k-quivers is monoidal with unit object E and $\underline{Hom}_q$ is an internal Hom-functor, i.e. we have a functorial bijection*

$$\mathsf{Hom}(U \otimes V, W) \xrightarrow{\sim} \mathsf{Hom}(U, \underline{Hom}_q(V, W))$$

for all augmented k-quivers U, V, W. The same assertions hold for the categories of graded and that of differential graded augmented k-quivers.

Proof. We define a map

$$\Phi : \mathsf{Hom}(U \otimes V, W) \xrightarrow{\sim} \mathsf{Hom}(U, \underline{Hom}_q(V, W)).$$

Let $F : U \otimes V \to W$ be a morphism of augmented k-quivers. We define

$$\Phi F : U \to \underline{Hom}_q(V, W)$$

as follows: For an object u of U, we define $(\Phi F)(u)$ to be the morphism of k-quivers $V \to W$ which sends an object v of V to $F(u, v)$ and a morphism $\beta : v_1 \to v_2$ of V to $F(\eta_u \otimes \beta)$, where we write η_u for $\eta_{u,u}$. Then $(\Phi F)(u)$ is compatible with the unit and the augmentation since $F(\eta_u \otimes \eta_v) = \eta_{F(u,v)}$ and

$$\varepsilon_{F(u,v)} F(\eta_u \otimes \beta) = (\varepsilon_u \otimes \varepsilon_v)(\eta_u \otimes \beta) = \varepsilon_v(\beta)\ ,$$

where we write ε_u instead of $\varepsilon_{u,u}$. Now let u_1, u_2 be objects of U and $\alpha : u_1 \to u_2$ a morphism. We define a morphism $(\Phi F)(\alpha) : (\Phi F)(u_1) \to (\Phi F)(u_2)$ of $\underline{Hom}_q(V, W)$

by sending a morphism $\beta : v_1 \to v_2$ of V to $F(\alpha \otimes \beta) : F(u_1, v_1) \to F(u_2, v_2)$. We have

$$\varepsilon_{F(u_1,v_1),F(u_2,v_2)}(F(\alpha \otimes \beta)) = \varepsilon_{u_1,u_2}(\alpha)\varepsilon_{v_1,v_2}(\beta)$$

since F is compatible with the augmentation. This shows that $(\Phi F)(\alpha)$ lies indeed in $\underline{Hom}_q(V, W)$.

We define a map $\Psi : \mathsf{Hom}(U, \underline{Hom}_q(V, W)) \to \mathsf{Hom}(U \otimes V, W)$. Let G be a morphism of augmented quivers from U to $\underline{Hom}_q(V, W)$. We define ΨG as follows: For objects u of U and v of V, we put $(\Psi G)(u, v) = (Gu)(v)$. For a morphism $\alpha \otimes \beta : (u_1, v_1) \to (u_2, v_2)$ of $U \otimes V$, we define

$$(\Psi G)(\alpha \otimes \beta) = (G\alpha)(\beta).$$

Let us check that ΨG is a morphism of augmented k-quivers. We have

$$(\Psi G)(\eta_u \otimes \eta_v) = (G\eta_u)(\eta_v).$$

Since $G : U \to \underline{Hom}_q(V, W)$ respects η, the morphism $G\eta_u$ equals η_{Gu} and this is given by $\beta \mapsto (Gu)(\beta)$. Since Gu respects η, we have $(Gu)(\eta_v) = \eta_{(Gu)(v)}$. Finally, we do obtain

$$(\Psi G)(\eta_u \otimes \eta_v) = \eta_{(Gu)(v)}.$$

Now, for morphisms $\alpha : u_1 \to u_2$ and $\beta : v_1 \to v_2$, we compute

$$\begin{aligned}\varepsilon((\Psi G)(\alpha \otimes \beta)) &= \varepsilon((G\alpha)(\beta)) \\ &= \varepsilon((G\alpha)(\eta\varepsilon(\beta))) = \varepsilon((G\alpha)(\eta(1)))\varepsilon(\beta)\ ,\end{aligned}$$

where the second last equality holds since $G\alpha$ vanishes on the kernel of ε and we have omitted the subscripts for better readability. Note that the scalar $\varepsilon((G\alpha)(\eta(1)))$ is the image of $G\alpha$ under the augmentation of $\underline{Hom}_q(V, W)$. Since G respects the augmentation, this scalar equals $\varepsilon(\alpha)$. So we have

$$\varepsilon((\Psi G)(\alpha \otimes \beta)) = \varepsilon(G\alpha)\varepsilon(\beta) = \varepsilon(\alpha)\varepsilon(\beta).$$

Thus ΨG respects the augmentation. We have shown that the maps Φ and Ψ are well-defined. Clearly they are inverse to each other. $\surd$

For an augmented k-quiver V, we denote by $\overline{V}$ the *reduction* of V, i.e. the k-quiver with the same objects, and with

$$\overline{V}(x, y) = \begin{cases} V(x, y) & \text{if } x \neq y \\ V(x, x)/\operatorname{im} \eta_{x,x} & \text{if } x = y. \end{cases}$$

The reduction functor is an equivalence from the category of augmented k-quivers to that of k-quivers but this equivalence is not monoidal for the naive tensor product of k-quivers.

The augmented k-quivers with one object form a full monoidal subcategory of the category of augmented k-quivers but this subcategory does not admit an internal Hom-functor.

5.2. Augmented cocomplete cocategories. A *cocategory* is a k-quiver C endowed with k-linear maps

$$\Delta : C(x, y) \to \bigoplus_{z \in \mathsf{obj}(C)} C(z, y) \otimes C(x, z)$$

for all objects x, y of C such that the natural coassociativity condition holds. It is *cocomplete* if each $f \in C(x, y)$ lies in the kernel of a sufficiently high iterate of

Δ. It is *counital* if it is endowed with a morphism of k-quivers $\eta : C \to k\,\mathsf{obj}\,C$ such that the two counit equations hold. An *augmented cocategory* is a counital cocategory C endowed with a morphism of counital cocategories $\varepsilon : k\,\mathsf{obj}(C) \to C$ such that the composition of η with ε is the identity of $k\,\mathsf{obj}(C)$. Then it is in particular an augmented k-quiver and the reduced quiver $\overline{C}$ becomes a cocategory. The augmented cocategory C is *cocomplete* if its reduction $\overline{C}$ is cocomplete. These definitions admit obvious *graded and differential graded variants.*

Clearly the unit object E of the category of augmented k-quivers admits a unique structure of (cocomplete) augmented cocategory. The tensor product $C_1 \otimes C_2$ of the k-quivers underlying two cocomplete augmented cocategories admits a natural structure of cocomplete augmented cocategory. For this tensor product, the category of cocomplete augmented cocategories becomes a monoidal category with unit object E.

Let V be an augmented k-quiver. The *tensor cocategory* TV has the same objects as V; and for any two objects x, y of TV, the k-module $(TV)\,(x, y)$ is the direct sum of $V(x, y)$ with the tensor products

$$\overline{V}(z_n, y) \otimes \overline{V}(z_{n-1}, z_n) \otimes \cdots \otimes \overline{V}(x, z_1)$$

where $n \geq 1$ and $(z_1, \ldots, z_n)$ runs through all sequences of objects of V. For an element $(f_n, \ldots, f_1)$, $n \geq 1$, of $\overline{TV}(x, y)$, one puts

$$\begin{aligned}\Delta(f_n, \ldots, f_1) = &\eta_{y,y} \otimes (f_n, \ldots, f_1) + \sum_{i=1}^{n} (f_n, \ldots, f_{i+1}) \otimes (f_i, \ldots, f_1) \\ &+ (f_n, \ldots, f_1) \otimes \eta_{x,x}.\end{aligned}$$

The counit (resp. the coaugmentation) of TV are obtained by composing the corresponding morphisms $V \to k\,\mathsf{obj}(V)$ (resp. $k\,\mathsf{obj}(V) \to V$) with the projection $pr_1 : TV \to V$ (resp. the inclusion $V \to TV$). The construction carries over to *augmented graded quivers* and *augmented differential graded quivers.*

Denote by aQu the category of augmented k-quivers and by aCo that of cocomplete augmented k-cocategories. The forgetful functor $F : \mathsf{aCo} \to \mathsf{aQu}$ is monoidal.

LEMMA 5.2. *The forgetful functor admits the right adjoint $V \mapsto TV$. More precisely, for each augmented cocategory C and each augmented k-quiver V, the map*

$$\mathsf{Hom}_{\mathsf{aCo}}(C, TV) \to \mathsf{Hom}_{\mathsf{aQu}}(FC, V)\ ,\ f \mapsto pr_1 \circ f$$

is bijective. Its inverse takes a morphism g to the unique morphism of augmented cocategories $G : C \to TV$ whose reduction sends $c \in \overline{C}$ to

$$\sum_{n=1}^{\infty} g^{\otimes n} \Delta^{(n)}(c).$$

The analogous assertions hold in the graded and in the differential graded setup.

THEOREM 5.3. *The monoidal category of cocomplete augmented cocategories admits an internal Hom-functor $\underline{Hom}_c$. Moreover, if C is a cocomplete augmented cocategory, then $\underline{Hom}_c(C, TV) = T\underline{Hom}_q(FC, V)$ for each augmented dg k-quiver V. The analogous assertions hold in the graded and in the differential graded setting.*

PROOF. We have to show that the functor $?\otimes C$ admits a right adjoint, i.e. that the functor $\mathsf{Hom}(?\otimes C, C')$ is representable for each C'. If $C' = TV$, then

$$\begin{aligned}\mathsf{Hom}(?\otimes C, C') &= \mathsf{Hom}(?\otimes C, TV) = \mathsf{Hom}(F?\otimes FC, V)\\ &= \mathsf{Hom}(F?, \underline{Hom}_q(FC, V)) = \mathsf{Hom}(?, T\underline{Hom}_q(FC,V)).\end{aligned}$$

If C' is arbitrary, we can write it as the equalizer of a pair of morphisms between cofree objects:

$$TV_1 \rightrightarrows TV_2.$$

Then we obtain the representing object $\underline{Hom}_c(C, C')$ as the equalizer of the pair

$$\underline{Hom}_c(?, TV_1) \rightrightarrows \underline{Hom}_c(?, TV_2).$$

$\surd$

5.3. Explicit description of $\underline{Hom}_c(C, TV)$ and the action morphism. Let C be a cocomplete augmented cocategory and V an augmented k-quiver. For $D = TV$, we will describe $\underline{Hom}_c(C, D)$ and the adjunction morphism (=action morphism)

$$\Phi : \underline{Hom}_c(C, D) \otimes C \to D.$$

These generalize V. Lyubashenko's constructions [**28**]. The objects of

$$\underline{Hom}_c(C, TV) = T\underline{Hom}_c(FC, V).$$

are the same as those of $\underline{Hom}_q(FC, V)$. Thus they are in bijection with the morphisms of augmented k-quivers f from C to V. The morphism η of $T\underline{Hom}_q(FC, V)$ is obtained by composing that of $\underline{Hom}_q(FC, V)$ with the canonical inclusion and dually, the morphism ε by composing that of $\underline{Hom}_q(FC, V)$ with the projection. If f and g are two objects, a morphism from f to g in $T\underline{Hom}_q(FC, V)$ is given by a tensor product of $n \geq 0$ composable morphisms of $\underline{Hom}_q(FC, V)$. For $n = 0$, this means that $f = g$ and that the morphism is a multiple of $\eta_{f,f}(1) = f$; for $n \geq 1$, the morphism is a tensor

$$(\varphi_n, \ldots, \varphi_1) = \varphi_n \otimes \cdots \otimes \varphi_1$$

of morphisms φ_i from f_{i-1} to f_i in the reduction $\overline{\underline{Hom}_q(FC, V)}$, where $f_0 = f$ and $f_n = g$. Thus, φ_i belongs to

$$\prod_{x,y\in \mathsf{obj}(C)} \mathsf{Hom}_k(C(x,y), V(f_{i-1}(x), f_i(y))$$

and the composition

$$\varepsilon_{f_{i-1}(x), f_i(y)} \circ (\varphi_i)_{x,y}$$

vanishes for all x, y. The image under Δ of such a morphism $(\varphi_n, \ldots, \varphi_1)$ is

$$\sum_{i=0}^{n} (\varphi_n, \ldots, \varphi_{i+1}) \otimes (\varphi_i, \ldots, \varphi_1).$$

Here, the terms for $i = 0$ and $i = n$ are respectively

$$(\varphi_n, \ldots, \varphi_1) \otimes f \text{ and } g \otimes (\varphi_n, \ldots, \varphi_1)\,,$$

by definition of the morphism η for $\underline{Hom}_q(FC, V)$. When we apply an iterate $\Delta^{(N)}$ of Δ to $(\varphi_n, \ldots, \varphi_1)$, we obtain sums of tensor products of N factors each of which may be reduced to some f_i. For example, we get, among others, the term

$$f_7 \otimes (\varphi_7) \otimes f_6 \otimes (\varphi_6, \varphi_5) \otimes f_4 \otimes f_4 \otimes (\varphi_4, \varphi_3, \varphi_2) \otimes f_1 \otimes (\varphi_1)$$

for $n = 7$ and $N = 9$.

The adjunction morphism $\Phi : \underline{Hom}_c(C, TV) \otimes C \to TV$ is the unique morphism of cocomplete augmented cocategories whose image under the forgetful functor makes the square

$$\begin{tikzcd} T\underline{Hom}_q(FC, V) \otimes C \arrow[r] \arrow[d] & TV \arrow[d] \\ \underline{Hom}_q(FC, V) \otimes C \arrow[r] & V \end{tikzcd}$$

commutative in the category of augmented k-quivers. Thus, the morphism Φ takes an object (f, x) of the tensor product $T\underline{Hom}_q(FC, V) \otimes C$ to $f(x)$ and a morphism $(\varphi_n, \ldots, \varphi_1) \otimes c$ to the image in $\bigoplus_{N \geq n} \overline{V}^{\otimes N}$ of the sum of the

$$(f_n^{\otimes r_n} \otimes \varphi_n \otimes f_{n-1}^{\otimes r_{n-1}} \otimes \cdots \otimes f_1^{\otimes r_1} \otimes \varphi_1 \otimes f_0^{\otimes r_0}) \circ \Delta^{(N)}(c)$$

where $N = n + \sum r_i$. Note that this sum has indeed only finitely many non zero terms: Since C is cocomplete, $\Delta^{(M)}(c)$ vanishes in $\overline{C}^{\otimes M}$ for some $M \gg 0$. Therefore, $\Delta^{(N)}(c)$ vanishes in $C^{\otimes p} \otimes \overline{C}^{\otimes M} \otimes C^{\otimes q}$ for all $N = p + M + q$, $p, q \geq 0$. Now for large enough N, each term of the image of $(\varphi_n, \ldots, \varphi_1)$ under $\Delta^{(N)}$ will contain a chain of M consecutive f_i's and therefore induce a map from $C^{\otimes p} \otimes \overline{C}^{\otimes M} \otimes C^{\otimes q}$ to $V^{\otimes p} \otimes \overline{V}^{\otimes M} \otimes V^{\otimes q}$. Hence the image of

$$\Delta^{(N)}(\varphi_n, \ldots, \varphi_1) \circ \Delta^{(N)}(c)$$

in $\overline{V}^{\otimes N}$ vanishes.

Note that if C is the unit object E, the action morphism $\underline{Hom}_c(E, D) \otimes E \to D$ is the identity. More generally, if C is non empty, then E is a retract of C and the action morphism $\underline{Hom}_c(C, D) \otimes C \to D$ admits a section. Dually, the coaction morphism $D \to \underline{Hom}_c(C, D \otimes C)$ admits a retraction in this case.

5.4. The composition morphism. Let C be a cocomplete augmented cocategory and V, W augmented k-quivers. Since the functor $? \otimes C$ is left adjoint to $\underline{Hom}_c(?, C)$, there is a unique morphism, *the composition morphism*,

$$\zeta : \underline{Hom}_c(TV, TW) \otimes \underline{Hom}_c(C, TV) \longrightarrow \underline{Hom}_c(C, TW)$$

which yields a commutative square

$$\begin{tikzcd} \underline{Hom}_c(TV, TW) \otimes \underline{Hom}_c(C, TV) \otimes C \arrow[r, "\zeta \otimes C"] \arrow[d, "\mathbf{1} \otimes \Phi"'] & \underline{Hom}_c(C, TW) \otimes C \arrow[d, "\Phi"] \\ \underline{Hom}_c(TV, TW) \otimes TV \arrow[r, "\Phi"] & TW. \end{tikzcd}$$

The composition morphism is constructed as the unique morphism such that the following square commutes in the category of augmented k-quivers

$$\begin{tikzcd} T\underline{Hom}_q(TV, W) \otimes T\underline{Hom}_q(C, V) \arrow[r] \arrow[d] & T\underline{Hom}_q(C, W) \arrow[d] \\ \underline{Hom}_q(TV, W) \otimes T\underline{Hom}_q(C, V) \arrow[r] & \underline{Hom}_q(C, W), \end{tikzcd}$$

where the bottom arrow is described as follows: a pair (f, f') of morphisms is sent to the composition $f \circ f'$. A morphism $\varphi' \otimes (\varphi_n, \ldots, \varphi_1)$ is sent to the morphism

$$c \mapsto \sum \varphi' \circ (f_n^{\otimes r_n} \otimes \varphi_n \otimes \cdots \otimes f_i^{\otimes r_i} \otimes \varphi_i \otimes f_{i-1}^{\otimes r_{i-1}} \otimes \cdots \otimes \varphi_1 \otimes f_0^{\otimes r_0}) \circ \Delta^{(N)}(c)$$

where N is the number of arguments of φ' and $n + \sum r_i = N$.

5.5. The comparison morphism. Let C be a cocomplete augmented cocategory and W an augmented k-quiver. The *comparison morphism* is the composition

$$F\underline{Hom}_c(C, TW) \to \underline{Hom}_q(FC, F\underline{Hom}_c(C, TW) \otimes FC) \to \underline{Hom}_q(FC, FTW)$$

of the coaction morphism for $? \otimes FC$ with the action morphism for $? \otimes C$. This morphism induces the natural injection in the sets of objects. It takes a morphism $(\varphi_n, \ldots, \varphi_1)$, $n \geq 1$, of $T\underline{Hom}_q(C, W)$ to the morphism of $\underline{Hom}_q(FC, FTW)$ which sends $c \in C(x, y)$ to the image in $\bigoplus_{N \geq n} \overline{W}^{\otimes N}$ of the sum of the

$$(f_n^{\otimes r_n} \otimes \varphi_n \otimes f_{n-1}^{\otimes r_{n-1}} \otimes \cdots \otimes f_1^{\otimes r_1} \otimes \varphi_1 \otimes f_0^{\otimes r_0}) \circ \Delta^{(N)}(c)$$

where $N = n + \sum r_i$. We have seen above that this sum has only finitely many non zero terms. If the map $C \to \overline{C}^{\otimes n}$ induced by the iterated comultiplication is surjective for all $n \geq 2$, then the comparison morphism is clearly injective on the sets of morphisms. This happens for example if $C = TV$ for some augmented k-quiver V. The comparison morphism is compatible with compositions.

5.6. The differential graded case. Let adgCo denote the category of cocomplete augmented dg coalgebras and agCo the category of cocomplete augmented graded coalgebras. The functor

$$\mathsf{For} : \mathsf{adgCo} \to \mathsf{agCo}$$

which forgets the differentials is monoidal. Thus, if A, B are objects of adgCo, the adjunction morphism

$$\underline{Hom}_q(A, B) \otimes A \to B$$

yields a morphism

$$\mathsf{For}\, \underline{Hom}_q(A, B) \otimes \mathsf{For}\, A \to \mathsf{For}\, B$$

and hence a morphism

$$\mathsf{For}\, \underline{Hom}_q(A, B) \to \underline{Hom}_q(\mathsf{For}\, A, \mathsf{For}\, B).$$

LEMMA 5.4. *The above morphism induces isomorphisms in the morphism spaces,* i. e. $\mathsf{For}\, \underline{Hom}_q(A, B)$ *is the full subcocategory of* $\underline{Hom}_q(\mathsf{For}\, A, \mathsf{For}\, B)$ *whose objects are the morphisms of* agCo *which commute with the differential.*

PROOF. It suffices to prove this when B is cofree on some augmented differential graded k-quiver V. Then we have

$$\mathsf{For}\, \underline{Hom}_q(A, TV) = \mathsf{For}\, T\underline{Hom}_q(A, V) = T\, \mathsf{For}\, \underline{Hom}_q(A, V)$$

where $\underline{Hom}_q(A, V)$ is the internal $\underline{Hom}_q$-object of the category of augmented differential graded k-quivers and we have also denoted by For the forgetful functor from dg k-quivers to graded k-quivers. Now it is clear that we have a full morphism of graded quivers

$$\mathsf{For}\, \underline{Hom}_q(A, V) \to \underline{Hom}_q(\mathsf{For}\, A, \mathsf{For}\, V)$$

and this implies the assertion. $\surd$

5.7. A_∞-functor categories. For an augmented graded k-quiver V, denote by SV the augmented graded k-quiver with the same objects whose reduction has the shifted k-modules $S\overline{V}(x,y)$ as morphism spaces. An *augmented A_∞-category* is an augmented graded k-quiver A whose bar construction $BA = TSA$ is endowed with a graded coalgebra differential so that it becomes a cocomplete augmented dg cocategory. Equivalently, the reduction $\overline{A}$ is endowed with a structure of A_∞-category without units in the sense of Kontsevich-Soibelman [**25**, 4.1]. For example, if $A = k \oplus \overline{A}$ is an ordinary augmented associative algebra, then we view it as an A_∞-category with one object by endowing

$$BA = TSA = k \oplus \bigoplus_{n\geq 1} (S\overline{A})^{\otimes n}$$

with the unique coalgebra differential such that $d(x \otimes y) = xy$ for $x, y \in \overline{A}$. Then the underlying complex of BA is

$$k \leftarrow \overline{A} \leftarrow \overline{A} \otimes \overline{A} \leftarrow \cdots$$

and $H_*(BA) = \mathsf{Tor}^A_*(k,k)$.

Let A, A' be two augmented A_∞-categories. By theorem 5.3, the cocategory $\underline{Hom}_c(BA, BA')$ is (canonically) cofree. We define the *A_∞-functor category* $\mathsf{Fun}_\infty(A, A')$ by

$$\underline{Hom}_c(BA, BA') = B(\mathsf{Fun}_\infty(A, A')).$$

It is an augmented A_∞-category. By definition, its objects are the *A_∞-functors* from A to A'. According to theorem 5.3 and the above lemma, the underlying graded category of $\underline{Hom}_c(BA, BA')$ is a full subcategory of

$$\underline{Hom}_c(BA, TSA') = T\underline{Hom}_q(BA, SA')\,,$$

Thus, the underlying graded quiver of $S\,\mathsf{Fun}_\infty(A, A')$ is a full subquiver of

$$\underline{Hom}_q(BA, SA').$$

Suppose that A and A' are ordinary augmented algebras. Then the objects of $\mathsf{Fun}_\infty(A, A')$ identify with the morphisms of augmented dg coalgebras $BA \to BA'$. For any two objects f, g, the morphism space

$$S\,\mathsf{Fun}_\infty(A, A')(f, g)$$

is

$$\mathsf{Hom}(k, S\overline{A'}) \oplus \mathsf{Hom}(\overline{BA}, S\overline{A'})$$

for $f \neq g$ and

$$k \oplus \mathsf{Hom}(k, S\overline{A'}) \oplus \mathsf{Hom}(\overline{BA}, S\overline{A'})$$

for $f = g$. *If $f \neq g$*, then the morphisms $f \to g$ are in canonical bijection with the space of *(f,g)-coderivations* $D : BA \to BA'$, *i. e.* the homogeneous linear maps satisfying

$$\Delta \circ D = (f \otimes D + D \otimes g) \circ \Delta.$$

Note that $\mathsf{Fun}_\infty(A, A')$ has infinitely many objects, in general, even though A and A' only have one. If $A = A'$ and $f = g$ is the identity functor then

$$\mathsf{Fun}_\infty(A, A)(\mathbf{1}, \mathbf{1})/k$$

is the reduced Hochschild complex

$$\overline{A} \to \mathsf{Hom}(\overline{A}, \overline{A}) \to \mathsf{Hom}(\overline{A} \otimes \overline{A}, \overline{A}) \to \ldots \to \mathsf{Hom}(\overline{A}^{\otimes p}, \overline{A}) \to \ldots$$

where $\overline{A}$ appears in degree -1.

If A, A', A'' are three augmented A_∞-categories, the composition morphism

$$\underline{Hom}_c(BA', BA'') \otimes \underline{Hom}_c(BA, BA') \to \underline{Hom}_c(BA, BA'')$$

yields a morphism

$$B\,\mathsf{Fun}_\infty(A', A'') \otimes B\,\mathsf{Fun}_\infty(A, A') \to B\,\mathsf{Fun}_\infty(A, A''),$$

which we view as an 'A_∞-bifunctor'

$$\mathsf{Fun}_\infty(A', A'') \times \mathsf{Fun}_\infty(A, A') \to \mathsf{Fun}_\infty(A, A'').$$

Let us show how this bifunctor yields the bialgebra structure on the bar construction of the Hochschild complex.

We need the following remark on cocategories: Let $\mathcal{C}$ be an augmented k-cocategory and x and object of $\mathcal{C}$. Then the endomorphism space $\mathcal{C}(x,x)$ is not, in general, a coalgebra in a natural way, because the comultiplication

$$\Delta : \mathcal{C}(x,x) \to \bigoplus_y \mathcal{C}(y,x) \otimes \mathcal{C}(x,y)$$

does not take values in $\mathcal{C}(x,x) \otimes \mathcal{C}(x,x)$. However, if we define $\mathsf{End}(x)$ as the set of all $c \in \mathcal{C}(x,x)$ such that $\Delta^{(N)}(c)$ lies in $\mathcal{C}(x,x)^{\otimes N}$ for all $N \geq 2$, then $\mathsf{End}(x)$ becomes an augmented coalgebra, which we call the *endomorphism coalgebra of* x.

Now let A be an ordinary augmented algebra and consider the cocategory

$$\mathcal{C} = \underline{Hom}_c(BA, BA) = B\,\mathsf{Fun}_\infty(A, A).$$

Then the endomorphism coalgebra of the identity functor $\mathsf{End}(\mathbf{1})$ is the bar construction $B(C(A,A))$, on the reduced Hochschild complex $C(A,A)$, *i. e.*

$$\overline{A} \to \mathsf{Hom}(\overline{A}, \overline{A}) \to \mathsf{Hom}(\overline{A} \otimes \overline{A}, \overline{A}) \to \ldots \to \mathsf{Hom}(\overline{A}^{\otimes p}, \overline{A}) \to \ldots$$

with $\overline{A}$ appearing in degree 0. The composition bifunctor

$$B\,\mathsf{Fun}_\infty(A, A) \otimes B\,\mathsf{Fun}_\infty(A, A) \to B\,\mathsf{Fun}_\infty(A, A)$$

then yields an associative multiplication on the endomorphism coalgebra

$$\mathsf{End}(\mathbf{1}) = B(C(A,A))$$

so that it becomes a bialgebra. Using the explicit description of the composition morphism in section 5.4 one checks that this bialgebra structure is given by the brace operations [**1**] [**18**] [**11**] [**12**].

5.8. Modules and bimodules. Let $\mathcal{C}(k)$ denote the dg category of complexes of vector spaces and $\mathcal{C}(k)^+$ the augmented dg category whose reduction is $\mathcal{C}(k)$. Let A be an augmented A_∞-category. By definition, the category of (left) A_∞-modules over A is the category

$$\mathsf{Fun}_\infty(A, \mathcal{C}(k)^+).$$

It is in fact a dg category (since the target category is a dg category). If A' is another augmented A_∞-category, then we can consider

$$\mathsf{Fun}_\infty(A', \mathsf{Fun}_\infty(A, \mathcal{C}(k)^+)).$$

From the canonical isomorphism

$$\underline{Hom}_c(U, \underline{Hom}_c(V, W)) \xrightarrow{\sim} \underline{Hom}_c(U \otimes V, W)$$

in the category of cocomplete augmented dg cocategories, we obtain

$$\mathsf{Fun}_\infty(A', \mathsf{Fun}_\infty(A, \mathcal{C}(k)^+) \xrightarrow{\sim} \underline{Hom}_c(BA' \otimes BA, \mathcal{C}(k)^+),$$

and the last term is the category of A_∞-bimodules. Equivalences of this type are quite unpleasant to prove directly, *cf.* for example Lemme 5.3.0.1 of [**26**].

References

[1] H. J. Baues, *The double bar and cobar constructions*, Compositio Math. **43** (1981), no. 3, 331–341.

[2] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[3] David Benson, Henning Krause, and Stefan Schwede, *Realizability of modules over Tate cohomology*, Trans. Amer. Math. Soc. **356** (2004), no. 9, 3621–3668 (electronic).

[4] Yuri Bespalov, V. V. Lyubashenko, and Oleksandr Manzyuk, *Closed multicategory of (pretriangulated) A_∞-categories*, in progress, 2006.

[5] A. A. Beĭlinson, *Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69.

[6] A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[7] Joseph Chuang and Alastair King, *The A_∞-structure on the Yoneda algebra of a monomial algebra*, preliminary notes.

[8] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[9] Gunnar Floystad, *Koszul duality and equivalences of categories*, arXiv:math.RA/0012264.

[10] Kenji Fukaya, *Morse homotopy, A^∞-category, and Floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993) (Seoul), Lecture Notes Ser., vol. 18, Seoul Nat. Univ., 1993, pp. 1–102.

[11] Ezra Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), Israel Math. Conf. Proc., vol. 7, Bar-Ilan Univ., Ramat Gan, 1993, pp. 65–78.

[12] Ezra Getzler and J. D. S. Jones, *Operads, homotopy algebra, and iterated integrals for double loop spaces*, hep-th/9403055.

[13] Ezra Getzler and John D. S. Jones, *A_∞-algebras and the cyclic bar complex*, Illinois J. Math. **34** (1990), no. 2, 256–283.

[14] Vladimir Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), no. 10, 3291–3323.

[15] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[16] Johannes Huebschmann, *The homotopy type of $F\Psi^q$. The complex and symplectic cases*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 487–518.

[17] ———, *The mod-p cohomology rings of metacyclic groups*, J. Pure Appl. Algebra **60** (1989), no. 1, 53–103.

[18] T. V. Kadeishvili, *The structure of the $A(\infty)$-algebra, and the Hochschild and Harrison cohomologies*, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR **91** (1988), 19–27.

[19] T. V. Kadeišvili, *On the theory of homology of fiber spaces*, Uspekhi Mat. Nauk **35** (1980), no. 3(213), 183–188, International Topology Conference (Moscow State Univ., Moscow, 1979).

[20] Bernhard Keller, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl. **3** (2001), no. 1, 1–35 (electronic).

[21] ———, *Addendum to: "Introduction to A-infinity algebras and modules" [Homology Homotopy Appl.* **3** *(2001), no. 1, 1–35;*, Homology Homotopy Appl. **4** (2002), no. 1, 25–28 (electronic).

[22] Maxim Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.

[23] ———, *Triangulated categories and geometry*, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellaïche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona, 1998.

[24] Maxim Kontsevich and Yan Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud., vol. 21, Kluwer Acad. Publ., Dordrecht, 2000, pp. 255–307.

[25] ———, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publishing, River Edge, NJ, 2001, pp. 203–263.

[26] Kenji Lefèvre-Hasegawa, *Sur les A_∞-catégories*, Thèse de doctorat, Université Denis Diderot – Paris 7, November 2003, available at B. Keller's homepage.

[27] D. M. Lu, J. H. Palmieri, Q. S. Wu, and J. J. Zhang, *A_∞-algebras for ring theorists*, Proceedings of the International Conference on Algebra, vol. 11, 2004, pp. 91–128.

[28] Volodymyr Lyubashenko, *Category of A_∞-categories*, Homology Homotopy Appl. **5** (2003), no. 1, 1–48 (electronic).

[29] Dag Madsen, *Ph. d. thesis*, NTNU, Trondheim, 2002.

[30] J. Peter May, *Matric Massey products*, J. Algebra **12** (1969), 533–568.

[31] Barry Mitchell, *Rings with several objects*, Advances in Math. **8** (1972), 1–161.

[32] Michael Penkava and Albert Schwarz, *A_∞ algebras and the cohomology of moduli spaces*, Lie groups and Lie algebras: E. B. Dynkin's Seminar, Amer. Math. Soc. Transl. Ser. 2, vol. 169, Amer. Math. Soc., Providence, RI, 1995, pp. 91–107.

[33] Alain Prouté, *Algèbres différentielles fortement homotopiquement associatives*, Thèse d'État, Université Paris VII, 1984.

[34] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.

[35] Jeremy Rickard, *Morita theory for derived categories*, J. London Math. Soc. **39** (1989), 436–456.

[36] Raphaël Rouquier, *Dimensions of triangulated categories*, arXiv:math.CT/0310134.

[37] Paul Seidel, *Fukaya categories and deformations*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 351–360.

[38] V. A. Smirnov, *Homology of fiber spaces*, Uspekhi Mat. Nauk **35** (1980), no. 3(213), 227–230, International Topology Conference (Moscow State Univ., Moscow, 1979).

[39] James Dillon Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. **108** (1963), 293–312.

[40] Jim Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*, Quantum groups (Leningrad, 1990), Lecture Notes in Math., vol. 1510, Springer, Berlin, 1992, pp. 120–137.

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