BIMODULE COMPLEXES VIA STRONG HOMOTOPY ACTIONS

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To Professor Klaus W. Roggenkamp on the occasion of his sixtieth birthday

Abstract. We present a new and explicit method for lifting a tilting complex to a bimodule complex. The key ingredient of our method is the notion of a strong homotopy action in the sense of Stasheff.

1. Introduction

Let A and B be (associative, unital) algebras over a commutative ring k. Denote by $\mathsf{Mod}\,A$ the category of (right) A-modules. Suppose that P is a B-module endowed with an algebra morphism

$$A \to \mathsf{Hom}_{\mathsf{Mod}\,B}(P, P).$$

Then P becomes an A-B-bimodule and we have the tensor functor

$$? \otimes_A P : \mathsf{Mod}\,A \to \mathsf{Mod}\,B\ ,$$

which takes the free A-module A_A to P. This is the basic fact which allows us to construct Morita equivalences $\mathsf{Mod}\,A \xrightarrow{\sim} \mathsf{Mod}\,B$.

Now let $\mathcal{D}B = \mathcal{D}\,\mathsf{Mod}\,B$ be the (unbounded) derived category of the category of B-modules. Suppose that we have a complex $T \in \mathcal{D}B$ and an algebra morphism

$$A \to \mathsf{Hom}_{\mathcal{D}B}(T, T).$$

It is not clear whether this map comes from an action of A on the *components* of T, even after replacing T by an isomorphic object of $\mathcal{D}B$. Therefore, the (derived) tensor product by T not well-defined and the analogy with the case of module categories seems to break down. Nevertheless, for complexes T satisfying the 'Toda condition'

$$\mathsf{Hom}_{\mathcal{D}B}(T, T[-n]) = 0\ , \quad \text{for all } n > 0\ ,$$

J. Rickard succeeded [8] in constructing a triangle functor $F : \mathcal{D}^- A \to \mathcal{D}B$ taking A_A to T, where $\mathcal{D}^- A \subset \mathcal{D}A$ denotes the subcategory of right bounded complexes. This construction was at the basis of the proof of his 'Morita theorem for derived categories'. Later he showed [9] that if F restricts to an equivalence between the bounded derived categories (and suitable flatness hypotheses hold), then after replacing T by an isomorphic object, it *is possible* to lift the A-action to the components of T. In his proof, he used the functor F constructed in [8].

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In [3] and [4], we gave an a priori construction of a lift (up to isomorphism) of T to a complex of bimodules X (under suitable flatness hypotheses). This made it possible to *define* $F = \mathbf{L}(? \otimes_A X)$ and to give a new proof [10, Ch. 8] of Rickard's Morita theorem.

Thus, up to now, there have been two constructions of a bimodule complex X from a complex T as above. Neither of them is very explicit: the first one [9] uses the functor F; the second one [3] uses resolutions over differential graded algebras.

In this paper, we present a new construction, which is surprisingly explicit. In fact, if we assume that T is a right bounded complex of projective B-modules, then essentially the only data we need are homotopies $H(f)$ such that

$$f = d \circ H(f) + H(f) \circ d$$

for each morphism of complexes $f : T \to T[-n]$, $n > 0$. The details of the construction are concisely presented in section 9. Its relation to Rickard's original construction is explained in section 10. We also prove a uniqueness result which improves on [9] and [3].

The essential new ingredient of our method is the notion of a *strong homotopy action* ($=A_\infty$-action) due to Stasheff [11], [12] and Kadeishvili [2], and recently popularized again by Kontsevich [6], [7]. The present article is self-contained but the interested reader may find more information and references on strong homotopy methods in [5]. We will show that if A is projective over k and T is right bounded with projective components and satisfies the Toda condition, then the 'action up to homotopy' of A on T may be enriched to a strong homotopy action. It is remarkable that this can be done without changing the underlying complex of T. In a second step, we show that each strong homotopy action on a complex K yields a strict action on a larger (but quasi-isomorphic) complex K'. In fact, K' may be viewed as a 'perturbed Hochschild resolution' of the complex K.

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Contents

2. The main theorem

Let k be a commutative ring and B an (associative, unital) k-algebra. Let T be a complex of (right) B-modules. Let A be another k-algebra. A *strict action* of A on T is an homomorphism (preserving the unit)

$$A \to \mathsf{Hom}_{\mathcal{C}B}(T, T) ,$$

where $\mathcal{C}B$ is the category of complexes of B-modules. Equivalently, a strict action is the datum of a complex of A-B-bimodules whose restriction to B equals T. An *homotopy action* of A on T is an homomorphism

$$\alpha : A \to \mathsf{Hom}_{\mathcal{H}B}(T, T) ,$$

where $\mathcal{H}B$ is the homotopy category of right B-modules. The pair (T, α) is then an *homotopy module*. If T and T' are endowed with homotopy actions α and α', a morphism of complexes $f : T \to T'$ is *compatible* with these actions if $f \circ \alpha(a)$ is homotopic to $\alpha'(a) \circ f$ for all $a \in A$. Then f is also called a *morphism of homotopy modules*.

Clearly, each strict action yields an homotopy action. The converse is false, in general. However, we have the

Theorem 2.1. *Let k be a commutative ring and A, B two (associative, unital) k-algebras. Let T be a complex of right B-modules endowed with an homotopy action by A. Suppose that A is projective as a k-module, and that T is right bounded with projective components and satisfies*

$$\mathsf{Hom}_{\mathcal{H}B}(T, T[-n]) = 0 \text{ for all } n > 0. \tag{2.1}$$

a) *There is a right bounded complex of projective B-modules X endowed with a strict action by A and a quasi-isomorphism $\varphi : T \to X$ of complexes of B-modules compatible with the homotopy actions by A.*
b) *If $\varphi : T \to X$ and $\varphi' : T \to X'$ are two quasi-isomorphisms as in a), then there is a unique isomorphism $\psi : X \to X'$ in the derived category of A-B-bimodules such that we have $\psi \circ \varphi = \varphi'$ in the homotopy category of B-modules.*

For the case where T is a tilting complex, the theorem was proved by J. Rickard in [9]. The general case is proved in [3]. We give a new proof which, compared to these previous approaches, is much more explicit. We illustrate this by two special cases. The general method is concisely presented in section 9.

3. First special case

Suppose that the assumptions of theorem 2.1 hold. Let us assume that T has non vanishing components at most in degrees 0 and 1:

$$T = (\ldots \to 0 \to T_1 \to T_0 \to 0 \to \ldots).$$

We will construct a quasi-isomorphism $\varphi : T \to X$ as in theorem 2.1 a).

Since A is projective over k, we can find a k-linear map $\widetilde{\alpha} : A \to \mathsf{Hom}_{\mathcal{C}B}(T,T)$ lifting the given homotopy action $\alpha : A \to \mathsf{Hom}_{\mathcal{H}B}(T,T)$. Now we define $m_2 : A \otimes T \to T$ by

$$m_2(a,x) = (\widetilde{\alpha}(a))(x)\ ,\ a \in A, x \in T.$$

Now we have to take into account the non-associativity of m_2: Consider the square

$$\begin{array}{ccc} A \otimes A \otimes T & \stackrel{\mathbf{1}_A \otimes m_2}{\to} & A \otimes T \\ m_A \otimes \mathbf{1}_T \downarrow & & \downarrow m_2 \\ A \otimes T & \stackrel{m_2}{\to} & T\ , \end{array}$$

where m_A denotes the multiplication of A. The square becomes commutative in the homotopy category. Hence there is a morphism of graded B-modules

$$m_3 : A \otimes A \otimes T \to T$$

homogeneous of degree -1 such that we have

$$m_2(ab,x) - m_2(a, m_2(b,x)) = -m_3(a,b,d(x)) - d(m_3(a,b,x))$$

for all $a, b \in A$ and $x \in T$. We construct a complex $\widetilde{X}$ as follows: The underlying graded module of $\widetilde{X}$ is

$$A \otimes A \otimes A \otimes T[2] \quad \oplus \quad A \otimes A \otimes T[1] \quad \oplus \quad A \otimes T.$$

The differential is given by

$$\begin{aligned} d(a,b,c,x) &= -ab \otimes c \otimes x + a \otimes bc \otimes x \\ &\qquad + a \otimes m_3(b,c,x) - a \otimes b \otimes m_2(c,x) + a \otimes b \otimes c \otimes d(x)\ , \\ d(a,b,x) &= -ab \otimes x + a \otimes m_2(b,x) - a \otimes b \otimes d(x)\ , \\ d(a,x) &= a \otimes d(x)\ , \end{aligned}$$

where $a, b, c \in A$, $x \in T$. We define the complex X to be the truncation

$$X = \tau_{\leq 1} \widetilde{X} = (\widetilde{X}_1 / \operatorname{im} d_2 \to \widetilde{X}_0).$$

We define the morphism of complexes $f : T \to \widetilde{X}$ by $f(x) = 1 \otimes x$. This morphism is compatible with the homotopy actions by A. Indeed, if we define the graded morphism $f_2 : A \otimes T \to \widetilde{X}$ of degree -1 by

$$f_2(a \otimes x) = 1 \otimes a \otimes x \in A \otimes A \otimes T[1]\ ,\ a \in A, x \in T\ ,$$

then we have

$$d(f_2(a,x)) + f_2(a,d(x)) = f(m_2(a,x)) - a\,f(x)\ ,\ a \in A\ ,\ x \in T.$$

By composition, f yields a morphism $T \to X$. Its homotopy class is the required quasi-isomorphism φ. Since $T \to \widetilde{X}$ is compatible with the homotopy action by A and $\widetilde{X} \to X$ is a morphism of complexes of bimodules, the morphism φ is compatible with the homotopy action by A.

4. Second special case

Suppose that the assumptions of theorem 2.1 hold. Let us assume that T has non vanishing components at most in degrees 0, 1 and 2:

$$T = (\ldots \to 0 \to T_2 \to T_1 \to T_0 \to 0 \to \ldots).$$

We will construct a complex of bimodules X and a quasi-isomorphism $\varphi : X \to T$ as in theorem 2.1 a). For this, we construct a morphisms m_2, m_3 as in section 3. Now consider the graded morphism $c : A^{\otimes 3} \otimes T \to T$ defined by

$$c(a,b,c,x) = m_3(ab,c,x) - m_3(a,bc,x) + m_3(a,b,m_2(c,x)) - m_2(a,m_3(b,c,x)).$$

A computation shows that c defines a morphism of complexes $A^{\otimes 3} \otimes T \to T[-1]$. By the (2.1), there exists a graded morphism $m_4 : A^{\otimes 3} \otimes T \to T$ homogeneous of degree -2 such that we have

$$c(a,b,c,x) = m_4(a,b,c,d_T(x)) - d_T(m_4(a,b,c,x))\ ,$$

$a, b, c \in A$, $x \in T$. We define $\widetilde{X}$ to be the complex whose underlying graded module is

$$\bigoplus_{i=1}^{4} A^{\otimes i} \otimes T[i-1]$$

and whose differential is given by

$$\begin{aligned}
d(a_0,\ldots,a_3,x) &= -(a_0a_1,a_2,a_3,x) + (a_0,a_1a_2,a_3,x) - (a_0,a_1,a_2a_3,x)\\
&\quad + (a_0,m_4(a_0,a_1,a_2,a_3,x)) - (a_0,a_1,m_3(a_2,a_3,x))\\
&\quad + (a_0,a_1,a_2,m_2(a_3,x)) - (a_0,a_1,a_2,a_3,d(x))\\
d(a_0,a_1,a_2,x) &= -(a_0a_1,a_2,x) + (a_0,a_1a_2,x)\\
&\quad + (a_0,m_3(a_1,a_2,x)) - (a_0,a_1,m_2(a_2,x)) + (a_0,a_1,a_2,d(x))\ ,\\
d(a_0,a_1,x) &= -(a_0a_1,x) + (a_0,m_2(a_1,x))\ ,\\
d(a_0,x) &= (a_0,d(x)).
\end{aligned}$$

We define $f : T \to \widetilde{X}$ by $x \mapsto 1 \otimes x$ and we define X to be the truncation

$$X = \tau_{\leq 2}\widetilde{X}.$$

The homotopy class of the composition $T \to \widetilde{X} \to X$ is the required morphism φ. As in section 3, one checks that φ is compatible with the homotopy actions by A.

5. Proof of uniqueness

We will prove part b) of the main theorem. This could be done by strong homotopy methods as well. The following argument is shorter but less explicit.

Let us first observe that φ and φ' are homotopy equivalences of complexes of B-modules. So there is a unique morphism $f : X \to X'$ of $\mathcal{H}B$ such that $f \circ \varphi = \varphi'$ in the homotopy category of B-modules. Of course, f is a morphism of homotopy modules. We have to show that it lifts to a unique morphism $X \to X'$ in the derived category of A-B-bimodules $\mathcal{D}(A^{op} \otimes B)$. Let us compute $\mathsf{Hom}_{\mathcal{D}(A^{op}\otimes B)}(X, X')$. The complex X is quasi-isomorphic to the (sum) total complex of its Hochschild resolution:

$$\ldots \to A \otimes A^{\otimes p} \otimes X \to \ldots \to A \otimes X \to 0\,,\ p \geq 0.$$

This total complex is right bounded and its components are projective over $A^{op} \otimes B$ since A is projective over k and the components of X are projective over B. So we can compute $\mathbf{R}\,\mathsf{Hom}^\bullet_{A-B}(X, X')$ by applying $\mathsf{Hom}^\bullet_{A-B}(?, X')$ to the total complex of the Hochschild resolution. Using the isomorphism

$$\mathsf{Hom}^\bullet_{A-B}(A \otimes A^{\otimes p} \otimes X, X') = \mathsf{Hom}^\bullet_B(A^{\otimes p} \otimes X, X')$$

we find that $\mathbf{R}\,\mathsf{Hom}^\bullet_{A-B}(X, X')$ isomorphic to the product total complex of the following double complex D

$$\mathsf{Hom}^\bullet_B\,(X, X') \to \mathsf{Hom}^\bullet_B\,(A \otimes X, X') \to \mathsf{Hom}^\bullet_B\,(A \otimes A \otimes X, X') \to \ldots.$$

We have to to compute $H^0\,\mathsf{Tot}^\Pi\,D$. For this, we first truncate the columns of D: For a complex of k-modules K, let

$$\tau_{\geq 0}K = (\ldots \to 0 \to 0 \to K^0/\operatorname{im} d^{-1} \to K^1 \to K^2 \to \ldots).$$

Let $D_{\geq 0}$ be the double complex obtained by applying $\tau_{\geq 0}$ to each column of D and let $D_{<0}$ be the kernel of $D \to D_{\geq 0}$. We claim that $D_{<0}$ is acyclic. Indeed, the homology of the p-th column of $D_{<0}$ in degree $-q$ is isomorphic to

$$\mathsf{Hom}_{\mathcal{H}\mathcal{B}}(A^{\otimes p} \otimes X, X'[-q]).$$

This vanishes for $-q < 0$ by the projectivity of A over k and assumption (2.1). Hence each column of $D_{<0}$ is acyclic. Moreover, $D_{<0}$ is concentrated in the right half plane. We claim that the product total complex $\mathsf{Tot}^\Pi\,D_{<0}$ is acyclic. Indeed, this complex is the inverse limit of the sequence of the complexes $A_p = \mathsf{Tot}^\Pi\,F_pD_{<0}$, $p \geq 0$, associated with the column filtration $F_pD_{<0}$. The A_p are acyclic by induction on p. Each map $A_{p+1} \to A_p$ is surjective in each component. It follows that it is surjective in the boundaries and hence in the cycles (which equal the boundaries since the A_p are acyclic). Therefore the inverse limit of the A_p is acyclic, i.e. $D_{<0}$ is acyclic. So the morphism $D \to D_{\geq 0}$ induces a quasi-isomorphism in the product total complexes. Hence it is enough to compute $H^0\,\mathsf{Tot}^\Pi\,D_{\geq 0}$. It is straightforward to check that this group is canonically isomorphic to the group of morphisms of homotopy modules $X \to X'$.

6. From homotopy actions to strong homotopy actions

6.1. **Lifting homotopy actions.** Suppose that k is a commutative ring, A and B are associative unital k-algebras, and L is a $\mathbf{Z}$-graded B-module. A *strong homotopy action* [2] of A on L is the datum of graded (B-linear) morphisms

$$m_n : A^{\otimes n-1} \otimes L \to L$$

defined for $n \geq 1$ and homogeneous of degree $2-n$ such that for each $n \geq 1$ and all $a_i \in A$, $x \in L$, we have

$$\begin{aligned} 0 = & \sum_{i=1}^{n-2} (-1)^{i-1} m_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n-1}, x) + \\ & \sum_{l=1}^{n} (-1)^{n-l} m_{n-l+1}(a_1, \ldots, a_{n-l}, m_l(a_{n-l+1}, \ldots, a_{n-1}, x)). \end{aligned} \tag{6.1}$$

Note that if L has non vanishing components only in degrees $0, \ldots, N$, then m_n vanishes for $n > N+2$. It is instructive to consider the cases $n = 1, 2, 3$ of (6.1): For $n = 1$, we obtain

$$0 = m_1 m_1$$

so that (L, m_1) is a complex. For $n = 2$, we have

$$0 = -m_2(a, m_1(x)) + m_1(m_2(a, x)) \ , \ a \in A, x \in L \ ,$$

so that $x \mapsto m_2(a, x)$ is a morphism of complexes for all $a \in A$. For $n = 3$, $a, b \in A$, $x \in L$, we have

$$0 = m_3(a, b, m_1(x)) + m_2(ab, x) - m_2(a, m_2(b, x)) + m_1(m_3(a, b, x)) \ , \tag{6.2}$$

which expresses the fact that m_2 is an associative operation up to an homotopy given by m_3.

Theorem 6.1. *Suppose that L is a graded B-module endowed with three graded morphisms $m_i : A^{\otimes i-1} \otimes L \to L$, $1 \leq i \leq 3$, homogeneous of degree $2-i$ and satisfying* (6.1) *for $n \leq 3$. Suppose that we have*

$$\mathsf{Hom}_{\mathcal{H}B}\,(A^{\otimes n} \otimes L, L[2-n]) = 0 \quad \text{for all } n \geq 3. \tag{6.3}$$

Then the triple m_1, m_2, m_3 may be completed to a strong homotopy action $m_n, n \geq 1$, of A on L.

In the next subsection, we will set up the dictionary between strong homotopy and differential coalgebra. We will then prove the theorem in 6.3 using this dictionary.

6.2. **Differential coalgebra.** Passing from strong homotopy notions to differential coalgebra notions is a classical device, cf. [11], [12], [1]. In this subsection, we adapt it to our needs.

Let

$$C = T(A[1]) = k \oplus A[1] \oplus (A[1] \otimes A[1]) \oplus \ldots$$

be the graded tensor algebra over the graded k-module $A[1]$. It becomes a graded coalgebra for the comultiplication defined by

$$\begin{aligned}\Delta(a_1,\dots,a_n) =& 1\otimes(a_1,\dots,a_n)+\sum_{i=1}^{n-1}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n)\\ &+(a_1,\dots,a_n)\otimes 1.\end{aligned}$$

The graded coalgebra $T(A[1])$ admits a unique graded endomorphism b of degree $+1$ which satisfies

$$\begin{aligned}b(x_1,x_2)&=x_1x_2 \text{ for } x_1,x_2\in A\,,\\ b(x_1,\dots,x_i)&=0 \text{ for } i\neq 2\,,\end{aligned}$$

and which is a coderivation:

$$\Delta\circ b=(b\otimes \mathbf{1}_C+\mathbf{1}_C\otimes b)\circ\Delta.$$

Here and elsewhere, we use the *graded tensor product*: For graded maps f, g and homogeneous elements x, y, we have

$$(f\otimes g)(x\otimes y)=(-1)^{|g||x|}f(x)\otimes g(y).$$

where the bars indicate the degree. Explicitly, the formula for b is

$$b(x_1,\dots,x_n)=\sum_{i=1}^{n-1}(-1)^{i-1}(x_1,\dots,x_ix_{i+1},\dots,x_n)\,,\ n\geq 2.$$

We have $b^2=0$ so that (C,d) is a differential graded coalgebra.

Now let L be a $\mathbf{Z}$-graded B-module. Let X be the graded B-module $C\otimes L$. So we have

$$X=L\oplus(A[1]\otimes L)\oplus\cdots\oplus(A[1]^{\otimes i}\otimes L)\oplus\cdots\ ,\ i\in\mathbf{N}.$$

The graded module $X=C\otimes L$ becomes a cofree graded comodule over C for the comultiplication induced from that of C. So we have

$$\begin{aligned}\delta(a_1,\dots,a_{n-1},x)=1\otimes(a_1,\dots,a_{n-1},x)+&a_1\otimes(a_2,\dots,a_{n-1},x)+\dots\\ &+(a_1,\dots,a_{n-1})\otimes x.\end{aligned}$$

A *coderivation* of X of degree e is a graded endomorphism b of degree e such that

$$\delta\circ b=b_C\otimes\mathbf{1}_X+\mathbf{1}_C\otimes b.$$

Let $\varepsilon: X\to L$ be the projection. Then the map $b\mapsto\varepsilon\circ b$ is a bijection from the set of degree e coderivations of X to the set of graded morphisms of degree e from X to L. Let us describe the inverse map: Let $b: X\to L$ be a graded morphism of degree e. It is given by its components (homogeneous of degree e)

$$b_i: A[1]^{\otimes i-1}\otimes L\to L\,,\ i\geq 1.$$

The corresponding coderivation is given by

$$
\begin{aligned}
b(a_1,\ldots,a_{n-1},x) = & \sum_{i=1}^{n-2}(-1)^{e(i-1)}(a_1,\ldots,a_ia_{i+1},\ldots,a_{n-1},x) \\
& + \sum_{l=1}^{n}(-1)^{e(n-l)}(a_1,\ldots,a_{n-l},b_l(a_{n-l+1},\ldots,a_{n-1},x)) ,
\end{aligned} \tag{6.4}
$$

where $a_i \in A$, $x \in L$.

Let U be a C-comodule. Since X is cofree, the map $f \mapsto \varepsilon \circ f$ is a bijection from the degree e comodule morphisms $f : U \to X$ to the degree e graded k-linear morphisms $U \to L$.

Let b be a degree 1 coderivation of $C \otimes X$. Then we have

$$(\mathbf{1}_C \otimes b)(b_C \otimes \mathbf{1}_X) = -(b_C \otimes \mathbf{1}_X)(\mathbf{1}_C \otimes b)$$

as morphisms $C \otimes X \to C \otimes X$. Since $b_C^2 = 0$, we deduce that

$$b^2 \circ \delta = (\mathbf{1}_C \otimes b + b_C \otimes \mathbf{1}_X)^2 \circ \delta = (\mathbf{1}_C \otimes b^2) \circ \delta.$$

It follows that $b^2 : X \to X$ is a morphism of C-comodules. In particular, we have $b^2 = 0$ iff $\varepsilon \circ b^2 = 0$. Thanks to (6.4), this last equality translates into

$$
\begin{aligned}
0 = & \sum_{i=1}^{n-2}(-1)^{(i-1)}b_{n-1}(a_1,\ldots,a_ia_{i+1},\ldots,a_{n-1},x) \\
& + \sum_{l=1}^{n}(-1)^{(n-l)}b_{n-l+1}(a_1,\ldots,a_{n-l},b_l(a_{n-l+1},\ldots,a_{n-1},x)) ,
\end{aligned} \tag{6.5}
$$

for all $n \geq 1$. If we compare (6.1) to (6.5), we see that the map $(m_n) \mapsto (b_n)$ defined by

$$b_n(a_1,\ldots,a_{n-1},x) = m_n(a_1,\ldots,a_{n-1},x)$$

is a bijection between strong homotopy actions of A on L and degree 1 comodule differentials on X. Note that m_i is a graded map $A^{\otimes i-1} \otimes L \to L$ of degree $2-i$ whereas b_i is a graded map $A[1]^{\otimes i-1} \otimes L \to L$ of degree 1.

Now assume that b is a degree 1 coderivation of X. Let us analyse the equation $b^2 = 0$ in terms of the components $b_i : A[1]^{\otimes i-1} \otimes L \to L$. If we identify them with their extensions to coderivations, then the equation $b^2 = 0$ translates into

$$0 = b_1b_n + b_2b_{n-1} + \ldots + b_nb_1 \tag{6.6}$$

for all $n \geq 1$. Note that for each $p \geq 0$, the right hand side takes $A[1]^{\otimes n+p} \otimes L$ to $A[1]^{\otimes p} \otimes L$. The equation $b^2 = 0$ holds iff, for all $n \geq 1$, the right hand side of (6.6) induces the zero map $A[1]^{\otimes n-1} \otimes L \to L$. Indeed, in this case we have $\varepsilon \circ b^2 = 0$.

6.3. **Proof of theorem 6.1.** We use the notations of the previous two subsections. If we identify graded maps $A^{\otimes i-1} \otimes L \to L$ with their extensions to coderivations

$X \to X$, then we are given coderivations b_1, b_2, b_3 such that we have

$$\begin{aligned} 0 =& b_1^2\,, \\ 0 =& b_1 b_2 + b_2 b_1\,, \\ 0 =& b_1 b_3 + b_2 b_2 + b_3 b_1. \end{aligned}$$

We have to construct b_i, $i \geq 4$, such that

$$b_1 b_n + b_2 b_{n-1} + \cdots b_{n-1} b_2 + b_n b_1 = 0 \tag{6.7}$$

for all $n \geq 4$. Suppose that $N \geq 4$ and that $b_1, \ldots, b_{N-1}$ have been constructed such that (6.7) holds for all $n \leq N-1$. We are looking for b_N such that

$$0 = b_1 b_N + b_N b_1 + (b_2 b_{N-1} + b_3 b_{N-2} + \cdots + b_{N-1} b_2).$$

Put

$$c = b_2 b_{N-1} + b_3 b_{N-2} + \cdots + b_{N-1} b_2.$$

Let $X_N \subset X$ be the C-subcomodule

$$L \oplus (A[1] \otimes L) \oplus \ldots \oplus (A[1]^{\otimes N-1} \otimes L).$$

Note that b_i takes X_N to X_{N-i+1}. In particular, c takes X_N to $X_1 = L$ and vanishes on X_{N-1}. So it induces a graded morphism of degree 2

$$X_N / X_{N-1} = A[1]^{\otimes N-1} \otimes L \to L.$$

We only have to show that this is a morphism of complexes: Indeed, by our assumption on L, it will then have to be nullhomotopic. The extension of an homotopy to a coderivation yields the required morphism b_N. To show that the morphism induced by c commutes with the differential b_1, we define $B = b_1 + \cdots + b_{N-1}$. Since (6.7) holds for $n \leq N-1$, we have

$$B^2 \equiv_N c \tag{6.8}$$

where $\equiv_N$ denotes the equality of the restrictions to X_N. We conclude that B^2 vanishes on X_{N-1} and takes X_N to $L \subset X_N$. This implies that $B^2 B \equiv_N B^2 b_1$ and $B B^2 \equiv_N b_1 B^2$. Therefore we have $B^2 b_1 \equiv_N b_1 B^2$ and finally $c b_1 \equiv_N b_1 c$.

6.4. **Morphisms of strong homotopy actions.** Suppose that k is a commutative ring, A and B are associative unital k-algebras, and L and M are *strong homotopy modules*, i.e. $\mathbf{Z}$-graded B-modules endowed with strong homotopy actions by A. A *morphism of strong homotopy modules* $f : L \to M$ is a sequence of graded (B-linear) morphisms

$$f_i : A^{\otimes i-1} \otimes L \to M$$

homogeneous of degree $1-i$ such that for each $n \geq 1$, we have

$$\begin{aligned} & \sum_{l=1}^{n} m_l(a_1, \ldots, a_{l-1}, f_{n-l+1}(a_l, \ldots, a_{n-1}, x)) \\ = \; & \sum_{i=1}^{n-2} (-1)^{i-1} f_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n-1}, x) \\ & + \sum_{l=1}^{n} (-1)^{n-l} f_{n-l+1}(a_1, \ldots, a_{n-l}, m_l(a_{n-l+1}, \ldots, a_{n-1}, x)) \,, \end{aligned} \tag{6.9}$$

for all $a_i \in A$, $x \in L$. For $n = 1$, this specializes to

$$m_1 f_1 = f_1 m_1$$

so that f_1 is a morphism of complexes. For $n = 2$, we obtain

$$m_1(f_2(a_1, x)) + m_2(a_1, f_1(x)) = f_1(m_2(a_1, x)) - f_2(a_1, m_1(x)) \,, \; a_1 \in A \,, \; x \in L \,, \tag{6.10}$$

which means that for each $a_1 \in A$, the morphism f_1 commutes with the left multiplication by a_1 up to the homotopy $x \mapsto f_2(a_1, x)$.

A morphism $f : L \to M$ of strong homotopy modules is *nullhomotopic* if there exists an *homotopy from* f *to* 0, i.e. a family

$$h_i : A^{\otimes i-1} \otimes L \to M \,, \; i \geq 1 \,,$$

of graded morphisms homogeneous of degree $-i$ such that for each $n \geq 1$, we have

$$\begin{aligned} f_n = & \sum_{l=1}^{n} (-1)^{l-1} m_l(a_1, \ldots, a_{l-1}, h_{n-l+1}(a_l, \ldots, a_{n-1}, x)) \\ & + \sum_{i=1}^{n-2} (-1)^{i-1} h_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n-1}, x) \\ & + \sum_{l=1}^{n} (-1)^{n-l} h_{n-l+1}(a_1, \ldots, a_{n-l}, m_l(a_{n-l+1}, \ldots, a_{n-1}, x)) \,, \end{aligned} \tag{6.11}$$

For $n = 1$, this equation becomes

$$f_1 = m_1 \, h_1 + h_1 \, m_1 \,,$$

which means that f_1 is nullhomotopic. Two morphisms between strong homotopy modules are *homotopic* if their difference is nullhomotopic.

We extend our dictionary between strong homotopy and differential graded coalgebra: Let $X = C \otimes L$ and $Y = C \otimes M$ be the differential graded comodules associated with L and M, in analogy with subsection 6.2. It is easy to check that the map $f \mapsto \varepsilon \circ f$, where $\varepsilon : Y \to M$ is the canonical projection, is a bijection from the set of comodule morphisms to the set of morphisms of graded modules $X \to M$ and that under this bijection, the morphisms of differential comodules correspond exactly to the morphisms of strong homotopy modules. If f is a nullhomotopic morphism of differential graded comodules, the map $h \mapsto \varepsilon \circ h$ also induces a bijection from the set of homotopies from f to 0 to the set of homotopies from $\varepsilon \circ f$ to 0.

7. From strong homotopy actions to strict actions

Let k be a commutative ring and A and B (associative, unital) k-algebras. Let L and M be strong homotopy modules (cf. 6.4). We define a complex of k-modules $\mathsf{H\overset{\infty}{o}m}^\bullet(L,M)$ as follows: its pth component is the k-module of sequences of graded (B-linear) morphisms

$$f_n : A^{\otimes n-1} \otimes L \to M\ ,\ n \geq 1\ ,$$

of degree $p+1-n$; its differential maps a sequence f_n in $\mathsf{H\overset{\infty}{o}m}^p(L,M)$ to the sequence g_n defined by

$$\begin{aligned} g_n = & \sum_{l=1}^{n} (-1)^{p(l-1)} m_l(a_1, \ldots, a_{l-1}, f_{n-l+1}(a_l, \ldots, a_{n-1}, x)) \\ & - (-1)^p \sum_{i=1}^{n-2} (-1)^{i-1} f_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n-1}, x) \\ & - (-1)^p \sum_{l=1}^{n} (-1)^{n-l} f_{n-l+1}(a_1, \ldots, a_{n-l}, m_l(a_{n-l+1}, \ldots, a_{n-1}, x))\ , \end{aligned} \tag{7.1}$$

Lemma 7.1. *The square of the above differential vanishes. The group of zero cycles of $\mathsf{H\overset{\infty}{o}m}^\bullet(L,M)$ identifies with the group of morphisms of strong homotopy modules $L \to M$. Zero boundaries correspond exactly to the nullhomotopic morphisms.*

Proof. We use the correspondence of 6.4: The set $\mathsf{H\overset{\infty}{o}m}^p(L,M)$ is in bijection with the set of graded comodule morphisms $f : C \otimes L \to C \otimes M$ of degree p and the differential corresponds to the map

$$f \mapsto b \circ f - (-1)^p f \circ b.$$

This shows that we have a well-defined complex. The rest follows upon inspection of (6.9) and (6.11). $\surd$

Let Shmod denote the category of strong homotopy A-modules over B and let Bimod denote the category of complexes of A-B-bimodules. We have an obvious functor

$$R : \mathsf{Bimod} \to \mathsf{Shmod}$$

which maps a complex of A-B-bimodules to the underlying $\mathbf{Z}$-graded B-module endowed with the homotopy action given by the differential, the multiplication, and $m_n = 0$ for all $n \geq 3$. We will construct a left adjoint. We use the notations of 6.2. For $X \in \mathsf{Shmod}$, let LX be the complex whose underlying graded A-B-bimodule is $A \otimes C \otimes X$ and whose differential is

(7.2)

$$\begin{aligned} d(a_0, a_1, \ldots, a_{n-1}, x) = & - (a_0 a_1, a_2, \ldots, a_{n-1}, x) \\ & + \sum_{i=1}^{n-2} (-1)^{i-1} (a_0, \ldots, a_i a_{i+1}, \ldots, a_{n-1}, x) \\ & + \sum_{l=1}^{n} (-1)^{n-l} (a_0, a_1, \ldots, a_{n-l}, m_l(a_{n-l+1}, \ldots, a_{n-1}, x)). \end{aligned}$$

Lemma 7.2. *The square of the above differential vanishes.*

Proof. Define a differential on the graded module $A \otimes C$ by

$$d(a_0, a_1, \dots, a_{n-1}) = -(a_0 a_1, a_2, \dots, a_{n-1}) + a_0 \otimes d_C(a_1, a_2, \dots, a_{n-1}).$$

It is not hard to check that its square vanishes. On the other hand, $C \otimes X$ is endowed with the differential of (6.4). Now the morphism

$$\mathbf{1}_A \otimes \Delta \otimes \mathbf{1}_X : A \otimes C \otimes X \to (A \otimes C) \otimes (C \otimes X)$$

defines an isomorphism onto a graded submodule and the differential given on $A \otimes C \otimes X$ is induced by the one on $(A \otimes C) \otimes (C \otimes X)$.

$\surd$

We define

$$\varphi : \mathsf{H\overset{\infty}{o}m}^\bullet(X, RY) \xrightarrow{\sim} \mathsf{Hom}^\bullet(LX, Y)$$

as the composition

$$\mathsf{H\overset{\infty}{o}m}^\bullet(X, RY) = \mathsf{Hom}_k^\bullet(C \otimes X, Y) \xrightarrow{\sim} \mathsf{Hom}_A^\bullet(A \otimes C \otimes X, Y) = \mathsf{Hom}^\bullet(LX, Y).$$

More explicitly, a morphism of strong homotopy modules f corresponds to a morphism of graded modules $f : C \otimes X \to Y$. By definition, the image of f under φ maps $a \otimes c \otimes x$ to $af(c, x)$. Of course, φ is an isomorphism of graded k-modules. Its inverse maps g to $c \otimes x \mapsto g(1 \otimes c \otimes x)$. Note that the fact that A has a unit is crucial for this.

Lemma 7.3. *The isomorphism φ is compatible with the differentials. In particular, the functors L and R are adjoints and induce a pair of adjoint functors in the homotopy categories.*

Proof. The claim follows from the equalities

$$\begin{aligned}
&d(\varphi(f)(a_0, \dots, a_{n-1}, x)) \\
&\qquad = a_0 d(f(a_1, \dots, a_{n-1}, x)) \\
&\varphi(f)(d(a_0, \dots, a_{n-1}, x)) \\
&\qquad = -(a_0 a_1) f(a_2, \dots, a_{n-1}, x) \\
&\qquad\quad + \sum_{i=1}^{n-2} (-1)^{i-1} a_0 f(a_1, \dots, a_i a_{i+1}, \dots a_{n-1}, x) \\
&\qquad\quad + \sum_{l=1}^{n} (-1)^{n-l} a_0 f(a_1, \dots, a_{n-l}, m_l(a_{n-l+1}, \dots, a_{n-1}, x)) \\
&\varphi(d(f))(a_0, \dots, a_{n-1}, x) \\
&\qquad = a_0 d(f(a_1, \dots, a_{n-1}, x)) - (-1)^p a_0 (a_1 f(a_2, \dots a_{n-1}, x)) \\
&\qquad\quad - (-1)^p \sum_{i=1}^{n-2} (-1)^{i-1} a_0 f(a_1, \dots, a_i a_{i+1}, \dots a_{n-1}, x) \\
&\qquad\quad - (-1)^p \sum_{l=1}^{n} (-1)^{n-l} a_0 f(a_1, \dots, a_{n-l}, m_l(a_{n-l+1}, \dots, a_{n-1}, x)).
\end{aligned}$$

$\surd$

The functors R and L induce a pair of adjoint functors between the homotopy categories of Shmod and Bimod. A *quasi-isomorphism* of Shmod is a morphism $f : X \to Y$ such that f_1 is a quasi-isomorphism of the underlying complexes. Then clearly the functor R preserves quasi-isomorphisms. Since A is projective over k, the functor L also preserves quasi-isomorphisms. Hence if we define the derived categories $\mathcal{D}\,\mathsf{Shmod}$ and $\mathcal{D}\,\mathsf{Bimod}$ to be the localizations of the homotopy categories with respect to the quasi-isomorphisms, then L and R induce a pair of adjoint functors between the derived categories:

$$\begin{array}{c} \mathcal{D}\,\mathsf{Bimod} \\ L \uparrow\downarrow R \\ \mathcal{D}\,\mathsf{Shmod} \end{array}$$

Let Y be a complex of A-B-bimodules. It is easy to check that $LRY = A \otimes T(A[1]) \otimes Y$ is isomorphic to the Hochschild resolution of Y and that the adjunction morphism

$$LRY = A \otimes T(A[1]) \otimes Y \to Y$$

identifies with the augmentation of the Hochschild resolution. In particular, the adjunction is a quasi-isomorphism. It follows that the functor $R : \mathcal{D}\,\mathsf{Bimod} \to \mathcal{D}\,\mathsf{Shmod}$ is fully faithful. If X is a strong homotopy module, the adjunction morphism $X \to RLX = A \otimes T(A[1]) \otimes X$ is the morphism of strong homotopy modules whose component in degree i is the morphism

$$f_i : A^{\otimes i-1} \otimes X \to A \otimes A[1]^{\otimes i-1} \otimes X \subset A \otimes T(A[1]) \otimes X$$

given by

$$(a_1, \dots, a_{i-1}) \otimes x \mapsto 1 \otimes (a_1, \dots, a_{i-1}) \otimes x.$$

We say that X is *H-unital* if the adjunction morphism is a quasi-isomorphism. We deduce the

Proposition 7.4. *The functor*

$$R : \mathcal{D}\,\mathsf{Bimod} \to \mathcal{D}\,\mathsf{Shmod}$$

is an equivalence onto the full subcategory of H-unital strong homotopy modules. Its inverse is induced by the functor L.

For the applications, we need a criterion for H-unitality:

Lemma 7.5. *Let X be a strong homotopy module. Then the following are equivalent:*

(i) *X is H-unital.*
(ii) *The morphism of complexes of k-modules $m_2(1,?) : X \to X$ induces the identity in homology.*

Proof. Suppose that (i) holds. The square

$$\begin{array}{ccc} X & \to & RLX \\ m_2(1,?) \downarrow & & \downarrow m_2(1,?) \\ X & \to & RLX \end{array}$$

commutes in the homotopy category of complexes of k-modules thanks to (6.10). By assumption, the adjunction morphism $X \to RLX$ is a quasi-isomorphism. The right vertical arrow is the identity (since the A-module structure on $RLX = A \otimes C \otimes X$ is induced from that of A). So if we apply the homology functor to the diagram, we see that (ii) holds.

Suppose that (ii) holds. Consider the filtrations

$$F_p RLX = A \otimes X \oplus (A \otimes A[1] \otimes X) \oplus \cdots \oplus (A \otimes A[1]^{\otimes p} \otimes X)\ ,\ p \geq 0\ ,$$

and

$$F_p X = X\ ,\ p \geq 0.$$

The morphism $f : X \to RLX$, $x \mapsto 1 \otimes x$, is compatible with the filtrations. The E_1-term of the spectral sequence associated with $F_p RLX$ is the Hochschild resolution of the graded A-module $H^* X$. By our assumption, this module is *unital* and thus the Hochschild resolution is quasi-isomorphic to the module $H^* X$ and the map $x \mapsto 1 \otimes x$ induces a quasi-isomorphism. It follows that f induces an isomorphism in the E_2-terms of the spectral sequences. Since the filtrations are bounded below and exhaustive, the spectral sequences converge (by the classical convergence theorem [13, 5.5.1]) and f is a quasi-isomorphism. $\surd$

8. Proof of existence

We prove part a) of theorem 2.1. We put $m_1 = d : T \to T$ and construct m_2, m_3 as in section 3. Since A is projective over k, the vanishing condition (6.3) follows from (2.1). Hence by theorem 6.1, the triple m_1, m_2, m_3 may be completed to a strong homotopy action $m_n, n \geq 1$, of A on T. Let us denote by $\widetilde{T} \in \mathsf{Shmod}$ the corresponding strong homotopy module, cf. 6.4. In the homotopy action of A on T, the unit of A acts by the identity, so that $m_2(1, ?) : \widetilde{T} \to \widetilde{T}$ is homotopic to the identity. By lemma 7.5, the strong homotopy module $\widetilde{T}$ is H-unital. So by proposition 7.4, it comes from a complex of bimodules. More precisely, the canonical morphism of strong homotopy modules

$$f : \widetilde{T} \to RL\widetilde{T}$$

is a quasi-isomorphism. This means that f_1 is a quasi-isomorphism, which, by (6.10), is compatible with the homotopy actions of A on $\widetilde{T}$ and $L\widetilde{T}$. We put $X = L\widetilde{T} = A \otimes T(A[1]) \otimes T$ and $\varphi = f_1$.

Note that if T is a bounded complex, we can truncate X to a bounded complex, as we did in sections 3 and 4.

9. Summary of the construction

For the convenience of the reader, we sum up the explicit procedure that we propose for turning a tilting complex into a complex of bimodules.

Suppose that the hypotheses of theorem 2.1 are satisfied: k is a commutative ring, A, B are two (associative, unital) k-algebras and T is a complex of right B-modules endowed with an homotopy action by A. We suppose that A is projective as a k-module, and that T is right bounded with projective components and satisfies

(9.1) $$\mathsf{Hom}_{\mathcal{H}B}(T, T[-n]) = 0 \text{ for all } n > 0.$$

In a first step, we define homogeneous graded B-module morphisms

$$m_n : A^{\otimes n} \otimes T \to T \; , \; n \geq 1 \; ,$$

of degree $2-n$. The construction works by induction on n. The cases $n=1$, $n=2$ and $n=3$ have to be treated separately.

The case $n = 1$**.** By definition, the morphism m_1 equals the differential of the complex T.

The case $n = 2$**.** Since A is projective over k, we can find a k-linear map $\widetilde{\alpha} : A \to \mathsf{Hom}_{\mathcal{C}B}(T, T)$ lifting the given homotopy action $\alpha : A \to \mathsf{Hom}_{\mathcal{H}B}(T, T)$. Now we define $m_2 : A \otimes T \to T$ by

$$m_2(a, x) = (\widetilde{\alpha}(a))(x) \; , \; a \in A, x \in T.$$

The case $n = 3$**.** To define m_3, we consider the square

$$\begin{array}{rcl} A \otimes A \otimes T & \stackrel{\mathbf{1}_A \otimes m_2}{\to} & A \otimes T \\ m_A \otimes \mathbf{1}_T \downarrow & & \downarrow m_2 \\ A \otimes T & \stackrel{m_2}{\to} & T \; , \end{array}$$

where m_A denotes the multiplication of A. The square becomes commutative in the homotopy category. Hence there is a morphism of graded B-modules

$$m_3 : A \otimes A \otimes T \to T$$

homogeneous of degree -1 such that we have

$$m_2(ab, x) - m_2(a, m_2(b, x)) = -m_3(a, b, d(x)) - d(m_3(a, b, x))$$

for all $a, b \in A$ and $x \in T$.

The case $n > 3$**.** Suppose that $m_1, \ldots, m_{n-1}$ have been constructed. Consider the morphism of graded B-modules

$$c : A^{\otimes n-1} \otimes T \to T$$

of degree $3-n$ defined by

(9.2) $$\begin{aligned} c(a_1, \ldots, a_{n-1}, x) = & \sum_{i=1}^{n-2} (-1)^{i-1} m_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n-1}, x) + \\ & \sum_{l=2}^{n-1} (-1)^{n-l} m_{n-l+1}(a_1, \ldots, a_{n-l}, m_l(a_{n-l+1}, \ldots, a_{n-1}, x)). \end{aligned}$$

It was shown in section 6.3 that c is a morphism of complexes of degree $3-n$. Now by our assumptions on T and A, all morphisms of complexes of B-modules of degree < 0 from $A^{\otimes n-1} \otimes T$ to T are nullhomotopic. We can therefore choose a graded morphism $m_n : A^{\otimes n-1} \otimes T \to T$ of degree $2-n$ such that

$$0 = c + m_1 \, m_n - (-1)^n \, m_n \circ (\mathbf{1}^{\otimes n-1} \otimes m_1).$$

This ends the inductive construction of m_n.

In a second step, we define a bimodule complex X quasi-isomorphic to T as a complex of right B-modules and which 'strictifies' the homotopy action of A on T. The underlying graded A-B-bimodule of X is defined by

$$X^n = \bigoplus_{-p+q=n} A \otimes A^{\otimes p} \otimes T^q \ , \ n \in \mathbf{Z}.$$

The differential of X is defined by

$$\begin{aligned} d(a_0, a_1, \ldots, a_{p-1}, x) = &- (a_0 a_1, a_2, \ldots, a_{p-1}, x) \\ &+ \sum_{i=1}^{p-2} (-1)^{i-1} (a_0, \ldots, a_i a_{i+1}, \ldots, a_{p-1}, x) \\ &+ \sum_{l=1}^{p} (-1)^{p-l} (a_0, a_1, \ldots, a_{p-l}, m_l(a_{p-l+1}, \ldots, a_{p-1}, x)). \end{aligned}$$

It is shown in lemma 7.2 that we do have $d^2 = 0$ for this differential and in section 8 that X is quasi-isomorphic to T as a complex of right B-modules and 'strictifies' the homotopy action on T.

10. Comparison with Rickard's construction

We keep the notations and assumptions of section 9 and assume moreover that T is a tilting complex [8]. Let us recall Rickard's construction [9] for replacing the complex T by a bimodule complex X: One starts by proving that $A^{op} \otimes T$ is a tilting complex over $A^{op} \otimes B$ and that its endomorphism ring is $A^{op} \otimes A$. Then, according to the main result of [8], there is a canonical functor

$$F : \mathcal{D}^-(A^{op} \otimes A) \to \mathcal{D}^-(A^{op} \otimes B)$$

sending $A^{op} \otimes A$ to $A^{op} \otimes T$. One then puts

$$X = F(A)$$

where A is viewed as a complex of bimodules concentrated in degree 0. To actually compute $F(A)$, one proceeds as follows: First, we have to replace A by a projective resolution over $A^{op} \otimes A$. Let us take the Hochschild resolution:

$$A \otimes A \leftarrow A \otimes A \otimes A \leftarrow \ldots \leftarrow A^{\otimes p+2} \otimes A \leftarrow \ldots \quad .$$

Now, using the action up to homotopy of A on T, we deduce a complex

$$A \otimes T \leftarrow A \otimes A \otimes T \leftarrow \ldots \leftarrow A^{\otimes p+1} \otimes T \leftarrow \ldots , \qquad (*)$$

whose components and differentials belong to the homotopy category of $A^{op} \otimes B$-modules, and more precisely to its additive subcategory generated by the tilting complex $A^{op} \otimes T$. In [8], Rickard shows how such a complex may be lifted to a (filtered) complex of $A^{op} \otimes B$-modules. This complex is $X = F(A)$. What the construction presented in section 9 achieves is precisely a lift of $(*)$ to the complex X.

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