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DERIVING DG CATEGORIES

Bernhard Keller[1]

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ABSTRACT. We investigate the (unbounded) derived category of a differential **Z**-graded category (=DG category). As a first application, we deduce a 'triangulated analogue' (4.3) of a theorem of Freyd's [5, Ex. 5.3 H] and Gabriel's [6, Ch. V] characterizing module categories among abelian categories. After adapting some homological algebra we go on to prove a 'Morita theorem' (8.2) generalizing results of [19] and [20]. Finally, we develop a formalism for Koszul duality [1] in the context of DG augmented categories.

SUMMARY

We give an account of the contents of this paper for the special case of DG algebras. Let k be a commutative ring and A a *DG (k-)algebra*, i.e. a **Z**-graded k-algebra

$$A = \coprod_{p\in\mathbf{Z}} A^p$$

endowed with a differential d of degree 1 such that

$$d(ab) = (da)b + (-1)^p a(db)$$

for all $a \in A^p$, $b \in A$. A *DG (right) A-module* is a **Z**-graded A-module $M = \coprod_{p\in\mathbf{Z}} M^p$ endowed with a differential d of degree 1 such that

$$d(ma) = (dm)a + (-1)^p m(da)$$

for all $m \in M^p$, $a \in A$. A *morphism of DG A-modules* is a homogeneous morphism of degree 0 of the underlying graded A-modules commuting with the differentials. The DG A-modules form an abelian *category* $\mathcal{C}A$. A morphism $f : M \to N$ of $\mathcal{C}A$ is *null-homotopic* if $f = dr + rd$ for some homogeneous morphism $r : M \to N$ of degree -1 of the underlying graded A-modules. The *homotopy category* $\mathcal{H}A$ has the same objects as $\mathcal{C}A$. Its morphisms are residue classes of morphisms of $\mathcal{C}A$ modulo null-homotopic morphisms. *It is a triangulated [23] category* (2.2). A *quasi-isomorphism* is a morphism of $\mathcal{C}A$ inducing isomorphisms in homology. The *derived category*

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$\mathcal{D}A$ is the localization [23] of $\mathcal{H}A$ with respect to the quasi-isomorphisms (4.1). It *has infinite direct sums.* Let $\mathcal{H}_p A$ be the smallest strictly (=closed under isomorphisms) full triangulated subcategory of $\mathcal{H}A$ containing A and closed under infinite direct sums. *Each DG A-module M is quasi-isomorphic to a module $\mathbf{p}M \in \mathcal{H}_p A$.* (3.1). *The canonical projection $\mathcal{H}A \to \mathcal{D}A$ restricts to an equivalence $\mathcal{H}_p A \to \mathcal{D}A$* (4.1). This is classical [11, VI, 10.2] for right bounded modules over negative DG algebras (i.e. $M^p = 0$ for all $p \gg 0$ and $A^p = 0$ for all $p > 0$).

The algebra A considered as a right DG A-module is *small in $\mathcal{D}A$*, i.e. the functor $(\mathcal{D}A)(A, ?)$ commutes with infinite direct sums. Moreover A is a *generator of $\mathcal{D}A$*, i.e. $\mathcal{D}A$ coincides with its smallest strictly full triangulated subcategory containing A and closed under infinite direct sums. Now suppose that $\mathcal{E}$ is a Frobenius category [9] with infinite direct sums and that the associated stable category $\underline{\mathcal{E}}$ admits a small generator X. Then *there is a DG algebra A and an S-equivalence $G : \underline{\mathcal{E}} \to \mathcal{D}A$ with $GX \xrightarrow{\sim} A$* (4.3). This is an analogue of Freyd's and Gabriel's characterization of module categories among abelian categories [5, Ex. 5.3 H] [6, Ch. V]. It suggests that in the study of triangulated categories, categories of DG modules might take the rôle that module categories play in the theory of abelian categories.

Let B and C be DG algebras. A *quasi-equivalence* $C \to B$ is a B-C-bimodule (i.e. a right-B-left-C-bimodule) E containing an element $e \in \mathrm{Z}^0 E$ such that the maps

$$B \to E\ ,\ b \mapsto eb \text{ and } C \to E\ ,\ c \mapsto ce$$

induce isomorphisms in homology. For example, if we are given a quasi-isomorphism $\varphi : C \to B$, we can take $E = {}_{\varphi} B_B$ and $e = 1$. Suppose that A is a DG algebra which is flat as a k-module. *There is an A-C-bimodule X such that*

$$\mathbf{L}(? \otimes_C X) : \mathcal{D}C \to \mathcal{D}A\ ,\ M \mapsto (\mathbf{p}M) \otimes_C X\ ,$$

is an equivalence iff C is quasi-equivalent to $B = \mathcal{H}om\,(T,T)$ for some module $T \in \mathcal{H}_p A$ which is a small generator of $\mathcal{D}A$ (8.2). Here $\mathcal{H}om\,(T,T)$ is the DG algebra whose nth component consists of the homogeneous graded morphisms $f : T \to T$ of degree n and whose differential maps f to $d \circ f - (-1)^n f \circ d$. It follows from ideas of Ravenel's [18] that *a DG A-module is small in $\mathcal{D}A$ iff it is contained in the smallest strictly full triangulated subcategory of $\mathcal{D}A$ containing A and closed under forming direct summands.* We reproduce A. Neeman's proof of this result [17, 2.2] in 5.3.

By applying suitable truncation functors to our DG algebras (9.1) we also generalize a result of [20] on realizing S-equivalences as derived functors (cf. also [13]).

Now suppose that *k is a field.* A *DG augmented algebra* is a DG algebra A endowed with a DG module $\overline{A}$ whose homology is isomorphic to k viewed as a DG k-module concentrated in degree 0. *There is a DG algebra A^* and an A-A^*-bimodule X such that $\mathbf{L}(X \otimes_A ?) : \mathcal{D}A^* \to \mathcal{D}A$ maps A^* to $\overline{A}$ and gives rise to an equivalence between the triangulated subcategories generated by A^* and $\overline{A}$* (10.2). We put $\overline{A^*} = \mathbf{R}\mathrm{Hom}_A(X, DA)$, where $DA = \mathrm{Hom}_k(A,k)$. Then $(A^*, \overline{A^*})$ is a DG augmented algebra called the *Koszul dual* (cf. [1]) of $(A, \overline{A})$. It is *unique* up to a quasi-equivalence compatible with the augmentation. For example, if $A = U(\mathfrak{G})$ for some Lie algebra $\mathfrak{G}$, then A^*

may be taken to be $\mathrm{Hom}_k(\Lambda\mathfrak{G}, k)$ with the shuffle product and the usual derivation (6.5). Let $A^\vee = DDA$. There is a canonical A^{**}-$A^\vee$-bimodule Y which in many cases gives rise to a quasi-equivalence $A^\vee \xrightarrow{\sim} A^{**}$ (10.3). We consider three special cases where $A^\vee$ is quasi-equivalent to A^{**} and $\mathcal{D}A$ is related to $\mathcal{D}A^*$ by a fully faithful embedding (10.5).

I am grateful to A. Neeman for pointing out Theorem 5.3 to me and calling my attention to his elegant proof in [17]. I thank the referee for his careful reading of the manuscript.

1. Graded categories and DG categories

1.1 Graded categories. Let k be a commutative ring. The tensor product over k will be denoted by $\otimes$. A *graded category* is a k-linear category $\mathcal{A}$ whose morphism spaces are $\mathbf{Z}$-graded k-modules

$$\mathcal{A}(A, B) = \coprod_{p \in \mathbf{Z}} \mathcal{A}(A, B)^p$$

such that the composition maps

$$\mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \to \mathcal{A}(A, C)$$

are homogeneous of degree 0, $\forall A, B, C \in \mathcal{A}$. A simple example is the *category* $\mathrm{Gra}\, k$ *of graded k-modules* $V = \coprod_{p \in \mathbf{Z}} V^p$ with

$$(\mathrm{Gra}\, k)(V, W)^p = \{ f \in \mathrm{Hom}_k(V, W) : f(V^q) \subset W^{p+q} \, , \ \forall q \}.$$

A graded category $\mathcal{A}$ is *concentrated in degree 0* if $\mathcal{A}(A, B)^p = 0$ for all $p \neq 0$, $A, B \in \mathcal{A}$. It is then completely determined by the k-linear *category* $\mathcal{A}^0$ having the same objects as $\mathcal{A}$ and the morphism spaces $\mathcal{A}^0(A, B) = \mathcal{A}(A, B)^0$.

If $\mathcal{A}$ and $\mathcal{B}$ are graded categories, a *graded functor* $F : \mathcal{A} \to \mathcal{B}$ is a k-linear functor whose associated maps

$$F(A, B) : \mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$$

are homogeneous of degree 0, $\forall A, B \in \mathcal{A}$.

Let $\mathcal{A}$ be a small graded category. The *opposite graded category* $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$, its morphism spaces are $\mathcal{A}^{\mathrm{op}}(A, B) = \mathcal{A}(B, A)$, and the composition is given by

$$\mathcal{A}^{\mathrm{op}}(A, B)^p \otimes \mathcal{A}^{\mathrm{op}}(B, C)^q \to \mathcal{A}^{\mathrm{op}}(A, C)^{p+q} \, , \ g \otimes f \mapsto (-1)^{pq} fg.$$

A *graded (right) $\mathcal{A}$-module* is a graded functor $M : \mathcal{A}^{\mathrm{op}} \to \mathrm{Gra}\, k$. For each $A \in \mathcal{A}$ we denote by $A^\wedge$ the *free $\mathcal{A}$-module* $\mathcal{A}(?, A)$. By definition

$$A^\wedge(f) g = (-1)^{pq} g \circ f \, , \ \forall f \in \mathcal{A}(C, B)^p \, , \ \forall g \in \mathcal{A}(B, A)^q .$$

We define $\mathcal{G}\mathcal{A}$ to be the *category* whose objects are graded $\mathcal{A}$-modules and whose morphism spaces $(\mathcal{G}\mathcal{A})(M, N)$ consist of the morphisms of functors $f : M \to N$ such that $fA : MA \to NA$ is homogeneous of degree 0 for each $A \in \mathcal{A}$.

If $\mathcal{A}$ is concentrated in degree 0, $\mathcal{GA}$ identifies with the category of sequences $(M_n)_{n\in\mathbf{Z}}$ of $\mathcal{A}^0$-modules (=k-linear contravariant functors from $\mathcal{A}^0$ to the category of k-modules).

We endow $\mathcal{GA}$ with the *shift* $M \mapsto M[1]$: By definition,

$$(M[1]A)^p = (MA)^{p+1} \text{ and } (M[1]a)(m) = (-1)^{pq}(Ma)(m)$$

for $a \in \mathcal{A}(B,A)^p$ and $m \in (MA)^q$. For a morphism $f : M \to N$ we put $(f[1]A)^p = (fA)^{p+1}$. The shift functor is clearly an autormorphism. Its nth iterate is denoted by $M \mapsto M[n]$, $n \in \mathbf{Z}$.

The graded *category* $\operatorname{Gra}\mathcal{A}$ has the same objects as $\mathcal{GA}$ and the morphisms spaces

$$(\operatorname{Gra}\mathcal{A})(M,N) \xrightarrow{\sim} \coprod_{p\in\mathbf{Z}} (\mathcal{GA})(M, N[p]).$$

The composition of morphisms produced by $f : M \to N[q]$ and $g : L \to M[p]$ is given by $f[p] \circ g$. We extend the shift functor to an automorphism of $\operatorname{Gra}\mathcal{A}$ in the obvious way.

1.2 Differential graded categories. A *differential graded category* (=DG category) is a graded category $\mathcal{A}$ whose morphism spaces are endowed with differentials d (i.e. homogeneous maps d of degree 1 with $d^2 = 0$) such that

$$d(fg) = (df)g + (-1)^p f(dg)\ ,\ \forall f \in \mathcal{A}(B,C)^p\ ,\ \forall g \in \mathcal{A}(A,B).$$

A simple example is the *category* $\operatorname{Dif} k$ *of differential k-modules* whose morphism spaces

$$(\operatorname{Dif} k)(V,W) \xrightarrow{\sim} (\operatorname{Gra} k)(V,W)$$

are endowed with the differential mapping $(f^p) \in (\operatorname{Gra} k)(V,W)^n$ to

$$(d \circ f^p - (-1)^n f^{p+1} \circ d).$$

If $\mathcal{A}$ and $\mathcal{B}$ are DG categories, a *DG functor* $F : \mathcal{A} \to \mathcal{B}$ is a graded functor such that $F(df) = d(Ff)$ for all morphisms f of $\mathcal{A}$. A *quasi-isomorphism* $F : \mathcal{A} \to \mathcal{B}$ is a DG functor inducing a bijection $\operatorname{obj}\mathcal{A} \to \operatorname{obj}\mathcal{B}$ and quasi-isomorphisms $\mathcal{A}(A,B) \to \mathcal{A}(FA,FB)$ for all $A, B \in \mathcal{A}$.

Let $\mathcal{A}$ be a small DG category. Its opposite $\mathcal{A}^{\mathrm{op}}$ is the opposite graded category of $\mathcal{A}$ endowed with the same differential as $\mathcal{A}$.

A *DG (right) $\mathcal{A}$-module* is a DG functor $M : \mathcal{A}^{\mathrm{op}} \to \operatorname{Dif} k$. Denote by $M|$ the underlying graded $\mathcal{A}$-module of M. The objects of the *DG category* $\operatorname{Dif}\mathcal{A}$ are the DG $\mathcal{A}$-modules, its morphism spaces are the graded k-modules

$$(\operatorname{Dif}\mathcal{A})(M,N) = (\operatorname{Gra}\mathcal{A})(M|, N|),$$

endowed with the differential given by

$$df = d \circ f - (-1)^p f \circ d,$$

for each homogeneous f of degree p. One easily verifies that this is well defined.

If $\mathcal{A}$ is concentrated in degree 0, DG $\mathcal{A}$-modules are in bijection with differential complexes of $\mathcal{A}^0$-modules.

For each $A \in \mathcal{A}$, the underlying graded module of the *free module* $A^\wedge$ is the free graded module associated with A. The differential of $A^\wedge(B)$ equals that of $\mathcal{A}(B, A)$. For each DG $\mathcal{A}$-module M and each $A \in \mathcal{A}$, the map

$$(\mathrm{Dif}\,\mathcal{A})(A^\wedge, M) \xrightarrow{\sim} M(A)\,,\ f \mapsto (fA)(\mathbf{1}_A).$$

is an isomorphism of DG k-modules ('Yoneda-isomorphism').

We lift the shift functor from graded modules to DG modules by defining the differential of $M[1]$ to be $-d[1]$, where $d : M \to M[1]$ is the differential of M.

2. Homotopy categories

2.1 k-linear structures. Let $\mathcal{A}$ be a DG category. The category $\mathcal{CA}$ (resp. $\mathcal{HA}$) has the same objects as $\mathrm{Dif}\,\mathcal{A}$. Its morphism spaces are

$$(\mathcal{CA})(M, N) = \mathrm{Z}^0\,(\mathrm{Dif}\,\mathcal{A})(M, N) \text{ resp. } (\mathcal{HA})(M, N) = \mathrm{H}^0(\mathrm{Dif}\,\mathcal{A})(M, N).$$

Thus the morphisms of $\mathcal{CA}$ are homogeneous of degree 0 and commute with the differential. The morphisms of $\mathcal{HA}$ are *residue classes* $\overline{f}$ of morphisms f of $\mathcal{CA}$ modulo *null-homotopic* morphisms, which by definition are of the form $dr + rd$ for some morphism $r : M \to N[-1]$ of $\mathcal{GA}$. We have a canonical projection functor $\mathcal{CA} \to \mathcal{HA}$. Two DG modules are *homotopy equivalent* if they become isomorphic in $\mathcal{HA}$. If $\mathcal{A}$ is concentrated in degree 0, $\mathcal{CA}$ (resp. $\mathcal{HA}$) identifies with the category (resp. the homotopy category) of differential complexes of $\mathcal{A}^0$-modules.

2.2 Exact and triangulated structures. We endow $\mathcal{CA}$ with an *exact structure* [16] by defining a conflation (=admissible short exact sequence [7, §9], [12, App. A]) to be a sequence

$$L \xrightarrow{i} M \xrightarrow{p} N$$

such that the underlying sequence of graded $\mathcal{A}$-modules is split short exact.

We endow $\mathcal{HA}$ with the *suspension functor* $S : \mathcal{HA} \to \mathcal{HA}\,,\ M \mapsto SM = M[1]$. We define a *triangle* of $\mathcal{HA}$ to be an S-sequence [14] isomorphic to some

$$L \xrightarrow{\overline{i}} M \xrightarrow{\overline{p}} N \xrightarrow{\overline{e}} SL\,,$$

where (i, p) is a conflation and $e = rds$, where r and s are chosen homogeneous morphisms of degree 0 such that $ps = \mathbf{1}_N$, $ri = \mathbf{1}_L$ and $rs = 0$.

Lemma.

a) $\mathcal{CA}$ is a Frobenius category [9].

b) $\mathcal{HA}$ is a triangulated category [23].

PROOF. a) Let $F : \mathcal{CA} \to \mathcal{GA}$ be the forgetful functor. For each $N \in \mathcal{GA}$, let $F_\rho N$ resp. $F_\lambda N$ be the DG $\mathcal{A}$-modules defined by

$$(F_\rho N)(A) = NA \oplus (NA)[1]\,, \quad d = \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}, \quad (F_\rho N)(a) = \begin{bmatrix} Na & 0 \\ dNa & (-1)^p Na \end{bmatrix}$$
$$(F_\lambda N)(A) = (NA)[-1] \oplus NA\,, \quad d = \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}, \quad (F_\lambda N)(a) = \begin{bmatrix} (-1)^p Na & 0 \\ (-1)^p dNa & Na \end{bmatrix},$$

where $A \in \mathcal{A}^{\mathrm{op}}$ and $a \in \mathcal{A}^{\mathrm{op}}(A,B)^p$. For each $M \in \mathcal{CA}$, define morphisms of DG $\mathcal{A}$-modules $\Phi M = [\mathbf{1}\ \ d]^t : M \to F_\rho F M$ and $\Psi M = [-d\ \ \mathbf{1}] : F_\lambda F M \to M$. We have bijections

$$(\mathcal{GA})(FM, N) \xrightarrow{\sim} (\mathcal{CA})(M, F_\rho N) \quad , \quad f \mapsto (F_\rho f)(\Phi M)$$
$$(\mathcal{GA})(N, FM) \xrightarrow{\sim} (\mathcal{CA})(F_\lambda N, M) \quad , \quad f \mapsto (\Psi M)(F_\lambda f).$$

Thus $F_\rho N$ is injective and $F_\lambda N$ is projective in $\mathcal{CA}$ for each $N \in \mathcal{GA}$. Since ΦM and ΨM fit into conflations

$$M \xrightarrow{\Phi M} F_\rho F M \longrightarrow M[1]\,,\ M[-1] \longrightarrow F_\lambda F M \xrightarrow{\Psi M} M\,,$$

we can conclude that $\mathcal{CA}$ has enough projectives and enough injectives. Moreover, M is itself projective (resp. injective) iff it is a direct summand of $F_\rho F M$ (resp. of $F_\lambda F M$). Since $F_\rho F M \xrightarrow{\sim} (F_\lambda F M)[1]$, we infer that M is projective iff it is injective. For later use, we introduce the notations $PM = F_\rho F M$ and $IM = F_\lambda F M$.

b) $\mathcal{HA}$ identifies with the stable category associated with $\mathcal{CA}$. Thus the assertion follows from [9, 9.4].

3. Resolution

3.1 P-resolutions. Let $\mathcal{A}$ be a DG category. Its *homology category* $\mathrm{H}^*\mathcal{A}$ is the graded category with the same objects as $\mathcal{A}$ and with the morphism spaces

$$(\mathrm{H}^*\mathcal{A})(A,B) = \coprod_{n \in \mathrm{Z}} \mathrm{H}^n \mathcal{A}(A,B).$$

We have a canonical functor $\mathrm{H}^* : \mathcal{CA} \to \mathrm{Gra}\, \mathrm{H}^*\mathcal{A}$ defined by

$$(\mathrm{H}^* M)(A) = \coprod_{n \in \mathrm{Z}} \mathrm{H}^n M(A).$$

It induces a functor

$$\mathcal{HA} \to \mathcal{G}\mathrm{H}^*\mathcal{A}$$

which will also be denoted by H^*.

A DG module N is *acyclic* if $\mathrm{H}^* N{=}0$. A DG module Q is *relatively projective* (cf. [15, X, §10]) if, in $\mathcal{CA}$, it is a direct summand of a direct sum of modules of the form $A\hat{}[n]$, $A \in \mathcal{A}$, $n \in \mathbf{Z}$. A DG module has *property (P)* if it is homotopy equivalent to a DG module P admitting a filtration

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \ldots F_p \subset F_{p+1} \ldots \subset P\,,\ p \in \mathbf{N}$$

in $\mathcal{CA}$ such that

(F1) P is the union of the F_p, $p \in \mathbf{N}$,

(F2) the inclusion morphism $F_{p-1} \subset F_p$ splits in $\mathcal{GA}$, $\forall\, p \in \mathbf{N}$,

(F3) the subquotient F_p/F_{p-1} is isomorphic in $\mathcal{CA}$ to a relatively projective module, $\forall\, p \in \mathbf{N}$.

Note that (F1) and (F2) imply that the following *sequence* $(*)$ is split exact in $\mathcal{GA}$ and hence produces a triangle in $\mathcal{HA}$

$$\coprod_{p \in \mathrm{N}} F_p \xrightarrow{\Phi} \coprod_{q \in \mathrm{N}} F_q \xrightarrow{\mathrm{can}} P \;;$$

here Φ has the components

$$F_p \xrightarrow{[1\; -\iota]^t} F_p \oplus F_{p+1} \xrightarrow{\mathrm{can}} \coprod_{q \in \mathrm{N}} F_q \;,\; \iota = \mathrm{incl}\,.$$

If $\mathcal{A}$ is concentrated in degree 0, a DG module P with (F1), (F2) and (F3) yields a complex of projective $\mathcal{A}^0$-modules. Conversely a *right bounded* complex of projective $\mathcal{A}^0$-modules gives rise to a DG module P with (F1), (F2) and (F3): Indeed, if $P^q = 0$ for $q > 0$, we can take $F_p = \coprod_{q > -p} P^q$.

THEOREM.

a) We have $(\mathcal{HA})(P, N) = 0$ for each acyclic N and each P with property (P).

b) For each $M \in \mathcal{HA}$ there is a triangle of $\mathcal{HA}$

$$\boldsymbol{p}M \to M \to \boldsymbol{a}M \to S\boldsymbol{p}M,$$

where $\boldsymbol{a}M$ is acyclic and $\boldsymbol{p}M$ has property (P).

c) Let

$$\ldots \to \overline{Q_n} \to \overline{Q_{n-1}} \to \ldots \to \overline{Q_1} \to \overline{Q_0} \to \mathrm{H}^* M \to 0$$

be a projective resolution of $\mathrm{H}^* M$ in $\mathcal{G}\mathrm{H}^*\mathcal{A}$ such that $\overline{Q_n} \xrightarrow{\sim} \mathrm{H}^* Q_n$ for a relatively projective $Q_n \in \mathcal{CA}$, $\forall\, n$. Then $\boldsymbol{p}M$ is homotopy equivalent to a module P admitting a filtration F_p with (F1), (F2) and such that $F_p/F_{p-1} \xrightarrow{\sim} Q_p[p]$ in $\mathcal{CA}$, $\forall\, p$.

We shall refer to $\boldsymbol{p}M$ as a *P-resolution* of M. If $\mathcal{A}$ is concentrated in degree 0, assertion c) implies that if M is a (possibly unbounded) complex of $\mathcal{A}^0$-modules and Q^p_* a given projective resolution of its pth homology, then M is quasi-isomorphic to a complex $\boldsymbol{p}M$ whose nth component is $\coprod_{p-q=n} Q^p_q$.

We define $\mathcal{H}_p\mathcal{A}$ to be the full *subcategory* of $\mathcal{HA}$ formed by the modules with property (P). Applying suitable Hom-functors to the triangle of b) and using a) we see that we have

$$(\mathcal{HA})(P, \boldsymbol{p}M) \xrightarrow{\sim} (\mathcal{HA})(P, M) \text{ and } (\mathcal{HA})(M, N) \xleftarrow{\sim} (\mathcal{HA})(\boldsymbol{a}M, N)$$

for all $P \in \mathcal{H}_p\mathcal{A}$ and all acyclic N. In particular, if $(\mathcal{HA})(M, N) = 0$ for each acyclic N, we have $0 = (\mathcal{HA})(M, \boldsymbol{a}M) \xleftarrow{\sim} (\mathcal{HA})(\boldsymbol{a}M, \boldsymbol{a}M)$, so that $\boldsymbol{a}M = 0$ and, by b), $\boldsymbol{p}M \xrightarrow{\sim} M$. Hence a DG module M lies in $\mathcal{H}_p\mathcal{A}$ iff $(\mathcal{H}\mathcal{A})(M,N)=0$ for each acyclic N. Therefore $\mathcal{H}_p\mathcal{A}$ is a triangulated subcategory of $\mathcal{H}\mathcal{A}$. The inclusion $\mathcal{H}_p\mathcal{A}\subset\mathcal{H}\mathcal{A}$ admits the right S-adjoint [14] $M\mapsto \boldsymbol{p}M$.

It follows from a) that each triangle

$$P\to M\to N\to P[1],$$

where N is acyclic and P has property (P), is canonically isomorphic to the triangle of b). If $(M_i)_{i\in I}$ is a family of modules, we can apply this to the triangle

$$\coprod \boldsymbol{p}M_i \to \coprod M_i \to \coprod \boldsymbol{a}M_i \to \coprod \boldsymbol{p}M_i[1]$$

to conclude that $\boldsymbol{p}$ and $\boldsymbol{a}$ commute with infinite direct sums.

Proof. a) The assertion holds for each P of the form $A^\wedge[n]$, $A\in\mathcal{A}$, $n\in\mathbf{Z}$, since

$$(\mathcal{H}\mathcal{A})(A^\wedge[n],N)=\mathrm{H}^0(\mathrm{Dif}\,\mathcal{A})(A^\wedge,N[-n])=\mathrm{H}^{-n}N(A)=0$$

for each acyclic N. Hence it holds for relatively projective P. It also holds if $F_p=P$ for $p\gg 0$ since such a P lies in the triangulated subcategory generated by the relatively projectives. In the general case, we apply $\mathcal{H}\mathcal{A}\,(?,N)$ to the triangle produced by the sequence $(*)$ and obtain an exact sequence

$$\prod_{q\in\mathrm{Z}}(\mathcal{H}\mathcal{A})(F_q,N)\leftarrow(\mathcal{H}\mathcal{A})(P,N)\leftarrow\prod_{p\in\mathrm{Z}}(\mathcal{H}\mathcal{A})(F_p[1],N).$$

Its outer terms vanish by the foregoing case.

b), c) Following [15, XII, 11] we endow $\mathcal{C}\mathcal{A}$ with another exact structure: Its class of conflations $\mathcal{E}$ consists of the sequences

$$L\to M\to N$$

such that

$$\begin{aligned} &0\to L(A)^n\to M(A)^n\to N(A)^n\to 0\\ \text{and}\quad &0\to \mathrm{H}^n L(A)\to\mathrm{H}^n M(A)\to\mathrm{H}^n N(A)\to 0\end{aligned}$$

are short exact sequences of k-modules, for all $A\in\mathcal{A}$, $n\in\mathbf{Z}$. This is equivalent to requiring that

$$\begin{aligned} &0\to L(A)^n\to M(A)^n\to N(A)^n\to 0\\ \text{and}\quad &0\to \mathrm{Z}^n L(A)\to\mathrm{Z}^n M(A)\to\mathrm{Z}^n N(A)\to 0\end{aligned}$$

be short exact for all $A\in\mathcal{A}$, $n\in\mathbf{Z}$. The isomorphisms

$$\begin{aligned}(\mathcal{C}\mathcal{A})(A^\wedge[-n],M)&=\mathrm{Z}^0(\mathrm{Dif}\,\mathcal{A})(A^\wedge,M[n])=\mathrm{Z}^n M(A)\\ (\mathcal{C}\mathcal{A})(PA^\wedge[-n],M)&=M(A)^n\end{aligned}$$

(2.2) show that if Q is relatively projective, then Q and PQ are $\mathcal{E}$-projective. It is also clear that for each module M we may find an $\mathcal{E}$-projective $Q'=Q\oplus PQ''$ and a morphism $p:Q'\to M$ inducing surjections

$$Q'(A)^n\to M(A)^n \text{ and } \mathrm{Z}^n Q'(A)\to\mathrm{Z}^n M(A)\,,\ \forall A\in\mathcal{A},\forall n\in\mathbf{Z}.$$

If $K \to Q'$ is a kernel of p in $\mathcal{CA}$, it is clear that $K \to Q' \to M$ is indeed a conflation. Thus, $\mathcal{CA}$ has enough $\mathcal{E}$-projectives and we can inductively construct an $\mathcal{E}$-resolution of M, i.e. an $\mathcal{E}$-acyclic complex [12, 4.1]

$$\ldots \to Q'_n \to Q'_{n-1} \to \ldots \to Q'_1 \to Q'_0 \xrightarrow{\varepsilon} M \to 0$$

with $\mathcal{E}$-projective $Q'_n = Q_n \oplus PQ''_n$, where Q_n and Q''_n are relatively projective. Under the hypotheses of c), we can refine this construction as follows: The map

$$(\mathcal{CA})\,(Q, M) \to (\mathcal{G}\mathrm{H}^*\mathcal{A})\,(\mathrm{H}^*Q, \mathrm{H}^*M)$$

is clearly surjective if Q is of the form $A^\wedge[n]$ for some $A \in \mathcal{A}$, $n \in \mathbf{Z}$. Hence it is surjective for relatively projective Q. We can therefore lift the given morphism $\overline{Q_0} \to \mathrm{H}^*M$ to a morphism $p : Q_0 \to M$ of $\mathcal{CA}$. Now we choose an $\mathcal{E}$-projective PQ''_0, with relatively projective Q''_0, and a morphism $q : PQ''_0 \to M$ inducing epimorphisms

$$PQ''_0(A)^n \to M(A)^n \ , \ \forall\, A \in \mathcal{A}, \forall\, n \in \mathbf{Z}.$$

Then

$$Q'_0 = Q_0 \oplus PQ''_0 \xrightarrow{[p\ q]} M$$

is the required deflation (=admissible epimorphism) with $\mathcal{E}$-projective Q'_0. Observe that, since PQ''_0 is null-homotopic, Q'_0 is homotopy equivalent to Q_0. Since $\mathrm{H}^* : \mathcal{CA} \to \mathcal{G}\mathrm{H}^*\mathcal{A}$ carries $\mathcal{E}$-conflations to short exact sequences, we can successively lift the given resolution of H^*M to an $\mathcal{E}$-acyclic sequence

$$\ldots \to Q'_n \to Q'_{n-1} \to \ldots \to Q'_1 \to Q'_0 \xrightarrow{\varepsilon} M \to 0$$

such that $Q'_n = Q_n \oplus PQ''_n$ for all $n \in \mathbf{N}$. If

$$K = (\ldots \to K^n \xrightarrow{d_K^n} K^{n+1} \to \ldots)\ , \ n \in \mathbf{Z}$$

is a differential complex over $\mathcal{CA}$, its *total module* $\mathrm{Tot}\, K$ has the underlying graded module

$$\coprod_{n \in \mathbf{Z}} K^n[-n]$$

and the differential

$$d = d_{K^n[-n]} + d_K^n.$$

Put

$$\boldsymbol{p}M = \mathrm{Tot}\,(\ldots \to Q'_m \to Q'_{m-1} \to \ldots \to Q'_1 \to Q'_0 \to 0 \to 0 \to \ldots)$$

and

$$F'_p = \mathrm{Tot}\,(\ldots \to 0 \to 0 \to Q'_p \to Q'_{p-1} \to \ldots \to Q'_1 \to Q'_0 \to 0 \to 0 \to \ldots)\ , \ p \geq 0.$$

Then $\boldsymbol{p}M$ with the filtration by the F'_p clearly satisfies (F1) and (F2), and $F'_p/F'_{p-1} = Q'_p[p]$, $\forall\, p$. By the lemma we will prove in 3.4, this implies that $\boldsymbol{p}M$ has property (P). The morphism

$\varepsilon : Q_0' \to M$ induces a morphism $\varphi : \boldsymbol{p}M \to M$. It remains to be shown that $\mathrm{H}^*\varphi$ is invertible or, equivalently, that

$$N = \mathrm{Tot}\,(\ldots \to Q_m' \to \ldots \to Q_1' \to Q_0' \to M \to 0 \to \ldots)$$

is acyclic. This follows from the lemma we will prove in 3.3 applied to each $N(A)$, $A \in \mathcal{A}$.

3.2 I-resolutions. We record without proof the following 'dual' of 3.1. Fix an injective generator E of the category of k-modules. For each $A \in \mathcal{A}$ define the $\mathcal{A}$-*module* $A^\vee$ by

$$B \mapsto (\mathrm{Dif}\, k)\,(\mathcal{A}\,(A, B), E),$$

where E is viewed as a DG k-module concentrated in degree 0. A DG $\mathcal{A}$-module is *relatively injective* if, in $\mathcal{CA}$, it is a direct summand of a direct product of modules $A^\vee[n]$, $A \in \mathcal{A}$, $n \in \mathbf{Z}$. A DG module has *property (I)* if it is homotopy equivalent to a DG module I admitting a filtration

$$I = F_0 \supset F_1 \supset \ldots \supset F_p \supset F_{p+1} \supset \ldots\,,\ p \in \mathbf{N}\,,$$

such that

(F1') the canonical morphism $I \to \varprojlim I/F_p$ is invertible,

(F2') the inclusion morphism $F_{p+1} \subset F_p$ splits in $\mathcal{GA}$ for all $p \in \mathbf{N}$,

(F3') the subquotient F_p/F_{p+1} is isomorphic in $\mathcal{CA}$ to a relatively injective module, $\forall\, p \in \mathbf{N}$.

By (F1') and (F2') the following *sequence* $(*')$ is split exact in $\mathcal{GA}$ and hence produces a triangle in $\mathcal{HA}$

$$I \xrightarrow{\mathrm{can}} \prod_{p\in\mathrm{N}} I/F_p \xrightarrow{\Phi'} \prod_{q\in\mathrm{N}} I/F_q\ ;$$

here Φ' has the components

$$\prod_{p\in\mathrm{N}} I/F_p \xrightarrow{\mathrm{can}} I/F_{q+1} \oplus I/F_q \xrightarrow{[-\pi\ \ 1]} I/F_q\ ,$$

where π is the canonical projection $I/F_{q+1} \to I/F_q$.

THEOREM.

a) We have $(\mathcal{HA})(N, I) = 0$ for each acyclic N and each I with property (I).

b) For each $M \in \mathcal{HA}$ there is a triangle of $\mathcal{HA}$

$$\boldsymbol{a}'M \to M \to \boldsymbol{i}M \to S\boldsymbol{a}'M\ ,$$

where $\boldsymbol{a}'M$ is acyclic and $\boldsymbol{i}M$ has property (I).

c) Let

$$0 \to \mathrm{H}^*M \to \overline{J_0} \to \overline{J_1} \to \ldots \to \overline{J_n} \to \overline{J_{n+1}} \to \ldots$$

be an injective resolution of H^*M *in* $\mathcal{G}\mathrm{H}^*\mathcal{A}$ *such that* $\overline{J_n} \xrightarrow{\sim} \mathrm{H}^*J_n$ *for a relatively injective* $J_n \in \mathcal{CA}$, $\forall\, n$. *Then* $\boldsymbol{i}M$ *is homotopy equivalent to a module* I *admitting a decreasing filtration* F_p *with (F1') and (F2') and such that* $F_p/F_{p+1} \xrightarrow{\sim} J_p[-p]$ *in* $\mathcal{CA}$ *for all* $p \in \mathbf{N}$.

3.3 Acyclic total complexes. Let

$$N = \coprod_{p,q\in \mathbf{Z}} N^{pq}$$

be a bigraded abelian group with commuting differentials d_I and d_{II} of bidegree $(1,0)$ and $(0,1)$, respectively. Let $\mathrm{Tot}\,N$ and $\widehat{\mathrm{Tot}}\,N$ be the differential graded groups with components

$$(\mathrm{Tot}\,N)^n = \coprod_{p+q=n} N^{pq} \text{ resp. } (\widehat{\mathrm{Tot}}\,N)^n = \prod_{p+q=n} N^{pq}\ ,\ n \in \mathbf{Z}\,,$$

and the differential given by

$$dt = d_I t + (-1)^p d_{II} t\ ,\ t \in N^{pq}.$$

For $r \in \mathbf{Z}$ denote by N^{*r} (resp. B^{*r}, Z^{*r}, H^{*r}) the differential graded groups with components

$$N^{nr} \text{ (resp. } \mathrm{Im}\, d_{II}^{n,r-1},\ \ \mathrm{Ker}\, d_{II}^{nr},\ \ \mathrm{Ker}\, d_{II}^{nr}/\mathrm{Im}\, d_{II}^{n,r-1}\ )\ ,\ n \in \mathbf{Z}\,,$$

and the differential induced by d_I.

Lemma. *If* N^{*r} *and* H^{*r} *are acyclic for all* $r \in \mathbf{Z}$, *then* $\mathrm{Tot}\,N$ *and* $\widehat{\mathrm{Tot}}\,N$ *are acyclic.*

Proof. If N^{*r} is acyclic for all $r \in \mathbf{Z}$, the same holds for the B^{*r}. Thus if N^{*r} and H^{*r} are acyclic for all $r \in \mathbf{Z}$, then so are the Z^{*r}. To prove that $\mathrm{Tot}\,N$ is acyclic we consider the differential bigraded subgroups $N_m \subset N$, $m \geq 1$, with $N_m^{*r} = 0$ for $r \notin [-m,m]$, $N_m^{*r} = N^{*r}$ for $r \in [-m, m-1]$, and $N_m^{*m} = Z^{*m}$. Clearly each $\mathrm{Tot}\,N_m$ admits a finite filtration with acyclic subquotients and hence is acyclic. Since we have

$$\mathrm{Tot}\,N \xleftarrow{\sim} \mathrm{Tot}\varinjlim N_m \xleftarrow{\sim} \varinjlim \mathrm{Tot}\,N_m\ ,$$

the assertion follows. Similarly, to prove that $\widehat{\mathrm{Tot}}\,N$ is acyclic, we consider the quotients Q_m of N, $m \geq 1$, with $Q_m^{*r} = 0$ for $r \notin [-m,m]$, $Q_m^{*r} = N_m^{*r}$ for $r \in [-m+1, m]$ and $Q_m^{*,-m} = B^{*,-m+1}$. As above, each $\widehat{\mathrm{Tot}}\,Q_m$ is acyclic and we have

$$\widehat{\mathrm{Tot}}\,N \xrightarrow{\sim} \widehat{\mathrm{Tot}}\varprojlim Q_m \xrightarrow{\sim} \varprojlim \widehat{\mathrm{Tot}}\,Q_m.$$

Moreover for each $m \geq 1$, the components of the canonical morphism

$$p_m : \widehat{\mathrm{Tot}}\,Q_{m+1} \to \widehat{\mathrm{Tot}}\,Q_m$$

are surjective. Therefore, p_m also induces surjections onto the groups $B^n \widehat{\mathrm{Tot}}\, Q_m = Z^n \widehat{\mathrm{Tot}}\, Q_m$, $n \in \mathbf{Z}$. By the Mittag-Leffler-criterion [8, 0_{III}, 13.1], $\widehat{\mathrm{Tot}}\, N$ is acyclic.

3.4 Adjusting limits. Let P' be a DG $\mathcal{A}$-module and

$$F'_0 \subset F'_1 \subset \ldots \subset F'_p \subset \ldots \subset P'$$

a filtration satisfying (F1) and (F2). Suppose that for each $p \geq 1$ a DG module Q_p and a homotopy equivalence $F'_p/F'_{p-1} \xrightarrow{\sim} Q_p$ are given.

LEMMA. *The DG module P' is homotopy equivalent to a DG module P admitting a filtration F_p satisfying (F1) and (F2) and such that F_p/F_{p-1} is isomorphic to Q_p in $\mathcal{CA}$, $\forall\, p$.*

PROOF. We will inductively construct a sequence

$$F_0 \subset F_1 \subset \ldots \subset F_p \subset \ldots$$

and a sequence of homotopy equivalences $\overline{f_p} : F'_p \to F_p$ such that the squares

$$\begin{array}{ccc} F'_p & \to & F'_{p+1} \\ \overline{f_p} \downarrow & & \downarrow \overline{f_{p+1}} \\ F_p & \to & F_{p+1} \end{array}$$

are commutative (in $\mathcal{HA}$), the sequence F_p satisfies (F2) and $F_p/F_{p-1} \xrightarrow{\sim} Q_p$ in $\mathcal{CA}$, $\forall\, p$. Of course, we put $F_0 = Q_0$ and let $\overline{f_0} : F'_0 \to F_0$ be the given homotopy equivalence. Suppose that the construction has been completed for all $p < n$. We have

$$\mathrm{Ext}_{\mathcal{CA}}(F'_n/F'_{n-1}, F'_{n-1}) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{CA}}(Q_n, F_{n-1}) ,$$

where $\mathrm{Ext}_{\mathcal{CA}}$ denotes classes of extensions in the exact category $\mathcal{CA}$ (2.2). We choose a conflation

$$F_{n-1} \to F_n \to Q_n$$

whose class corresponds to that of the given extension of F'_n/F'_{n-1} by F'_{n-1}. Then we have a commutative diagram

$$\begin{array}{ccccc} F'_{n-1} & \to F'_n \to & F'_n/F_{n-1} & \to & F'_{n-1}[1] \\ \overline{f_{n-1}} \downarrow & & \downarrow & & \downarrow \overline{f_{n-1}}[1] \\ F_{n-1} & \to F_n \to & Q_n & \to & F_{n-1}[1] \end{array}$$

We choose $\overline{f_n}$ so as to fit into the diagram. Now let P be the union of the F_p. Using the sequence $(*)$ of 3.1 we get triangles

$$\coprod_{p \in \mathbf{Z}} F'_p \xrightarrow{\Phi} \coprod_{q \in \mathbf{Z}} F'_q \longrightarrow P' \longrightarrow S \coprod_{p \in \mathbf{Z}} F'_p$$

$$\coprod_{p \in \mathbf{Z}} F_p \xrightarrow{\Phi} \coprod_{q \in \mathbf{Z}} F_q \longrightarrow P \longrightarrow S \coprod_{p \in \mathbf{Z}} F_p .$$

The $\overline{f_p}$ yield a commutative square

$$\begin{array}{ccc} \coprod_{p\in \mathbf{Z}} F'_p & \xrightarrow{\overline{\Phi}} & \coprod_{q\in \mathbf{Z}} F'_q \\ \overline{a}\downarrow & & \downarrow \overline{b} \\ \coprod_{p\in \mathbf{Z}} F_p & \xrightarrow{\overline{\Phi}} & \coprod_{q\in \mathbf{Z}} F_q \end{array}$$

where $\overline{a}$ and $\overline{b}$ are homotopy equivalences. Using axiom TR3 [23, Ch. I, §1] and the five lemma we see that P is homotopy equivalent to P'.

4. Derived categories and stable categories

4.1 Derived categories. Let $\mathcal{A}$ be a small DG category. Let Σ be the class of *quasi-isomorphisms* of $\mathcal{HA}$ (i.e. morphisms $\overline{s}$ such that $\mathrm{H}^*\overline{s}$ is invertible). By definition [11, Ch. VI, 10] the *derived category* of $\mathcal{A}$ is the localization $\mathcal{DA} = (\mathcal{HA})[\Sigma^{-1}]$ [23]. It follows from theorem 3.1 that the canonical functor $\mathcal{HA} \to \mathcal{DA}$ induces an equivalence $\mathcal{H}_p\mathcal{A} \to \mathcal{DA}$. If $\mathcal{A}$ is concentrated in degree 0, $\mathcal{DA}$ identifies with the unbounded derived category of the category of $\mathcal{A}^0$-modules. As in the case of the derived category of an exact category, one constructs [7, 12.3] a functor which completes the images in $\mathcal{DA}$ of pointwise short exact sequences of $\mathcal{CA}$ into triangles.

Since (infinite) direct sums of acyclic modules are acyclic, $\mathcal{DA}$ has direct sums, and the canonical functors $\mathcal{CA} \to \mathcal{HA} \to \mathcal{DA}$ commute with direct sums.

4.2 Small objects and generators. Let $\mathcal{A}$ be a small DG category and $\mathcal{T}$ a k-linear triangulated category with infinite direct sums. An object $X \in \mathcal{T}$ is *small* if $\mathcal{T}(X, ?)$ commutes with (infinite) direct sums. By the five lemma, if two vertices of a triangle of $\mathcal{T}$ are small, then so is the third one. Each $A^\wedge$ is small in $\mathcal{DA}$. Indeed, let $(M_i)_{i\in I}$ be a family of modules and $A \in \mathcal{A}$. Then

$$(\mathcal{DA})(A^\wedge, \coprod_{i\in I} M_i) \xrightarrow{\sim} \mathrm{H}^0 \coprod M_i(A) \xrightarrow{\sim} \coprod \mathrm{H}^0 M_i(A) \xrightarrow{\sim} \coprod_{i\in I} (\mathcal{DA})(A^\wedge, M_i).$$

Let $\mathcal{H}_p^b\mathcal{A}$ be the smallest strictly (=closed under ismorphisms) full triangulated subcategory of $\mathcal{H}_p\mathcal{A}$ containing the $A^\wedge$, $A \in \mathcal{A}$.

A set $\mathcal{X} \subset \mathcal{T}$ is a *set of generators* if $\mathcal{T}$ coincides with its smallest strictly full triangulated subcategory containing $\mathcal{X}$ and closed under direct sums. It follows from the sequence $(*)$ of 3.1 that the $A^\wedge$, $A \in \mathcal{A}$, form a set of generators for $\mathcal{DA}$.

Let $F, F' : \mathcal{DA} \to \mathcal{T}$ be two k-linear S-functors commuting with direct sums and $\mu : F \to F'$ a morphism of S-functors [14].

Lemma.

a) The restriction of F to $\mathcal{H}_p^b\mathcal{A}$ is fully faithful iff F induces bijections

$$(\mathcal{DA})(A^\wedge, B^\wedge[n]) \to \mathcal{T}(FA^\wedge, FB^\wedge[n])$$

for all $A, B \in \mathcal{A}$, $n \in \mathbf{Z}$.

b) F is fully faithful if $F|\mathcal{H}_p^b\mathcal{A}$ is fully faithful and $FA^\wedge$ is small for each $A \in \mathcal{A}$.

c) F is an equivalence iff $F|\mathcal{H}_p^b\mathcal{A}$ is fully faithful and the $FA^\wedge$, $A \in \mathcal{A}$, form a set of small generators for $\mathcal{T}$.

d) The morphism $\mu : F \to F'$ is invertible iff $\mu A^\wedge$ is invertible for each $A \in \mathcal{A}$.

PROOF. a) results from 'devissage' (cf. e.g. [9, 10.10]).

b) Let $A \in \mathcal{A}$. By the five lemma, the modules M such that the map

$$(\mathcal{D}\mathcal{A})(A^\wedge, M) \to \mathcal{T}(FA^\wedge, FM)$$

is bijective form a strictly full triangulated subcategory of $\mathcal{D}\mathcal{A}$. It contains all the generators $B^\wedge$, $B \in \mathcal{A}$, and is closed under infinite direct sums (since both, $A^\wedge$ and $FA^\wedge$, are small and F commutes with infinite direct sums). This subcategory therefore coincides with $\mathcal{D}\mathcal{A}$. The same argument shows that for fixed $M \in \mathcal{D}\mathcal{A}$, the map

$$(\mathcal{D}\mathcal{A})(L, M) \to \mathcal{T}(FL, FM)$$

is bijective for each $L \in \mathcal{D}\mathcal{A}$.

c) is now clear.

d) The DG modules M with invertible μM form a strictly full triangulated subcategory of $\mathcal{D}\mathcal{A}$ which moreover is closed under infinite direct sums. This subcategory equals $\mathcal{D}\mathcal{A}$ iff it contains the $A^\wedge$, $A \in \mathcal{A}$, as these form a set of generators for $\mathcal{D}\mathcal{A}$.

4.3 Stable categories. Let $\mathcal{E}$ be a k-linear Frobenius category [9] with (infinite) direct sums. Since $\mathcal{E}$ has enough injectives, it is clear that direct sums of conflations (=admissible short exact sequences) of $\mathcal{E}$ are conflations. Moreover, direct sums of injectives (=projectives in $\mathcal{E}$) are injective. In particular, the associated stable category $\underline{\mathcal{E}}$ is a triangulated category with infinite direct sums. Suppose that $\underline{\mathcal{E}}$ admits a set of small generators $\mathcal{X} \subset \underline{\mathcal{E}}$.

THEOREM. (cf. [5, Ex. 5.3 H]) *There is a DG category $\mathcal{A}$ and an S-equivalence $G : \underline{\mathcal{E}} \to \mathcal{D}\mathcal{A}$ giving rise to an equivalence between $\mathcal{X} \subset \underline{\mathcal{E}}$ and the full subcategory of $\mathcal{D}\mathcal{A}$ formed by the free modules $A^\wedge$, $A \in \mathcal{A}$.*

PROOF. Let $\widetilde{\mathcal{E}}$ be the category of acyclic [14, 1.5] differential complexes

$$P = (\ldots \to P^n \xrightarrow{d} P^{n-1} \to \ldots)\,,\; n \in \mathbf{Z}$$

with projective components $P^n \in \mathcal{E}$. Endow $\widetilde{\mathcal{E}}$ with the pointwise split short exact sequences. Then $\widetilde{\mathcal{E}}$ is a Frobenius category and it is easy to see that the functor $P \mapsto \mathrm{Z}^0 P$ induces an S-equivalence

$$G_1 : \underline{\widetilde{\mathcal{E}}} \to \underline{\mathcal{E}}.$$

For each $X \in \mathcal{X}$, choose $\widetilde{X} \in \widetilde{\mathcal{E}}$ with $\mathrm{Z}^0 \widetilde{X} \xrightarrow{\sim} X$. Let $\mathcal{A}$ be the DG category whose objects are the $\widetilde{X}$ and whose morphism spaces are

$$\mathcal{A}(\widetilde{X}, \widetilde{Y}) \xrightarrow{\sim} \mathcal{H}om(\widetilde{X}, \widetilde{Y}),$$

where for $P, Q \in \widetilde{\mathcal{E}}$, the DG k-module $\mathcal{H}om(P, Q)$ has the components

$$\prod_{p \in \mathbf{Z}} \mathcal{E}(P^p, Q^{n+p}), \ n \in \mathbf{Z},$$

and the differential given by $d(f^p) = (d \circ f^p - (-1)^n f^{p+1} \circ d)$. Note that

$$\underline{\widetilde{\mathcal{E}}}(P, S^n Q) \xrightarrow{\sim} \mathrm{H}^n \mathcal{H}om(P, Q).$$

It is clear that the composition of the exact functor

$$\widetilde{\mathcal{E}} \to \mathcal{C}\mathcal{A}, \ P \mapsto (\widetilde{X} \mapsto \mathcal{H}om(\widetilde{X}, P))$$

with the canonical projection $\mathcal{C}\mathcal{A} \to \mathcal{D}\mathcal{A}$ vanishes on projectives of $\widetilde{\mathcal{E}}$ (=null-homotopic complexes in $\widetilde{\mathcal{E}}$) and hence induces an S-functor

$$G_2 : \underline{\widetilde{\mathcal{E}}} \to \mathcal{D}\mathcal{A}.$$

For $\widetilde{X} \in \widetilde{\mathcal{X}}$ the module $G_2 \widetilde{X}$ is isomorphic to $\widetilde{X}^\wedge$, the free module associated with $\widetilde{X} \in \mathcal{A}$. If P_i, $i \in I$, is a family in $\widetilde{\mathcal{E}}$ and $\widetilde{X} \in \widetilde{\mathcal{X}}$, the nth homology of the morphism

$$\coprod \mathcal{H}om(\widetilde{X}, P_i) \to \mathcal{H}om(\widetilde{X}, \coprod P_i)$$

identifies with

$$\coprod \underline{\widetilde{\mathcal{E}}}(\widetilde{X}, S^n P_i) \to \underline{\widetilde{\mathcal{E}}}(\widetilde{X}, \coprod S^n P_i),$$

which is bijective since $\widetilde{X}$ is small in $\widetilde{\mathcal{E}}$. Hence G_2 commutes with direct sums. We have already seen that G_2 induces bijections

$$\underline{\widetilde{\mathcal{E}}}(\widetilde{X}, S^n \widetilde{Y}) \xrightarrow{\sim} \mathrm{H}^n \mathcal{H}om(\widetilde{X}, \widetilde{Y}) \xrightarrow{\sim} \mathrm{H}^n \mathcal{A}(\widetilde{X}, \widetilde{Y}) \xrightarrow{\sim} (\mathcal{D}\mathcal{A})(G_2 \widetilde{X}, S^n G_2 \widetilde{Y}), \ \widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{X}}, \ n \in \mathbf{Z}.$$

By the argument of 4.2 b), we conclude that G_2 is fully faithful. The essential image of G_2 contains the generators $A^\wedge$, $A \in \mathcal{A}$, of $\mathcal{D}\mathcal{A}$. So G_2 is essentially surjective. We let G be the composition of G_2 with an S-quasi-inverse of G_1.

5. Small objects

Let $\mathcal{A}$ be a small DG category. Each free module $A^\wedge$, $A \in \mathcal{A}$, is small in $\mathcal{D}\mathcal{A}$, and so are the objects of the smallest strictly full triangulated subcategory of $\mathcal{D}\mathcal{A}$ containing the $A^\wedge$, $A \in \mathcal{A}$, and closed under forming direct summands. Ravenel's ideas [18] imply that this subcategory *coincides* with the full subcategory of small objects of $\mathcal{D}\mathcal{A}$. In 5.3, we give A. Neeman's proof [17, 2.2] of Ravenel's result.

5.1 Homotopy limits and small objects. Let $\mathcal{T}$ be a triangulated category with (infinite) sums. Let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \to X_p \xrightarrow{f_p} X_{p+1} \to \ldots\,,\ p \in \mathbf{N}$$

be a sequence of morphisms of $\mathcal{T}$. Let there be given a *homotopy limit* of the sequence, i.e. an object X with morphisms $\psi_p : X_p \to X$ fitting into a triangle

$$\coprod X_p \xrightarrow{\Phi} \coprod X_q \xrightarrow{\Psi} X \to S \coprod X_p\,,$$

where Φ is defined as in 3.1 and Ψ has the components ψ_q. Note that a homotopy limit is unique up to non-unique isomorphism.

Let $M \in \mathcal{T}$ be small. Then $\mathcal{T}(M, ?)$ commutes with direct sums and thus transforms the above triangle into the long exact sequence

$$\ldots \to \coprod \mathcal{T}(M, X_p) \xrightarrow{\Phi_*} \coprod \mathcal{T}(M, X_q) \xrightarrow{\Psi_*} \mathcal{T}(M, X) \to \ldots$$

It is easy to see that $(S\Phi)_*$ is injective. We therefore have an isomorphism

$$\varinjlim \mathcal{T}(M, X_p) \xrightarrow{\sim} \operatorname{Cok} \Phi_* \xrightarrow{\sim} \mathcal{T}(M, X).$$

5.2 Brown's representability theorem. Keep the hypotheses of 5.1 and assume that $\mathcal{T}$ admits a set of small generators $\mathcal{X}$. For completeness we include a proof of the following

THEOREM. [3] *A cohomological functor $F : \mathcal{T} \to (\mathcal{A}b)^{\mathrm{op}}$ is representable iff it commutes with direct sums.*

REMARK. More precisely, the proof will show that each such F is represented by the homotopy limit of a sequence

$$X_0 \xrightarrow{f_0} X_1 \to \ldots \to X_p \xrightarrow{f_p} X_{p+1} \to \ldots\,,\ p \in \mathbf{N}\,,$$

where X_0 as well as the cone (=third corner of a triangle) over each f_p is an (infinite) sum of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$. In particular, each $M \in \mathcal{T}$ is the homotopy limit of such a sequence, as we see by taking $F = \mathcal{T}(?, M)$.

PROOF. We have to prove that the condition is sufficient. Let $\mathcal{X}^+$ be the class of direct sums of objects $S^n X$, $n \in \mathbf{Z}$, $X \in \mathcal{X}$. For each $M \in \mathcal{T}$ put $M^\wedge = \mathcal{T}(M, ?)$. Since $\mathcal{X}$ is a set, there is an $X_0 \in \mathcal{X}^+$ and a morphism $\pi_0 : X_0^\wedge \to F$ inducing a surjection

$$X_0^\wedge(S^n X) \to F S^n X$$

for all $X \in \mathcal{X}$, $n \in \mathbf{Z}$. We will inductively construct a sequence

$$X_0 \xrightarrow{f_0} X_1 \to \ldots \to X_p \xrightarrow{f_p} X_{p+1} \to \ldots\,,\ p \in \mathbf{N}\,,$$

and morphisms $\pi_{p+1} : X_{p+1}^\wedge \to F$ such that $\pi_{p+1} f_p^\wedge = \pi_p$. Suppose that for some $p \geq 0$ we have constructed X_p and π_p. Choose $Z_p \in \mathcal{X}^+$ admitting a morphism $\rho_p : Z_p \to X_p$ which induces a surjection

$$Z_p^\wedge(S^n X) \to \operatorname{Ker} \pi_p(S^n X)$$

for all $X \in \mathcal{X}$, $n \in \mathbf{Z}$. Define X_{p+1} by the triangle

$$Z_p \xrightarrow{\rho_p} X_p \xrightarrow{f_p} X_{p+1} \to SZ_p.$$

Since we have an exact sequence

$$FZ_p \xleftarrow{F\rho_p} FX_p \leftarrow FX_{p+1}$$

and by definition $\pi_p \rho_p^\wedge = 0$, we can choose $\pi_{p+1} : X_{p+1}^\wedge \to F$ such that $\pi_{p+1} f_p^\wedge = \pi_p$. Define X_∞ by the triangle

$$\coprod_{p \in \mathbf{N}} X_p \xrightarrow{\Phi} \coprod_{q \in \mathbf{N}} X_q \xrightarrow{\Psi} X_\infty \to S \coprod_{p \in \mathbf{N}} X_p \ ,$$

where Φ has the components

$$X_p \xrightarrow{[1 \ -f_p]^t} X_p \oplus X_{p+1} \xrightarrow{\text{can}} \coprod_{q \in \mathbf{N}} X_q .$$

Since $F : \mathcal{T} \to (\mathcal{A}b)^{\text{op}}$ commutes with direct sums, it takes sums of $\mathcal{T}$ to products of $\mathcal{A}b$. Thus we have an exact sequence

$$\prod_{p \in \mathbf{N}} FX_p \leftarrow \prod_{q \in \mathbf{N}} FX_q \leftarrow FX_\infty \ ,$$

which shows that there is a morphism $\pi_\infty : X_\infty^\wedge \to F$ such that $\pi_\infty \Psi_q^\wedge = \pi_q^\wedge$ for all $q \in \mathbf{N}$. By an easy diagram chase we see that π_∞ induces an isomorphism

$$\mathcal{T}(S^n X, X_\infty) \to FS^n X$$

for all $X \in \mathcal{X}$, $n \in \mathbf{Z}$. Since $\mathcal{X}$ generates $\mathcal{T}$, we can conclude that π_∞ is an isomorphism.

5.3 Small objects. Keep the hypotheses of 5.2. If $\mathcal{U}$ and $\mathcal{V}$ are classes of objects of $\mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the class of objects X occuring in a triangle

$$U \to X \to V \to SU$$

with $U \in \mathcal{U}$, $V \in \mathcal{V}$. The octahedral axiom implies that the operation $*$ is associative. The objects of $\mathcal{X} * \mathcal{X} * \ldots * \mathcal{X}$ (n factors) are called *extensions of length n of objects of $\mathcal{X}$*. The following theorem and its proof can be found in [17, 2.2].

THEOREM. [18] [17] *Each small object of $\mathcal{T}$ is a direct summand of an extension of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$.*

REMARKS. a) We will of course apply the theorem to the case where $\mathcal{T}$ is the derived category of a DG algebra $\mathcal{A}$ and where $\mathcal{X}$ consists of the free modules $A^\wedge$, $A \in \mathcal{A}$.

b) One can adapt the proof of [19, 6.3] to show that, if $\mathcal{A}$ is a negative DG category, i.e. $\mathcal{A}(A,B)^n = 0$ for all $n > 0$, $A, B \in \mathcal{A}$, then each small object of $\mathcal{DA}$ is an extension of $\mathcal{DA}$-direct summands of finite sums of free modules $A^\wedge$, $A \in \mathcal{A}$.

Proof. [17] Let M be a small object of $\mathcal{T}$. Choose a sequence

$$X_0 \xrightarrow{f_0} X_1 \to \ldots \to X_p \xrightarrow{f_p} X_{p+1} \to \ldots\,,\ p \in \mathbf{N}\,,$$

as in remark 5.2. By 5.1 we have an isomorphism

$$\varinjlim \mathcal{T}(M, X_p) \xrightarrow{\sim} \mathcal{T}(M, M).$$

In particular, the identity of M factors through some X_p, which means that M is a direct summand of X_p. Now X_p is an extension of sums of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$. So we can apply the following lemma to $Z' = 0$ and $Z = X_p$ to obtain the commutative square

$$\begin{array}{ccc} M' & \to & M \\ \downarrow & & \downarrow \\ 0 & \to & X_p\,, \end{array}$$

where the cone on the first line is an extension M'' of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$. Since $M \to X_p$ is a (split) monomorphism, the morphism $M' \to M$ vanishes and thus M is a direct summand of M''.

Lemma. [17, 2.3] *Let $M \in \mathcal{T}$ be small and let $c : Z' \to Z$ be a morphism whose mapping cone is an extension of (infinite) sums of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$. Then each diagram*

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ Z' & \xrightarrow{c} & Z \end{array}$$

may be completed to a commutative square

$$\begin{array}{ccc} M' & \xrightarrow{b} & M \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{c} & Z \end{array}$$

such that the cone over b is an extension of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$.

Proof. By assumption the cone Z'' over c is an extension of sums of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$. We proceed by induction on the length l of Z''. If we have $l = 1$, then Z'' is itself a sum of objects $S^n X$, $X \in \mathcal{X}$, $n \in \mathbf{Z}$. By the smallness of Y, the composition $M \to Z \to Z''$ factors through a finite subsum $M'' \subset Z''$. We find the required square by completing

$$\begin{array}{ccccccc} & & M & \to & M'' & & \\ & & \downarrow & & \downarrow & & \\ Z' & \xrightarrow{c} & Z & \to & Z'' & \to & SZ' \end{array}$$

to a morphism of triangles

$$\begin{array}{ccccccc} M' & \xrightarrow{b} & M & \to & M'' & \to & SM' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z' & \xrightarrow{c} & Z & \to & Z'' & \to & SZ'. \end{array}$$

If we have $l > 1$, then Z'' occurs in a triangle

$$Z''_0 \to Z'' \to Z''_1 \to SZ''_0$$

where both, Z''_0 and Z''_1, are of length $< l$. By forming an octahedron over

$$Z \to Z'' \to Z''_1$$

we see that c is the composition of two morphisms c_0 and c_1 whose cones are Z''_0 and Z''_1. By the induction hypothesis we have a commutative diagram

$$\begin{array}{ccccc} M' & \xrightarrow{b_0} & M_1 & \xrightarrow{b_1} & M \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \xrightarrow{c_0} & Z_1 & \xrightarrow{c_1} & Z \,, \end{array}$$

where the cones of b_0 and b_1 are extensions of objects of $\mathcal{X}$. By the octahedral axiom the same holds for $b = b_1 b_0$.

6. Standard functors

6.1 Hom and tensor. Let $\mathcal{A}$ and $\mathcal{B}$ be small DG categories. The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is the DG category whose objects are the pairs (A, B) of objects $A \in \mathcal{A}$, $B \in \mathcal{B}$, and whose morphism spaces are

$$(\mathcal{A} \otimes \mathcal{B})\,((A, B), (A', B')) \xrightarrow{\sim} \mathcal{A}\,(A, A') \otimes \mathcal{B}\,(B, B').$$

The *composition* of $\mathcal{A} \otimes \mathcal{B}$ is given by the formula

$$(f' \otimes g')(f \otimes g) = (-1)^{pq} f' f \otimes g' g$$

for $f \in \mathcal{A}\,(A, A')^p$ and $g' \in \mathcal{B}\,(B', B'')^q$.

Let X be an *$\mathcal{A}$-$\mathcal{B}$-bimodule*, i.e. a module over $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$. It gives rise to a pair of adjoint DG functors

$$\begin{array}{c} \mathrm{Dif}\,\mathcal{A} \\ T_X \uparrow\downarrow H_X \\ \mathrm{Dif}\,\mathcal{B} \end{array}$$

which are defined as follows

$$\begin{array}{rcl} (H_X M)(B) & = & (\mathrm{Dif}\,\mathcal{A})\,(X(?, B), M) \\ (T_X N)(A) & = & \mathrm{Cok}\,(\coprod_{B, C \in \mathcal{B}} NC \otimes \mathcal{B}\,(B, C) \otimes X(A, B) \xrightarrow{\nu} \coprod_{B \in \mathcal{B}} NB \otimes X(A, B))\,, \end{array}$$

where $\nu(n \otimes f \otimes x) = (Nn)(f) \otimes x - n \otimes X(A, f)(x)$. Observe that for each $B \in \mathcal{B}$ we have $T_X B^{\wedge} \xrightarrow{\sim} X(?, B)$ since

$$(\mathrm{Dif}\,\mathcal{A})\,(T_X B^{\wedge}, M) = (\mathrm{Dif}\,\mathcal{B})\,(B^{\wedge}, H_X M) = (H_X M)(B) = (\mathrm{Dif}\,\mathcal{A})\,(X(?, B), M)$$

for each $M \in \mathrm{Dif}\,\mathcal{A}$. For brevity, we put $X^B = X(?, B)$.

The functors H_X and T_X induce a pair of adjoint functors between $\mathcal{HA}$ and $\mathcal{HB}$ which will also be denoted by H_X and T_X. We denote by $\mathbf{L}T_X$ the *left derived functor* of T_X, i.e. the composition

$$\mathcal{DB} \to \mathcal{H}_p\mathcal{B} \xrightarrow{T_X} \mathcal{HA} \to \mathcal{DA}\ ,\ N \mapsto T_X\boldsymbol{p}N.$$

Observe that $\mathbf{L}T_X$ commutes with direct sums since $\boldsymbol{p}$ and T_X do.

LEMMA.

a) $\mathbf{L}T_X$ *is an equivalence iff the morphisms* $\mathcal{B}(B,C) \to (\mathrm{Dif}\,\mathcal{A})(X^B, X^C)$ *induce isomorphisms in homology,* $\forall\, B, C \in \mathcal{B}$*, and the* X^B*,* $B \in \mathcal{B}$*, form a set of small generators for* $\mathcal{DA}$*.*

b) A morphism $X \to X'$ *of* $\mathcal{A}$*-*$\mathcal{B}$*-bimodules is a quasi-isomorphism iff the induced morphism* $\mathbf{L}T_X \to \mathbf{L}T_{X'}$ *is invertible.*

c) Suppose that X *has property (P) over* $\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}$*. If* $\mathcal{A}$ *is* k*-flat, then* T_X *preserves acyclicity. If* $\mathcal{B}$ *is* k*-projective, then* T_X *preserves property (P). If* k *is a field then* $T_X N$ *has property (P) for each DG* $\mathcal{B}$*-module* N*.*

PROOF. a) follows from 4.2 c), and b) from 4.2 d). It suffices to prove c) for the case where $X = (A', B')^\wedge$ for some $(A', B') \in \mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}$. Then we have $T_X N = N(B') \otimes_k \mathcal{A}(A', ?)$. So the first two assertions are clear. To prove the last one, we fix an acyclic DG $\mathcal{A}$-module M and observe that

$$(\mathrm{Dif}\,\mathcal{A})(T_X N, M) \xrightarrow{\sim} (\mathrm{Dif}\,k)(N(B'), M(A')).$$

Since k is a field, $M(A')$ is even null-homotopic. Hence we have $(\mathcal{HA})(T_X N, M) = 0$, and the assertion follows from 3.1.

EXAMPLE. Let $F : \mathcal{B} \to \mathcal{A}$ be a DG functor and put $X(A, B) = \mathcal{A}(A, FB)$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$. Then clearly $X^B = (FB)^\wedge$. Hence $\mathbf{L}T_X$ is an equivalence iff $\mathrm{H}^*F : \mathrm{H}^*\mathcal{A} \to \mathrm{H}^*\mathcal{B}$ is an equivalence.

6.2 Right projective bimodules. We keep the assumptions of 6.1 and assume in addition that X^B has property (P) for each $B \in \mathcal{B}$. Since

$$(H_X M)(B) = (\mathrm{Dif}\,\mathcal{A})(X^B, M)\,,$$

it follows from theorem 3.1 that $H_X M$ is acyclic for each acyclic M. The induced functor $\mathcal{DA} \to \mathcal{DB}$ will be denoted by $\mathbf{R}H_X$. We have

$$(\mathcal{HA})(T_X P, M) = (\mathcal{HB})(P, H_X M) = 0$$

whenever P has property (P) and M is acyclic. By 3.1 we conclude that T_X preserves property (P). Using this we see that

$$(\mathcal{DA})(\mathbf{L}T_X N, M) = (\mathcal{HA})(T_X\boldsymbol{p}N, M) = (\mathcal{HB})(\boldsymbol{p}N, H_X M) = (\mathcal{DB})(N, \mathbf{R}H_X M)\,,$$

i.e. that $\mathbf{R}H_X$ is a right adjoint of $\mathbf{L}T_X$.

Now define a $\mathcal{B}$-$\mathcal{A}$-module $X^\top$ by

$$X^\top(B,A) = (\mathrm{Dif}\,\mathcal{A})\,(X^B, A^\wedge).$$

For each $M \in \mathrm{Dif}\,\mathcal{A}$, we have a canonical morphism $T_{X^\top}M \to H_X M$.

LEMMA.

a) The morphism $\mathbf{L}T_{X^\top}M \to \mathbf{R}H_X M$ is invertible for all $M \in \mathcal{H}_p^b\mathcal{A}$. It is invertible for all M iff the X^B are small in $\mathcal{D}\mathcal{A}$, $\forall\, B \in \mathcal{B}$.

b) If $\mathbf{L}T_X : \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{A}$ is an equivalence, its quasi-inverse is isomorphic to $\mathbf{L}T_{X^\top}$.

PROOF. a) The morphism is clearly invertible for free M. By 'devissage' it is invertible for $M \in \mathcal{H}_p^b\mathcal{A}$. Since H_X commutes with infinite direct sums iff the X^B are small, the second assertion follows from 4.2 d).

b) If $\mathbf{L}T_X$ is an equivalence then so is $\mathbf{R}H_X$. In particular, $\mathbf{R}H_X$ commutes with direct sums. The assertion now follows from a) and 4.2 d).

EXAMPLE. Keep the notations of example 6.1. If $\mathbf{L}T_X$ is an equivalence, a quasi-inverse is given by $\mathbf{L}T_{X^\top}$.

6.3 Flat targets. We keep the assumptions of 6.1 and assume in addition that $\mathcal{A}$ is *k-flat*, i.e. $\mathcal{A}(A,B)$ is a flat k-module, $\forall\, A, B \in \mathcal{A}$. Let $\boldsymbol{p}X$ be a P-resolution of X over $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$. Note that for $B \in \mathcal{B}$ the $\mathcal{A}$-module $(\boldsymbol{p}X)^B$ need not have property (P) (unless $\mathcal{B}(B', B)$ is projective over k for each $B' \in \mathcal{B}$). In particular, the canonical morphism $\boldsymbol{p}(X^B) \to (\boldsymbol{p}X)_B$ of $\mathcal{H}\mathcal{A}$ need not be a quasi-isomorphism.

LEMMA.

a) We have $\mathbf{L}T_X N \xrightarrow{\sim} T_{\boldsymbol{p}X}N$ for each $N \in \mathcal{D}\mathcal{B}$.

b) Let $\mathcal{C}$ be another DG category and Y a $\mathcal{B}$-$\mathcal{C}$-bimodule. We have $\mathbf{L}T_X\mathbf{L}T_Y \xrightarrow{\sim} \mathbf{L}T_Z$, where $Z = T_{\boldsymbol{p}X}Y$.

PROOF. a) By 6.1 b) we have $\mathbf{L}T_{\boldsymbol{p}X} \xrightarrow{\sim} \mathbf{L}T_X$. So we only have to show that $\mathbf{L}T_{\boldsymbol{p}X}N \xrightarrow{\sim} T_{\boldsymbol{p}X}N$ for each $N \in \mathcal{D}\mathcal{B}$. It is enough to check that $T_{\boldsymbol{p}X}N$ is acyclic for each acyclic N. Now $T_{\boldsymbol{p}X}N$ inherits from $\boldsymbol{p}X$ a complete filtration which splits in $\mathcal{G}\mathcal{A}$ and has subquotients $T_Q N$, where Q is relative projective. So it is enough to show that $T_F N$ is acyclic for each $F = (A', B')^\wedge$, $(A', B') \in \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}$. But

$$(T_F N)(A) \xrightarrow{\sim} \mathcal{A}(A, A') \otimes N(B').$$

b) follows from a) and the fact that $T_{\boldsymbol{p}X}T_Y \xrightarrow{\sim} T_Z$ as functors $\mathrm{Dif}\,\mathcal{C} \to \mathrm{Dif}\,\mathcal{A}$.

6.4 Tensor functors and DG functors. Let $\mathcal{A}$ and $\mathcal{B}$ be small DG categories. Let $F : \operatorname{Dif}\mathcal{B} \to \operatorname{Dif}\mathcal{A}$ be an arbitrary DG functor. Its *left derived functor* is the composition

$$\mathcal{DB} \to \mathcal{H}_p\mathcal{B} \xrightarrow{F} \mathcal{HA} \to \mathcal{DA}\ ,\ N \mapsto F\boldsymbol{p}N$$

Let X be the bimodule $X(A,B) = (FB^\wedge)(A) = (\operatorname{Dif}\mathcal{A})(A^\wedge, FB^\wedge)$. For each $\mathcal{B}$-module N, the canonical morphism

$$NB \xrightarrow{\sim} (\operatorname{Dif}\mathcal{B})(B^\wedge, N) \to (\operatorname{Dif}\mathcal{A})(FB^\wedge, FN) = (\operatorname{Dif}\mathcal{A})(X(?,B), FN) = (H_X FN)(B)$$

comes from a natural morphism $N \to H_X FN$. By adjunction, we obtain $T_X N \to FN$. The induced morphism

$$\mathbf{L}T_X N \to \mathbf{L}FN$$

is clearly invertible for $N = B^\wedge[n]$, $B \in \mathcal{B}$, $n \in \mathbf{Z}$. This implies the first assertion of the following lemma. The second one follows from lemma 4.2.

Lemma. *The canonical morphism*

$$\mathbf{L}T_X N \to \mathbf{L}FN$$

is invertible for each $N \in \mathcal{H}_p^b\mathcal{B}$. It is invertible for all $N \in \mathcal{DB}$ iff $\mathbf{L}F$ commutes with direct sums.

6.5 Example: Lie algebra cohomology. Let R be a k-algebra with 1 and L a (k,R)-Lie algebra [21, §2], i.e. L is a Lie algebra over R, and R is endowed with a left L-module structure such that

$$[X, rY] = (Xr)Y + r[X,Y]$$

for all $X, Y \in L$, $r \in R$. In addition, we assume that L is projective as an R-module. For example this holds for the $(\mathbf{R}, C^\infty(M))$-Lie algebra formed by the C^∞-vector fields on a C^∞-manifold M [21, §4]. Let the Lie algebra Z be the semi-direct product of L by R and let A be the 'universal algebra of differential operators generated by R and L': A is an associative k-algebra endowed with a k-linear morphism $\iota : Z \to A$ which is universal for the properties

$$\iota([U,V]) = [\iota(U), \iota(V)] \text{ and } \iota(rU) = \iota(r)\iota(U)$$

for all $U, V \in Z$, $r \in R$. The canonical Z-action on R uniquely extends to an A-module structure. Let ε denote the map $A \to R, a \mapsto a.1$.

Let E be the graded exterior R-algebra over L and let X be the differential complex with components $X^n = A \otimes_R E^{-n}$ and the differential [21, §4]

$$\begin{aligned} d(a \otimes X_1 \wedge \ldots \wedge X_n) &= \sum_{i=1}^{n} (-1)^{i-1} a X_i \otimes X_1 \wedge \ldots \widehat{X_i} \ldots \wedge X_n \\ &\quad + \sum_{j<k} (-1)^{j+k} a \otimes [X_j, X_k] \wedge X_1 \wedge \ldots \widehat{X_j} \ldots \widehat{X_k} \ldots \wedge X_n . \end{aligned}$$

The complex X together with the augmentation $\varepsilon : X^0 \to R$ is a projective resolution of the left A-module R [21, §4]. The corresponding quasi-isomorphism $X \to R$ will also be denoted by ε.

Let B be the DG R-module $(\mathrm{Dif}\, A^{\mathrm{op}})\,(X, R)$. We will freely make use of the identifications

$$B = (\mathrm{Dif}\, A^{\mathrm{op}})\,(X, R) = \mathrm{Hom}_A(A \otimes_R E, R) = \mathrm{Hom}_R(E, R).$$

Endowed with the 'shuffle product' B becomes a DG algebra [10, §9] : Recall that for $f \in B^p$, $g \in B^q$, and $n = p + q$, one puts

$$(fg)(X_1 \wedge \ldots \wedge X_n) = \sum \sigma_{ij} f(X_{i_1}, \ldots, X_{i_p})\, g(X_{j_1}, \ldots, X_{j_q})\,,$$

where σ_{ij} is the parity of the permutation

$$1 \mapsto i_1\,,\ \ldots\,,\ p \mapsto i_p\,,\ p+1 \mapsto j_1\,,\ \ldots\,,\ p+q \mapsto j_q\,,$$

and the sum ranges over all tuples i, j with $i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$ and $\{1, \ldots, p+q\} = \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\}$.

Let $f \in B^p$. We define a DG left B-module structure on X by putting $f.(a \otimes X_1 \wedge \ldots \wedge X_n) = 0$ for $p > n$ and, with the same notations as for the shuffle product,

$$f(a \otimes X_1 \wedge \ldots \wedge X_n) = \sum \sigma_{ij}\, a \otimes f(X_{i_1}, \ldots, X_{i_p})\, X_{j_1} \wedge \ldots \wedge X_{j_q}$$

for $p < n$ and $p + q = n$. It is clear that the actions of A and B on X commute among each other and agree on R so that X becomes an A^{op}-B-bimodule. Note that $X|\,A^{\mathrm{op}}$ has property (P) (3.1).

Lemma.

a) The functors $\mathbf{L}T_X : \mathcal{D}B \to \mathcal{D}A^{\mathrm{op}}$ and $\mathbf{R}H_X$ induce quasi-inverse S-equivalences between $\mathcal{H}_p^b B$ and the full triangulated subcategory of $\mathcal{D}A^{\mathrm{op}}$ generated by R.

b) If L is finitely generated over R, then $\mathbf{L}T_X : \mathcal{D}B \to \mathcal{D}A^{\mathrm{op}}$ is fully faithful and $\mathbf{R}H_X \xleftarrow{\sim} \mathbf{L}T_{X^\top}$.

Proof. a) By 4.2 a) we have to check that the morphism of complexes

$$\lambda : B \to (\mathrm{Dif}\, A^{\mathrm{op}})\,(X, X)$$

mapping f to left multiplication by f is a quasi-isomorphism. By definition the composition of λ with

$$\varepsilon_* : (\mathrm{Dif}\, A^{\mathrm{op}})\,(X, X) \to (\mathrm{Dif}\, A^{\mathrm{op}})\,(X, R)$$

is the identity. Since $\varepsilon : X \to R$ is a quasi-isomorphism and X has property (P), ε_* is a quasi-isomorphism. Hence so is λ.

b) If L is finitely generated, $X|\,A^{\mathrm{op}}$ is a bounded complex of finitely generated projective A-modules. In particular, X is small in $\mathcal{D}A^{\mathrm{op}}$. The assertion now follows from 4.2 b) and 6.2 a).

6.6 Example: Bar resolution. Let $\mathcal{A}$ be a small DG category. Let $\widetilde{Y}$ be the bar resolution [4, IX, §6] of $\mathcal{A}$, i.e. the complex of $\mathcal{A}$-$\mathcal{A}$-bimodules with $\widetilde{Y}(A,B)^n = 0$ for $n > 0$ and

$$\widetilde{Y}^{-n}(A,C) = \coprod_{B_0,\ldots,B_n} \mathcal{A}(B_0,C) \otimes \mathcal{A}(B_1,B_0) \otimes \ldots \otimes \mathcal{A}(B_n,B_{n-1}) \otimes \mathcal{A}(A,B_n)\,,\ n \geq 0$$

endowed with the differential d of degree 1 with

$$d(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}$$

Let Y be the total module of $\widetilde{Y}$ (cf. the proof of 3.1). Define I to be the $\mathcal{A}$-$\mathcal{A}$-bimodule $I(A,B) = \mathcal{A}(A,B)$. By [4, IX, §6] we have a quasi-isomorphism $\varepsilon : Y \to I$ induced by the composition map

$$\coprod_{B_0} \mathcal{A}(B_0,C) \otimes \mathcal{A}(A,B_0) \to \mathcal{A}(A,C).$$

The maps

$$\widetilde{Y}^{-n} \to \coprod_{p+q=n} \widetilde{Y}^{-p} \otimes \widetilde{Y}^{-q}$$

given by

$$a_0 \otimes \ldots \otimes a_{n+1} \mapsto (a_0 \otimes \ldots \otimes a_p \otimes 1 \otimes 1 \otimes a_{p+1} \otimes \ldots \otimes a_{n+1})$$

yield a morphism

$$\Delta : Y \to Y \circ Y,$$

where by definition $? \circ Y = T_Y$. We have commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \circ Y \\ \| & & \downarrow Y \circ \varepsilon \\ Y & \xrightarrow{\mathrm{can}} & Y \circ I \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \circ Y \\ \| & & \downarrow \varepsilon \circ Y \\ Y & \xrightarrow{\mathrm{can}} & I \circ Y \end{array} \qquad \begin{array}{ccccc} Y & \xrightarrow{\Delta} & Y \circ Y & \xrightarrow{Y \circ \Delta} & Y \circ (Y \circ Y) \\ \| & & & & \downarrow \mathrm{can} \\ Y & \xrightarrow{\Delta} & Y \circ Y & \xrightarrow{\Delta \circ Y} & (Y \circ Y) \circ Y. \end{array}$$

Now let $\mathcal{B}$ be a set of DG $\mathcal{A}$-modules. The above diagrams ensure that we can make $\mathcal{B}$ into a DG category by requiring that

$$\mathcal{B}(L,M) \xrightarrow{\sim} (\mathrm{Dif}\,\mathcal{A})(Y \circ L, M)\,,$$

that the identity $\mathbf{1}_L^{\mathcal{B}}$ corresponds to the composition

$$Y \circ L \xrightarrow{\varepsilon \circ L} I \circ L \xrightarrow{\mathrm{can}} L\,,$$

and that the composition of two morphisms of $\mathcal{B}$ coming from $g : Y \circ L \to M$ and $f : Y \circ M \to N$ is given by the composition

$$Y \circ L \xrightarrow{\Delta \circ L} (Y \circ Y) \circ L \xrightarrow{\mathrm{can}} Y \circ (Y \circ L) \xrightarrow{Y \circ g} Y \circ M \xrightarrow{f} N.$$

We then have a canonical $\mathcal{A}$-$\mathcal{B}$-bimodule $X(A,L) := (Y \circ L)(A)$, where the action of $g : Y \circ L \to M$ is given by the composition

$$Y \circ L \xrightarrow{\Delta \circ L} (Y \circ Y) \circ L \xrightarrow{\mathrm{can}} Y \circ (Y \circ Y) \xrightarrow{Y \circ g} Y \circ M.$$

Now *suppose that k is a field.* Then each $\widetilde{Y}^n$ is relatively projective over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$. Since Y admits the filtration $F^p = \coprod_{n \geq -p} \widetilde{Y}^n$, it has property (P) over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$. Using 6.1 b) and c) we infer that the composition η

$$Y \circ M \xrightarrow{\varepsilon \circ M} I \circ M \xrightarrow{\mathrm{can}} M$$

is a P-resolution for each DG $\mathcal{A}$-module M. Therefore the morphism

$$\eta_* : (\mathrm{Dif}\,\mathcal{A})\,(Y \circ L, Y \circ M) \to (\mathrm{Dif}\,\mathcal{A})\,(Y \circ L, M)\;,\; L, M \in \mathcal{B}\,,$$

is a quasi-isomorphism. And so is the canonical morphism

$$\mathcal{B}\,(L, M) \to (\mathrm{Dif}\,\mathcal{A})\,(X^L, X^M) = (\mathrm{Dif}\,\mathcal{A})\,(Y \circ L, Y \circ M)$$

since it has η_* as a left inverse. Using 4.2 we infer the

LEMMA.

a) The restriction of $\mathbf{L}T_X$ to $\mathcal{H}_p^b\mathcal{B}$ is fully faithful.

b) If each $L \in \mathcal{B}$ is small in $\mathcal{D}\mathcal{A}$, then $\mathbf{L}T_X$ is fully faithful.

c) $\mathbf{L}T_X$ is an equivalence iff the objects of $\mathcal{B}$ form a set of small generators for $\mathcal{D}\mathcal{A}$.

7. Quasi-functors and lifts

7.1 Quasi-functors. Let $\mathcal{A}$ and $\mathcal{B}$ be small DG categories. Denote by $\underline{\mathcal{A}}$ the full subcategory of $\mathcal{D}\mathcal{A}$ whose objects are the $A\hat{}$, $A \in \mathcal{A}$, and by $\mathbf{Z}\underline{\mathcal{A}}$ the full subcategory whose objects are the $A\hat{}[n]$, $n \in \mathbf{Z}$, $A \in \mathcal{A}$. Note that we have

$$(\mathbf{Z}\underline{\mathcal{A}})\,(A\hat{}[n], B\hat{}[m]) = \mathrm{H}^{m-n}\mathcal{A}\,(A, B)$$

for all $A, B \in \mathcal{A}$, $n, m \in \mathbf{Z}$.

Let X be an $\mathcal{A}$-$\mathcal{B}$-bimodule. By definition, X is a *quasi-functor* $\mathcal{B} \to \mathcal{A}$ if it satisfies the conditions of the following lemma. Note that in this case $\mathbf{L}T_X$ gives rise to a functor $\mathbf{Z}\underline{\mathcal{B}} \to \mathbf{Z}\underline{\mathcal{A}}$ and hence to a functor $\mathrm{H}^*\mathcal{B} \to \mathrm{H}^*\mathcal{A}$.

LEMMA. *The following are equivalent*

i) $\mathbf{L}T_X$ gives rise to a functor $\underline{\mathcal{B}} \to \underline{\mathcal{A}}$.

ii) For each $B \in \mathcal{B}$ the functor $(\mathcal{D}\mathcal{A})\,(?, X^B)$ is representable by an object of $\underline{\mathcal{A}}$.

iii) For each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ and an element $x_B \in \mathrm{Z}^0X(A, B)$ such that for each $A' \in \mathcal{A}$ the morphism

$$\mathcal{A}\,(A', A) \to X(A', B)\;,\; f \mapsto X(f, B)(x_B)$$

induces isomorphisms in homology.

PROOF. Exercise.

Suppose for example that $\mathcal{A}$ and $\mathcal{B}$ are concentrated in degree 0. Then $\mathcal{A}^0$ is equivalent to $\underline{\mathcal{A}}$. Thus by i), a quasi-functor X yields a functor $F^0 : \mathcal{B}^0 \to \mathcal{A}^0$; hence a functor $F : \mathcal{B} \to \mathcal{A}$. It is easy to see that in $\mathcal{D}(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}})$, X is isomorphic to the bimodule $(A, B) \mapsto \mathcal{A}(A, FB)$.

7.2 Quasi-equivalences. Keep the hypotheses of 7.1. By definition, X is a *quasi-equivalence* if the conditions of the following lemma hold. In this case $\mathcal{B}$ is *quasi-equivalent* to $\mathcal{A}$.

LEMMA. *The following are equivalent*

i) $\mathbf{L}T_X$ *is an equivalence giving rise to an equivalence* $\underline{\mathcal{B}} \to \underline{\mathcal{A}}$.

ii) $\mathbf{L}T_X$ *gives rise to equivalences* $\mathbf{Z}\underline{\mathcal{B}} \to \mathbf{Z}\underline{\mathcal{A}}$ *and* $\underline{\mathcal{B}} \to \underline{\mathcal{A}}$.

iii) There is a subset $D \subset \mathcal{A} \times \mathcal{B}$ *projecting onto* $\mathcal{A}$ *as well as onto* $\mathcal{B}$, *and for each* $(A, B) \in D$ *there is an element* $x_{AB} \in \mathrm{Z}^0 X(A, B)$ *such that the morphisms*

$$\mathcal{A}(A', A) \to X(A', B) \quad , \quad f \mapsto X(f, B)(x_{AB})$$
$$\mathcal{B}(B, B') \to X(A, B') \quad , \quad g \mapsto X(A, g)(x_{AB})$$

induce isomorphisms in homology for each $A' \in \mathcal{A}$, $B' \in \mathcal{B}$.

PROOF. Exercise.

EXAMPLE. Each DG functor $F : \mathcal{B} \to \mathcal{A}$ inducing an equivalence $\mathrm{H}^* F : \mathrm{H}^*\mathcal{B} \to \mathrm{H}^*\mathcal{A}$ yields a quasi-equivalence $X(A, B) = \mathcal{A}(A, FB)$. If $\mathcal{A}$ and $\mathcal{B}$ are concentrated in degree 0, each quasi-equivalence comes from an equivalence $F : \mathcal{B} \to \mathcal{A}$.

REMARK. If k is a field, 'quasi-equivalence' is an equivalence relation (6.1 c and 6.2 b imply reflexivity; 6.3 b implies transitivity).

7.3 Lifts. Let $\mathcal{A}$ be a small DG category. Let $\mathcal{U} \subset \mathcal{D}\mathcal{A}$ be a full small subcategory and $\mathbf{Z}\mathcal{U} \subset \mathcal{D}\mathcal{A}$ the full subcategory whose objects are the $U[n]$, $U \in \mathcal{U}$, $n \in \mathbf{Z}$. A *lift* of $\mathcal{U}$ is a DG category $\mathcal{B}$ together with an $\mathcal{A}$-$\mathcal{B}$-bimodule X such that $\mathbf{L}T_X$ gives rise to equivalences $\mathbf{Z}\underline{\mathcal{B}} \xrightarrow{\sim} \mathbf{Z}\mathcal{U}$ and $\underline{\mathcal{B}} \xrightarrow{\sim} \mathcal{U}$.

$$\begin{array}{ccc} \mathbf{Z}\underline{\mathcal{B}} & \xrightarrow{\sim} & \mathbf{Z}\mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{D}\mathcal{B} & \xrightarrow{\mathbf{L}T_X} & \mathcal{D}\mathcal{A}. \end{array}$$

EXAMPLES. With the notations of 6.5, (B, X) is a lift of $\mathcal{U} = \{R\}$. — If k is a field, any $\mathcal{U} \subset \mathcal{D}\mathcal{A}$ may be lifted using the bar resolution of 6.6.

The definition of a lift implies in particular that $\mathbf{L}T_X$ induces an equivalence from $\mathcal{H}_p^b\mathcal{B}$ onto the triangulated subcategory of $\mathcal{D}\mathcal{A}$ generated by $\mathcal{U}$ (4.2 a). If X^B has property (P) for each $B \in \mathcal{B}$, a quasi-inverse is induced by $\mathbf{R}H_X$. Indeed, if $M \in \mathcal{H}_p^b\mathcal{B}$, we have

$$(\mathcal{D}\mathcal{B})\,(S^n B^\wedge, \mathbf{R}H_X \mathbf{L}T_X M) \xrightarrow{\sim} (\mathcal{D}\mathcal{A})\,(\mathbf{L}T_X S^n B^\wedge, \mathbf{L}T_X M) \xleftarrow{\sim} (\mathcal{D}\mathcal{B})\,(S^n B^\wedge, M)$$

since $\mathbf{L}T_X$ is fully faithful on $\mathcal{H}_p^b\mathcal{B}$. This means that $\mathbf{R}H_X\mathbf{L}T_X M \leftarrow M$ is invertible.

We see from 6.1 that $\mathbf{L}T_X$ is itself an equivalence iff the objects of $\mathcal{U}$ form a system of small generators for $\mathcal{D}\mathcal{A}$.

If $\mathcal{U}$ is given, we can always construct a *standard lift* by taking $\mathcal{B}$ to be the full subcategory of $\mathrm{Dif}\,\mathcal{A}$ formed by chosen objects $\boldsymbol{p}U$, $U \in \mathcal{U}$, and X to be the bimodule

$$(A, \boldsymbol{p}U) \mapsto (\boldsymbol{p}U)(A)\,,\ \boldsymbol{p}U \in \mathcal{B}, A \in \mathcal{A}.$$

Now let $(\mathcal{B}, X)$ be any lift of $\mathcal{U}$ such that X^B has property (P) for each $B \in \mathcal{B}$. Let $\mathcal{C}$ be a DG category and $F : \mathrm{Dif}\,\mathcal{C} \to \mathrm{Dif}\,\mathcal{A}$ a DG functor such that $\mathbf{L}F : \mathcal{D}\mathcal{C} \to \mathcal{D}\mathcal{A}$ induces a functor $\underline{\mathcal{C}} \to \mathcal{U}$.

$$\begin{array}{ccccc} & & \mathcal{D}\mathcal{C} & \leftarrow & \underline{\mathcal{C}} \\ & & \downarrow \mathbf{L}F & & \downarrow \\ \mathcal{D}\mathcal{B} & \xrightarrow{\mathbf{L}T_X} & \mathcal{D}\mathcal{A} & \leftarrow & \mathcal{U} \\ \uparrow & & \uparrow & & \\ \underline{\mathcal{B}} & \rightarrow & \mathcal{U} & & \end{array}$$

Lemma. *Put* $Y(B, C) = (H_X F C^\wedge)(B)$.

a) $\mathbf{L}T_Y$ *induces a functor* $\underline{\mathcal{C}} \to \underline{\mathcal{B}}$ *; hence* Y *is a quasi-functor. It is a quasi-equivalence if* $\mathbf{L}F$ *induces an equivalence* $\mathbf{Z}\underline{\mathcal{C}} \to \mathbf{Z}\mathcal{U}$.

b) *There is a canonical morphism*

$$\mathbf{L}T_X\mathbf{L}T_Y M \to \mathbf{L}FM\ ,$$

which is invertible for $M \in \mathcal{H}_p^b\mathcal{C}$. *It is invertible for arbitrary* $M \in \mathcal{D}\mathcal{C}$ *iff* $\mathbf{L}F$ *commutes with direct sums.*

c) *If* $(\mathcal{C}, Z)$ *is a lift of* $\mathcal{U}$ *and* $F = T_Z$, *then* Y *is a quasi-equivalence* $\mathcal{C} \to \mathcal{B}$ *and we have* $\mathbf{L}T_X\mathbf{L}T_Y \xrightarrow{\sim} \mathbf{L}T_Z$. *If moreover* Z_C *has property (P) for each* $C \in \mathcal{C}$, *then* $\mathbf{R}H_Y\mathbf{R}H_X \xrightarrow{\sim} \mathbf{R}H_Z$.

Remark. In 10.3 we will need the following fact. Suppose that F, T_X and T_Y all preserve acyclicity so that their derived functors are isomorphic to the functors induced by them. Then the morphism of b) is produced by the composition

$$T_X T_Y \xrightarrow{T_X\alpha} T_X H_X F \xrightarrow{\Phi F} F$$

which is even defined as a morphism of DG functors. Here $\alpha : T_Y \to H_X F$ denotes the canonical morphism constructed in 6.4, and Φ the adjunction morphism.

PROOF. a) Consider the functor $G = H_X \circ F : \mathrm{Dif}\mathcal{C} \to \mathrm{Dif}\mathcal{B}$. We have $\mathbf{L}G = \mathbf{R}H_X \mathbf{L}F$. So $\mathbf{L}G$ induces a functor $\underline{\mathcal{C}} \to \underline{\mathcal{B}}$. By definition we have $Y(B,C) = (GC^\wedge)(B)$. Hence we have a morphism $T_Y \to G$ such that $\mathbf{L}T_Y M \to \mathbf{L}GM$ is invertible for each $M \in \mathcal{H}_p^b\mathcal{C}$ (6.4). So $\mathbf{L}T_Y$ induces a functor $\underline{\mathcal{C}} \to \underline{\mathcal{B}}$. We have morphisms

$$\mathbf{L}T_X\mathbf{L}T_Y \to \mathbf{L}T_X\mathbf{L}G = \mathbf{L}T_X\mathbf{R}H_X\mathbf{L}F \to \mathbf{L}F$$

which are invertible on $\mathcal{H}_p^b\mathcal{C}$. Thus $\mathbf{L}T_X$ induces an equivalence $\mathbf{Z}\underline{\mathcal{C}} \to \mathbf{Z}\underline{\mathcal{B}}$ iff $\mathbf{L}F$ induces an equivalence $\mathbf{Z}\underline{\mathcal{C}} \to \mathbf{Z}\mathcal{U}$. The second assertion now follows from 7.2.

b) follows from the proof of a) and 4.2 d).

The first two assertions of c) are immediate form a) and b). The last assertion is clear since if $\mathbf{L}T_Y$ is an equivalence and $\mathbf{L}T_X\mathbf{L}T_Y \xrightarrow{\sim} \mathbf{L}T_Z$, then $\mathbf{R}H_Y\mathbf{R}H_X$ is right adjoint to $\mathbf{L}T_Z$.

7.4 On the unicity of lifts. Keep the hypotheses of 7.3 and assume in addition that $\mathcal{A}$ is k-flat. Since X^B has property (P), $\forall\, B \in \mathcal{B}$, we have a well defined pair of adjoint functors

$$H_X^! : \mathcal{D}(\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}}) \to \mathcal{D}(\mathcal{B} \otimes \mathcal{C}^{\mathrm{op}}) \quad , \quad Z \mapsto H_X Z$$
$$T_X^! : \mathcal{D}(\mathcal{B} \otimes \mathcal{C}^{\mathrm{op}}) \to \mathcal{D}(\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}}) \quad , \quad Y \mapsto T_X\boldsymbol{p}Y$$

LEMMA. *For each $Y \in \mathcal{D}(\mathcal{B} \otimes \mathcal{C}^{\mathrm{op}})$ we have*

$$\mathbf{L}T_X\mathbf{L}T_Y \xrightarrow{\sim} \mathbf{L}T_Z\ ,$$

where $Z = T_X^! Y$. Moreover $T_X^!$ induces an equivalence between the full subcategories

$$\{Y : \mathbf{L}T_Y \text{ gives rise to a functor } \underline{\mathcal{C}} \to \underline{\mathcal{B}}\} \subset \mathcal{D}(\mathcal{B} \otimes \mathcal{C}^{\mathrm{op}})$$
$$\text{and } \{Z : \mathbf{L}T_Z \text{ gives rise to a functor } \underline{\mathcal{C}} \to \mathcal{U}\} \subset \mathcal{D}(\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}}).$$

PROOF. We have $T_X\boldsymbol{p}Y \xleftarrow{\sim} T_{\boldsymbol{p}X}\boldsymbol{p}Y$ by 6.1 b) and $T_{\boldsymbol{p}X}\boldsymbol{p}Y \xrightarrow{\sim} T_{\boldsymbol{p}X}Y$ by the k-flatness of $\mathcal{A}$ (6.3 a). So we have $T_X^! Y \xrightarrow{\sim} T_{\boldsymbol{p}X}Y$. By 6.3 b) this implies the first assertion. Since $\mathbf{L}T_X$ gives rise to a functor $\underline{\mathcal{B}} \to \mathcal{U}$, we infer that $T_X^!$ induces indeed a functor between the given subcategories. Suppose that $\mathbf{L}T_Y$ gives rise to a functor $\underline{\mathcal{C}} \to \underline{\mathcal{B}}$. We have to show that the canonical morphism $\boldsymbol{p}Y \to H_X T_X \boldsymbol{p}Y$ of $\mathcal{H}(\mathcal{B} \otimes \mathcal{C}^{\mathrm{op}})$ is a quasi-isomorphism. But we have already seen that $H_X T_X \boldsymbol{p}Y \xrightarrow{\sim} H_X T_{\boldsymbol{p}X} Y$, and on the other hand, for each $B \in \mathcal{B}$, we have

$$(\boldsymbol{p}Y)_B \xrightarrow{\sim} Y_B \xrightarrow{\sim} H_X T_X \boldsymbol{p}(Y_B) \xrightarrow{\sim} H_X T_{\boldsymbol{p}X} Y_B\ ,$$

where we use 6.3 a) for the third isomorphism and the fact that $Y_B \in \mathcal{U}$ for the second one. Now suppose that $\mathbf{L}T_Z$ gives rise to a functor $\underline{\mathcal{C}} \to \underline{\mathcal{A}}$. We have to show that the canonical morphism $T_X\boldsymbol{p}(H_X Z) \to Z$ of $\mathcal{D}(\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}})$ is invertible. As above we have $T_X\boldsymbol{p}(H_X Z) \xrightarrow{\sim} T_{\boldsymbol{p}X} H_X Z$ and

$$Z_C \xleftarrow{\sim} T_X\boldsymbol{p}H_X Z_C \xleftarrow{\sim} T_{\boldsymbol{p}X} H_X Z_C\ ,$$

where we use $Z_C \in \mathcal{U}$ for the first isomorphism and 6.3 a) for the second one.

8. Application: Derived equivalences

8.1 Arbitrary targets. Let $\mathcal{A}$ and $\mathcal{C}$ be small DG categories.

THEOREM. *Assertion i) implies ii), and ii) implies iii).*

i) There is a DG functor $H : \mathrm{Dif}\mathcal{C} \to \mathrm{Dif}\mathcal{A}$ *such that* $\mathbf{L}H : \mathcal{DC} \to \mathcal{DA}$ *is an equivalence.*

ii) $\mathcal{C}$ *is quasi-equivalent to a full DG subcategory* $\mathcal{B}$ *of* $\mathrm{Dif}\mathcal{A}$ *whose objects have property (P) and form a set of small generators for* $\mathcal{DA}$.

iii) There are a DG category $\mathcal{B}$ *and DG functors*

$$\mathrm{Dif}\mathcal{C} \xrightarrow{G} \mathrm{Dif}\mathcal{B} \xrightarrow{F} \mathrm{Dif}\mathcal{A}$$

such that $\mathbf{L}G$ *and* $\mathbf{L}F$ *are equivalences.*

PROOF. *i) implies ii):* By 6.4 we have $\mathbf{L}H \xrightarrow{\sim} \mathbf{L}T_Z$ for some $\mathcal{A}$-$\mathcal{C}$-bimodule Z. So $(\mathcal{C}, Z)$ is a lift of $\mathcal{U} = \{\mathbf{L}HC^\wedge : C \in \mathcal{C}\}$. Take $\mathcal{B}$ to be a standard lift of $\mathcal{U}$. The assertion then follows from 7.3 c) and 4.2 c).

ii) implies iii): By 7.2 we have an equivalence $\mathbf{L}T_X : \mathcal{DC} \to \mathcal{DB}$ and by 7.3 an equivalence $\mathbf{L}F : \mathcal{DB} \to \mathcal{DA}$.

8.2 Flat targets. Let $\mathcal{A}$ and $\mathcal{C}$ be small DG categories and assume that $\mathcal{A}$ is k-flat.

THEOREM. *The following are equivalent*

i) There is an $\mathcal{A}$-$\mathcal{C}$*-bimodule* X *such that* $\mathbf{L}T_X : \mathcal{DC} \to \mathcal{DA}$ *is an equivalence.*

ii) $\mathcal{C}$ *is quasi-equivalent to a full DG subcategory* $\mathcal{B}$ *of* $\mathrm{Dif}\mathcal{A}$ *whose objects have property (P) and form a set of small generators for* $\mathcal{DA}$.

PROOF. i) implies ii) by 8.1. Conversely, ii) implies i) by 8.1 iii), 6.4 and 6.3 b).

REMARK. Recall from section 5 that a DG module is small in $\mathcal{DA}$ iff it is contained in the smallest strictly full triangulated subcategory of $\mathcal{DA}$ containing the free modules and closed under forming direct summands.

9. Application: Stalk categories

9.1 Modules over $\mathrm{H}^0\mathcal{A}$. Let $\mathcal{A}$ be a small DG category. Let $\mathrm{H}^0\mathcal{A}$ (resp. $\tau^{\leq 0}\mathcal{A}$) be the DG category with the same objects as $\mathcal{A}$ and with the morphism spaces

$$(\mathrm{H}^0\mathcal{A})(A,B) = \mathrm{H}^0\mathcal{A}(A,B)\,,\ A,B \in \mathcal{A},$$

viewed as DG k-modules concentrated in degree 0 (resp.

$$(\tau^{\leq 0}\mathcal{A})(A,B) = \tau^{\leq 0}\mathcal{A}(A,B)\,,\ A,B\in\mathcal{A},$$

where $\tau^{\leq 0}K$ denotes the subcomplex C of K with $C^n = 0$ for $n>0$, $C^0 = Z^0K$, and $C^n = K^n$ for $n<0$). We have the obvious functors

$$\mathrm{H}^0\mathcal{A} \xleftarrow{\pi} \tau^{\leq 0}\mathcal{A} \xrightarrow{\iota} \mathcal{A}.$$

As in example 6.1, they yield functors

$$\mathcal{D}\mathrm{H}^0\mathcal{A} \xleftarrow{\mathbf{L}T_X} \mathcal{D}\tau^{\leq 0}\mathcal{A} \xrightarrow{\mathbf{L}T_Y} \mathcal{D}\mathcal{A}\,,$$

where $X(A,B) = (\mathrm{H}^0\mathcal{A})(A,\pi B)$ and $Y(A,B) = \mathcal{A}(A,\iota B)$. The functor $\mathbf{L}T_X$ is an equivalence iff $\mathcal{A}$ satisfies the 'Toda-condition' (cf. [22])

$$\mathrm{H}^n\mathcal{A}(A,B) = 0\,,\ \forall\, n<0\,,\ \forall\, A,B\in\mathcal{A}.$$

In this case (example 6.2), we have a canonical functor from $\mathcal{D}\mathrm{H}^0\mathcal{A}$ to $\mathcal{D}\mathcal{A}$ given simply by the composition

$$\mathcal{D}\mathrm{H}^0\mathcal{A} \xrightarrow{\mathbf{L}T_{X^\top}} \mathcal{D}\tau^{\leq 0}\mathcal{A} \xrightarrow{\mathbf{L}T_Y} \mathcal{D}\mathcal{A}.$$

If $\mathcal{A}$ is k-flat, this simplifies to

$$\mathcal{D}\mathrm{H}^0\mathcal{A} \xrightarrow{\mathbf{L}T_Z} \mathcal{D}\mathcal{A}\,,$$

where Z is the $\mathcal{A}$-$\mathrm{H}^0\mathcal{A}$-bimodule $T_{\boldsymbol{p}Y}X^\top$ (6.3 b).

9.2 Equivalences. Let $\mathcal{B}$ be a small k-linear category. We identify $\mathcal{B}$ with a DG category concentrated in degree 0. Let $\mathcal{A}$ be an arbitrary small DG category.

THEOREM. (cf. [19], [12]) *The following are equivalent*

i) There are DG categories $\mathcal{A}_1$, $\mathcal{A}_2$ and DG functors

$$\mathrm{Dif}\,\mathcal{B} \xrightarrow{F_3} \mathrm{Dif}\,\mathcal{A}_2 \xrightarrow{F_2} \mathrm{Dif}\,\mathcal{A}_1 \xrightarrow{F_1} \mathrm{Dif}\,\mathcal{A}$$

such that $\mathbf{L}F_1$, $\mathbf{L}F_2$ and $\mathbf{L}F_3$ are equivalences.

ii) There is an S-equivalence $\mathcal{D}\mathcal{B} \xrightarrow{\sim} \mathcal{D}\mathcal{A}$.

iii) $\mathcal{B}$ is equivalent to a full subcategory $\mathcal{U}$ of $\mathcal{D}\mathcal{A}$ whose objects form a set of small generators and satisfy $(\mathcal{D}\mathcal{A})(U,V[n]) = 0$ for all $n\neq 0$, $U,V\in\mathcal{U}$.

REMARK. We refer to [19, 6.4] for more precise information in the case where $\mathcal{A}$ and $\mathcal{B}$ are rings.

PROOF. By 4.2 c), ii) implies iii). To prove that iii) implies i), let $\mathcal{A}_1$ be a full subcategory of $\mathrm{Dif}\,\mathcal{A}$ consisting of chosen objects $\boldsymbol{p}U$, $U\in\mathcal{U}$. Let $F_1 = T_X$ where $X(A,A_1) = A_1(A)$. By 6.1,

$\mathbf{L}F_1$ is an equivalence. By the assumption on $\mathcal{U}$ we have $\mathrm{H}^n\mathcal{A}_1(A,B)=0$ for $n\neq 0$ and arbitrary $A,B\in\mathcal{A}_1$, and $\mathrm{H}^0\mathcal{A}_1$ is equivalent to $\mathcal{B}$. Now the assertion is clear from 9.1.

Using 6.3 b) and 6.4 we find the

COROLLARY. (cf. [20]) *If $\mathcal{A}$ is k-flat, the following are equivalent*

i) There is an $\mathcal{A}$-$\mathcal{B}$-bimodule X such that $\mathbf{L}T_X:\mathcal{DB}\to\mathcal{DA}$ is an equivalence.

ii) There is an S-equivalence $\mathcal{DB}\to\mathcal{DA}$.

iii) $\mathcal{B}$ is equivalent to a full subcategory $\mathcal{U}$ of $\mathcal{DA}$ whose objects form a set of small generators and satisfy $(\mathcal{DA})(U,V[n])=0$ for all $n\neq 0$, $U,V\in\mathcal{U}$.

REMARK. We refer to [20] for more precise information in the case where $\mathcal{A}$ and $\mathcal{B}$ are rings. A straightforward construction of the bimodule in this case is given in [13].

10. APPLICATION: KOSZUL DUALITY FOR DGA CATEGORIES

10.1 Preliminaries. Suppose that k is a field. Define the *functor* $D:\mathrm{Dif}\,k\to\mathrm{Dif}\,k$ by

$$DM=(\mathrm{Dif}\,k)(M,k)\,,$$

where k is viewed as a DG k-module concentrated in degree 0. Let $\mathcal{A}$ be a DG k-category. For each $A\in\mathcal{A}$ we define the *$\mathcal{A}$-module* $A^\vee$ by

$$A^\vee(B)=D\mathcal{A}(A,B)\,,\ B\in\mathcal{A}.$$

For each DG module M and each $A\in\mathcal{A}$ we have a *canonical isomorphism* of DG k-modules

$$\begin{array}{rcl}(\mathrm{Dif}\mathcal{A})(M,A^\vee) & \xrightarrow{\sim} & DM(A)\\ \varphi & \mapsto & (m\mapsto((\varphi A)(m))(\mathbf{1}_A)).\end{array}$$

In particular, we have a canonical morphism

$$\mathcal{A}(A,B)\to DD\mathcal{A}(A,B)\xrightarrow{\sim} DA^\vee(B)\xrightarrow{\sim}(\mathrm{Dif}\mathcal{A})(A^\vee,B^\vee)\,,$$

which is a quasi-isomorphism if $\dim\mathrm{H}^n\mathcal{A}(A,B)<\infty$ for each $n\in\mathbf{Z}$. So in this case the *full subcategory* $\mathcal{A}^\vee$ of $\mathrm{Dif}\mathcal{A}$ formed by the $A^\vee$, $A\in\mathcal{A}$, is quasi-equivalent to $\mathcal{A}$.

Fix $A\in\mathcal{A}$. To compute $(\mathcal{DA})(?,A^\vee)$, we first remark that if N is acyclic, we have

$$(\mathcal{HA})(N,A^\vee)=\mathrm{H}^0DN(A)=0.$$

Therefore

$$(\mathcal{DA})(M,A^\vee)\xrightarrow{\sim}(\mathcal{HA})(\mathbf{p}M,A^\vee)\xleftarrow{\sim}(\mathcal{HA})(M,A^\vee)\xrightarrow{\sim}\mathrm{H}^0DM(A)\,,$$

and in particular $\mathrm{H}^n\mathcal{A}^\vee(A^\vee, B^\vee) \xrightarrow{\sim} (\mathcal{D}\mathcal{A})(A^\vee, B^\vee[n])$. So if we define the $\mathcal{A}$-$\mathcal{A}^\vee$-*bimodule* $X_\vee$ by $(A, B^\vee) \mapsto B^\vee(A)$, then $(\mathcal{A}^\vee, X_\vee)$ is a lift (7.3) of $\{A^\vee : A \in \mathcal{A}\} \subset \mathcal{D}\mathcal{A}$.

10.2 The Koszul dual. Suppose from now on that $\mathcal{A}$ is an *augmented* DG category (=DGA category) i.e.

a) Distinct objects of $\mathcal{A}$ are non-isomorphic.

b) For each $A \in \mathcal{A}$ a DG module $\overline{A}$ is given such that $\mathrm{H}^0\overline{A}(A) \xrightarrow{\sim} k$ and $\mathrm{H}^n\overline{A}(B) = 0$ whenever $n \neq 0$ or $B \neq A$.

Now let $(\mathcal{A}^*, X)$ be a lift (7.3) of $\{\overline{A} : A \in \mathcal{A}\} \subset \mathcal{D}\mathcal{A}$. After deleting some objects from $\mathcal{A}^*$ we may (and will) assume that we have a bijection $A \mapsto A^*$ between the objects of $\mathcal{A}$ and those of $\mathcal{A}^*$ such that $\mathbf{L}T_X A^{*\wedge} \xrightarrow{\sim} \overline{A}$ for each $A \in \mathcal{A}$. By 6.3 a) we also may (and will) assume that X has property (P) as a bimodule. Since k is a field, this implies in particular that $X(?, A^*)$ has property (P) for each $A^* \in \mathcal{A}^*$ (6.1 c). Hence the functors H_X and T_X both preserve acyclicity and induce a pair of adjoint functors between $\mathcal{D}\mathcal{A}^*$ and $\mathcal{D}\mathcal{A}$, which will also be denoted by T_X and H_X.

We make $\mathcal{A}^*$ into an augmented DG category by putting

$$\overline{A^*} = H_X A^\vee .$$

This is a good definition since indeed

$$\begin{aligned}
\mathrm{H}^n\overline{A^*}(B^*) &\xrightarrow{\sim} (\mathcal{D}\mathcal{A}^*)(B^{*\wedge}, \overline{A^*}[n]) \xrightarrow{\sim} (\mathcal{D}\mathcal{A}^*)(B^{*\wedge}, H_X A^\vee[n]) \\
&\xrightarrow{\sim} (\mathcal{D}\mathcal{A})(T_X B^{*\wedge}, A^\vee[n]) \xrightarrow{\sim} (\mathcal{D}\mathcal{A})(\overline{B}, A^\vee[n]) \\
&\xrightarrow{\sim} \mathrm{H}^n D\overline{B}(A).
\end{aligned}$$

We define $\mathcal{A}^*$ with the $\overline{A^*}$, $A^* \in \mathcal{A}^*$, to be the *Koszul dual* of the DGA category $\mathcal{A}$ (cf. [1]). We sum up our notations in the diagram

$$\begin{array}{ccc}
\overline{A} & \mathcal{D}\mathcal{A} & A^\vee \\
\uparrow & T_X \uparrow\downarrow H_X & \downarrow \\
A^{*\wedge} & \mathcal{D}\mathcal{A}^* & \overline{A^*}.
\end{array}$$

If $\mathcal{B}$ is another DGA category, a quasi-functor $Y : \mathcal{B} \to \mathcal{A}$ is *compatible with the augmentations* if $H_Y\overline{A} \xrightarrow{\sim} \overline{B}$ whenever $T_Y B^\wedge \xrightarrow{\sim} A^\wedge$.

By 7.3 c) the Koszul dual is determined by the above construction up to a quasi-equivalence *compatible* with the augmentation, i.e. if X' and $\mathcal{A}^{*\prime}$ result from different choices made in the construction, there is an $\mathcal{A}^{*\prime}$-$\mathcal{A}^*$-bimodule Y having property (P) such that $T_Y : \mathcal{D}\mathcal{A}^* \to \mathcal{D}\mathcal{A}^{*\prime}$ satisfies $T_{X'}T_Y \xrightarrow{\sim} T_X$, $T_Y A^{*\wedge} \xrightarrow{\sim} A^{*\prime\wedge}$ and

$$H_Y\overline{A^{*\prime}} \xrightarrow{\sim} H_Y H_{X'} A^\vee \xrightarrow{\sim} H_X A^\vee \xrightarrow{\sim} \overline{A^*}$$

for each $A \in \mathcal{A}$.

The Koszul dual defined in [2] is quasi-equivalent to the full subcategory of Dif$\mathcal{A}^*$ formed by the $A^{*\wedge}[n(A)]$, where $n : \mathcal{A} \to \mathbf{Z}$ is a given 'weight function' for $\mathcal{A}$. Note that the morphism spaces of this category simply identify with the shifted spaces

$$\mathcal{A}^*\,(A^*, B^*)[n(B) - n(A)]\ ,\ A, B \in \mathcal{A}.$$

EXAMPLES. a) Let $\mathfrak{G}$ be a k-Lie algebra and $U(\mathfrak{G})$ its universal enveloping algebra. In the notations of 6.5 (with $R = k$), the Koszul dual of $A = U(\mathfrak{G})$ is quasi-equivalent to B.

b) Let V be a finite-dimensional k-vector space, DV its dual over k, ΛDV the exterior algebra on DV, and SV the graded symmetric algebra on V. View $A = \Lambda DV$ as a DG algebra concentrated in degree 0, and $B = SV$ as a DG algebra with the components $B^n = S^n V$ and vanishing differential. Define (commuting) right and left A-actions on ΛV by

$$\begin{aligned} v^*.(v_1 \wedge \ldots \wedge v_n) &= \sum_{i=1}^{n} (-1)^{i+1} v^*(v_i) v_1 \wedge \ldots \widehat{v_i} \ldots \wedge v_n \\ (v_1 \wedge \ldots \wedge v_n).v^* &= \sum_{i=1}^{n} (-1)^{n+i} v^*(v_i) v_1 \wedge \ldots \widehat{v_i} \ldots \wedge v_n. \end{aligned}$$

Endow the graded A-B-bimodule $X = SV \otimes \Lambda V$ with the differential

$$d : X^p \to X^{p+1}\ ,\ x \mapsto (-1)^p \sum_{i=1}^{n} (v_i \otimes v_i^*) x\ ,$$

where the v_i, $1 \le i \le n$, form a basis of V and (v_i^*) is the dual basis. Then (B, X) is a lift of the trivial A-module k. Hence the Koszul dual of A is quasi-equivalent to B.

c) Let V be a finite-dimensional k-vector space, $I \subset \mathbf{Z}$ an interval and $\mathcal{A}_I$ the DG category concentrated in degree 0 whose objects are the $i \in I$ and whose morphism spaces are the

$$\mathcal{A}_I\,(i, j) = S^{j-i} V$$

concentrated in degree 0. For each $i \in I$ let $\bar{\imath}$ be the DG $\mathcal{A}_I$-module concentrated in degree 0 with $\bar{\imath}(j) = k$ for $i = j$ and $\bar{\imath}(j) = 0$ for $i \neq j$. Let $\mathcal{B}_I$ be the DG category whose objects are the symbols i^*, $i \in I$ and whose morphism spaces are the stalk complexes

$$\mathcal{B}_I\,(i^*, j^*) = (\Lambda^{i-j} DV)[j - i].$$

Let X_I be the $\mathcal{A}_I$-$\mathcal{B}_I$-bimodule given by

$$X_I(i, j^*)^n = \Lambda^{-n} V \otimes S^{n+j-i} V$$

endowed with the differential given by left multiplication by $\sum_{i=1}^{n} v_i^* \otimes v_i$, where the v_i, $1 \le i \le n$, form a basis of V and (v_i^*) is the dual basis. Then $(\mathcal{B}_I, X_I)$ is a lift of $\{\bar{\imath} : i \in I\} \subset \mathcal{D}\mathcal{A}_I$. So the Koszul dual of $\mathcal{A}_I$ is quasi-equivalent to $\mathcal{B}_I$. Clearly, the modules $i^\vee$, $i \in I$, are the unions of their finite-dimensional submodules and the functor $i^\wedge \mapsto i^\vee$ is an equivalence. It therefore follows from the lemma on the 'symmetric' case (10.5) that the Koszul dual of $\mathcal{B}_I$ is quasi-equivalent to $\mathcal{A}_I$.

10.3 The double dual. The composition of H_X with the functor $T_{X_\vee} : \mathcal{DA}^\vee \to \mathcal{DA}$ of 10.1 induces a functor $\underline{\mathcal{A}^\vee} \to \{\overline{A^*} : A \in \mathcal{A}\} \subset \mathcal{DA}^*$. Thus (7.3 a), we have a quasi-functor $Y : \mathcal{A}^\vee \to \mathcal{A}^{**}$, which is a quasi-equivalence iff the restriction of $H_X : \mathcal{DA} \to \mathcal{DA}^*$ to the subcategory formed by the $A^\vee[n]$, $A \in \mathcal{A}$, $n \in \mathbf{Z}$, is fully faithful.

$$\begin{array}{ccc} & T_{X_\vee} & \\ \mathcal{DA}^\vee & \longrightarrow & \mathcal{DA} \\ T_Y \downarrow & & T_X \uparrow\downarrow H_X \\ \mathcal{DA}^{**} & \longrightarrow & \mathcal{DA}^*. \\ & T_{X_*} & \end{array}$$

We endow $\mathcal{A}^\vee$ with the augmentation defined by

$$\overline{A^\vee}(B^\vee) = D(\mathrm{Dif}\,\mathcal{A})\,(\overline{A}, B^\vee) \xrightarrow{\sim} DD\overline{A}(B).$$

Lemma. *The quasi-functor $Y : \mathcal{A}^\vee \to \mathcal{A}^{**}$ is compatible with the augmentations.*

Proof. Let $(\mathcal{A}^{**}, X_*)$ be the chosen lift for the $\overline{A^*}$, $A \in \mathcal{A}$. Recall that we assume that X_* has property (P) as a bimodule. Fix $A \in \mathcal{A}$. We have to show that $\overline{A^\vee} \xrightarrow{\sim} H_Y\overline{A^{**}}$. By definition $H_Y\overline{A^{**}} = H_Y H_{X_*} A^{*\vee}$. We will show that $H_Y H_{X_*} A^{*\vee} \xrightarrow{\sim} \overline{A^\vee}$ by explicitly exhibiting a quasi-isomorphism. For short we write ${}^\vee(?,?)$ for $(\mathrm{Dif}\,\mathcal{A}^\vee)(?,?)$, ... We have the following series of morphisms of DG k-modules, functorial in $B^\vee \in \mathcal{A}^\vee$

$$\begin{aligned} (H_Y H_{X_*} A^{*\vee})(B^\vee) &\xrightarrow{\sim} {}^\vee(B^{\vee\wedge}, H_Y H_{X_*} A^{*\vee}) \xrightarrow{\sim} {}^*(T_{X_*} T_Y B^{\vee\wedge}, A^{*\vee}) \\ &\xrightarrow{\sim} D\,{}^*(A^{*\wedge}, T_{X_*} T_Y B^{\vee\wedge}) \leftarrow D\,{}^*(A^{*\wedge}, H_X T_{X_\vee} B^{\vee\wedge}). \end{aligned}$$

The last arrow is induced by the morphism

$$T_{X_*} T_Y \to H_X T_{X_\vee}$$

of DG functors $\mathrm{Dif}\,\mathcal{A}^\vee \to \mathrm{Dif}\,\mathcal{A}^*$ exhibited in remark 7.3. It is a quasi-isomorphism since $B^{\vee\wedge} \in \mathcal{H}_p^b\mathcal{A}^\vee$ (7.3 b). We continue the series of morphisms:

$$\begin{aligned} D\,{}^*(A^{*\wedge}, H_X T_{X_\vee} B^{\vee\wedge}) &\xrightarrow{\sim} D\,(T_X A^{*\wedge}, T_{X_\vee} B^{\vee\wedge}) \\ &\xrightarrow{\sim} D\,(T_X A^{*\wedge}, B^\vee) \end{aligned}$$

since by construction $T_{X_\vee} B^{\vee\wedge} \xrightarrow{\sim} X_\vee(?, B^\vee) \xrightarrow{\sim} B^\vee$ in $\mathrm{Dif}\,\mathcal{A}$. Now since $T_X A^{*\wedge}$ is quasi-equivalent to $\overline{A}$, we have a quasi-isomorphism

$$D\,(T_X A^{*\wedge}, B^\vee) \leftarrow D\,(\overline{A}, B^\vee).$$

By definition the last term is isomorphic to $\overline{A^\vee}(B^\vee)$.

10.4 Properties of $\mathcal{A}^*$. Let M be a DG $\mathcal{A}$-module and $n \in \mathbf{N}$. By definition we have $\mathrm{sdim}\,M \le n$ (resp. $\mathrm{pdim}\,M \le n$, resp. $\mathrm{idim}\,M \le n$) if there is a sequence

$$0 = M_{-1} \to M_0 \to M_1 \to M_2 \to \ldots \to M_n = M$$

of morphisms of $\mathcal{DA}$ such that in each triangle

$$M_{i-1} \to M_i \to Q_i \to M_{i-1}[1]\,,\ 0 \le i \le n\,,$$

the module Q_i is isomorphic to a finite direct sum of modules of the form $\overline{A}[n]$ (resp. $A^\wedge[n]$, resp. $A^\vee[n]$), $A \in \mathcal{A}$, $n \in \mathbf{Z}$. The (possibly infinite) numbers $\operatorname{sdim} M$, $\operatorname{pdim} M$ and $\operatorname{idim} M$ are referred to as the *semi-simple, the projective, and the injective dimension* of M, respectively.

Let $\nu : \operatorname{Dif}\mathcal{A} \to \operatorname{Dif}\mathcal{A}$ be the functor defined by

$$(\nu M)(A) = D(\operatorname{Dif}\mathcal{A})\,(M, A^\wedge).$$

For example, we have $\nu A^\wedge = A^\vee$ by the definition of $A^\vee$ for each $A \in \mathcal{A}$. We have a natural transformation

$$D(\operatorname{Dif}\mathcal{A})\,(M,N) \to (\operatorname{Dif}\mathcal{A})\,(N, \nu M)$$

which is defined as follows: Given a linear form φ on $(\operatorname{Dif}\mathcal{A})\,(M,N)$ and an $f \in (\operatorname{Dif}\mathcal{A})\,(A^\wedge, N) \xleftarrow{\sim} N(A)$, the associated linear form on $(\operatorname{Dif}\mathcal{A})\,(M, A^\wedge)$ maps g to $\varphi(fg)$. Clearly this is an isomorphism for $M = B^\wedge[n]$, $B \in \mathcal{A}$, $n \in \mathbf{Z}$, and therefore a quasi-isomorphism for $M \in \mathcal{H}_p^b\mathcal{A}$.

Lemma.

a) If $\operatorname{sdim} M < \infty$ and $\operatorname{pdim} M < \infty$ then $H_X \mathbf{L}\nu M \xrightarrow{\sim} (\mathbf{L}\nu) H_X M$ in $\mathcal{DA}^*$.

b) For each $A \in \mathcal{A}$ we have
1) $\operatorname{pdim} \overline{A^*} \le \operatorname{sdim} A^\vee$ 2) $\operatorname{sdim} A^{*\wedge} \le \operatorname{idim} \overline{A}$
3) $\operatorname{idim} \overline{A^*} \le \operatorname{sdim} A^\wedge$ 4) $\operatorname{sdim} A^{*\vee} \le \operatorname{pdim} \overline{A}$

Proof. a) Since $\operatorname{sdim} M < \infty$, we have $T_X M \in \mathcal{H}_p^b\mathcal{A}^*$ and $M \xrightarrow{\sim} T_X N$ for $N \xrightarrow{\sim} H_X M$. We assume that N (and hence $T_X N$) has property (P). We have to show that $H_X \nu T_X N \xrightarrow{\sim} \nu N$. We write ${}^*(?,?)$ and $(?,?)$ instead of $(\operatorname{Dif}\mathcal{A}^*)\,(?,?)$ and $(\operatorname{Dif}\mathcal{A})\,(?,?)$. We have the following series of quasi-isomorphisms functorial in $A^* \in \mathcal{A}^*$

$$(H_X \nu T_X N)(A^*) \to {}^*(A^{*\wedge}, H_X \nu T_X N) \to (T_X A^{*\wedge}, \nu T_X N).$$

Since $T_X N \in \mathcal{H}_p^b\mathcal{A}$ and $N \in \mathcal{H}_p^b\mathcal{A}^*$, we also have the following quasi-isomorphisms:

$$(T_X A^{*\wedge}, \nu T_X N) \to D(T_X N, T_X A^{*\wedge}) \to D\,{}^*(N, A^{*\wedge}) = (\nu N)(A^*).$$

b) Assertions 1) and 2) are trivial since $H_X \overline{B} \xrightarrow{\sim} B^{*\wedge}$, $B \in \mathcal{A}$, and $H_X A^\vee = \overline{A}$, $A \in \mathcal{A}$. For 3) we use that

$$\overline{A^*} \xrightarrow{\sim} H_X A^\vee \xrightarrow{\sim} H_X \nu A^\wedge \xrightarrow{\sim} (\mathbf{L}\nu) H_X A^\wedge$$

if $\operatorname{sdim} A^\wedge < \infty$, and $B^{*\vee} = (\mathbf{L}\nu) H_X \overline{B}$ for each $B \in \mathcal{A}$. For 4) we use that $A^{*\vee} \xrightarrow{\sim} \mathbf{L}\nu H_X \overline{A} \xrightarrow{\sim} H_X \mathbf{L}\nu \overline{A}$ if $\operatorname{pdim} \overline{A} < \infty$ and $\overline{B^*} = H_X \mathbf{L}\nu B^\wedge$ for each $B \in \mathcal{A}$.

10.5 Three special cases. We consider three cases where $\mathcal{A}^\vee$ is quasi-equivalent to $\mathcal{A}^{**}$, and there is a fully faithful embedding relating $\mathcal{DA}$ and $\mathcal{DA}^*$.

Lemma. (The 'finite' case) *Suppose that $\operatorname{pdim} \overline{A} < \infty$ and $\operatorname{sdim} A^\wedge < \infty$ for all $A \in \mathcal{A}$.*

a) sdim $A^{*\vee} < \infty$ *and idim* $\overline{A^*} < \infty$ *for all* $A^* \in \mathcal{A}^*$.

b) T_X *and* H_X *are quasi-inverse equivalences between* $\mathcal{D}\mathcal{A}^*$ *and* $\mathcal{D}\mathcal{A}$.

c) We have quasi-equivalences $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee \xrightarrow{\sim} \mathcal{A}^{**}$.

EXAMPLES. a) The category $\mathcal{B}_I$ of 10.2 c) for finite I.

b) Let Λ be a finite-dimensional k-algebra of finite global dimension all of whose simple modules are one-dimensional. We take $\mathcal{A}$ to be the k-linear category formed by chosen representatives of the indecomposable projective A-modules and for each $A \in \mathcal{A}$ we take $\overline{A}$ to be the head of A.

PROOF. a) holds by 10.4 b).

b) Since $\operatorname{pdim} \overline{A} < \infty$, we have $\overline{A} \in \mathcal{H}_p^b\mathcal{A}$ for each $A \in \mathcal{A}$. Moreover, since $\operatorname{sdim} B^\wedge < \infty$, the triangulated subcategory generated by the $\overline{A}$ contains each $B^\wedge$, $B \in \mathcal{A}$. Hence the $\overline{A}$, $A \in \mathcal{A}$, form a system of small generators for $\mathcal{D}\mathcal{A}$ and the assertion follows from 6.1 a) and 6.2.

c) Since H_X is fully faithful, $\mathcal{A}^\vee$ is quasi-equivalent to $\mathcal{A}^{**}$ (10.3). Since $\operatorname{sdim} A^\wedge < \infty$ for all $A \in \mathcal{A}$, we have

$$\infty > \dim \mathrm{H}^n A^\wedge(B) = \dim \mathrm{H}^n \mathcal{A}(A, B)$$

for all $A, B \in \mathcal{A}$ so that $\mathcal{A} \to \mathcal{A}^\vee$ is a quasi-equivalence (example 7.2).

LEMMA. (The 'exterior' case) *Suppose that sdim* $A^\wedge < \infty$ *and sdim* $A^\vee < \infty$ *for all* $A \in \mathcal{A}$.

a) pdim $\overline{A^*} < \infty$ *and idim* $\overline{A^*} < \infty$ *for each* $A^* \in \mathcal{A}^*$.

b) T_X *and* H_X *induce quasi-inverse equivalences between* $\mathcal{H}_p^b\mathcal{A}^*$ *and the smallest full triangulated subcategory of* $\mathcal{D}\mathcal{A}$ *containing the* $\overline{A}$, $A \in \mathcal{A}$.

c) $T_{X^\top} : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{A}^*$ *is fully faithful.*

d) We have quasi-equivalences $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee \xrightarrow{\sim} \mathcal{A}^{**}$.

REMARK. Part b) yields theorem 16 of [2].

EXAMPLES. a) Example 10.2 b).

b) The category $\mathcal{B}_I$ of example 10.2 c).

c) If Λ is a finite-dimensional algebra of arbitrary global dimension with one-dimensional simples, we can proceed as in example b) of the 'finite case'.

PROOF. a) holds by 10.4 b). By the definition of 'lift' (7.3) we have b).

c) Let $\mathcal{T}$ be the full triangulated subcategory of $\mathcal{D}\mathcal{A}$ generated by the $\overline{A}$, $A \in \mathcal{A}$. The restriction of H_X to $\mathcal{T}$ is fully faithful (7.3). Since $\mathcal{H}_p^b\mathcal{A}$ is contained in $\mathcal{T}$, H_X is fully faithful on $\mathcal{H}_p^b\mathcal{A}$, and $H_X A^\wedge$ lies in $\mathcal{H}_p^b\mathcal{A}^*$ for each $A \in \mathcal{A}$. In particular, $H_X A^\wedge$ is small for each $A \in \mathcal{A}$. Since $T_{X^\top}$ agrees with H_X on $\mathcal{H}_p^b\mathcal{A}$ (6.2 a), the assertion follows from 4.2 b).

d) Since the $A^\vee$, $A \in \mathcal{A}$, lie in $\mathcal{T}$, $\mathcal{A}^\vee$ is quasi-equivalent to $\mathcal{A}^{**}$. Since the $A^\wedge$, $A \in \mathcal{A}$, lie in $\mathcal{T}$, we have

$$\infty > \dim \mathrm{H}^n A^\wedge(B) = \dim \mathrm{H}^n \mathcal{A}(B, A)$$

for all $A, B \in \mathcal{A}$, so that $\mathcal{A} \to \mathcal{A}^\vee$ is a quasi-equivalence (example 7.1).

LEMMA. (The 'symmetric' case) *Suppose that* $pdim\,\overline{A} < \infty$ *and* $idim\,\overline{A} < \infty$ *for all* $A \in \mathcal{A}$.

a) $sdim\,A^{*\wedge} < \infty$ *and* $sdim A^{*\vee} < \infty$ *for all* $A^* \in \mathcal{A}^*$.

b) T_X *and* H_X *induce quasi-inverse equivalences between* $\mathcal{H}_p^b\mathcal{A}^*$ *and the smallest full triangulated subcategory of* $\mathcal{D}\mathcal{A}$ *containing the* $\overline{A}$, $A \in \mathcal{A}$.

c) $T_X : \mathcal{D}\mathcal{A}^* \to \mathcal{D}\mathcal{A}$ *is fully faithful.*

d) We have a quasi-equivalence $\mathcal{A}^\vee \to \mathcal{A}^{**}$ *if each* $B^\vee$, $B \in \mathcal{A}$, *lies in the smallest triangulated subcategory of* $\mathcal{D}\mathcal{A}$ *closed under direct sums and containing the* $\overline{A}$, $A \in \mathcal{A}$.

EXAMPLES. In example 10.2 a), we have $\mathrm{pdim}\,\overline{A} < \infty$ and $\mathrm{idim}\,\overline{A} < \infty$ if $\mathfrak{G}$ is finite-dimensional. This also holds in 10.2 c). For 10.2 c) the assumption of d) is satisfied as well.

PROOF. a) holds by 10.4 b). By the definition of 'lift' (7.3) we have b).

c) and d): By 4.2 b), T_X is fully faithful. So T_X induces an equivalence onto its image, which is precisely the smallest strictly full triangulated subcategory containing the $\overline{A}$, $A \in \mathcal{A}$, and closed under direct sums. A quasi-inverse is induced by H_X. Thus the restriction of H_X to the subcategory of $\mathcal{D}\mathcal{A}$ formed by the $B^\vee$, $B \in \mathcal{A}$, is fully faithful. Now d) follows by 10.3.

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B. Keller, Univ. Paris 7, U.F.R. de Math., Tour 45/55, 5e étage,
2, pl. Jussieu, 75251 Paris, France
keller@mathp7.jussieu.fr