A-INFINITY ALGEBRAS IN REPRESENTATION THEORY

BERNHARD KELLER

Abstract. We give a brief introduction to A_∞-algebras and show three contexts in which they appear in representation theory: the study of Yoneda algebras and Koszulity, the description of categories of filtered modules and the description of triangulated categories.

Contents

1. Definitions, the bar construction, the minimality theorem

1.1. **A-infinity algebras and morphisms.** We refer to [11] for a list of references and a topological motivation for the following definition: Let k be a field. An A_∞-*algebra over* k is a $\mathbf{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbf{Z}} A^p$$

endowed with graded maps (=homogeneous k-linear maps)

$$m_n : A^{\otimes n} \to A \, , \; n \geq 1 \, ,$$

of degree $2-n$ satisfying the following relations

- We have $m_1 m_1 = 0$, i.e. (A, m_1) is a differential complex.
- We have
$$m_1 m_2 = m_2 (m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$
as maps $A^{\otimes 2} \to A$. Here $\mathbf{1}$ denotes the identity map of the space A. So m_1 is a (graded) derivation with respect to the multiplication m_2.
- We have
$$\begin{aligned} &m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) \\ &\qquad = m_1 m_3 + m_3 (m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1) \end{aligned}$$

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as maps $A^{\otimes 3} \to A$. Note that the left hand side is the associator for m_2 and that the right hand side may be viewed as the boundary of m_3 in the morphism complex $\mathsf{Hom}^{\bullet}{}_k(A^{\otimes 3}, A)$. This implies that m_2 is associative up to homotopy.

- More generally, for $n \geq 1$, we have
$$\sum (-1)^{r+st}\, m_u\,(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$
where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$.

More signs appear when these formulas are applied to elements by the Koszul sign rule
$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$$
for homogeneous maps f, g and homogeneous elements x, y, where the vertical bars denote the degree. In general, an A_∞-algebra is not associative, but its homology H^*A is a (graded) associative algebra for the multiplication induced by m_2. If $A = A^0$, i.e. $A^p = 0$ for all $p \neq 0$, then A is an ordinary associative algebra (but not necessarily unital) and conversely, each graded associative algebra may be viewed as an A_∞-algebra whose compositions m_i vanish for $i \geq 3$.

A *morphism of A_∞-algebras* $f : A \to B$ is a family
$$f_n : A^{\otimes n} \to B$$
of graded maps of degree $1 - n$ such that

- we have $f_1 m_1 = m_1 f_1$, i.e. f_1 is a morphism of complexes,
- we have
$$f_1 m_2 = m_2\,(f_1 \otimes f_1) + m_1 f_2 + f_2\,(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)\,,$$
which means that f_1 commutes with the multiplication m_2 up to a homotopy given by f_2,
- more generally, for $n \geq 1$, we have
$$\sum (-1)^{r+st}\, f_u\,(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum (-1)^s\, m_r\,(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r})$$
where the first sum runs over all decompositions $n = r + s + t$, we put $u = r+1+t$, and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \cdots + i_r$; the sign on the right hand side is given by
$$s = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1).$$

The morphism f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism. It is *strict* if $f_i = 0$ for all $i \neq 1$. The *identity morphism* is the strict morphism f such that f_1 is the identity of A. The *composition* of two morphisms $f : B \to C$ and $g : A \to B$ is given by
$$(f \circ g)_n = \sum (-1)^s f_r \,\circ\, (g_{i_1} \otimes \cdots \otimes g_{i_r})$$
where the sum and the sign are as in the defining identity. It is not hard to check we do obtain a category (especially if we disregard the signs). A more conceptual proof of this uses the *bar construction*.

1.2. **The bar construction.** Following [23] and [7] we will restate the definitions of A_∞-algebras and their morphisms in a more efficient way. Let V be a $\mathbf{Z}$-graded vector space and let

$$\overline{T}V = k \oplus V \oplus V^{\otimes 2} \oplus \cdots$$

be the reduced tensor algebra (in the category of graded spaces). We make $\overline{T}V$ into a graded coalgebra via the comultiplication

$$\Delta : \overline{T}V \to \overline{T}V \otimes \overline{T}V$$

defined by

$$\Delta(v_1, \ldots, v_n) = \sum_{1 \leq i \leq n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n) \, .$$

Note that $\Delta(v) = 0$ and $\Delta(v_1, v_2) = v_1 \otimes v_2$. It is then easy to see that each graded map $b : \overline{T}V \to V$ of degree 1 uniquely lifts to a graded coderivation $b : \overline{T}V \to \overline{T}V$ of degree 1, where *coderivation* means that we have

$$\Delta b = (b \otimes \mathbf{1} + \mathbf{1} \otimes b) \, \Delta.$$

To see that this is plausible, one should remember that algebra derivations $\overline{T}V \to \overline{T}V$ are in bijection with maps $V \to \overline{T}V$. The component of b that maps $V^{\otimes n}$ to $V^{\otimes u}$ is

$$\sum \mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t}$$

where $r+s+t = n$ and $r+1+t = u$. Thus we have a bijection between coderivations $b : \overline{T}V \to \overline{T}V$ of degree 1 and families of maps $b_n : V^{\otimes n} \to V$, $n \geq 1$, of degree 1.

Now let A be another $\mathbf{Z}$-graded space and let SA be its suspension: $(SA)^p = A^{p+1}$. Define a bijection between the families of maps $b_n : (SA)^{\otimes n} \to SA$ of degree 1 and maps $m_n : A^{\otimes n} \to A$, $n \geq 1$, by the commutative square

$$\begin{array}{ccc} (SA)^{\otimes n} & \xrightarrow{b_n} & SA \\ s^{\otimes n} \uparrow & & \uparrow s \\ A^{\otimes n} & \xrightarrow{m_n} & A \end{array} \quad ,$$

where $s : A \to SA$ is the canonical map $a \mapsto a$ of degree -1. Note that m_n is of degree $-n + 1 - (-1) = 2 - n$. Then we have the following easy

Lemma. *The following are equivalent*

(i) *The maps $m_n : A^{\otimes n} \to A$ yield an A_∞-algebra structure on A.*
(ii) *The coderivation $b : \overline{T}SA \to \overline{T}SA$ satisfies $b^2 = 0$.*
(iii) *For each $n \geq 1$, we have*

$$\sum b_u \, (\mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t}) = 0 \, ,$$

where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$.

Note that no signs appear in the last formula. The lemma yields a bijection between A_∞-algebra structures on A and coalgebra differentials on $\overline{T}SA$. It is also easy to check that if A and B are A_∞-algebras, then morphisms of A_∞-algebras $f : A \to B$ are in bijection with morphisms of dg coalgebras $\overline{T}SA \to \overline{T}SB$.

1.3. **The minimality theorem.**

Theorem (Kadeishvili [6], see also [5], [21], [19], [2], [4], [15]). *Let A be an A_∞-algebra. Then the homology H^*A has an A_∞-algebra structure such that*

1) *we have $m_1 = 0$ and m_2 is induced by m_2^A and*
2) *there is a quasi-isomorphism of A_∞-algebras $H^*A \to A$ lifting the identity of H^*A.*

Moreover, this structure is unique up to (non unique) isomorphism of A_∞-algebras.

Let us introduce some terminology: An A_∞-algebra with vanishing m_1 is called *minimal*. In the situation of the theorem, the A_∞-algebra H^*A is called a *minimal model* for A. The A_∞-algebra A is called *formal* if its minimal model can be chosen to be an ordinary associative graded algebra, i.e. such that $m_n = 0$ for all $n \geq 3$.

As a consequence, we obtain that if B is an ordinary associative algebra with 1 and M a B-module, then $\mathsf{Ext}_B^*(M, M)$ carries an A_∞-algebra structure canonical up to non unique A_∞-isomorphism. Indeed, choose a projective resolution $P \to M$ of M. Then the morphism complex $A = \mathsf{Hom}^\bullet{}_B(P, P)$ is a differential graded algebra whose homology identifies with the Yoneda algebra $\mathsf{Ext}_B^*(M, M)$. In particular, A is an A_∞-algebra and its homology $\mathsf{Ext}_B^*(M, M)$ carries the A_∞-structure of the minimal model of A.

1.4. **Inductive construction of the higher compositions.** Let us suppose that A is a dg algebra. We will show how to construct the first four higher compositions m_i on H^*A and the first four components f_i of an A_∞-quasi-isomorphism $f : H^*A \to A$. We will denote all composition maps with i arguments by m_i. In particular, $m_1 : A \to A$ is the differential of A and $m_2 : A \otimes A \to A$ the multiplication, and all higher $m_i : A^{\otimes i} \to A$, $i \geq 3$, vanish. We wish to construct the $m_i : (H^*A)^{\otimes i} \to H^*A$.

1) Let $m_1 : H^*A \to H^*A$ be zero. Choose f_1 as a morphism of complexes $(H^*A, 0) \to A$ such that H^*f_1 identifies with the identity of H^*A.
2) Let $m_2 : H^*A \otimes H^*A \to H^*A$ be the multiplication induced by that of A. Then by definition, the morphisms $m_2\,(f_1 \otimes f_1)$ and $f_1\, m_2$ are homotopic as morphisms of complexes from $H^*A \otimes H^*A$ to A. We choose $f_2 : H^*A \otimes H^*A \to A$ as a morphism of complexes of degree -1 such that
$$f_1 m_2 = m_1 f_2 + m_2\,(f_1 \otimes f_1).$$
3) Let
$$\Phi_3 = m_2\,(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2\,(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}).$$
Then Φ_3 is a morphism of complexes $(H^*A)^{\otimes 3} \to A$ of degree -1. Since $f_1 : H^*A \to A$ is a quasi-isomorphism, there is a morphism of complexes $m_3 : (H^*A)^{\otimes 3} \to H^*A$ such that $f_1\, m_3$ is homotopic to Φ_3. So there is a graded morphism $f_3 : (H^*A)^{\otimes 3} \to A$ such that
$$f_1\, m_3 = m_1\, f_3 + \Phi_3.$$
4) Let
$$\begin{aligned}\Phi_4 \;=\; & m_2\,(f_1 \otimes f_3 - f_2 \otimes f_2 + f_3 \otimes f_1) + f_2\,(\mathbf{1} \otimes m_3 + m_3 \otimes \mathbf{1}) \\ & - f_3\,(\mathbf{1} \otimes \mathbf{1} \otimes m_2 - \mathbf{1} \otimes m_2 \otimes \mathbf{1} + m_2 \otimes \mathbf{1} \otimes \mathbf{1}).\end{aligned}$$

Then $\Phi_4 : (H^*A)^{\otimes 4} \to A$ is a morphism of complexes of degree -2. Thus there is a morphism of complexes $m_4 : (H^*A)^{\otimes 4} \to H^*A$ such that $f_1 m_4$ is homotopic to Φ_4. Hence we can find a graded morphism $f_4 : (H^*A)^{\otimes 4} \to A$ of degree -3 such that

$$f_1 m_4 = m_1 f_4 + \Phi_4.$$

2. Yoneda algebras, Koszulity and filtered modules

2.1. **Motivation.** Let B be an associative algebra with 1 and let $M_1, \ldots, M_r$ be (right) B-modules. Consider the full subcategory $\mathcal{F} = \mathsf{filt}(M_1, \ldots, M_r)$ of the category $\mathsf{Mod}\, B$ of B-modules whose objects are the modules admitting a finite filtration whose subquotients are among the M_i. Note that morphisms in this category are not required to respect the filtrations. Let $\mathcal{D}\,\mathsf{Mod}\, B$ denote the (unbounded) derived category of the category of B-modules and $\mathcal{T} = \mathsf{tria}(M_1, \ldots, M_r)$ its full triangulated subcategory generated by $M_1, \ldots, M_r$. Put $M = \bigoplus M_i$.

Question: Can we reconstruct $\mathcal{F}$ and $\mathcal{T}$ from the Yoneda algebra

$$\mathsf{Ext} = \mathsf{Ext}^*_B(M, M) \quad ?$$

The answer is clearly no. For example, if we take $B = k[x]/(x^n)$, and $M = k$, then the graded algebra Ext identifies with the algebra $k[u, v]/(v^2)$, $|u| = 2$, $|v| = 1$, for all $n \geq 3$, but the categories $\mathcal{F}$ are non-equivalent for different values of n. However, the Yoneda algebra *admits an A_∞-algebra structure and this structure allows us to reconstruct both $\mathcal{F}$ and $\mathcal{T}$*. More precisely,

1) there is a description of $\mathcal{F}$ using only

$$\mathsf{Ext}^0 \oplus \mathsf{Ext}^1 \oplus \mathsf{Ext}^2$$

and the 'restriction' of the m_i, $i \geq 2$, to this truncation. This description generalizes the description of finite-length modules over a finite-dimensional algebra as representations of a quiver with relations.

2) There is an analogous description of $\mathcal{T}$ which uses all of the information present in the A_∞-algebra structure on Ext.

Below we will give the description of $\mathcal{F}$. We refer to [11] for an analogous description of $\mathcal{T}$. A different description will be given below in section 3.

2.2. **The Yoneda algebra of the simples.** Let k be a field, Q a finite quiver and I an admissible ideal of kQ. Put $B = kQ/I$ and let $M = S$ be the sum of the canonical representatives of the isoclasses of the simple B-modules.

Proposition 1. a) *The A_∞-algebra $A = \mathsf{Ext}^*_B(S, S)$ is formal (i.e. A_∞-isomorphic to an algebra with $m_i = 0$ for all $i \geq 3$) if and only if B is a Koszul algebra.*

b) *As an A_∞-algebra, $\mathsf{Ext}^*_B(S, S)$ is generated by its components in degree 0 and 1.*

In a), we say that B is Koszul if B is isomorphic to kQ/I', where I' is a homogeneous ideal (all arrows of Q are of degree 1) such that the graded algebra kQ/I' is Koszul (cf. [1]). In b), we claim that $\mathsf{Ext}^*_B(S, S)$ coincides with its smallest subspace closed under all operations m_i and containing the components of degree 0 and 1.

We now describe the part of the A_∞-structure on $\mathsf{Ext}^*_B(S,S)$ which only involves the components of degree 0 and 1. For this, let $J = \langle kQ_1 \rangle$ be the ideal generated by the arrows of Q. Let D denote the dual over k. Recall that we have a canonical map

$$\bigoplus_{i=1}^{\infty} \mathsf{Ext}^1_B(S,S)^{\otimes i} \to DJ$$

and a canonical isomorphism

$$\mathsf{Ext}^2_B(S,S) \xrightarrow{\sim} D(I/JI+IJ).$$

Let s be a k-linear section of the surjection $I \to I/(JI+IJ)$. By composing the inclusion $I \subset J$ with s, we obtain a map $I/(JI+IJ) \to J$ whose image is a space of representatives for the irreducible relations. The top horizontal arrow of the following square is the dual of this map

$$\begin{array}{ccc} DJ & \to & D(I/IJ+JI) \\ \uparrow & & \uparrow \\ \bigoplus_{i=1}^{\infty} \mathsf{Ext}^1_B(S,S)^{\otimes i} & \xrightarrow{\mu} & \mathsf{Ext}^2_B(S,S) \end{array}$$

The map μ is defined by the commutativity of this square.

Proposition 2. *The component*

$$\mu_i : \mathsf{Ext}^1_B(S,S)^{\otimes i} \to \mathsf{Ext}^2_B(S,S)$$

equals the restriction of m_i to $Ext^1_B(S,S)^{\otimes i}$ for some choice of minimal model structure m_i, $i \geq 1$, on $Ext^*_B(S,S)$.

In particular, we see that the 'restriction' to

$$\mathsf{Ext}^0_B(S,S) \oplus \mathsf{Ext}^1_B(S,S) \oplus \mathsf{Ext}^2_B(S,S)$$

of the A_∞-structure on $\mathsf{Ext}^*_B(S,S)$ yields a quiver with relations for B and hence determines the category of finite-dimensional B-modules.

2.3. **Reconstruction.** As announced in 2.1, we will now show how to reconstruct the category $\mathcal{F}$ from the A_∞-algebra $\mathsf{Ext}^*_B(M,M)$. More precisely, we will reconstruct it from the more natural A_∞-category associated with the M_i. An A_∞-*category* $\mathcal{A}$ is the datum of

- a class of objects $\mathsf{obj}(\mathcal{A})$,
- for all $A, B \in \mathcal{A}$ (we write $A \in \mathcal{A}$ instead of $A \in \mathsf{obj}(\mathcal{A})$), a $\mathbf{Z}$-graded vector space $\mathsf{Hom}_{\mathcal{A}}(A,B)$, often *denoted by* (A,B),
- for all $n \geq 1$ and all $A_0, \ldots, A_n \in \mathcal{A}$, a graded map

  $$m_n : (A_{n-1}, A_n) \otimes (A_{n-2}, A_{n-1}) \otimes \cdots \otimes (A_0, A_1) \to (A_0, A_n)$$

  of degree $2-n$

such that for each $n \geq 1$ and all $A_0, \ldots, A_n \in \mathcal{A}$, we have the identity

$$\sum (-1)^{r+st}\, m_u\, (\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$

of maps

$$(A_{n-1}, A_n) \otimes (A_{n-2}, A_{n-1}) \otimes \cdots \otimes (A_0, A_1) \to (A_0, A_n)\,,$$

where the sum runs over all decompositions $n = r+s+t$ and we put $u = r+1+t$.

For example, the datum of an A_∞-category $\mathcal{A}$ with one object $*$ is equivalent to the datum of the endomorphism A_∞-algebra $\mathsf{Hom}_{\mathcal{A}}(*,*)$. Another class of examples is provided by dg 'categories without identities': these are exactly the A_∞-categories whose composition maps m_n vanish for all $n \geq 3$.

Note that for two reasons, an A_∞-category is not a category in general : firstly, the composition m_2 may not be associative; secondly, there may not be identity morphisms.

A variant of the minimality theorem 1.3 yields a canonical (up to non unique isomorphism) structure of minimal A_∞-category on the graded category $\mathcal{E}xt$ with objects $M_1, \ldots, M_r$ and morphism spaces

$$\mathcal{E}xt\,(M_i, M_j) = \mathsf{Ext}^*_B(M_i, M_j).$$

We will reconstruct $\mathcal{F} = \mathsf{filt}(M_1, \ldots, M_r)$ from $\mathcal{E}xt$. In fact, as it will turn out, we only need the truncations

$$\mathsf{Ext}^0_B(M_i, M_j) \oplus \mathsf{Ext}^1_B(M_i, M_j) \oplus \mathsf{Ext}^2_B(M_i, M_j)$$

of the morphism spaces and only the restriction of the A_∞-structure to these truncations.

Let $\mathcal{A}$ be a minimal A_∞-category with $\mathcal{A}\,(A, B)^p = 0$ for all $p < 0$ and all $A, B \in \mathcal{A}$. We will construct a 'category without units' $\mathsf{twst}\,\mathcal{A}$ called the *category of twisted stalks*. It will turn out that $\mathsf{twst}\,\mathcal{E}xt$ is equivalent to $\mathcal{F}$. A twisted stalk over $\mathcal{A}$ is a pair M consisting of a sequence $A_1, \ldots, A_n$ of objects of $\mathcal{A}$ and of a matrix $\delta = (\delta_{ij})$ of elements

$$\delta_{ij} \in \mathcal{A}(A_i, A_j)^1$$

satisfying $\delta_{ij} = 0$ for all $i \geq j$ and

$$0 = \sum_{t=2}^{\infty} (-1)^{\frac{t(t-1)}{2}} m_t(\delta, \ldots, \delta) = -m_2(\delta, \delta) - m_3(\delta, \delta, \delta) + \ldots.$$

Here we still denote by m_i the natural extension of m_i to matrices with coefficients in the morphisms spaces of $\mathcal{A}$. One should think of M as an object with a filtration whose subquotients are the A_i.

A *morphism of twisted stalks* $M \to M'$ is given by a matrix $f = (f_{ij})$ of elements

$$f_{ij} \in \mathcal{A}\,(A_j, A'_i)^0$$

satisfying

$$\begin{aligned} 0 \quad = \quad & -m_2(\delta', f) + m_2(f, \delta) - m_3(\delta', \delta', f) + m_3(\delta', f, \delta) - m_3(f, \delta, \delta) + \ldots \\ & \pm m_n(\delta', \ldots, \delta', f, \delta, \ldots, \delta) \ldots \end{aligned}$$

The *composition* of two morphisms $f : M' \to M''$ and $g : M \to M'$ is given by

$$\begin{aligned} & m_2(f, g) + m_3(f, g, \delta) - m_3(f, \delta', g) + m_3(\delta'', f, g) + \ldots \\ & \pm m_n(\delta'', \ldots, \delta'', f, \delta', \ldots, \delta', g, \delta, \ldots, \delta) + \ldots \end{aligned}$$

(for the signs cf. [11, 7.6]).

Lemma. *The composition of* $\mathsf{twst}\,\mathcal{A}$ *is associative.*

The lemma is easy to check modulo the signs. To prove it rigorously, one can use an analogue of the bar construction, cf. [14].

Proposition. *The category of twisted stalks* $\mathsf{twst}\,\mathcal{E}xt$ *is equivalent to* $\mathcal{F}$ *by a functor that sends the object* $(M_i, \delta = 0)$ *to* M_i, $1 \leq i \leq r$.

In the case where $M_1, \dots, M_r$ are the simple modules over a finite-dimensional algebra B as in 2.2, this proposition reduces to the description of the category of finite-length modules over B as representations of a quiver with relations. An example with some non simple M_i is given in [11, 7.9].

3. Description of triangulated categories

We will explain how each 'algebraic' triangulated category with a generator becomes equivalent to the derived category of an A_∞-category. A class of conjectural examples is provided by Kontsevich's homological interpretation of mirror symmetry.

Let A be an A_∞-algebra. An *A_∞-module* over A is a $\mathbf{Z}$-graded space M endowed with maps

$$m_n^M : M \otimes A^{\otimes n} \to M \ , \ n \geq 1 \ ,$$

homogeneous of degree $2-n$ which satisfy the 'same' identities as those in the definition of an A_∞-algebra (some of the m_i have to be interpreted as m_i^M). For example, if A is concentrated in degree 0, each complex of A^0-modules gives rise to an A_∞-module over A (but not all A_∞-modules are obtained in this way).

For an arbitrary A_∞-algebra, the structures of A_∞-module on a graded vector space M are in bijection with the comodule differentials on the induced comodule $M \otimes_k TSA$, where $TSA = k \oplus \overline{T}SA$ is the cofree coaugmented coalgebra on SA endowed with the differential of the sum of the complexes k and $\overline{T}SA$. By definition, the *morphisms of A_∞-modules* $M \to N$ are in bijection with the morphisms of dg comodules $M \otimes TSA \to N \otimes TSA$. Concretely, a morphism of A_∞-modules $M \to N$ is given by a family of graded maps $f_i : M \otimes A^{\otimes i} \to N$, $i \geq 1$, homogeneous of degree $1-i$ satisfying certain conditions. The first of these states that f_1 is a morphism of complexes. By definition, f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism.

An A_∞-module M is *H-unital* [24] if the corresponding comodule $M \otimes_k TSA$ has vanishing homology. If A is a unital associative algebra concentrated in degree 0 and M a unital module over A, then $M \otimes TSA$ coincides with the augmented bar resolution of M over A and hence is acyclic. So M is H-unital in this case.

An A_∞-algebra is *H-unital* if it is H-unital as a module over itself. In contrast to more naive notions of unitality, H-unitality is preserved under quasi-isomorphisms. In particular, A is H-unital iff its minimal model is H-unital.

From now on, we suppose that A is an H-unital A_∞-algebra. The *derived category* $\mathcal{D}_\infty A$ has as objects the H-unital A_∞-modules. Its morphisms are obtained from morphisms of A_∞-modules by formally inverting all quasi-isomorphisms. One can show [14] that $\mathcal{D}_\infty A$ has a natural triangulated structure. Moreover if A is an associative unital algebra (or dg algebra) and $\mathcal{D}A$ its derived category, the canonical functor

$$\mathcal{D}A \to \mathcal{D}_\infty A$$

is an equivalence.

The *perfect derived category* $\mathcal{D}_{per}A$ is by definition the smallest triangulated subcategory of $\mathcal{D}A$ containing the free module A and stable under forming direct factors. Thus if A is an ordinary unital associative algebra, then $\mathcal{D}_{per}A$ is the subcategory of perfect complexes in $\mathcal{D}A$. So in this case, it identifies with the homotopy category of bounded complexes of finitely generated projective modules. In the general case, an analogous description of $\mathcal{D}_{per}A$ can be given using twisted objects in a way similar to twisted stalks (2.3), cf. [11]. In fact, $\mathcal{D}_{per}A$ is equivalent to the closure under direct factors of the 0-homology category of the A_∞-category of twisted objects over A.

An *algebraic triangulated category* is a triangulated category $\mathcal{T}$ which is triangle equivalent to the stable category of a Frobenius category.

Theorem. *Let $\mathcal{T}$ be a k-linear algebraic triangulated category with split idempotents. Suppose that $\mathcal{T}$ is generated by an object U, i.e. $\mathcal{T}$ coincides with its smallest full triangulated subcategory containing U and closed under forming direct summands. Then the graded algebra*

$$A = \bigoplus_{i \in \mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(U, U[i])$$

admits a structure of H-unital A_∞-algebra with $m_1 = 0$ and m_2 the natural multiplication such that there is a triangle equivalence

$$\mathcal{D}_{per} A \to \mathcal{T}$$

mapping A to U.

For example, if $\mathcal{T}$ is the bounded derived category of the category of coherent sheaves on a smooth projective variety X and U is a tilting sheaf (or a tilting complex) on X, then the algebra A is concentrated in degree 0 and all the higher m_i vanish for degree reasons. In this case, A is a finite-dimensional algebra of finite global dimension, $\mathcal{D}_{per} A$ coincides with the bounded derived category of finitely generated A-modules and we obtain an equivalence

$$\mathcal{D}^b A \to \mathcal{D}^b(\mathsf{coh}\, X).$$

If B is a finite-dimensional algebra of the form kQ/I, $\mathcal{T}$ the bounded derived category of finite-dimensional B-modules and U the sum of the simple modules, then the A_∞-algebra A of the theorem identifies with the Yoneda algebra $\mathsf{Ext}^*_B(S, S)$, $S = U$, with its (almost) canonical A_∞-structure. We obtain a triangle equivalence

$$\mathcal{D}_{per} A \to \mathcal{D}^b B.$$

The right hand side may be thought of as a description of the derived category in terms of projective modules, whereas the left hand side is a description in terms of simple modules.

The theorem above has an obvious variant where A_∞-algebras are replaced by A_∞-categories:

Theorem. *Let $\mathcal{T}$ be a k-linear algebraic triangulated category with split idempotents and $\mathcal{U} \subset \mathcal{T}$ a full subcategory such that $\mathcal{T}$ coincides with its smallest triangulated subcategory containing $\mathcal{U}$ and stable under forming direct factors. Then there is an H-unital A_∞-category $\mathcal{A}$ such that there is a bijection $A \to U_A$ from the objects of $\mathcal{A}$ to those of $\mathcal{U}$ and a triangle equivalence*

$$\mathcal{D}_{per} \mathcal{A} \to \mathcal{T}$$

mapping the free $\mathcal{A}$-module $\mathsf{Hom}_{\mathcal{A}}(?, A)$ to U_A.

In his lecture [12] at the ICM in 1994, M. Kontsevich gave a conjectural class of examples: Let X be a smooth projective complex variety admitting a mirror symmetric dual Y (in particular, ω_X is then trivial). Kontsevich conjectured that there is a triangle equivalence

$$\mathcal{D}_{per} \mathcal{F}(Y) \to \mathcal{D}^b(\mathsf{coh}\, X)$$

where $\mathcal{F}(Y)$ is the 'Fukaya category' of Y. The variety Y is a symplectic variety, the objects of $\mathcal{F}(Y)$ are Lagrangian subvarieties $\mathcal{L} \subset Y$ endowed with additional structure, morphism spaces of $\mathcal{F}(Y)$ should be Floer homology groups and higher compositions given by Donaldson products. No rigorous definition seems to be known until now.

However, Kontsevich's conjecture was proved by Polishchuk and Zaslov [18] for a suitably adapted version of $\mathcal{F}(Y)$ in the case where X is an elliptic curve (and so Y is a symplectic torus).

Kontsevich's conjecture implies that the group of symplectomorphisms of Y embeds into the automorphism group of the derived category $\mathcal{D}^b(\mathsf{coh}\, X)$. The existence of many automorphisms belonging conjecturally to the image of this embedding was checked in work by R. P. Horja [3] and by P. Seidel and R. Thomas [20].

Refined versions of Kontsevich's conjecture avoid many of the analytic problems that plagued his earlier approach. We refer to Kontsevich-Soibelman [13] for details.

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UFR de Mathématiques, UMR 7586 du CNRS, Case 7012, Université Paris 7, 2, place Jussieu, 75251 Paris Cedex 05, France

E-mail address: `keller@math.jussieu.fr`
`www.math.jussieu.fr/~keller`