Derived categories and their uses

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1 Introduction

1.1 Historical remarks

Derived categories are a 'formalism for hyperhomology' [61]. Used at first only by the circle around Grothendieck they have now become wide-spread in a number of subjects beyond algebraic geometry, and have found their way into graduate text books [33] [38] [44] [62].

According to L. Illusie [32], derived categories were invented by A. Grothendieck in the early sixties. He needed them to formulate and prove the extensions of Serre's duality theorem [55] which he had announced [24] at the International Congress in 1958. The essential constructions were worked out by his pupil J.-L. Verdier who, in the course of the year 1963, wrote down a summary of the principal results [56]. Having at his disposal the required foundations Grothendieck exposed the duality theory he had conceived of in a huge manuscript [25], which served as a basis for the seminar [29] that Hartshorne conducted at Harvard in the autumn of the same year.'

Derived categories found their first applications in duality theory in the coherent setting [25] [29] and then also in the étale [60] [13] and in the locally compact setting [57] [58] [59] [22].

At the beginning of the seventies, Grothendieck-Verdier's methods were adapted to the study of systems of partial differential equations by M. Sato [53] and M. Kashiwara [37]. Derived categories have now become the standard language of microlocal analysis (cf. [38] [46] [52] or [6]). Through Brylinski-Kashiwara's proof of the Kazhdan-Lusztig conjecture [9] they have penetrated the representation theory of Lie groups and finite Chevalley groups [54] [4]. In this theory, a central role is played by certain abelian subcategories of derived categories which are modeled on the category of perverse sheaves [2], which originated in the sheaf-theoretic interpretation [14] of intersection cohomology [20] [21].

In their fundamental papers [1] and [3], Beilinson and Bernstein-Gelfand used derived categories to establish a beautiful relation between coherent sheaves on projective space and representations of certain finite-dimensional algebras. Their constructions had numerous generalizations [16] [34] [35] [36] [10]. They also led D. Happel to a systematic investigation of the derived category of a finite-dimensional algebra [26] [27]. He realized that derived categories provide the proper setting for the so-called tilting theory [7] [28] [5]. This theory subsequently reached its full scope when it was generalized to 'Morita theory' for derived categories of module categories [48] [50] (cf. also [39] [40] [41]). Morita theory has further widened the range of applications of derived categories. Thus, Broué's conjectures on representations of finite groups [8] are typical of the synthesis of precision with generality that can be achieved by the systematic use of this language.

1.2 Motivation of the principal constructions

Grothendieck's key observation was that the constructions of homological algebra do not barely yield cohomology groups but in fact complexes with a certain indeterminacy. To make this precise, he defined a *quasi-isomorphism* between two complexes over an abelian category $\mathcal{A}$ to be a morphism of complexes $s : L \to M$ inducing an isomorphism $\mathrm{H}^n(s) : \mathrm{H}^n(L) \to \mathrm{H}^n(M)$ for each $n \in \mathbf{Z}$. The result of a homological construction is then a complex which is 'well defined up to quasi-isomorphism'. To illustrate this point, let us recall the definition of the left derived functors $\mathrm{Tor}_n^A(M,N)$, where A is an associative ring with 1, N a (fixed) left A-module and M a right A-module. We choose a resolution

$$\ldots \to P^i \to P^{i+1} \to \ldots \to P^{-1} \to P^0 \to M \to 0$$

(i.e. a quasi-isomorphism $P \to M$) with projective right A-modules P^i. Then we consider the complex $P \otimes_A N$ obtained by applying $? \otimes_A N$ to each term P^i, and 'define' $\mathrm{Tor}_n^A(M,N)$ to be the $(-n)$th cohomology group of $P \otimes_A N$. If $P' \to M$ is another resolution, there is a morphism of resolutions $P \to P'$ (i.e. morphism of complexes compatible with the augmentations $P \to M$ and $P' \to M$), which is a homotopy equivalence. The induced morphism $P \otimes_A N \to P' \otimes_A N$ is still a homotopy equivalence and a fortiori a quasi-isomorphism. We thus obtain a system of isomorphisms between the $\mathrm{H}^{-n}(P \otimes_A N)$, and we can give a more canonical definition of $\mathrm{Tor}_n^A(M,N)$ as the (inverse) limit of this system. Sometimes it is preferable to 'compute' $\mathrm{Tor}_n^A(M,N)$ using flat resolutions. If $F \to M$ is such a resolution, there exists a morphism of resolutions $P \to F$. This is no longer a homotopy equivalence but still induces a quasi-isomorphism $P \otimes_A N \to F \otimes_A N$. So we have $\mathrm{H}^{-n}(P \otimes_A N) \xrightarrow{\sim} \mathrm{H}^{-n}(F \otimes_A N)$. However, the construction yields more, to wit the family of complexes $F \otimes_A N$ indexed by all flat resolutions F, which forms a single class with respect to quasi-isomorphism. More precisely, any two such complexes $F \otimes_A N$ and $F' \otimes_A N$ are linked by quasi-isomorphisms $F \otimes_A N \leftarrow P \otimes_A N \to F' \otimes_A N$. The datum of this class is of course richer than that of the $\mathrm{Tor}_n^A(M,N)$. For example, if A is a flat algebra over a commutative ring k, it allows us to recover $\mathrm{Tor}_n^A(M, N \otimes_k X)$ for each k-module X.

Considerations like these must have led Grothendieck to define the *derived category* $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ by 'formally adjoining inverses of all quasi-isomorphisms' to the *category* $\mathbf{C}(\mathcal{A})$ *of complexes* over $\mathcal{A}$. So the objects of $\mathbf{D}(\mathcal{A})$ are complexes and its morphisms are deduced from morphisms of complexes by 'abstract localization'. The right (resp. left) 'total derived functors' of an additive functor $F : \mathcal{A} \to \mathcal{B}$ will then have to be certain 'extensions' of F to a functor $\mathbf{R}F$ (resp. $\mathbf{L}F$) whose composition with $\mathrm{H}^n(?)$ should yield the traditional functors $\mathbf{R}^n F$ (resp. $\mathbf{L}^n F$).

It was Verdier's observation that one obtains a convenient description of the morphisms of $\mathbf{D}(\mathcal{A})$ by a 'calculus of fractions' if, in a first step, one passes to the *homotopy category* $\mathbf{H}(\mathcal{A})$, whose objects are complexes and whose morphisms are homotopy classes of morphisms of complexes. In a second step, the derived category is defined as the localisation of $\mathbf{H}(\mathcal{A})$ with respect to all quasi-isomorphisms. The important point is that in $\mathbf{H}(\mathcal{A})$ the (homotopy classes of) quasi-isomorphisms $M' \leftarrow M$ (resp. $L \leftarrow L'$) starting (resp. ending) at a fixed complex form a *filtered* category ${}_M\Sigma$ (resp. Σ_L). We have

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(L,M) = \varinjlim_{\Sigma_M} \mathrm{Hom}_{\mathrm{H}(\mathcal{A})}(L,M') = \varinjlim_{{}_L\Sigma} \mathrm{Hom}_{\mathrm{H}(\mathcal{A})}(L',M).$$

The elements of the two right hand members are intuitively interpreted as 'left fractions' $s^{-1}f$ or 'right fractions' $g\,t^{-1}$ associated with diagrams

$$L \xrightarrow{f} M' \xleftarrow{s} M \quad \text{or} \quad L \xleftarrow{t} L' \xrightarrow{g} M$$

of $\mathbf{H}(\mathcal{A})$. This also leads to a simple definition of the derived functors: Examples suggest that the derived functors $\mathbf{R}F$ and $\mathbf{L}F$ can not, in general, be defined on all of $\mathcal{D}\mathcal{A}$. Following Deligne [12, 1.2] we define the *domain* of $\mathbf{R}F$ (resp. $\mathbf{L}F$) to be the full subcategory of $\mathbf{D}(\mathcal{A})$ formed by the complexes M (resp. L) such that

$$\varinjlim_{\Sigma_M} FM' \quad (\text{resp.} \quad \varprojlim_{{}_L\Sigma} FL' \quad)$$

exists in $\mathbf{D}(\mathcal{B})$ and is preserved by *all* functors starting from the category $\mathbf{D}(\mathcal{B})$. For such an M (resp. L) we put

$$\mathbf{R}F\,M := \varinjlim_{\Sigma_M} FM' \quad (\text{resp. } \mathbf{L}F\,L := \varprojlim_{{}_L\Sigma} FL' \quad).$$

The functors thus constructed satisfy the universal property by which Grothendieck-Verdier originally [61] defined derived functors. When they exist, they enjoy properties which apparently do not follow directly from the universal property.

1.3 On the use of derived categories

Any relation formulated in the language of derived categories and functors gives rise to assertions formulated in the more traditional language of cohomology groups, filtrations, spectral sequences ... . Of course, these can frequently be proved without explicitly mentioning derived categories so that we may wonder why we should make the effort of using this more abstract language. The answer is that the simplicity of the phenomena, hidden by the notation in the old language, is clearly apparent in the new one. The example of the Künneth relations [61] serves to illustrate this point:

Let X and Y be compact spaces, R a commutative ring with 1, and $\mathcal{F}$ and $\mathcal{G}$ sheaves of R-modules on X and Y, respectively. If $R = \mathbf{Z}$, and either $\mathcal{F}$ or $\mathcal{G}$ is torsion-free, we have split short exact sequences

$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(X,\mathcal{F}) \otimes_R \mathrm{H}^q(Y,\mathcal{G}) \to \mathrm{H}^n(X \times Y, \mathcal{F} \otimes_R \mathcal{G}) \tag{1}$$

$$\to \bigoplus_{p+q=n+1} \mathrm{Tor}^{\mathbf{Z}}(\mathrm{H}^p(X,\mathcal{F}), \mathrm{H}^q(Y,\mathcal{G})) \to 0.$$

When the ring R is more complicated, for example $R = \mathbf{Z}/l^r\mathbf{Z}$, l prime, $r > 1$, and if we make no hypothesis on $\mathcal{F}$ or $\mathcal{G}$, then in the traditional language we only have two spectral sequences with isomorphic abutments whose initial terms are

$$'K_2^{p,q} = \bigoplus_{r+s=q} \mathrm{Tor}^R_{-p}(\mathrm{H}^r(X,\mathcal{F}), \mathrm{H}^s(Y,\mathcal{G}))\,, \tag{2}$$

$$''K_2^{p,q} = \mathrm{H}^p(X \times Y, \mathcal{T}or^R_{-q}(\mathcal{F},\mathcal{G})) \tag{3}$$

However, these spectral sequences and the isomorphism between their abutments are just the consequence and the imperfect translation of the following relation in the derived category of R-modules

$$\mathbf{R}\Gamma\,(X,\mathcal{F}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma\,(Y,\mathcal{G}) \xrightarrow{\sim} \mathbf{R}\Gamma\,(X \times Y, \mathcal{F} \otimes^{\mathbf{L}}_R \mathcal{G})\,, \tag{4}$$

where $\mathbf{R}\Gamma\,(X, ?)$ denotes the right derived functor of the global section functor and $\otimes^{\mathbf{L}}_R$ denotes the left derived functor of the tensor product functor. (We suppose that X and Y are spaces of finite cohomological dimension, for example, finite cell complexes.) The members of (4) are complexes of R-modules which are well determined up to quasi-isomorphism, and whose cohomology groups are the abutment of the spectral sequences (2) and (3), respectively. Of course, formula (4) still holds when $\mathcal{F}$ and $\mathcal{G}$ are (suitably bounded) complexes of sheaves on X and Y. The formula is easy to work with in practice and also allows us to formulate commutativity and associativity properties when there are several factors.

Extension of scalars leads to an analogous formula in the derived categories: If S is an R-algebra and $\mathcal{F}$ a sheaf of R-modules on X, we have the relation

$$\mathbf{R}\Gamma\,(X,\mathcal{F}) \otimes^{\mathbf{L}}_R S \xrightarrow{\sim} \mathbf{R}\Gamma\,(X, \mathcal{F} \otimes^{\mathbf{L}}_R S).$$

Metaphorically speaking, one can say that *naïve formulas which are false in the traditional language become true in the language of derived categories and functors.*

2 Outline of the Chapter

The machinery needed to define a derived category in full generality tends to obscure the simplicity of the phenomena. We therefore start in section 3 with the example of the derived category of a module category. The same construction applies to any abelian category with enough projectives.

The class of abelian categories is not closed under many important constructions. Thus the category of projective objects or the category of filtered objects of an abelian category are no longer abelian in general. This leads us to working with exact categories in the sense of Quillen [47]. We recall their definition and the main examples in section 4.

Heller's stable categories [30] provide an efficient approach [26] to the homotopy category. They also yield many other important examples of triangulated categories, and, more generally, of suspended categories (cf. section 7). We give Heller's construction in section 6. It is functorial in the sense that exact functors give rise to 'stable functors'. The notion of a triangle functor (=S-functor [42] =exact functor [12]) appears as the natural axiomatization of this concept. Triangle functors, equivalences and adjoints are presented in section 8.

In section 9, we recall basic facts on the localization of categories from [15]. These are then specialized to triangulated categories in section 10. Proofs for the results of these sections may be found in [29] [6] [38].

In section 11, we formulate Verdier's definition of the derived category [56] in the context of exact categories.

In section 12, we give a sufficient condition for an inclusion of exact categories to induce an equivalence of their derived categories. This is a key result since it corresponds to the theorem on the existence and unicity of injective resolutions in classical homological algebra.

In sections 13, 14, and 15, we develop the theory of derived functors following Deligne. Derived functors are constructed using a 'generalized calculus of fractions'. This approach makes it possible to easily deduce fundamental results on restrictions, adjoints and compositions in the generality they deserve. Proofs for some non-trivial lemmas of these sections may be found in [12].

3 The derived category of a module category

For basic module theoretic notions and terminology we refer to [31, I, IV]. We shall sometimes write $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y in a category $\mathcal{C}$.

Let R be an associative ring with 1 and denote by $\operatorname{Mod} R$ the category of right R-modules. By definition, the *objects* of $\mathbf{D}^b(\operatorname{Mod} R)$, the *derived category of* $\operatorname{Mod} R$, are the chain complexes

$$P = (\ldots \to P_n \xrightarrow{d_n^P} P_{n-1} \to \ldots)$$

of projective right R-modules P_n, $n \in \mathbf{Z}$, such that we have $P_n = 0$ for all $n \ll 0$ and $\mathrm{H}_n(P) = 0$ for all $n \gg 0$, where $\mathrm{H}_n(P)$ denotes the nth homology module of P. If P and Q are such complexes a *morphism* $P \to Q$ of $\mathbf{D}^b(\operatorname{Mod} R)$ is given by the equivalence class $\overline{f}$ of a morphism of complexes $f : P \to Q$ modulo the subgroup of *null-homotopic morphisms*, i.e. those with components of the form

$$d_{n+1}^Q r_n + r_{n-1} d_n^P$$

for some family of R-module homomorphisms $r_n : P_n \to Q_{n-1}$, $n \in \mathbf{Z}$. The *composition* of morphisms of $\mathbf{D}^b(\operatorname{Mod} R)$ is induced by the composition of morphisms of complexes.

The category of R-modules is related to its derived category by a *canonical embedding*: The canonical functor $\operatorname{can} : \operatorname{Mod} R \to \mathbf{D}^b(\operatorname{Mod} R)$ sends an R-module M to the complex

$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to 0 \to 0 \to \ldots$$

given by a chosen projective resolution of M. If $f : M \to N$ is an R-module homomorphism, $\operatorname{can} f$ is the uniquely determined homotopy class of morphisms of complexes $g : \operatorname{can} M \to \operatorname{can} N$ such that $\mathrm{H}_0(g)$ is identified with f.

We endow $\mathbf{D}^b(\operatorname{Mod} R)$ with the endofunctor S, called the *suspension functor* (or shift functor), and defined by

$$(SP)_n = P_{n-1}, \;\; d_n^{SP} = -d_{n-1}^P,$$

on the objects $P \in \mathbf{D}^b(\operatorname{Mod} R)$ and by $S\overline{f} = \overline{g}$, $g_n = f_{n-1}$, on morphisms $\overline{f}$.

We omit the symbol can from the notations to state the fundamental formula

$$\operatorname{Hom}_D(M, S^n N) \xrightarrow{\sim} \operatorname{Ext}_R^n(M, N)\,,\; n \in \mathbf{N}\,, \tag{5}$$

where $\operatorname{Hom}_D(,)$ denotes morphisms in the derived category and M, N are R-modules. This isomorphism is compatible with the product structures in the sense that the composition

$$L \xrightarrow{\overline{g}} S^m M \xrightarrow{S^m \overline{f}} S^{m+n} N$$

corresponds to the 'splicing product' [31, IV,Ex. 9.3] of the n-extension determined by $\overline{f}$ with the m-extension determined by $\overline{g}$.

Example 3.1 : Fields. Suppose that $R = k$ is a (skew) field. Then it is not hard to see that each $P \in \mathbf{D}^b(\operatorname{Mod} k)$ is isomorphic to a finite sum of objects $S^n M$, $M \in \operatorname{Mod} k$, $n \in \mathbf{Z}$. Moreover, by formula (5) there are no non-trivial morphisms from $S^i M$ to $S^j N$ unless $i = j$, and $\operatorname{Hom}_D(S^i M, S^i N) \xrightarrow{\sim} \operatorname{Hom}_k(M, N)$. Thus, $\mathbf{D}^b(\operatorname{Mod} k)$ is equivalent to the category of $\mathbf{Z}$-graded k-vector spaces with finitely many non-zero components. The equivalence is realized by the homology functor $P \mapsto \mathrm{H}_*(P)$.

Example 3.2 : Hereditary rings. Suppose that R is hereditary (i.e. submodules of projective R-modules are projective). For example, we can take for R a principal domain or the ring of upper triangular $n \times n$-matrices over a field. Then, as in example 3.1, each $P \in \mathbf{D}^b(\operatorname{Mod} R)$ is isomorphic to a finite sum of objects $S^n M$, $M \in \operatorname{Mod} R$, $n \in \mathbf{Z}$. Formula (5) shows that

$$\operatorname{Hom}_D(S^i M, S^j N) = \begin{cases} 0 & j \neq i, i+1 \\ \operatorname{Hom}_R(M, N) & j = i \\ \operatorname{Ext}^1_R(M, N) & j = i+1. \end{cases}$$

Example 3.3 : Dual numbers. Let k be a commutative field and let $R = k[\delta]/(\delta^2)$ be the ring of dual numbers over k. The complexes

$$\ldots \to 0 \to A \xrightarrow{\delta} A \xrightarrow{\delta} \ldots \xrightarrow{\delta} A \xrightarrow{\delta} A \to 0 \to \ldots$$

with non-zero components in degrees $0, \ldots, N$, $N \geq 1$, have non-zero homology in degrees 0 and N, only, but they do not admit non-trivial decompositions as direct sums in $\mathbf{D}^b(\operatorname{Mod} R)$.

4 Exact categories

We refer to [45] for basic category theoretic notions and terminology. A category which is equivalent to a small category will be called *svelte*.

A pair of morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in an additive category is *exact* if i is a kernel of p and p a cokernel of i.

An *exact category* [47] is an additive category $\mathcal{A}$ endowed with a class $\mathcal{E}$ of exact pairs closed under isomorphism and satisfying the following axioms Ex0–Ex2$^{\mathrm{op}}$ [39]. The *deflations* (resp. *inflations*) mentioned in the axioms are by definition the morphisms p (resp. i) occurring in pairs (i, p) of $\mathcal{E}$. We shall refer to such pairs as *conflations*.

Ex0 The identity morphism of the zero object is a deflation.

Ex1 A composition of two deflations is a deflation.

Ex1$^{\mathrm{op}}$ A composition of two inflations is an inflation.

Ex2 Each diagram

$$\begin{array}{ccc} & & C' \\ & & \downarrow c \\ B & \xrightarrow{p} & C\,, \end{array}$$

where p is a deflation, may be completed to a cartesian square

$$\begin{array}{ccc} B' & \xrightarrow{p'} & C' \\ b \downarrow & & \downarrow c \\ B & \xrightarrow{p} & C\,, \end{array}$$

where p' is a deflation.

Ex2$^{\mathrm{op}}$ Each diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ a \downarrow & & \\ A' & & \end{array}$$

where i is an inflation, may be completed to a cocartesian square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ a \downarrow & & \downarrow b \\ A' & \xrightarrow{i'} & B'\,, \end{array}$$

where i' is an inflation.

An *abelian category* is an exact category such that each morphism f admits a factorization $f = ip$, where p is a deflation and i an inflation. In this case, the class of conflations coincides with the class of all exact pairs.

If $\mathcal{A}$ and $\mathcal{B}$ are exact categories, an *exact functor* $\mathcal{A} \to \mathcal{B}$ is an additive functor taking conflations of $\mathcal{A}$ to conflations of $\mathcal{B}$.

A *fully exact subcategory* of an exact category $\mathcal{A}$ is a full additive subcategory $\mathcal{B} \subset \mathcal{A}$ which is *closed under extensions*, i.e. if it contains the end terms of a conflation of $\mathcal{A}$, it also contains the middle term. Then $\mathcal{B}$ endowed with the conflations of $\mathcal{A}$ having their terms in $\mathcal{B}$ is an exact category, and the inclusion $\mathcal{B} \subset \mathcal{A}$ is a fully faithful exact functor.

Example 4.1 : Module categories and their fully exact subcategories. Let R be an associative ring with 1. The category $\operatorname{Mod} R$ of right R-modules endowed with all short exact sequences is an abelian category. The classes of free, projective, flat, injective, finitely generated, ... modules all form fully exact subcategories of $\operatorname{Mod} R$.

In general, *any svelte exact category may be embedded as a fully exact subcategory of some module category* [47] [39]. As a consequence [39], in any argument involving only a finite diagram and such notions as deflations, inflations, conflations, it is legitimate to suppose that we are operating in a fully exact subcategory of a category of modules.

Example 4.2 : Additive categories. Let $\mathcal{A}$ be an additive category. Endowed with all split short exact sequences $\mathcal{A}$ becomes an exact category.

Example 4.3 : The category of complexes. Let $\mathcal{A}$ be an additive category. Denote by $\mathbf{C}(\mathcal{A})$ the category of differential complexes

$$\ldots \to A^n \xrightarrow{d_A^n} A^{n+1} \to \ldots$$

over $\mathcal{A}$. Endow $\mathbf{C}(\mathcal{A})$ with the class of all pairs (i,p) such that (i^n, p^n) is a split short exact sequence for each $n \in \mathbf{Z}$. Then $\mathbf{C}(\mathcal{A})$ is an exact category.

Example 4.4 : k-split sequences. Let k be a commutative ring and R an associative k-algebra. Endowed with the sequences whose restrictions to k are split short exact the category $\operatorname{Mod} R$ of section 3 becomes an exact category.

Example 4.5 : Filtered objects. Let $\mathcal{A}$ be an exact category. The objects of the *filtered category* $\mathbf{F}(\mathcal{A})$ are the sequences of inflations

$$A = (\ldots \to A^p \xrightarrow{j_A^p} A^{p+1} \to \ldots)\ ,\ p \in \mathbf{Z}\ ,$$

of $\mathcal{A}$ such that $A^p = 0$ for $p \ll 0$ and $\operatorname{Cok} j_A^p = 0$ for all $p \gg 0$. The morphisms from A to $B \in \mathbf{F}(\mathcal{A})$ bijectively correspond to sequences $f^p \in \mathcal{A}(A^p, B^p)$ such that $f^{p+1} j_A^p = j_B^p f^p$ for all $p \in \mathbf{Z}$. The sequences whose components are conflations of $\mathcal{A}$ form an exact structure on $\mathbf{F}(\mathcal{A})$. Note that if $\mathcal{A}$ contains a non-zero object, then $\mathbf{F}(\mathcal{A})$ is not abelian (even if $\mathcal{A}$ is).

Example 4.6 : Banach spaces. Let $\mathcal{A}$ be the category of complex Banach spaces. The axioms for an exact structure are satisfied by the sequences which are short exact as sequences of complex vector spaces.

5 Exact categories with enough injectives

Let $\mathcal{A}$ be an exact category. An object $I \in \mathcal{A}$ is *injective* (resp. *projective*) if the sequence

$$\mathcal{A}(B, I) \xrightarrow{i^*} \mathcal{A}(A, I) \to 0 \quad (\text{resp. } \mathcal{A}(P, B) \xrightarrow{p_*} \mathcal{A}(P, C) \to 0\ )$$

is exact for each conflation (i,p) of $\mathcal{A}$. *We assume from now on that $\mathcal{A}$ has enough injectives*, i.e. that each $A \in \mathcal{A}$ admits a conflation

$$A \xrightarrow{i_A} IA \xrightarrow{p_A} SA$$

with injective IA. If $\mathcal{A}$ also has enough projectives (i.e. for each $A \in \mathcal{A}$ there is a deflation $P \to A$ with projective P), and the classes of projectives and injectives coincide, we call $\mathcal{A}$ a *Frobenius category*.

Example 5.1 : Module categories. The category of modules over an associative ring R with 1 has enough projectives and injectives. Projectives and injectives coincide for example if R is the group ring of a finite group over a commutative field.

Example 5.2 : Additive categories. In example (4.2), each object is injective and projective, and we can take i_A to be the identity of A for each $A \in \mathcal{A}$.

Example 5.3 : The category of complexes. In example (4.3), we define

$$(IA)^n = A^n \oplus A^{n+1}\ ,\ d^n_{IA} = \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}\ ,\ (i_A)^n = \begin{bmatrix} \mathbf{1} \\ d^n_A \end{bmatrix}$$
$$(SA)^n = A^{n+1}\ ,\ d^n_{SA} = -d^{n+1}_A\ ,\ p^n_A = [-d^n_A\ \ \mathbf{1}].$$

It is easy to see that IA is injective in $\mathbf{C}(\mathcal{A})$. Now the inflation i_A splits iff A is homotopic to zero. Thus, *a complex is injective in* $\mathbf{C}(\mathcal{A})$ *iff it is homotopic to zero.* Since the complexes IA, $A \in \mathbf{C}(\mathcal{A})$, are also projective, $\mathbf{C}(\mathcal{A})$ *is a Frobenius category.*

Example 5.4 : k-split exact sequences. In example (4.4) we can take for i_M the canonical injection

$$M \to \mathrm{Hom}_k(R, M)\ ,\ m \mapsto (r \mapsto rm).$$

If $R = k[G]$ for a finite group G, the fully exact subcategory of $\mathrm{Mod}\, R$ formed by finitely generated k-free R-modules is a Frobenius category.

Example 5.5 : Filtered objects. In example (4.5) it is not hard to see that $\mathbf{F}(\mathcal{A})$ has enough injectives iff $\mathcal{A}$ has, and in this case the injectives of $\mathbf{F}(\mathcal{A})$ are the filtered objects with injective components [39]. Similarly, $\mathbf{F}(\mathcal{A})$ has enough projectives iff $\mathcal{A}$ has, and in this case the projectives of $\mathbf{F}(\mathcal{A})$ are the filtered objects of A with projective components and such that j^p_A splits for all $p \in \mathbf{Z}$.

Example 5.6 : Banach spaces. As a consequence of the Hahn-Banach theorem, the one-dimensional complex Banach space is injective for the category of example (4.6). More generally, the space of bounded functions on a discrete topological space is injective. There are enough injectives since each Banach space identifies with a closed subspace of the space of bounded functions on the unit sphere of its dual with the discrete topology.

6 Stable categories

Keep the notations and hypotheses of section 5. The *stable category* $\underline{\mathcal{A}}$ associated with $\mathcal{A}$ has the same objects as $\mathcal{A}$. A *morphism* of $\underline{\mathcal{A}}$ is the equivalence class $\overline{f}$ of a morphism $f : A \to B$ of $\mathcal{A}$ modulo the subgroup of morphisms factoring through an injective of $\mathcal{A}$. The *composition* of $\underline{\mathcal{A}}$ is induced by that of $\mathcal{A}$.

Example 6.1 : The homotopy category. The *homotopy category* $\mathbf{H}(\mathcal{A})$ of an additive category $\mathcal{A}$ is by definition the stable category of the category $\mathbf{C}(\mathcal{A})$ of complexes over $\mathcal{A}$ (cf. example 4.3). So the objects of $\mathbf{H}(\mathcal{A})$ are complexes over $\mathcal{A}$ and the morphisms are homotopy classes of morphisms of complexes, by example (5.3).

The stable category is an additive category and the projection functor $\mathcal{A} \to \underline{\mathcal{A}}$ is an additive functor. However, in general, $\underline{\mathcal{A}}$ does not carry an exact structure making the projection functor into an exact functor. Nonetheless, in order to keep track of the conflations of $\mathcal{A}$, we can endow $\underline{\mathcal{A}}$ with the following 'less rigid' structure:

First, we complete the assignment $A \mapsto SA$ to an endofunctor of $\underline{\mathcal{A}}$ by putting $S\overline{f} = \overline{h}$, where h is any morphism fitting into a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & IA & \xrightarrow{p_A} & SA \\ f\downarrow & & \downarrow g & & \downarrow h \\ B & \xrightarrow{i_B} & IB & \xrightarrow{p_B} & SB. \end{array}$$

Indeed, by the injectivity of IB, such diagrams exist. Clearly $\overline{h}$ does not depend on the choice of g.

Secondly, we associate with each conflation $\varepsilon = (i, p)$ of $\mathcal{A}$ a sequence

$$A \xrightarrow{\overline{i}} B \xrightarrow{\overline{p}} C \xrightarrow{\partial\varepsilon} SA$$

called a *standard triangle* and defined by requiring the existence of a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \| & & \downarrow g & & \downarrow e \\ A & \xrightarrow{i_A} & IA & \xrightarrow{p_A} & SA. \end{array} \quad , \ \partial\varepsilon = \overline{e}\,,$$

Again, g exists by the injectivity of IA, and $\overline{e}$ is independent of the choice of g.

If $\mathcal{C}$ is an arbitrary category endowed with an endofunctor $S : \mathcal{C} \to \mathcal{C}$, an *$S$-sequence* is a sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

of $\mathcal{C}$ and a *morphism of S-sequences* from (u, v, w) to (u', v', w') is a commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & SX \\ x\downarrow & & \downarrow & & \downarrow & & \downarrow Sx \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & SX'. \end{array}$$

With this terminology, we define a *triangle* of $\underline{\mathcal{A}}$ to be an S-sequence isomorphic to a standard triangle. A *morphism of triangles* is a morphism of the underlying S-sequences. Note that the standard triangle construction defines a *functor* from the category of conflations to the category of triangles.

Theorem 6.2 *The category $\underline{\mathcal{A}}$ endowed with the suspension functor S and the above triangles satisfies the following axioms SP0-SP4.*

SP0 Each S-sequence isomorphic to a triangle is itself a triangle.

SP1 For each object X, the S-sequence

$$0 \to X \xrightarrow{\mathbf{1}_X} X \to S0$$

is a triangle.

SP2 If (u, v, w) is a triangle, then so is $(v, w, -Su)$.

SP3 If (u, v, w) and (u', v', w') are triangles and x, y morphisms such that $yu = u'x$, then there is a morphism z such that $zv = v'y$ and $(Sx)w = w'z$.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & SX \\ x\downarrow & & y\downarrow & & z\downarrow & & \downarrow Sx \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & SX'. \end{array}$$

SP4 For each pair of morphisms

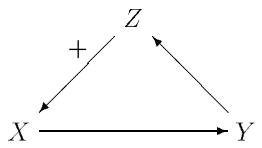
$$X \xrightarrow{u} Y \xrightarrow{v} Z$$

there is a commutative diagram

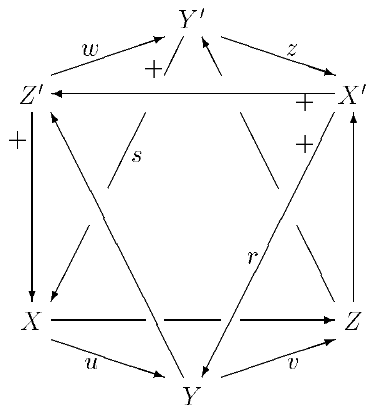
$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{x} & Z' & \rightarrow & SX \\
\| & & v \downarrow & & \downarrow w & & \| \\
X & \rightarrow & Z & \xrightarrow{y} & Y' & \xrightarrow{s} & SX \\
 & & \downarrow & & \downarrow t & & \downarrow Su \\
 & & X' & \xrightarrow{\mathbf{1}} & X' & \xrightarrow{r} & SY \\
 & & r \downarrow & & \downarrow & & \\
 & & SY & \xrightarrow{Sx} & SZ', & &
\end{array}$$

where the first two rows and the two central columns are triangles.

We refer to [26] for a proof of the theorem in the case where $\mathcal{A}$ is a Frobenius category. Property SP4 can be given a more symmetric form if we represent a morphism $X \rightarrow SY$ by the symbol $X \xrightarrow{+} SY$ and write a triangle in the form



With this notation, the diagram of SP4 can be written as an octahedron in which 4 faces represent triangles. The other 4 as well as two of the 3 squares 'containing the center' are commutative.



7 Suspended categories and triangulated categories

A *suspended category* [42] is an additive category $\mathcal{S}$ with an additive endofunctor $S : \mathcal{S} \rightarrow \mathcal{S}$ called the *suspension functor* and a class of S-sequences called *triangles* and satisfying the axioms SP0-SP4 of section 6.

A *triangulated category* is a suspended category whose suspension functor is an equivalence.

By theorem 6.2, the stable category of an exact category $\mathcal{A}$ with enough injectives is a suspended category. If $\mathcal{A}$ is even a Frobenius category, it is easy to see that $\underline{\mathcal{A}}$ is triangulated.

Example 7.1 : The mapping cone. Let $\mathcal{A}$ be an additive category. The homotopy category $\mathbf{H}(\mathcal{A})$ is triangulated (6.1). Here the suspension functor is even an automorphism. Axiom SP4 implies that for each morphism of complexes $f : X \rightarrow Y$, there is a triangle

$$X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \xrightarrow{\overline{h}} SX.$$

Concretely, we can construct Z as the *mapping cone* Cf over f. It is defined as the cokernel of the conflation $[i_X \;\; f]^t : X \to IX \oplus Y$ and hence fits into a diagram

$$\begin{array}{ccccc} X & \xrightarrow{[i_X \;\; f]^t} & IX \oplus Y & \xrightarrow{[k \;\; g]} & Cf \\ \| & & \downarrow [\mathbf{1} \;\; 0] & & \downarrow h \\ X & \xrightarrow{i_X} & IX & \xrightarrow{p_X} & SX. \end{array}$$

The standard triangle provided by this diagram is clearly isomorphic to $(\overline{f}, \overline{g}, \overline{h})$. Explicitly

$$(Cf)^n = X^{n+1} \oplus Y^n \quad , \quad d^n_{Cf} = \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix}$$

$$g^n = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix} \quad , \quad h^n = [-\mathbf{1} \;\; 0].$$

The following properties of a suspended category $\mathcal{S}$ are easy consequences of the axioms. Proofs may be found in [29], [6], [38].

a) Each morphism $u : X \to Y$ can be embedded into a triangle (u, v, w).

b) For each triangle
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$
and each $V \in \mathcal{S}$, the induced sequence
$$\mathcal{S}(X,V) \leftarrow \mathcal{S}(Y,V) \leftarrow \mathcal{S}(Z,V) \leftarrow \mathcal{S}(SX,V) \leftarrow \mathcal{S}(SY,V) \ldots$$
is exact. In particular, $vu = wv = (Su)w = 0$.

c) If in axiom SP3 the morphisms x and y are invertible, then so is z.

d) If (u, v, w) and (u', v', w') are triangles, then so is
$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{v \oplus v'} Z \oplus Z' \xrightarrow{w \oplus w'} S(X \oplus X').$$

e) If
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$
is a triangle, the sequence
$$0 \to Y \xrightarrow{v} Z \xrightarrow{w} SX \to 0$$
is split exact iff $u = 0$.

f) For an arbitrary choice of the triangles starting with u, v and vu in axiom SP4, there are morphisms w and z such that the second central column is a triangle and the whole diagram is commutative.

Now suppose that $\mathcal{S}$ *is a triangulated category.* Then in addition we have

g) If $(v, w, -Su)$ is a triangle of $\mathcal{S}$, then so is (u, v, w).

h) If (u, v, w) and (u', v', w') are triangles and y, z morphisms such that $zv = v'y$, then there is a morphism x such that $yu = u'x$ and $(Sx)w = w'z$.

i) For each triangle
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$
and each $V \in \mathcal{T}$ the induced sequence
$$\mathcal{T}(V,X) \to \mathcal{T}(V,Y) \to \mathcal{T}(V,Z) \to \mathcal{S}(V,SX) \to \mathcal{S}(V,SY) \to \ldots$$
is exact.

This implies in particular that our notion of triangulated category coincides with that of [2, 1.1]

8 Triangle functors

We shall denote all suspension functors by the same letter S.

Let $\mathcal{S}$ and $\mathcal{T}$ be two suspended categories. A *triangle functor* from $\mathcal{S}$ to $\mathcal{T}$ is a pair consisting of an additive functor $F : \mathcal{S} \to \mathcal{T}$ and a morphism of functors $\alpha : FS \to SF$ such that

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{(\alpha X)(Fw)} SFX$$

is a triangle of $\mathcal{T}$ for each triangle (u, v, w) of $\mathcal{S}$. This implies that α is invertible, as we see by considering the case $Y = 0$ and using property b) of section 7.

Example 8.1 : Triangle functors induced by exact functors. Let $\mathcal{A}$ and $\mathcal{B}$ be two exact categories with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor preserving injectives. Then F induces an additive functor $\underline{F} : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$. For each $A \in \mathcal{A}$ define αA to be the class of a morphism a fitting into a commutative diagram

$$\begin{array}{ccccc}
FA & \xrightarrow{Fi_A} & FIA & \xrightarrow{Fp_A} & FSA \\
\| & & \downarrow & & \downarrow a \\
FA & \xrightarrow{i_{FA}} & IFA & \xrightarrow{p_{FA}} & SFA.
\end{array}$$

Then $(\underline{F}, \alpha)$ is a triangle functor $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$. This construction transforms compositions of exact functors to the compositions of the corresponding triangle functors.

A *morphism of triangle functors* $(F, \alpha) \to (G, \beta)$ is a morphism of functors $\mu : F \to G$ such that the square

$$\begin{array}{ccc}
FS & \xrightarrow{\alpha} & SF \\
\mu S \downarrow & & \downarrow S\mu \\
GS & \xrightarrow{\beta} & SG
\end{array}$$

is commutative. A triangle functor $(F, \alpha) : \mathcal{S} \to \mathcal{T}$ is a *triangle equivalence* if there exists a triangle functor $(G, \beta) : \mathcal{T} \to \mathcal{S}$ such that the *composed triangle functors* $(GF, (\beta F)(G\alpha))$ and $(FG, (\alpha G)(F\beta))$ are isomorphic to the *identical triangle functors* $(\mathbf{1}_{\mathcal{S}}, \mathbf{1}_S)$ and $(\mathbf{1}_{\mathcal{T}}, \mathbf{1}_S)$ respectively.

Lemma 8.2 *A triangle functor (F, α) is a triangle equivalence iff F is an equivalence of the underlying categories.*

Let $(R, \rho) : \mathcal{S} \to \mathcal{T}$ and $(L, \lambda) : \mathcal{T} \to \mathcal{S}$ be two triangle functors such that L is left adjoint to R. Let $\Psi : \mathbf{1}_{\mathcal{T}} \to RL$ and $\Phi : LR \to \mathbf{1}_{\mathcal{S}}$ be two 'compatible' adjunction morphisms, i.e. we have $(\Phi L)(L\Psi) = \mathbf{1}_L$ and $(R\Phi)(\Psi R) = \mathbf{1}_R$. For $X \in \mathcal{T}$ and $Y \in \mathcal{S}$, denote by $\mu(X, Y)$ the canonical bijection

$$\mathcal{S}(LX, Y) \to \mathcal{T}(X, RY), \ f \mapsto (Rf)(\Psi X).$$

Then it is not hard to see that the following conditions are equivalent

i) $\lambda = (\Phi SL)(L\rho^{-1}L)(LS\Psi)$ iii) $\Phi S = (S\Phi)(\lambda R)(L\rho)$
ii) $\rho^{-1} = (RS\Phi)(R\lambda R)(\Psi SR)$ iv) $S\Psi = (\rho L)(R\lambda)(\Psi S)$
v) The following diagram is commutative

$$\begin{array}{ccccc}
\mathcal{S}(LX, Y) & \xrightarrow{S} & \mathcal{S}(SLX, SY) & \xrightarrow{\lambda^*} & \mathcal{S}(LSX, SY) \\
\mu(X, Y) \downarrow & & & & \downarrow \mu(SX, SY) \\
\mathcal{T}(X, RY) & \xrightarrow{S} & \mathcal{T}(SX, SRY) & \xleftarrow{\rho_*} & \mathcal{T}(SX, RSY).
\end{array}$$

If they are fulfilled, we say that Φ and Ψ are *compatible triangle adjunction morphisms* and that (L, λ) is a *left triangle adjoint* of (R, ρ).

Lemma 8.3 *Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories, $(R, \rho) : \mathcal{S} \to \mathcal{T}$ a triangle functor, L a left adjoint of R, $\Phi : LR \to \mathbf{1}_{\mathcal{S}}$ and $\Psi : \mathbf{1}_{\mathcal{T}} \to RL$ compatible adjunction morphisms and $\lambda = (\Phi SL)(L\rho^{-1}L)(LS\Psi)$. Then (L, λ) is a triangle functor and is a left triangle adjoint of (R, ρ).*

A proof is given in [40, 6.7].

Example 8.4 : Infinite sums of triangles. Let $\mathcal{T}$ be a triangulated category and I a set. Suppose that each family $(X_i)_{i\in I}$ admits a direct sum $\bigoplus_{i\in I} X_i$ in $\mathcal{T}$. This amounts to requiring that the diagonal functor

$$D : \mathcal{T} \to \prod_{i\in I} \mathcal{T} ,$$

which with each object $X \in \mathcal{T}$ associates the constant family with value X, admits a left adjoint. Now the product category admits a canonical triangulated structure with suspension functor $S(X_i) = (SX_i)$, and $(D, \mathbf{1})$ is a triangle functor. Thus, by the lemma,

$$\bigoplus : \prod_{i\in I} \mathcal{T} \to \mathcal{T}$$

can be completed to a triangle functor. Loosely speaking this means that sums of families of triangles indexed by I are still triangles.

9 Localization of categories

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we will denote by $\mathcal{H}om\,(\mathcal{C},\mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$. Note that in general, the morphisms between two functors do not form a set but only a 'class'. A category $\mathcal{C}$ will be called *large* to point out that the morphisms between fixed objects are not assumed to form a set.

Let $\mathcal{C}$ be a category and Σ a class of morphisms of $\mathcal{C}$. There always exists [15, I, 1] a large category $\mathcal{C}[\Sigma^{-1}]$ and a functor $Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ which is 'universal' among the functors making the elements of Σ invertible, that is to say that, for each category $\mathcal{D}$, the functor

$$\mathcal{H}om\,(Q,\mathcal{D}) : \mathcal{H}om\,(\mathcal{C}[\Sigma^{-1}],\mathcal{D}) \to \mathcal{H}om\,(\mathcal{C},\mathcal{D})$$

induces an isomorphism onto the full subcategory of functors making the elements of Σ invertible.

Now suppose that Σ *admits a calculus of left fractions*, i.e.

F1 The identity of each object is in Σ.

F2 The composition of two elements of Σ belongs to Σ.

F3 Each diagram

$$X' \xleftarrow{s} X \xrightarrow{f} Y$$

with $s \in \Sigma$ can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ s\downarrow & & \downarrow t \\ X' & \xrightarrow{f'} & Y' \end{array}$$

with $t \in \Sigma$.

F4 If f, g are morphisms and there exists $s \in \Sigma$ such that $fs = gs$, then there exists $t \in \Sigma$ such that $tf = tg$.

Then the category $\mathcal{C}[\Sigma^{-1}]$ admits the following simple description: The *objects* of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$. The *morphisms* $X \to Y$ of $\mathcal{C}[\Sigma^{-1}]$ are the equivalence classes of diagrams

$$X \xrightarrow{f} Y' \xleftarrow{s} Y ,$$

where by definition (s, f) is *equivalent* to (t, g) if there exists a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d] & \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & Y''' & Y \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\ & Y'' \arrow[u] & \end{tikzcd}$$

such that $u \in \Sigma$. Let $(s|f)$ denote the equivalence class of (s,f). We define the *composition* of $\mathcal{C}[\Sigma^{-1}]$ by

$$(s|f) \circ (t|g) = (s'\,t|g'\,f)\ ,$$

where s' and g' fit into the following commutative diagram, which exists by F3:

$$\begin{tikzcd} & & Z'' & & \\ & Y' \arrow[ur, "g'"] & & Z' \arrow[ul, "s'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z. \arrow[ul, "t"'] \end{tikzcd}$$

One easily verifies that $\mathcal{C}[\Sigma^{-1}]$ is indeed a category, that the quotient functor

$$Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]\ ,\ X \mapsto X\ ,\ f \mapsto (\mathbf{1}|f)$$

makes the elements of Σ invertible (the inverse of $(\mathbf{1}|s)$ is $(s|\mathbf{1})$), and that it does have the universal property stated above (cf. [15]).

If Σ also *admits a calculus of right fractions* (i.e. the duals of F1-F4 are satisfied), the dual of the above construction yields a category, which, by the universal property, is canonically isomorphic to $\mathcal{C}[\Sigma^{-1}]$.

Now let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory. Denote by $\Sigma \cap \mathcal{B}$ the class of morphisms of $\mathcal{B}$ lying in Σ. We say that $\mathcal{B}$ *is right cofinal in* $\mathcal{C}$ *with respect to* Σ, if for each morphism $s : X' \to X$ of Σ with $X' \in \mathcal{B}$, there is a morphism $m : X \to X''$ such that the composition ms belongs to $\Sigma \cap \mathcal{B}$. The *left* variant of this property is defined dually.

Lemma 9.1 *The class $\Sigma \cap \mathcal{B}$ admits a calculus of left fractions. If $\mathcal{B}$ is right cofinal in $\mathcal{C}$ w.r.t. Σ, the canonical functor*

$$\mathcal{B}[(\Sigma \cap \mathcal{B})^{-1}] \to \mathcal{C}[\Sigma^{-1}]$$

is fully faithful.

10 Localization of triangulated categories

Let $\mathcal{T}$ be a triangulated category and $\mathcal{N} \subset \mathcal{T}$ a *full suspended subcategory*, i.e. a full additive subcategory such that $S\mathcal{N} \subset \mathcal{N}$ and $\mathcal{N}$ is *closed under extensions*, i.e. if the terms X and Z of a triangle (X,Y,Z) belong to $\mathcal{N}$, then so does Y. We say that $\mathcal{N}$ is a *full triangulated subcategory* if we also have $\Sigma^{-1}\mathcal{N} \subset \mathcal{N}$.

Let Σ be the class of morphisms s of $\mathcal{T}$ occurring in a triangle

$$N \to X \xrightarrow{s} X' \to SN\ ,$$

with $N \in \mathcal{N}$.

Lemma 10.1 *The class Σ is a multiplicative system with $S\Sigma \subset \Sigma$. Moreover, if, in the setting of axiom SP3 (section 6), the morphisms x and y belong to Σ, then z may be found in Σ. If $\mathcal{N}$ is a full triangulated subcategory of $\mathcal{T}$, we have $S^{-1}\Sigma \subset \Sigma$.*

The localization $\mathcal{T}[\Sigma^{-1}]$ is an additive category and the quotient functor $Q : \mathcal{T} \to \mathcal{T}[\Sigma^{-1}]$ an additive functor (by [15, I, 3.3]). We endow it with the *suspension functor* S induced by $S : \mathcal{T} \to \mathcal{T}$. We declare the *triangles* of $\mathcal{T}[\Sigma^{-1}]$ to be those S-sequences which are isomorphic to images of triangles of $\mathcal{T}$ under the quotient functor.

By SP1 and SP2, the morphisms $N \to 0$ with $N \in \mathcal{N}$ belong to Σ. Thus the quotient functor annihilates $\mathcal{N}$. We define

$$\mathcal{T}/\mathcal{N} := \mathcal{T}[\Sigma^{-1}].$$

If $\mathcal{S}$ is a suspended category, denote by $\mathcal{H}om_{tria}(\mathcal{T}, \mathcal{S})$ the large category of triangle functors from $\mathcal{T}$ to $\mathcal{S}$.

Proposition 10.2 *The category $\mathcal{T}/\mathcal{N}$ endowed with the above structure becomes a suspended category and $(Q, \mathbf{1}) : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ a triangle functor. For each suspended category $\mathcal{S}$, the functor*

$$\mathcal{H}om_{tria}(Q, \mathcal{S}) : \mathcal{H}om_{tria}(\mathcal{T}/\mathcal{N}, \mathcal{S}) \to \mathcal{H}om_{tria}(\mathcal{T}, \mathcal{S})$$

induces an isomorphism onto the full subcategory of triangle functors annihilating $\mathcal{N}$. If $\mathcal{N} \subset \mathcal{T}$ is a full triangulated subcategory, then $\mathcal{T}/\mathcal{N}$ is triangulated.

Let $\mathcal{S} \subset \mathcal{T}$ be a full triangulated subcategory. If

$$N \to X \to X' \to SN$$

is a triangle with $N \in \mathcal{N}$ and $X, X' \in \mathcal{S}$, then N lies in $\mathcal{S} \cap \mathcal{N}$ as an extension of X by $S^{-1}X'$. So the multiplicative system of $\mathcal{S}$ defined by $\mathcal{S} \cap \mathcal{N}$ coincides with $\Sigma \cap \mathcal{S}$.

Lemma 10.3 *If each morphism $N \to X'$ with $N \in \mathcal{N}$ and $X' \in \mathcal{S}$ admits a factorization $N \to N' \to X'$ with $N' \in \mathcal{N} \cap \mathcal{S}$, then $\mathcal{S}$ is right cofinal w.r.t. Σ. In particular, the canonical functor $\mathcal{S}/\mathcal{S} \cap \mathcal{N} \to \mathcal{T}/\mathcal{N}$ is fully faithful.*

11 Derived categories

Let $\mathcal{A}$ be an exact category (cf. section 4). A complex N over $\mathcal{A}$ is *acyclic in degree n* if d_N^{n-1} factors as

$$\begin{tikzcd} N^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr, "p^{n-1}"'] & & N^n \\ & Z^{n-1} \arrow[ur, "i^{n-1}"'] & \end{tikzcd}$$

where p^{n-1} is a cokernel for d^{n-2} and a deflation, and i^{n-1} is a kernel for d^n and an inflation. The complex N is *acyclic* if it is acyclic in each degree.

Example 11.1 : $\mathcal{A}$ abelian. Then N is acyclic in degree n iff $\mathrm{H}^n(N) = 0$.

Example 11.2 : Null-homotopic complexes. Let R be an associative ring with 1 and $e \in R$ an idempotent. Let $\mathcal{A}$ be the exact category of free R-modules (cf. example 4.1). The 'periodic' complex

$$\dots \xrightarrow{1-e} R \xrightarrow{e} R \xrightarrow{1-e} R \xrightarrow{e} \dots$$

is acyclic iff Ker e and Ker $(1-e)$ are free R-modules. Note, however, that this complex is always null-homotopic. If $\mathcal{A}$ is any exact category it is easy to see that the following are equivalent

i) Each null-homotopic complex is acyclic.

ii) Idempotents split in $\mathcal{A}$, i.e. Ker e and Ker $(1-e)$ exist for each idempotent $e : A \to A$ of $\mathcal{A}$.

iii) The class of acyclic complexes is closed under isomorphism in $\mathbf{H}(\mathcal{A})$.

Denote by $\mathcal{N}$ the full subcategory of $\mathbf{H}(\mathcal{A})$ formed by the complexes which are isomorphic to acyclic complexes.

Lemma 11.3 *$\mathcal{N}$ is a full triangulated subcategory of $\mathbf{H}(\mathcal{A})$.*

The morphisms $\overline{s}$ of $\mathbf{H}(\mathcal{A})$ occurring in triangles $N \to X \xrightarrow{\overline{s}} X' \to SN$ with $N \in \mathcal{N}$ are called *quasi-isomorphisms*. If $\mathcal{A}$ is abelian, a morphism $\overline{s}$ is a quasi-isomorphism if and only if $\mathrm{H}^n(\overline{s})$ is invertible for each $n \in \mathbf{Z}$. By definition (cf. section 10) the multiplicative system Σ associated with $\mathcal{N}$ is formed by all quasi-isomorphisms. The *derived category of $\mathcal{A}$* is the localization (cf. section 10)

$$\mathbf{D}(\mathcal{A}) := \mathbf{H}(\mathcal{A})/\mathcal{N} = \mathbf{H}(\mathcal{A})[\Sigma^{-1}].$$

Example 11.4 : The abelian case. If $\mathcal{A}$ is abelian, this definition of $\mathbf{D}(\mathcal{A})$ is identical with Verdier's [56].

Example 11.5 : The split case. If each conflation of $\mathcal{A}$ splits, we have $\mathcal{N} = 0$ and $\mathbf{H}(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A})$.

Let

$$\varepsilon : X \xrightarrow{i} Y \xrightarrow{p} Z$$

be a sequence of complexes over $\mathcal{A}$ such that (i^n, p^n) is a conflation for each $n \in \mathbf{Z}$. We will associate with ε a functorial triangle of $\mathbf{D}(\mathcal{A})$ which coincides with the image of $(\overline{\imath}, \overline{p}, \partial\varepsilon)$ if (i^n, p^n) is a split conflation for all $n \in \mathbf{Z}$ (cf. section 5). Form a commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{[i_X \ \ i]^t} & IX \oplus Y & \xrightarrow{[k \ \ g]} & Ci \\
\| & & \downarrow [0 \ \mathbf{1}] & & \downarrow s \\
X & \xrightarrow{i} & Y & \xrightarrow{p} & Z,
\end{array}$$

where Ci is the mapping cone of example 7.1. In the notations used there, the triangle determined by ε is then

$$X \xrightarrow{Q\overline{\imath}} Y \xrightarrow{Q\overline{p}} Z \xrightarrow{Q\overline{h}\,(Q\overline{s})^{-1}} SX\ ,$$

where Q is the quotient functor and $(Qs)^{-1}$ is well defined by the

Lemma 11.6 *The morphism $\overline{s}$ is a quasi-isomorphism.*

Let $\mathbf{C}^+(\mathcal{A})$, $\mathbf{C}^-(\mathcal{A})$ and $\mathbf{C}^b(\mathcal{A})$ be the full subcategories of $\mathbf{C}(\mathcal{A})$ formed by the complexes A such that $A^n = 0$ for all $n \ll 0$, resp. $n \gg 0$, resp. all $n \gg 0$ and all $n \ll 0$. Let $\mathbf{H}^+(\mathcal{A})$, $\mathbf{H}^-(\mathcal{A})$ and $\mathbf{H}^b(\mathcal{A})$ be the images of these subcategories in $\mathbf{H}(\mathcal{A})$. Note that these latter subcategories are not closed under isomorphism in $\mathbf{H}(\mathcal{A})$. Nevertheless it is clear that their closures under isomorphism form full suspended subcategories (cf. section 10) of $\mathbf{H}(\mathcal{A})$. For $* \in \{+, -, b\}$ we put

$$\mathbf{D}^*(\mathcal{A}) := \mathbf{H}^*(\mathcal{A})/\mathbf{H}^*(\mathcal{A}) \cap \mathcal{N}.$$

Note that we have canonical isomorphisms

$$\mathbf{H}^+(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathbf{H}^-(\mathcal{A})^{\mathrm{op}} \quad \mathbf{D}^+(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathbf{D}^-(\mathcal{A})^{\mathrm{op}}$$

mapping a complex A to the complex B with $B^n = A^{-n}$ and $d_B^n = d_A^{-n-1}$.

Lemma 11.7 *The canonical functors*

$$\mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{A})\ ,\ * \in \{+, -, b\}\ ,$$

induce equivalences onto the full subcategories of $\mathbf{D}(\mathcal{A})$ formed by the complexes which are acyclic in degree n for all $n \ll 0$, resp. $n \gg 0$, resp. all $n \gg 0$ and all $n \ll 0$. The subcategory $\mathbf{H}^+(\mathcal{A})$ (resp. $\mathbf{H}^-(\mathcal{A})$) is right (resp. left) cofinal in $\mathbf{H}(\mathcal{A})$ w.r.t. the class of quasi-isomorphisms. The subcategory $\mathbf{H}^b(\mathcal{A})$ is right cofinal in $\mathbf{H}^-(\mathcal{A})$ w.r.t. the class of quasi-isomorphisms.

12 Derived categories of fully exact subcategories

Let $\mathcal{A}$ be an exact category and $\mathcal{B} \subset \mathcal{A}$ a fully exact subcategory (cf. 4). Consider the conditions

C1 For each $A \in \mathcal{A}$ there is a conflation $A \to B \to A'$ with $B \in \mathcal{B}$.

C2 For each conflation $B \to A \to A'$ of $\mathcal{A}$ with $B \in \mathcal{B}$, there is a commutative diagram

$$\begin{array}{ccccc} B & \to & A & \to & A' \\ \| & & \downarrow & & \downarrow \\ B & \to & B' & \to & B'' \end{array}$$

whose second row is a conflation of $\mathcal{B}$.

Note that C2 is implied by C1 together with the following stronger condition: For each conflation $B \to B' \to A''$ of $\mathcal{A}$ with B and B' in $\mathcal{B}$, we have $A'' \in \mathcal{B}$.

Theorem 12.1 (cf. [39, 4.1])

a) Suppose C1 holds. Then for each left bounded complex A over $\mathcal{A}$, there is a quasi-isomorphism $A \to B$ for some left bounded complex B over $\mathcal{B}$. In particular, the canonical functor $\mathbf{D}^+(\mathcal{B}) \to \mathbf{D}^+(\mathcal{A})$ is essentially surjective.

b) Suppose C2 holds. Then the category $\mathbf{H}^+(\mathcal{B})$ is right cofinal in $\mathbf{H}^+(\mathcal{A})$ w.r.t. the class of quasi-isomorphisms. In particular, the canonical functor $\mathbf{D}^+(\mathcal{B}) \to \mathbf{D}^+(\mathcal{A})$ is fully faithful.

Example 12.2 : Injectives. If $\mathcal{A}$ has enough injectives (cf. 5), conditions C1 and C2 are obviously satisfied for the full subcategory $\mathcal{B} = \mathcal{I}$ formed by the injectives of $\mathcal{A}$ and endowed with the split conflations. Thus we have

$$\mathbf{D}^+(\mathcal{A}) \xleftarrow{\sim} \mathbf{D}^+(\mathcal{I}) \xleftarrow{\sim} \mathbf{H}^+(\mathcal{I}).$$

Example 12.3 : Noetherian modules. Let R be a right noetherian ring and $\operatorname{mod} R$ the category of noetherian R-modules. The dual of Condition C2 is clearly satisfied for the fully exact subcategory $\operatorname{mod} R \subset \operatorname{Mod} R$. Thus the functor

$$\mathbf{D}^-(\operatorname{mod} R) \to \mathbf{D}^-(\operatorname{Mod} R)$$

is fully faithful.

Example 12.4 : Filtered objects. Let $\mathcal{E}$ be an exact category and $\mathcal{A}$ the category of sequences

$$A = (\ldots \to A^p \xrightarrow{f_A^p} A^{p+1} \to \ldots)$$

of morphisms of $\mathcal{E}$ with $A^p = 0$ for all $p \ll 0$ and f_A^p invertible for all $p \gg 0$. Let $\mathcal{B} = \mathbf{F}(\mathcal{E})$ be the category of filtered objects over $\mathcal{E}$ (cf. example 4.5). It is not hard to prove that $\mathcal{B}$ viewed as a fully exact subcategory of $\mathcal{A}$ satisfies the duals of C1 and C2.

13 Derived functors, restrictions, adjoints

Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories, and $\mathcal{M} \subset \mathcal{S}$ and $\mathcal{N} \subset \mathcal{T}$ full triangulated subcategories (cf. section 10). Let $(F, \varphi) : \mathcal{S} \to \mathcal{T}$ be a triangle functor. We do *not* assume that $F\mathcal{M} \subset \mathcal{N}$. Hence in general, F will not induce a functor $\mathcal{S}/\mathcal{M} \to \mathcal{T}/\mathcal{N}$. Nevertheless there often exists an 'approximation' to such an induced functor, namely a triangle functor $\mathbf{R}F : \mathcal{S}/\mathcal{M} \to \mathcal{T}/\mathcal{N}$ and a morphism of triangle functors can $: QF \to (\mathbf{R}F)Q$. We follow P. Deligne's approach [12] to the construction of $\mathbf{R}F$.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{T} \arrow[d, "Q"] \arrow[dl, phantom, "\overset{\text{can}}{\swarrow}"] \\
\mathcal{S}/\mathcal{M} \arrow[r, "\mathbf{R}F"'] & \mathcal{T}/\mathcal{N}
\end{tikzcd}$$

Let Σ be the multiplicative system associated with $\mathcal{M}$ (cf. section 10). Let Y be an object of $\mathcal{S}/\mathcal{M}$. We define a contravariant functor $\mathbf{r}FY$ from $\mathcal{T}/\mathcal{N}$ to the category of abelian groups as follows: The value of $\mathbf{r}FY$ at $X \in \mathcal{T}/\mathcal{N}$ is formed by the equivalence classes $(f|s)$ of pairs

$$X \xrightarrow{f} FY' , \quad Y' \xleftarrow{s} Y ,$$

where $f \in (\mathcal{T}/\mathcal{N})(X, FY')$ and $s \in \Sigma$. Here, two pairs (f, s) and (g, t) are considered *equivalent* if there are commutative diagrams of $\mathcal{T}/\mathcal{N}$ and $\mathcal{S}$

$$\begin{tikzcd}
 & FY' \arrow[d, "Fv"] \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & FY''' \\
 & FY'' \arrow[u, "Fw"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[d, "v"'] & \\
Y''' & Y \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\
Y'' \arrow[u, "w"] &
\end{tikzcd}$$

such that $u \in \Sigma$. We say that $\mathbf{R}FY$ is *defined at* Y if $\mathbf{r}FY$ is a representable functor. In this case, we *define* $\mathbf{R}FY$ to be a representative of $\mathbf{r}FY$. So $\mathbf{R}FY$ is an object of $\mathcal{T}/\mathcal{N}$ endowed with an isomorphism

$$(\mathcal{T}/\mathcal{N})(?, \mathbf{R}FY) \xrightarrow{\sim} \mathbf{r}FY.$$

The datum of such an isomorphism is equivalent to the following more explicit data:

- For each $s : Y \to Y'$ of Σ, we have a morphism $\rho_s : FY' \to \mathbf{R}FY$ such that $\rho_u(Fv) = \rho_s$ whenever $u = vs$ belongs to Σ.
- There is some $s_0 : Y \to Y_0'$ and a morphism $\sigma : \mathbf{R}FY \to FY_0'$ such that $(\mathbf{1}_{FY'}|s) = (\sigma\rho_s|s_0)$ for each $s : Y \to Y'$.

In fact, if the isomorphism is given, ρ_s corresponds to the class $(\mathbf{1}_{FY'}|s)$ and $\mathbf{1}_{\mathbf{R}FY}$ to $(\sigma|s_0)$. Conversely, if the ρ_s, s_0, and σ are given, the associated isomorphism maps $g : X \to (\mathbf{R}F)Y$ to $(\sigma g\,|s_0)$, and its inverse maps $(f|s)$ to $\rho_s f$.

If we view Y as an object of $\mathcal{S}$, then, by definition, the *canonical morphism* $\text{can} : QFY \to (\mathbf{R}F)QY$ equals ρ_s for $s = \mathbf{1}_Y$.

Let $\alpha = (t|g)$ be a morphism $Y \to Z$ of $\mathcal{S}/\mathcal{M}$. We define the *morphism* $\mathbf{r}F\alpha : \mathbf{r}FY \to \mathbf{r}FZ$ by

$$\mathbf{r}F\alpha(f|s) = ((Fg')f\,|s't) ,$$

where s' and g' fit into a commutative diagram

$$\begin{tikzcd}
 & & FZ'' \\
 & FY' \arrow[ur, "Fg'"] & \\
X \arrow[ur, "f"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z'' & & \\
Y' \arrow[ur, "g'"] & & Z' \arrow[ul, "s'"'] & \\
 & Y \arrow[ul, "s"] \arrow[ur, "g"] & & Z \arrow[ul, "t"']
\end{tikzcd}$$

which exists by F3 (cf. section 9). One easily verifies that this makes $\mathbf{r}F$ into a functor from $\mathcal{S}/\mathcal{M}$ to the category of functors from $\mathcal{T}/\mathcal{N}$ to the category of abelian groups. Now suppose that $\mathbf{R}F$ is defined at Y and Z. We define *the morphism* $\mathbf{R}F\alpha$ by the commutative diagram

$$\begin{array}{ccc} (\mathcal{T}/\mathcal{N})(?,\mathbf{r}FY) & \xrightarrow{\sim} & \mathbf{r}FY \\ (\mathbf{R}F\alpha)_* \downarrow & & \downarrow \mathbf{r}F\alpha \\ (\mathcal{T}/\mathcal{N})(?,\mathbf{r}FZ) & \xrightarrow{\sim} & \mathbf{r}FZ. \end{array}$$

Thus $\mathbf{R}F$ becomes a *functor* $\mathcal{U} \to \mathcal{T}/\mathcal{N}$, where $\mathcal{U}$ denotes the full subcategory formed by the objects at which $\mathbf{R}F$ is defined. Suppose that $\mathbf{R}F$ is defined at $Y \in \mathcal{S}/\mathcal{M}$. The following chain of isomorphisms shows that $\mathbf{R}F$ is defined at SY and that $\varphi : FS \to SF$ yields a *canonical morphism* $\mathbf{R}\varphi : (\mathbf{R}F)S \to S(\mathbf{R}F)$

$$(\mathbf{r}F)(SY) \xleftarrow{\sim} \mathbf{r}(FS)(Y) \xrightarrow{\sim} \mathbf{r}(SF)(Y) \xleftarrow{\sim} (\mathcal{T}/\mathcal{N})(?, S\mathbf{R}FY).$$

Proposition 13.1 (cf. [12, 1.2]) *If*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

is a triangle of $\mathcal{S}/\mathcal{M}$ and $\mathbf{R}F$ is defined at X and Z, then it is defined at Y. In this case $(\mathbf{R}Fu, \mathbf{R}Fv, (\mathbf{R}\varphi)(X)\mathbf{R}Fw)$ is a triangle of $\mathcal{T}/\mathcal{N}$.

In particular, $\mathcal{U}$ is a triangulated subcategory of $\mathcal{S}/\mathcal{M}$ and $(\mathbf{R}F, \mathbf{R}\varphi) : \mathcal{U} \to \mathcal{T}/\mathcal{N}$ is a triangle functor. It is called the *right derived functor of* (F,φ) (with respect to $\mathcal{M}$ and $\mathcal{N}$).

Let I denote the inclusion of the preimage of $\mathcal{U}$ in $\mathcal{S}$.

Lemma 13.2 *The canonical morphism* can $: Q\,F\,I \to (\mathbf{R}F)\,Q\,I$ *is a morphism of triangle functors.*

The *left derived functor* $(\mathbf{L}F, \mathbf{L}\varphi)$ *of* (F,φ) is defined dually: For $X \in \mathcal{S}/\mathcal{M}$, one defines a covariant functor $\mathbf{l}FX$ whose value at $Y \in \mathcal{T}/\mathcal{N}$ is formed by the equivalence classes $(s|f)$ of pairs

$$X \xleftarrow{s} X' , \quad FX' \xrightarrow{f} Y ,$$

where $f \in (\mathcal{T}/\mathcal{N})(FX', Y)$ and $s \in \Sigma$ .... The *canonical morphism* can $: QFX \to \mathbf{L}F\,QX$ corresponds to the class $(\mathbf{1}_X|\mathbf{1}_{FX})$ ...

Example 13.3 : Induced functors. If we have $F\mathcal{M} \subset \mathcal{N}$, then $\mathbf{R}F$ and $\mathbf{L}F$ are isomorphic to the triangle functor $\mathcal{S}/\mathcal{M} \to \mathcal{T}/\mathcal{N}$ induced by F, and can $: QF \to \mathbf{R}FQ$ and can $: \mathbf{L}F\,Q \to QF$ are isomorphisms.

Suppose that (F', φ') is another triangle functor and $\mu : F \to F'$ a morphism of triangle functors. Then for each $Y \in \mathcal{S}/\mathcal{M}$, the morphism μ induces a morphism

$$\mathbf{r}\mu : \mathbf{r}FY \to \mathbf{r}F'Y$$

and hence a morphism $\mathbf{R}\mu : \mathbf{R}FY \to \mathbf{R}F'Y$ if both, $\mathbf{R}F$ and $\mathbf{R}F'$, are defined at Y. Note that the assignments $\mu \mapsto \mathbf{r}\mu$ and $\mu \mapsto \mathbf{R}\mu$ are compatible with compositions.

Lemma 13.4 *The morphism $\mathbf{R}\mu$ is a morphism of triangle functors between the restrictions of $\mathbf{R}F$ and $\mathbf{R}F'$ to the intersection of their domains.*

Keep the above hypotheses and let $\mathcal{U} \subset \mathcal{S}$ be a full triangulated subcategory which is right cofinal in $\mathcal{S}$ with respect to Σ. Denote by $I : \mathcal{U} \to \mathcal{S}$ the inclusion functor. Recall from lemma 10.3 that the induced functor $\mathbf{R}I : \mathcal{U}/(\mathcal{U} \cap \mathcal{M}) \to \mathcal{S}/\mathcal{M}$ is fully faithful.

Lemma 13.5 *Let $U \in \mathcal{U}$. Then* $\mathbf{R}F$ *is defined at* U *if and only if* $\mathbf{R}(F\,I)$ *is defined at* U *and in this case the canonical morphism*

$$\mathbf{R}(F\,I)(U) \to \mathbf{R}F\,\mathbf{R}I(U)$$

is invertible.

Keep the above hypotheses. Let $(R,\rho) : \mathcal{S} \to \mathcal{T}$ be a triangle functor and $(L,\lambda) : \mathcal{T} \to \mathcal{S}$ a left triangle adjoint (cf. section 8). Let $X \in \mathcal{T}/\mathcal{N}$ and $Y \in \mathcal{S}/\mathcal{M}$ be objects such that $\mathbf{L}L$ is defined at X and $\mathbf{R}R$ is defined at Y.

Lemma 13.6 *We have a canonical isomorphism*

$$\nu(X,Y) : \mathcal{S}/\mathcal{M}\,(\mathbf{L}LX, Y) \longrightarrow \mathcal{T}/\mathcal{N}\,(X, \mathbf{R}RY).$$

Moreover the diagram

$$\begin{array}{ccccc}
\mathcal{S}/\mathcal{M}\,(\mathbf{L}LX,Y) & \xrightarrow{S} & \mathcal{S}/\mathcal{M}\,(S\mathbf{L}LX,SY) & \xrightarrow{(\mathrm{L}\lambda)^*} & \mathcal{S}/\mathcal{M}\,(\mathbf{L}LSX,SY) \\
\nu(X,Y)\downarrow & & & & \downarrow \nu(SX,SY) \\
\mathcal{T}/\mathcal{N}\,(X,\mathbf{R}RY) & \xrightarrow{S} & \mathcal{T}/\mathcal{N}\,(SX,S\mathbf{R}RY) & \xleftarrow{(\mathrm{R}\rho)_*} & \mathcal{T}/\mathcal{N}\,(SX,\mathbf{R}RSY)
\end{array}$$

is commutative.

In particular, if $\mathbf{R}R$ and $\mathbf{L}L$ are defined everywhere, then $\mathbf{L}L$ is a left triangle adjoint of $\mathbf{R}R$ (cf. section 8).

14 Split objects, compositions of derived functors

Keep the hypotheses of section 13. An object Y of $\mathcal{S}$ is *F-split* with respect to $\mathcal{M}$ and $\mathcal{N}$ if $\mathbf{R}F$ is defined at Y and the canonical morphism $FY \to \mathbf{R}FY$ of $\mathcal{T}/\mathcal{N}$ is invertible.

Lemma 14.1 *The following are equivalent*

i) Y is F-split.

ii) For each morphism $s : Y \to Y'$ of Σ, the morphism QFs admits a retraction (=left inverse).

iii) For each morphism $f : M \to Y$ of $\mathcal{S}$ with $M \in \mathcal{M}$, the morphism Ff factors through an object of $\mathcal{N}$.

Let Y_0 be an object of $\mathcal{S}$. If there is a morphism $s_0 : Y_0 \to Y$ of Σ with F-split Y, then $\mathbf{R}F$ is defined at Y_0 and we have

$$\mathbf{R}FY_0 \xrightarrow{\sim} \mathbf{R}FY \xleftarrow{\sim} FY.$$

Indeed, this is clear since $\mathbf{r}F(s_0|\mathbf{1}_Y)$ provides an isomorphism $\mathbf{r}FY_0 \xrightarrow{\sim} \mathbf{r}FY$.

We say that $\mathcal{S}$ has *enough F-split objects* (with respect to $\mathcal{M}$ and $\mathcal{N}$) if, for each $Y_0 \in \mathcal{S}$, there is a morphism $s_0 : Y_0 \to Y$ of Σ with F-split Y. In this case $\mathbf{R}F$ is defined at each object of $\mathcal{S}/\mathcal{M}$.

Let $\mathcal{R}$ be another triangulated category, $\mathcal{L} \subset \mathcal{R}$ a full triangulated subcategory and $G : \mathcal{R} \to \mathcal{S}$ a triangle functor. Suppose that for each object Z_0 of $\mathcal{R}$, the multiplicative system defined by $\mathcal{L}$ contains a morphism $Z_0 \to Z$ such that Z is G-split and GZ is F-split.

Lemma 14.2 *The functor* $\mathbf{R}G$ *is defined on* $\mathcal{R}/\mathcal{L}$*, the functor* $\mathbf{R}F$ *is defined at each* $\mathbf{R}GZ_0$*,* $Z_0 \in \mathcal{R}/\mathcal{L}$*, and we have a canonical isomorphism of triangle functors*

$$\mathbf{R}(G\,F) \xrightarrow{\sim} \mathbf{R}G\,\mathbf{R}F.$$

15 Derived functors between derived categories

Let $\mathcal{A}$ and $\mathcal{C}$ be exact categories and $F : \mathcal{A} \to \mathcal{C}$ an additive (but not necessarily exact) functor. Clearly F induces a triangle functor $\mathbf{H}(\mathcal{A}) \to \mathbf{H}(\mathcal{C})$, which will be denoted by (F, φ). The construction of section 13 then yields the *right derived functor* $(\mathbf{R}F, \mathbf{R}\varphi)$ *of* F defined on a full triangulated subcategory of $\mathbf{D}(\mathcal{A})$ and taking values in $\mathbf{D}(\mathcal{C})$. Similarly for the *left derived functor* $(\mathbf{L}F, \mathbf{L}\varphi)$.

If $\mathcal{C}$ is abelian, one defines the n-th right (resp. left) derived functor of F by

$$\mathbf{R}^n FX = \mathrm{H}^n(\mathbf{R}FX) \quad (\text{resp.} \quad \mathbf{L}_n FX = \mathrm{H}^{-n}(\mathbf{L}FX) \quad) \, , \; n \in \mathbf{Z}.$$

Typically, $\mathbf{R}F$ is defined on $\mathbf{D}^+(\mathcal{A})$. Lemma 13.5 and Lemma 11 then show that the restriction of $\mathbf{R}F$ to $\mathbf{D}^+(\mathcal{A})$ coincides with the derived functor of the restriction of F to $\mathbf{H}^+(\mathcal{A})$.

An object $A \in \mathcal{A}$ is called *(right) F-acyclic* if A viewed as a complex concentrated in degree zero is a (right) F-split object of $\mathbf{H}(\mathcal{A})$. The following lemma is often useful for finding acyclic objects.

Lemma 15.1 *Let $\mathcal{B} \subset \mathcal{A}$ be a fully exact subcategory satisfying condition C2 of section 12 and such that the restriction of F to $\mathcal{B}$ is an exact functor. Then $\mathcal{B}$ consists of right F-acyclic objects.*

Example 15.2 : Injectives. If $\mathcal{B}$ is the subcategory of the injectives of $\mathcal{A}$, then each conflation of $\mathcal{B}$ splits. So any additive functor restricts to an exact functor on $\mathcal{B}$. Hence an injective object is F-acyclic for any additive functor F.

Let $\mathcal{A}c \subset \mathcal{A}$ be the full subcategory formed by the F-acyclic objects.

Lemma 15.3 *The category $\mathcal{A}c$ is a fully exact subcategory of $\mathcal{A}$ and satisfies condition C2 of section 12. The restriction of F to $\mathcal{A}c$ is an exact functor.*

Now suppose that $\mathcal{A}$ *admits enough (right) F-acyclic objects*, i.e. that for each $A \in \mathcal{A}$, there is a conflation

$$A \to B \to A'$$

with F-acyclic B. This means that $\mathcal{A}c$ satisfies condition C1 of section 12. Hence for each $X \in \mathbf{H}^+(\mathcal{A})$ there is a quasi-isomorphism $X \to X'$ with $X' \in \mathbf{H}^+(\mathcal{A}c)$.

Lemma 15.4 *The functor $\mathbf{R}F$ is defined on $\mathbf{D}^+(\mathcal{A})$. If X is a left bounded complex, we have $\mathbf{R}FX \xrightarrow{\sim} FX'$, where $X \to X'$ is a quasi-isomorphism with $X' \in \mathbf{H}^+(\mathcal{A}c)$. Each left bounded complex over $\mathcal{A}c$ is right F-split.*

Example 15.5 : Injectives. If $\mathcal{A}$ has enough injectives, it has enough F-acyclic objects for any additive functor F. The right derived functor is then computed by evaluating F on an 'injective resolution' X' of the complex X constructed with the aid of theorem 12.1.

Now let $R : \mathcal{A} \to \mathcal{C}$ be an additive functor and $L : \mathcal{C} \to \mathcal{A}$ a left adjoint. Suppose that $\mathcal{A}$ admits enough right R-acyclic objects and that $\mathcal{C}$ admits enough left L-acyclic objects. Then we have well defined derived functors $\mathbf{R}R : \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}(\mathcal{C})$ and $\mathbf{L}L : \mathbf{D}^-(\mathcal{C}) \to \mathbf{D}(\mathcal{A})$.

Lemma 15.6 *For $X \in \mathbf{D}^-(\mathcal{C})$ and $Y \in \mathbf{D}^+(\mathcal{A})$, we have a canonical isomorphism*

$$\nu(X, Y) : \mathbf{D}(\mathcal{A})(\mathbf{L}LX, Y) \xrightarrow{\sim} \mathbf{D}(\mathcal{C})(X, \mathbf{R}RY)$$

compatible with the suspension functors as in lemma 13.6.

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