DERIVED INVARIANCE OF HIGHER STRUCTURES ON THE HOCHSCHILD COMPLEX

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Abstract. We show that derived equivalences preserve the homotopy type of the (cohomological) Hochschild complex as a B_∞-algebra. More generally, we prove that, as an object of the homotopy category of B_∞-algebras, the Hochschild complex is contravariant with respect to fully faithful derived tensor functors. We also show that the Hochschild complexes of a Koszul algebra and its dual are homotopy equivalent as B_∞-algebras. In particular, their Hochschild cohomologies are isomorphic as algebras, which is a recent result by R.-O. Buchweitz [4], and as Lie algebras. Our methods also yield a derived invariant definition of the Hochschild complex of an exact category.

1. Introduction

Let k be a field. It is a theorem of D. Happel [13] and J. Rickard [27] that if two (associative, unital) k-algebras A and B have equivalent derived categories

$$\mathcal{D}A \xrightarrow{\sim} \mathcal{D}B\ ,$$

then there is an isomorphism of graded algebras between their Hochschild cohomologies $HH^*(A,A)$ and $HH^*(B,B)$. More precisely, if X is a complex of A-B-bimodules such that the total derived functor

$$?\otimes^{\mathsf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence, then there is a canonical algebra isomorphism $f_X : HH^*(B,B) \xrightarrow{\sim} HH^*(A,A)$ associated with X. It was shown in [16] that f_X also respects the Lie algebra structures given by the Gerstenhaber bracket. In this article, we refine and generalize these results: the cup product and the Lie bracket on $H^*(A,A)$ come from certain operations on the Hochschild complex $C(A,A)$ itself. In fact, this complex has many more operations, most of which do not descend to homology. The situation is described precisely by saying that $C(A,A)$ is a B_∞*-algebra* in the sense of GetzlerJones [8, 5.2], *i.e.* its bar construction $B(C(A,A))$ is endowed with a differential and a multiplication which, together with the canonical comultiplication, make $B(C(A,A))$ into a differential graded bialgebra. Let us consider the homotopy category of B_∞-algebras $\mathsf{Ho}(B_\infty)$, *i.e.* the category obtained from the category of B_∞-algebras by formally inverting all morphisms which induce quasi-isomorphisms in the underlying complexes. Our main result (3.2) is that in the above situation, the morphism f_X lifts to an isomorphism

$$\varphi_X : C(B,B) \xrightarrow{\sim} C(A,A)$$

in the homotopy category of B_∞-algebras. This refines the known invariance results. Moreover, we succeed in constructing a (not necessarily invertible) morphism φ_X under weaker hypotheses: for φ_X to be defined, it suffices that the functor

$$?\otimes^{\mathsf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$$

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induces a fully faithful functor on the subcategory of perfect complexes, or, equivalently, that the canonical morphism

$$A \to \mathsf{RHom}_B(X, X)$$

is a quasi-isomorphism (3.3). If this is the case and, moreover, the canonical morphism

$$B^{op} \to \mathsf{RHom}_{A^{op}}(X, X)$$

is a quasi-isomorphism, then φ_X is invertible. This suffices to show that if A is a Koszul algebra and $A^!$ its Koszul dual, viewed as a differential graded algebra with vanishing differential, then we have an isomorphism

$$C(A, A) \xrightarrow{\sim} C(A^!, A^!)$$

in the homotopy category of B_∞-algebras (3.5). In particular, this yields a new proof of R.-O. Buchweitz' theorem [4] that the Hochschild cohomology algebras of A and $A^!$ are isomorphic.

To prove these results, we work in the more general setting of dg categories (section 4). This proves to be extremely convenient thanks to the fact that each fully faithful dg functor

$$F : \mathcal{A} \to \mathcal{B}$$

induces a restriction map

$$F^* : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})$$

which is compatible with the B_∞-structures. Apart from this observation, our main technical tool is (the generalization to dg categories of) the homotopy bicartesian square (4.5)

$$\begin{tikzcd} C(T, T) \arrow[r] \arrow[d] & C(A, A) \arrow[d] \\ C(B, B) \arrow[r] & C(A, X, B) \end{tikzcd} \quad ,$$

where A and B are dg algebras, X is a dg A-B-bimodule,

$$T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

the obvious dg algebra of upper triangular matrices and $C(A, X, B)$ the canonical complex computing

$$\mathsf{RHom}_{A^{op} \otimes B}(X, X).$$

The long exact sequence associated with this square was discovered by D. Happel [13] (in the case where $A = k$) and has been further studied and generalized in [24], [11], [10], [9], [5], [3]. Similar ideas appear in work on automorphism groups by Guil-Asensio–Saorín [12].

Using this square and the functoriality of $\mathcal{A} \mapsto C(\mathcal{A}, \mathcal{A})$ with respect to fully faithful dg functors, we construct (4.6)

$$\varphi_X : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A}) \, ,$$

for suitable $\mathcal{A}$-$\mathcal{B}$-bimodules X, as a morphism of the homotopy category $\mathsf{Ho}(B_\infty)$. We prove its fundamental properties, notably its (partial) compatibility with tensor products of bimodules.

The generality of the construction in section 4 makes it possible to define (section 5) the Hochschild complex of a (small, k-linear) exact category $\mathcal{E}$ in the sense of Quillen [25] in a way which is invariant under derived equivalences: We define

$$C(\mathcal{E}, \mathcal{E}) = C(\mathcal{Q}, \mathcal{Q}) \, , \ \mathcal{Q} = \mathcal{C}^b(\mathcal{E}) / \mathcal{A}c^b(\mathcal{E})) \, ,$$

where $\mathcal{C}^b(\mathcal{E})/\mathcal{A}c^b(\mathcal{E})$ is the quotient dg category of the dg category of bounded complexes $\mathcal{C}^b(\mathcal{E})$ by the dg category of acyclic bounded complexes $\mathcal{A}c^b(\mathcal{E})$. The existence and uniqueness of such a quotient (in a suitable category) was proved in [19] and [6]. The Hochschild cohomology and deformation theory of *abelian* categories is being studied by T. Lowen [22]. The relation of our approach to her work remains to be elucidated.

2. B_∞-algebras

Let k be a commutative ring and C a $\mathbf{Z}$-graded k-module

$$C = \oplus_{p\in\mathbf{Z}} C^p.$$

The *suspension* $C[1]$ *of* C is the $\mathbf{Z}$-graded k-module with $C[1]^p = C^{p+1}$ for all $p \in \mathbf{Z}$. Let BC be the graded tensor k-coalgebra on the suspension $C[1]$ of C. We denote its comultiplication by Δ, its counit by η and its canonical augmentation by ε. A B_∞-*algebra* [8] [1] structure on C is the datum of a differential d and a multiplication μ on BC such that $(BC, d, \mu, \Delta, \varepsilon, \eta)$ is a differential graded bialgebra (it is then automatically a dg Hopf algebra).

The datum of the differential on BC translates into the datum of an A_∞-algebra structure on C (in particular, C itself carries a natural differential) while the multiplication μ corresponds to the datum of a family of morphisms

$$C^{\otimes l} \otimes C^{\otimes m} \to C\ ,\ l, m \geq 0\ ,$$

satisfying a series of compatibility conditions among each other and with the A_∞-structure.

Let A be an associative unital dg k-algebra and $C = C(A, A)$ its Hochschild complex. It was shown by Getzler and Jones [8] that C has a canonical B_∞-algebra structure: The underlying A_∞-structure on C is the canonical dg algebra structure on C (whose homology is the Hochschild cohomology algebra) whereas the multiplication μ corresponds to the brace operations [1], [15], [8, 5.2]. In particular, the B_∞-structure on $C(A, A)$ determines the Gerstenhaber bracket and the cup product on the Hochschild cohomology $HH^*(A, A)$.

A natural explanation for the presence of the B_∞-structure on the Hochschild complex $C(A, A)$ is provided by A_∞-category theory [20] [21] [23] [17]: Let $\mathcal{A}$ be the A_∞-category with one object whose A_∞-endomorphism algebra is A. Let $\mathsf{Fun}(\mathcal{A}, \mathcal{A})$ be the A_∞-category of A_∞-functors from $\mathcal{A}$ to $\mathcal{A}$ and let $\mathbf{1}$ be the identical A_∞-functor. Its A_∞-endomorphism algebra in $\mathsf{Fun}(\mathcal{A}, \mathcal{A})$ is naturally isomorphic to the dg algebra $C(A, A)$

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{A},\mathcal{A})}(\mathbf{1}, \mathbf{1}) = C(A, A)$$

and the associative multiplication on $BC(A, A)$ comes from the composition A_∞-bifunctor

$$\mathsf{Fun}(\mathcal{A}, \mathcal{A}) \times \mathsf{Fun}(\mathcal{A}, \mathcal{A}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{A}).$$

B_∞-algebras can be considered as the algebras over a certain asymmetric operad (whose components are free graded k-modules of finite total rank). Therefore [14], the category of B_∞-algebras admits the structure of a Quillen model category [26] [7] whose fibrations are the surjections and whose weak equivalences are the quasi-isomorphisms (*i.e.* the morphisms of B_∞-algebras $C \to C'$ which induce isomorphisms in the homology of the underlying complexes). We define the *homotopy category of* B_∞-*algebras* $\mathsf{Ho}(B_\infty)$ to be homotopy category associated with this model category, *i.e.* the localization of the category of B_∞-algebras with respect to the class of quasi-isomorphisms.

3. The invariance theorem

3.1. **Notation and terminology.** Let k be a commutative ring. For a dg k-algebra A, we denote its derived category [18] by $\mathcal{D}A$. By $\mathsf{per}\, A \subset \mathcal{D}A$ we denote the closure of the free A-module A under shifts, extensions and passage to direct factors. The subcategory $\mathsf{per}\, A$ thus [18, 5.3] consists precisely of the *perfect* (=compact=small) objects, *i.e.* the objects $X \in \mathcal{D}A$ such that the functor

$$\mathsf{Hom}_{\mathcal{D}A}(X, ?) : \mathcal{D}A \to \mathsf{Mod}\, k$$

commutes with infinite direct sums. A dg A-module P is *cofibrant* if, for each surjective quasi-isomorphism of dg A-modules $M \to M'$, each morphism $L \to M'$ lifts to a morphism $L \to M$. For example, if k is a field, each complex of vector spaces is a cofibrant dg k-module. If k is a commutative ring, each right bounded complex of projective k-modules is a cofibrant dg k-module. More generally, up to homotopy equivalence, the cofibrant dg A-modules are precisely those having the property (P) of [18, 3.1].

3.2. **The main theorem.** Let k be a commutative ring. Let A and B be dg algebras and let X be a dg A-B-bimodule. Suppose that the dg k-modules underlying A, B and X are cofibrant. Recall [18, 4.2] that the functor

$$? \otimes^{\mathsf{L}}_A X : \mathsf{per}\, A \to \mathcal{D}B$$

is fully faithful iff the canonical map

$$H^n A \to \mathsf{Hom}_{\mathcal{D}B}(X, X[n])$$

is an isomorphism for all $n \in \mathbf{Z}$ iff the canonical morphism

$$A \to \mathsf{RHom}_B(X, X)$$

is a quasi-isomorphism.

Theorem. *If the functors*

$$? \otimes^{\mathsf{L}}_A X : \mathsf{per}\, A \to \mathcal{D}B \text{ and } X \otimes^{\mathsf{L}}_B ? : \mathsf{per}(B^{op}) \to \mathcal{D}(A^{op})$$

are fully faithful, there is a canonical isomorphism

$$\varphi_X : C(B, B) \to C(A, A)$$

in the homotopy category of B_∞-algebras.

Corollary. *If the functor*

$$? \otimes^{\mathsf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence, there is a canonical isomorphism

$$\varphi_X : C(B, B) \to C(A, A)$$

in the homotopy category of B_∞-algebras.

The theorem and its corollary follow from the more general theorem 4.6 below.

3.3. **Functoriality.** In 4.6, we will construct a morphism φ_X with suitable functoriality properties for a larger class of dg bimodules X: With the above notations, assume only that the functor

$$? \otimes^{\mathsf{L}}_B X : \mathsf{per}\, A \to \mathcal{D}B$$

is fully faithful.

Theorem. *In the homotopy category of B_∞-algebras, there is a canonical morphism*

$$\varphi_X : C(B, B) \to C(A, A)$$

associated with X such that

a) *φ_X only depends on the isomorphism class of X in $\mathcal{D}(A^{op} \otimes B)$,*
b) *if the functor*

$$X \otimes^{\mathsf{L}}_B ? : \mathsf{per}(B^{op}) \to \mathcal{D}(A^{op})$$

is fully faithful, then φ_X is invertible,
c) *if $B = A$ and $X = A$ then φ_X is the identity,*
d) *if $X = Y \otimes^{\mathsf{L}}_C Z$ for a dg algebra C which is cofibrant over k and for dg bimodules ${}_AY_C$ and ${}_CZ_B$ cofibrant over k and such that the functors*

$$? \otimes^{\mathsf{L}}_A Y : \mathsf{per}\, A \to \mathcal{D}C \text{ and } ? \otimes^{\mathsf{L}}_C Z : \mathsf{per}\, C \to \mathcal{D}B$$

are fully faithful, then $\varphi_X = \varphi_Y \circ \varphi_Z$.

3.4. **A remark on compositions.** If A, B, C are dg algebras which are cofibrant over k and ${}_AY_C$ and ${}_CZ_B$ are dg bimodules cofibrant over k such that the functors

$$? \otimes^{\mathsf{L}}_A Y : \mathsf{per}\, A \to \mathcal{D}C \text{ and } ? \otimes^{\mathsf{L}}_C Z : \mathsf{per}\, C \to \mathcal{D}B$$

are fully faithful, it does not follow that the functor

$$? \otimes^{L}_A X : \mathsf{per}\, A \to \mathcal{D}B \ , \quad \text{where } X = Y \otimes^{\mathsf{L}}_C Z \ ,$$

is fully faithful. For example, let k be a field and consider $A = k[x]$, where x is of degree 0, $B = A$ and $C = k[\xi]$, where ξ is of degree 1 and $\xi^2 = 0$, $d\xi = 0$. Let Z be the bimodule whose underlying complex of $k[x]$-modules is

$$(k\xi' \oplus k) \otimes k[x]$$

where ξ' is of degree -1 and

$$d(\xi' \otimes 1) = 1 \otimes x \ , \ \xi.(\xi' \otimes 1) = 1 \otimes 1.$$

Then the functor $? \otimes^{\mathsf{L}}_C Z$ takes the free module $k[\xi]$ to k (concentrated in degree 0) and the trivial module k to the injective hull

$$E = k[x, x^{-1}]/xk[x]$$

of the trivial $k[x]$-module k. Clearly the restriction of $? \otimes^{\mathsf{L}}_C Z$ to $\mathsf{per}\, C$ is fully faithful. Let Y be the bimodule

$$E \otimes k[\xi]$$

with the differential determined by

$$d(x^{-p} \otimes 1) = x^{-p+1} \otimes \xi.$$

Then the functor $? \otimes^{\mathsf{L}}_A Y$ takes the free module $A = k[x]$ to the trivial module k. The restriction of $? \otimes^{\mathsf{L}}_A Y$ to $\mathsf{per}\, A$ is fully faithful. The functor

$$? \otimes^{L}_A X : \mathsf{per}\, A \to \mathcal{D}B \ , \quad \text{where } X = Y \otimes^{\mathsf{L}}_C Z$$

is isomorphic to the composition of $? \otimes^{\mathsf{L}}_C Y$ with $? \otimes^{\mathsf{L}}_B Z$. It takes the free module $k[x]$ to the injective module E. It is not fully faithful, since the endomorphism ring of E is the power series ring $k[[x]]$.

Note finally that both, Y and Z, satisfy the assumptions of theorem 3.3 b) so that $A = B$ and C do have homotopy equivalent Hochschild B_∞-algebras.

3.5. **Application to Koszul algebras.** In this section, we suppose that k is a field. We consider bigraded k-modules

$$M = \oplus_{p,q\in\mathbf{Z}} M_q^p .$$

The first index p denotes the 'differential grading': all differentials are of degree $(1,0)$. The second index q denotes the 'Adams grading'. For a bimodule M, we have the differential shift $M[1]$ and the Adams shift $M\langle 1\rangle$, which are defined respectively by

$$(M[1])_q^p = M_q^{p+1} \text{ and } (M\langle 1\rangle)_q^p = M_{q+1}^p .$$

Let

$$A = A_0 \oplus A_1 \ldots$$

be a positively Adams-graded k-algebra (concentrated in differential degree 0) such that A_0 is a separable k-algebra (i.e. A_0 is a projective $A_0 \otimes A_0^{op}$-module) and all A_p are finitely generated as right and as left A_0-modules. Suppose that in the category of graded A-modules (with degree 0 morphisms), the A-module A_0 admits a *linear projective resolution*, *i.e.* a projective resolution

$$\ldots \to P^{-p} \to P^{-p+1} \to \ldots \to P^{-1} \to P_0$$

such that P^{-p} is generated in degree p for all $p \in \mathbf{N}$. This means that A is a *Koszul algebra*, *cf.* for example [2]. Let $A^!$ be the dg algebra with vanishing differential whose pth component is

$$\mathsf{Ext}^p(A_0, A_0\langle -p\rangle)\ ,\ p \in \mathbf{Z}.$$

R.-O. Buchweitz has shown [4] that there is an isomorphism

$$HH^*(A,A) \xrightarrow{\sim} HH^*(A^!, A^!)$$

compatible with the cup product. We obtain the following stronger version of his result:

Theorem. *There is a canonical isomorphism*

$$\varphi : C(A,A) \xrightarrow{\sim} C(A^!, A^!)$$

in $\mathsf{Ho}(B_\infty)$. *In particular,* φ *induces an isomorphism*

$$HH^*(A,A) \xrightarrow{\sim} HH^*(A^!, A^!) \tag{3.5.1}$$

compatible with the cup product and the Gerstenhaber bracket.

Proof. Let X be the Koszul complex. It is a dg $A^!$-A-bimodule, which, as a right dg A-module, is a projective resolution of A_0. By the assumption that A_0 admits a linear resolution, the complex

$$\mathsf{RHom}_A(X, X\langle -q\rangle[q])$$

has its homology concentrated in degree 0 for all $q \in \mathbf{Z}$. By the definition of $A^!$, it follows that the canonical morphism

$$A_q^! \xrightarrow{\sim} \mathsf{Ext}^q(A_0, A_0\langle -q\rangle) \xrightarrow{\sim} \mathsf{RHom}_A(X, X\langle -q\rangle[q])$$

is a quasi-isomorphism for all $q \in \mathbf{Z}$. Now we claim that the canonical morphism

$$A_q^{op} \to \mathsf{RHom}_{(A^!)^{op}}(X, X\langle -q\rangle[q])$$

is also a quasi-isomorphism for all $q \in \mathbf{Z}$. This results from the fact that A^{op} is Koszul with Koszul dual $(A^!)^{op}$ and Koszul complex X viewed as a dg A^{op}-$(A^!)^{op}$-bimodule, *cf.* [2]. The assertion now follows from the 'Adams graded' variant of theorem 3.2. Alternatively, it follows from part b) of theorem 4.6 below applied to the dg category $\mathcal{A}$ with object set $\mathbf{Z}$ and morphism spaces

$$\mathcal{A}(i,j) = A_{j-i}\ ,\ i,j \in \mathbf{Z}\ ,$$

(concentrated in degree 0), the dg category $\mathcal{B}$ defined analogously using $A^!$ and the dg $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{X}$ defined by

$$\mathcal{X}(i,j) = X_{j-i}\ ,\ j \in \mathcal{A}, i \in \mathcal{B}.$$

$\surd$

4. Generalization and proof of the invariance theorem

4.1. **From algebras to categories.** Let k be a commutative ring and $\mathcal{A}$ a dg k-category. By $k[\mathcal{A}]$, we denote the dg algebra

$$\bigoplus_{A,B\in\mathcal{A}} \mathcal{A}\,(A,B)$$

with matrix multiplication and the natural differential. This algebra does not have a unit but for each finite family of elements a_i, $i \in I$, of $k[\mathcal{A}]$, there is an element $e \in k[\mathcal{A}]$ of the form

$$e = \mathbf{1}_{A_1} \oplus \cdots \oplus \mathbf{1}_{A_n}$$

for certain $A_1, \ldots, A_n$ of $\mathcal{A}$, such that $ea_i = a_i = a_i e$ for all $i \in I$. By $\mathsf{Modlu}\, k[\mathcal{A}]$, we denote the category of the right dg modules M over $k[\mathcal{A}]$ which are *locally unital*, i.e. for each finite family of elements m_i, $i \in I$, of M, there is an element e as above such that $me = m$. By $\mathsf{Mod}\,\mathcal{A}$, we denote the category of right dg modules over $\mathcal{A}$. Then we have a canonical equivalence

$$\mathsf{Mod}\,\mathcal{A} \to \mathsf{Modlu}\, k[\mathcal{A}]$$

which takes a module M to

$$\bigoplus_{A\in\mathcal{A}} M(A).$$

Its quasi-inverse takes a locally unital dg module L to the module

$$\mathcal{A}^{op} \to \mathsf{Mod}\, k\ ,\ A \mapsto L.\mathbf{1}_A.$$

This equivalence preserves quasi-isomorphisms. We define the derived category $\mathcal{D}k[\mathcal{A}]$ to be the localization of $\mathsf{Modlu}\, k[\mathcal{A}]$ with respect to the quasi-isomorphisms. Thus we obtain an equivalence of derived categories

$$\mathcal{D}\mathcal{A} \xrightarrow{\sim} \mathcal{D}k[\mathcal{A}].$$

4.2. **The Hochschild complex for categories and algebras.** Keep the notations of the preceding section and assume moreover that $\mathcal{A}$ is *cofibrant over* k, *i.e.* such that $\mathcal{A}\,(A,B)$ is cofibrant over k for all objects A, B of $\mathcal{A}$. The *Hochschild complex of* $\mathcal{A}$ is the product total complex $C(\mathcal{A},\mathcal{A})$ of the double complex with components which vanish for $p < 0$ and are equal to

$$\prod_{A_0,\ldots,A_p} \mathsf{Hom}_k(\mathcal{A}\,(A_{p-1},A_p) \otimes \cdots \otimes \mathcal{A}\,(A_0,A_1), \mathcal{A}\,(A_0,A_p))$$

for $p \geq 0$, where the product ranges over all sequences of p objects of $\mathcal{A}$. The differential of a p-cochain c is defined by the canonical formula. The Hochschild complex of $k[\mathcal{A}]$ is given by the product total complex $C(k[\mathcal{A}], k[\mathcal{A}])$ of the double complex with the components which vanish for $p < 0$ and are equal to

$$\mathsf{Hom}_k(k[\mathcal{A}]^{\otimes p}, k[\mathcal{A}])$$

for $p \geq 0$. We have an injective morphism of complexes

$$C(\mathcal{A},\mathcal{A}) \to C(k[\mathcal{A}], k[\mathcal{A}]). \tag{4.2.1}$$

Let $I_{\mathcal{A}}$ denote the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$(A,B) \mapsto \mathcal{A}\,(A,B)\ ,\ A, B \in \mathcal{A}.$$

Lemma. *The morphism 4.2.1 is a quasi-isomorphism of dg k-modules. Both sides are canonically isomorphic to*

$$\mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(I_{\mathcal{A}}, I_{\mathcal{A}}).$$

In particular, the nth homology of both sides is canonically isomorphic to

$$HH^n(\mathcal{A},\mathcal{A}) := \mathsf{Hom}_{\mathcal{D}(\mathcal{A}^{op}\otimes\mathcal{A})}(I_{\mathcal{A}}, I_{\mathcal{A}}[n]),\ n \in \mathbf{Z}.$$

Proof. The bar resolution of $I_{\mathcal{A}}$ is the sum total complex $BR(\mathcal{A})$ of the double complex with components which vanish for $p < 0$ and equal

$$\bigoplus_{A_0,\dots,A_p} \mathcal{A}(A_p,-)\otimes\mathcal{A}(A_{p-1},A_p)\otimes\cdots\otimes\mathcal{A}(A_0,A_1)\otimes\mathcal{A}(?,A_0)$$

for $p \geq 0$. The differential is given by the canonical formula. The canonical morphism $BR(\mathcal{A}) \to I_{\mathcal{A}}$ is a quasi-isomorphism and $BR(\mathcal{A})$ is cofibrant over $\mathcal{A}^{op}\otimes\mathcal{A}$. Thus, we can compute

$$\mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(I_{\mathcal{A}}, I_{\mathcal{A}})$$

as the image of $BR(\mathcal{A})$ under $\mathsf{Hom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(?, I_{\mathcal{A}})$. Now we have an isomorphism

$$\mathsf{Hom}(BR(\mathcal{A}), I_{\mathcal{A}}) \xrightarrow{\sim} C(\mathcal{A},\mathcal{A}).$$

The category $\mathsf{Mod}(\mathcal{A}^{op}\otimes\mathcal{A})$ is equivalent to $\mathsf{Modlu}(k[\mathcal{A}]^{op}\otimes k[\mathcal{A}])$ and the equivalence F takes $I_{\mathcal{A}}$ to the bimodule $k[\mathcal{A}]$. It takes $BR(\mathcal{A})$ to a complex $F(BR(\mathcal{A}))$ which naturally identifies with a quotient of the bar resolution $BR(k[\mathcal{A}])$ of $k[\mathcal{A}]$. We have a morphism of resolutions

$$BR(k[\mathcal{A}]) \to F(BR(\mathcal{A})).$$

The induced morphism

$$\mathsf{Hom}_{k[\mathcal{A}]^{op}\otimes k[\mathcal{A}]}(BR(k[\mathcal{A}]), k[\mathcal{A}]) \leftarrow \mathsf{Hom}_{k[\mathcal{A}]^{op}\otimes k[\mathcal{A}]}(F(BR(\mathcal{A})), k[\mathcal{A}])$$

identifies with the inclusion 4.2.1. It is a quasi-isomorphism of dg k-modules because both sides compute

$$\mathsf{RHom}_{k[\mathcal{A}]^{op}\otimes k[\mathcal{A}]}(k[\mathcal{A}], k[\mathcal{A}]).$$

$\surd$

4.3. **Functoriality of the categorical Hochschild complex.** Keep the notations of the previous paragraph and let $\mathcal{B}$ be another dg category which is cofibrant over k. Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful dg functor. The restriction along F yields an obvious morphism

$$F^* : C(\mathcal{B},\mathcal{B}) \to C(\mathcal{A},\mathcal{A})$$

which is compatible with the differential and, indeed, with the structure of B_∞-algebra, as one easily checks. Compositions of fully faithful functors clearly yield compositions of the restrictions. This functoriality property is a great advantage, which we gain in working with categories rather than with algebras.

The morphism F^* has a natural interpretation in the derived category: The restriction functor

$$\mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{B}) \to \mathcal{D}(\mathcal{A}^{op}\otimes\mathcal{A})$$

admits a fully faithful left adjoint, which we still denote by F. It takes $\mathcal{A}(B,?)\otimes\mathcal{A}(-,A)$ to $\mathcal{B}(FB,?)\otimes\mathcal{A}(-,FA)$. If we have a Hochschild n-cochain c of $\mathcal{B}$, it corresponds to a morphism $c : I_{\mathcal{B}} \to I_{\mathcal{B}}[n]$. Now we have a functorial isomorphism

$$M \otimes^{\mathsf{L}}_{\mathcal{B}} I_{\mathcal{B}} \xrightarrow{\sim} M$$

for each $M \in \mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{B})$, since $I_{\mathcal{B}}$ is the neutral object for the monoidal structure given by tensoring over $\mathcal{B}$. Thus c gives rise to a functorial morphism $Mc : M \to M[n]$. In particular, we obtain

$$(FI_{\mathcal{A}})c : FI_{\mathcal{A}} \to FI_{\mathcal{A}}[n].$$

By the full faithfulness of F, this yields a morphism $I_{\mathcal{A}} \to I_{\mathcal{A}}[n]$. This is the morphism that corresponds to the Hochschild cochain F^*c.

4.4. **Bimodule morphisms.** Let $\mathcal{A}$ and $\mathcal{B}$ be dg algebras which are cofibrant over k. Let X be an $\mathcal{A}$-$\mathcal{B}$-bimodule which is cofibrant over k. Let $C(\mathcal{A}, X, \mathcal{B})$ be the product total complex of the complex whose components vanish for $p < 0$ and equal

$$\prod \mathsf{Hom}_k(\mathcal{A}(A_{l-1}, A_l)\otimes\cdots\otimes\mathcal{A}(A_0, A_1)\otimes X(B_m, A_0)\otimes\mathcal{B}(B_{m-1}, B_m)\otimes\cdots\otimes\mathcal{B}(B_0, B_1), X)$$

for $p \geq 0$, where the inner product ranges over all objects $A_0, \ldots, A_l$ of $\mathcal{A}$ and all objects $B_0, \ldots, B_m$ of $\mathcal{B}$ and $l + m = p$. The differential is given by the same formula as the Hochschild differential. We define a morphism of complexes

$$\alpha : C(\mathcal{A}, \mathcal{A}) \to C(\mathcal{A}, X, \mathcal{B})$$

by taking a cochain

$$c \in C^p(\mathcal{A}, \mathcal{A})$$

to the map whose only non trivial components are the

$$\mathcal{A}\,(A_{p-1}, A_p) \otimes \ldots \otimes \mathcal{A}(A_0, A_1) \otimes X(B_0, A_0) \to X(B_0, A_p)\,,\ u \otimes x \mapsto c(u)x.$$

Similarly, we define a morphism

$$\beta : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, X, \mathcal{B}).$$

Let us interpret $C(\mathcal{A}, X, \mathcal{B})$ and the morphisms α and β in the derived categories: One easily checks that $C(\mathcal{A}, X, \mathcal{B})$ is isomorphic to

$$\mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{B}}(X, X).$$

Let us suppose (without restriction of generality) that X is cofibrant over $\mathcal{A}^{op} \otimes \mathcal{B}$. Then $X_{\mathcal{B}}$ is cofibrant over $\mathcal{B}$ (because $\mathcal{A}$ is cofibrant over k). The left action of $\mathcal{A}$ on X yields morphisms

$$\mathcal{A}(A, B) \to \mathsf{Hom}_{\mathcal{B}}(X(?, A), X(?, B))\,,\ A, B \in \mathcal{A}\,,$$

which yield a morphism of $\mathcal{A}$-$\mathcal{A}$-bimodules

$$\lambda : I_{\mathcal{A}} \to \mathsf{Hom}_{\mathcal{B}}(X, X) = \mathsf{RHom}_{\mathcal{B}}(X, X).$$

This induces a morphism

$$\lambda_* : \mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(I_{\mathcal{A}}, I_{\mathcal{A}}) \to \mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(I_{\mathcal{A}}, \mathsf{RHom}_{\mathcal{B}}(X, X)).$$

The right hand side is canonically isomorphic to $\mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{B}}(X, X)$. It is not hard to check that the following diagram is commutative:

$$\begin{tikzcd}
C(\mathcal{A}, \mathcal{A}) \arrow[rr, "\sim"] \arrow[d, "\alpha"'] & & \mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(I_{\mathcal{A}}, I_{\mathcal{A}}) \arrow[d, "\lambda_*"] \\
C(\mathcal{A}, X, \mathcal{B}) \arrow[r, "\sim"] & \mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{B}}(X, X) \arrow[r, "\sim"] & \mathsf{RHom}_{\mathcal{A}^{op}\otimes\mathcal{A}}(I_{\mathcal{A}}, \mathsf{RHom}_{\mathcal{B}}(X, X)).
\end{tikzcd}$$

It follows that α is a quasi-isomorphism if λ is an isomorphism in $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{A})$. A similar diagram links β to ρ_*, where

$$\rho : I_{\mathcal{B}^{op}} \to \mathsf{RHom}_{\mathcal{A}^{op}}(X, X)$$

is given by the right action of $\mathcal{B}$ on X.

4.5. **Triangular matrices.** We keep the notations of the previous paragraph. Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, X, \mathcal{B})$ be the dg category whose set of objects is the disjoint union of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and whose morphisms are defined by

$$\mathcal{G}(A, A') = \mathcal{A}(A, A') \, , \; \mathcal{G}(B, B') = \mathcal{B}(B, B') \, , \; \mathcal{G}(B, A) = X(B, A) \, , \; \mathcal{G}(A, B) = 0$$

for all objects A, A' of $\mathcal{A}$ and B, B' of $\mathcal{B}$. Note that we have fully faithful functors $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}$ and $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{G}$.

For example, if $\mathcal{A}$ and $\mathcal{B}$ each have one object and A and B denote the endomorphism algebras of the unique objects of $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{G}$ can be visualized as

$$\begin{tikzcd} 1 \arrow[loop above, "A"] & 2 \arrow[l, "X"'] \arrow[loop above, "B"] \end{tikzcd}$$

and $k[\mathcal{G}]$ is the algebra of upper triangular matrices

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \, , \; a \in A \, , \; b \in B \, , \; x \in X.$$

Recall that, in a triangulated category, a *homotopy bicartesian square* is a commutative square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] & Z \end{tikzcd}$$

endowed with a morphism $Z \to U[1]$ such that the sequence

$$U \longrightarrow V \oplus W \longrightarrow Z \longrightarrow U[1]$$

is a triangle.

Lemma. *The complex $C(\mathcal{G}, \mathcal{G})$ is isomorphic to the mapping cylinder on the morphism*

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} : C(\mathcal{A}, \mathcal{A}) \oplus C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, X, \mathcal{B}) \, ,$$

where α and β are the morphisms defined in 4.4. Therefore we have a homotopy bicartesian square

(4.5.1)
$$\begin{tikzcd} C(\mathcal{G}, \mathcal{G}) \arrow[r, "i_{\mathcal{A}}^*"] \arrow[d, "i_{\mathcal{B}}^*"'] & C(\mathcal{A}, \mathcal{A}) \arrow[d, "\alpha"] \\ C(\mathcal{B}, \mathcal{B}) \arrow[r, "\beta"'] & C(\mathcal{A}, X, \mathcal{B}) \end{tikzcd}$$

in the derived category of dg k-modules. In particular, we have a long exact sequence
(4.5.2)
$$\ldots \to HH^n\,(\mathcal{G}, \mathcal{G}) \to HH^n\,(\mathcal{A}, \mathcal{A}) \oplus HH^n\,(\mathcal{B}, \mathcal{B}) \to \mathsf{Hom}_{\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})}(X, X[n]) \to \ldots$$

The proof of the lemma is an easy computation left to the reader. We refer to the introduction for the history of the lemma. We stress that $i_{\mathcal{A}}^*$ and $i_{\mathcal{B}}^*$ are morphisms of B_∞-algebras, as we have seen in 4.3, and that α and β have simple interpretations in the derived categories, as seen in 4.4.

4.6. **The main theorem on the Hochschild complex of a dg category.** We keep the assumptions of paragraph 4.4. Assume that the functor

$$? \otimes^{\mathsf{L}}_{\mathcal{A}} X : \mathsf{per}\,\mathcal{A} \to \mathcal{D}\mathcal{B}$$

is fully faithful. Then the morphism λ

$$I_{\mathcal{A}} \to \mathsf{RHom}_{\mathcal{B}}(X, X)$$

of paragraph 4.4 is invertible and thus the morphism α is a quasi-isomorphism. By the bicartesian square 4.5.1, the morphism $i^*_{\mathcal{B}}$ is a quasi-isomorphism. Thus it becomes invertible in the homotopy category of B_∞-algebras. We define

$$\varphi_X = i^*_{\mathcal{A}} \circ (i^*_{\mathcal{B}})^{-1}.$$

Theorem. a) *φ_X only depends on the isomorphism class of X in $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$.*

b) *If the functor*

$$X \otimes^{L}_{\mathcal{B}} ? : \mathsf{per}(\mathcal{B}^{op}) \to \mathcal{D}(\mathcal{A}^{op})$$

is fully faithful, then φ_X is invertible. This holds in particular if

$$? \otimes^{\mathsf{L}}_{\mathcal{A}} X : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$$

is an equivalence.

c) *If $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful dg functor and the bimodule X is defined by*

$$X(B, A) = \mathcal{B}(B, FA)\ ,\ A \in \mathcal{A}\ ,\ B \in \mathcal{B}\ ,$$

then $\varphi_X = F^*$ in $\mathsf{Ho}(B_\infty)$. In particular, if $\mathcal{B} = \mathcal{A}$ and $X = I_{\mathcal{A}}$, then φ_X is the identity,

d) *If $\mathcal{C}$ is a dg category cofibrant over k and Y a dg $\mathcal{B}$-$\mathcal{C}$-bimodule cofibrant over k such that the functors*

$$? \otimes^{\mathsf{L}}_{\mathcal{C}} Y : \mathsf{per}\,\mathcal{B} \to \mathcal{D}\mathcal{C} \text{ and } ? \otimes^{\mathsf{L}}_{\mathcal{C}} Z : \mathsf{per}\,\mathcal{A} \to \mathcal{D}\mathcal{C}$$

are fully faithful, where $Z = X \otimes^{\mathsf{L}}_{\mathcal{B}} Y$, then $\varphi_Z = \varphi_X \circ \varphi_Y$.

Before proving the theorem, let us record the following useful consequence: Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor which is not necessarily fully faithful but which induces quasi-isomorphisms

$$\mathcal{A}(A, B) \to \mathcal{B}(FA, FB)\ ,\ A, B \in \mathcal{A}.$$

Let X_F be the bimodule

$$(A, B) \mapsto \mathcal{B}(B, FA)\ ,\ A, B \in \mathcal{A}.$$

Then the functor

$$? \otimes^{\mathsf{L}}_{\mathcal{A}} X_F : \mathsf{per}\,\mathcal{A} \to \mathcal{D}\mathcal{B}$$

takes $\mathcal{A}(?, A)$ to $\mathcal{B}(?, FA)$. Therefore it is fully faithful (and takes $\mathsf{per}\,\mathcal{A}$ to $\mathsf{per}\,\mathcal{B}$). Hence we have a well defined morphism

$$\varphi_F := \varphi_{X_F} : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})$$

in $\mathsf{Ho}(B_\infty)$. Moreover, if F induces an equivalence $H^0\mathcal{A} \to H^0\mathcal{B}$, then the associated tensor functor is and equivalence $\mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ and so φ_F is invertible.

If $G : \mathcal{B} \to \mathcal{C}$ is another functor inducing quasi-isomorphisms in the morphism complexes, then clearly GF also has this property, and

$$X_F \otimes^{\mathsf{L}}_{\mathcal{B}} X_G \xrightarrow{\sim} X_{G \circ F}$$

so that we have

$$\varphi_{GF} = \varphi_F \circ \varphi_G.$$

Proof. Let us prove the first statement of b). We note that under the assumption, the morphism β is a quasi-isomorphism. So by the homotopy bicartesian square 4.5.1, the morphism $i_{\mathcal{A}}^*$ is invertible and hence φ_X is invertible. Let us prove the second statement of b). For a dg $\mathcal{B}$-module M, denote by $\mathsf{Hom}_{\mathcal{B}}(M, \mathcal{B})$ the dg $\mathcal{B}^{op}$-module

$$B \mapsto \mathsf{Hom}_{\mathcal{B}}(M, \mathcal{B}(?, B)) \ , \ B \in \mathcal{B}.$$

Then the transposition functor

$$\mathsf{Tr}_{\mathcal{B}} = \mathsf{RHom}_{\mathcal{B}}(?, \mathcal{B}) : \mathcal{D}\mathcal{B} \to \mathcal{D}(\mathcal{B}^{op})^{op}$$

induces an equivalence

$$\mathsf{per}\,\mathcal{B} \xrightarrow{\sim} \mathsf{per}(\mathcal{B}^{op})^{op}.$$

Notice that the functor $\mathsf{RHom}_{\mathcal{B}}(X, ?) : \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{A}$ is an equivalence, since it is left adjoint to the equivalence $X \otimes^{\mathsf{L}}_{\mathcal{A}} ?$. For $Q \in \mathsf{per}\,\mathcal{B}$, we have natural isomorphisms

$$\begin{aligned} X_{\mathcal{B}} \otimes^{\mathsf{L}}_{\mathcal{B}} \mathsf{RHom}_{\mathcal{B}}(Q, \mathcal{B}) &\xrightarrow{\sim} \mathsf{RHom}_{\mathcal{B}}(Q, X) \\ &\xrightarrow{\sim} \mathsf{RHom}_{\mathcal{A}}(\mathsf{RHom}_{\mathcal{B}}(X, Q), \mathsf{RHom}_{\mathcal{B}}(X, X)) \\ &\xrightarrow{\sim} \mathsf{RHom}_{\mathcal{A}}(\mathsf{RHom}_{\mathcal{B}}(X, Q), \mathcal{A}). \end{aligned}$$

Here we use that Q is perfect for the first isomorphism, that $\mathsf{RHom}_{\mathcal{B}}(X, ?)$ is fully faithful for the second, and that $\mathsf{RHom}_{\mathcal{B}}(X, ?)$ takes $X(?, A)$ to $\mathcal{A}(?, A)$ for the third. We deduce that we have a natural isomorphism

$$(X \otimes^{\mathsf{L}}_{\mathcal{B}} ?) \circ \mathsf{Tr}_{\mathcal{B}} \xrightarrow{\sim} \mathsf{Tr}_{\mathcal{A}} \circ \mathsf{RHom}_{\mathcal{B}}(X, ?)$$

of functors $\mathsf{per}\,\mathcal{B} \to \mathcal{D}(\mathcal{A}^{op})^{op}$. This shows that the functor

$$X \otimes^{\mathsf{L}}_{\mathcal{B}} ? : \mathsf{per}(\mathcal{B}^{op}) \to \mathcal{D}(\mathcal{A}^{op})$$

is fully faithful. Let us now prove d) under the additional assumption that X is cofibrant over $\mathcal{B}$ and that $Z = X \otimes_{\mathcal{B}} Y$. The general case follows from a), which we will prove later. We consider the following diagram of dg categories:

$$\begin{tikzcd}
(\mathcal{A} \leftarrow \mathcal{C}) \arrow[dr] & & & \\
& (\mathcal{A} \leftarrow \mathcal{B} \leftarrow \mathcal{C}) & (\mathcal{B} \leftarrow \mathcal{C}) \arrow[l, "i_4"] & \mathcal{C} \arrow[l, "i_1"] \arrow[ulll, bend right, "i_3"'] \\
& (\mathcal{A} \leftarrow \mathcal{B}) \arrow[u] & \mathcal{B} \arrow[l, "i_2"] \arrow[u] & \\
& \mathcal{A} \arrow[u] \arrow[uuul, bend left] & &
\end{tikzcd}$$

Here, the symbol $\mathcal{A} \leftarrow \mathcal{B}$ denotes $\mathcal{G}(\mathcal{A}, X, \mathcal{B})$ and similarly for $\mathcal{B} \leftarrow \mathcal{C}$ and $\mathcal{A} \leftarrow \mathcal{C}$, which denotes $\mathcal{G}(\mathcal{A}, X \otimes_{\mathcal{B}} Y, \mathcal{C})$. The symbol $\mathcal{A} \leftarrow \mathcal{B} \leftarrow \mathcal{C}$ denotes the category $\mathcal{U}$ whose set of objects is the disjoint union of the objects of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and whose only possibly non zero morphisms are given by

$$\begin{gathered} \mathcal{U}(A, A') = \mathcal{A}(A, A') \ , \ \mathcal{U}(B, A) = X(B, A) \ , \ \mathcal{U}(C, A) = Z(C, A) \ , \\ \mathcal{U}(B, B') = \mathcal{B}(B, B') \ , \ \mathcal{U}(C, B) = Y(C, B) \ , \ \mathcal{U}(C, C') = \mathcal{C}(C, C'). \end{gathered}$$

All arrows of the diagram denote the obvious fully faithful dg functors. The diagram therefore yields a commutative diagram in $\mathsf{Ho}(B_\infty)$. The composition $\varphi_X \circ \varphi_Y$ is obtained from the arrows on the lower right rim whereas φ_Z comes from the long skew arrows. Note that i_1, i_2 and i_3 induce isomorphisms in $\mathsf{Ho}(B_\infty)$. We can conclude that $\varphi_Z = \varphi_X \circ \varphi_Y$ once we show that i_4 induces an isomorphism. For this, we reinterpret $\mathcal{U}$ as $\mathcal{G}(\mathcal{A}, U, \mathcal{B} \leftarrow \mathcal{C})$, where U is the bimodule with

$$U(B, A) = X(B, A) \text{ and } U(C, A) = Z(C, A).$$

The functor

$$? \otimes^{\mathsf{L}}_{\mathcal{A}} U : \operatorname{\mathsf{per}} \mathcal{A} \to \mathcal{D}(\mathcal{B} \leftarrow \mathcal{C})$$

equals the composition

$$\operatorname{\mathsf{per}} \mathcal{A} \xrightarrow{?\otimes^{\mathsf{L}}_{\mathcal{A}} X} \mathcal{D}\mathcal{B} \xrightarrow{i_{\mathcal{B}*}} \mathcal{D}(\mathcal{B} \leftarrow \mathcal{C})$$

Since both functors are fully faithful, so is $? \otimes^{\mathsf{L}}_{\mathcal{A}} U$. It follows that i_4 induces an invertible morphism in $\mathsf{Ho}(B_\infty)$ and we are done.

We deduce the last statement of c): Indeed, for $X = I_{\mathcal{A}}$, we have

$$\varphi_X = \varphi_{X \otimes^{\mathsf{L}}_{\mathcal{A}} X} = \varphi_X \circ \varphi_X.$$

Since φ_X is invertible, this implies that it is the identity. Let us now prove the first statement of c): With notations as above, we consider the following diagram of dg categories and fully faithful functors

$$\begin{tikzcd}
(\mathcal{A} \leftarrow \mathcal{B}) & \mathcal{B} \arrow[l] & \\
\mathcal{A} \arrow[u] & (\mathcal{A} \leftarrow \mathcal{A}) \arrow[ul, "(\mathbf{1}, F)"'] & \mathcal{A} \arrow[l] \arrow[ul, "F"'] \\
& \mathcal{A} \arrow[u] \arrow[ul, "\mathbf{1}"] &
\end{tikzcd}$$

The lower right angle yields $\varphi_{I_{\mathcal{A}}}$, which equals the identity, as we have just proved. The upper right angle yields φ_X for the bimodule X associated with F. The commutativity of the image of the diagram in $\mathsf{Ho}(B_\infty)$ implies that $\varphi_X = F^*$.

Before we prove a), let us first extend $F \mapsto \varphi_F$ to dg functors which are not necessarily fully faithful. Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor. Let X_F denote the associated bimodule defined by

$$X_F(B, A) = \mathcal{B}(B, FA).$$

Suppose that

$$? \otimes^{\mathsf{L}}_{\mathcal{A}} X_F : \operatorname{\mathsf{per}} \mathcal{A} \to \mathcal{D}\mathcal{B}$$

is fully faithful. For example, this is the case if F induces a fully faithful functor $H^*\mathcal{A} \to H^*\mathcal{B}$. We have a well defined morphism

$$\varphi_F := \varphi_{X_F} : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})$$

in $\mathsf{Ho}(B_\infty)$. We have just proved that if F is itself fully faithful, then $\varphi_F = F^*$. Now suppose that $G : \mathcal{B} \to \mathcal{C}$ is another dg functor such that

$$? \otimes^{\mathsf{L}}_{\mathcal{B}} X_G : \operatorname{\mathsf{per}} \mathcal{B} \to \mathcal{D}\mathcal{C}$$

is fully faithful. The bimodule X_F is cofibrant over $\mathcal{B}$ and we have

$$X_F \otimes_{\mathcal{B}} X_G = X_{G \circ F}.$$

Moreover, the associated functor $\operatorname{\mathsf{per}} \mathcal{A} \to \mathcal{D}\mathcal{C}$ is fully faithful since the functor

$$? \otimes^{\mathsf{L}}_{\mathcal{A}} X_F : \operatorname{\mathsf{per}} \mathcal{A} \to \mathcal{D}\mathcal{B}$$

takes $\operatorname{\mathsf{per}} \mathcal{A}$ to $\operatorname{\mathsf{per}} \mathcal{B}$. Therefore, it follows from the special case of c) which we have proved that

$$\varphi_{G \circ F} = \varphi_F \circ \varphi_G.$$

We conclude that φ_F is in particular defined and functorial in the dg functors $F : \mathcal{A} \to \mathcal{B}$ which induce fully faithful functors $H^*\mathcal{A} \to H^*\mathcal{B}$.

Let us now prove a). Let X be as in a) and let $f : X \to X'$ be a quasi-isomorphism. Then f yields an obvious dg functor

$$F : \mathcal{G}(\mathcal{A}, X, \mathcal{B}) \to \mathcal{B}(\mathcal{A}, X', \mathcal{B}) \, ,$$

which is not fully faithful, in general, but which induces a fully faithful functor in homology. Now we consider the following diagram of dg categories and dg functors:

$$\begin{tikzcd} & (\mathcal{A} \xleftarrow{X} \mathcal{B}) \arrow[d, "F"'] & \\ \mathcal{A} \arrow[ur] \arrow[dr] & & \mathcal{B} \arrow[ul] \arrow[dl] \\ & (\mathcal{A} \xleftarrow{X'} \mathcal{B}) & \end{tikzcd}$$

By what we have just seen, this diagram has a well defined and commutative image in $\mathsf{Ho}(B_\infty)$. In the image, the two left hand arrows become invertible, hence so does φ_F. The composition of the upper arrows yields φ_X and the composition of the lower arrows yields φ'_X. $\surd$

5. The Hochschild complex of an exact category

In this section, for simplicity, we will suppose that k is a field.

5.1. **Exact dg categories and their quotients.** We refer to [19, 2.1] for the notion of an exact dg category. The simplest example of such a category is the dg category of complexes over an additive category. If $\mathcal{A}$ is an exact dg category, it is in particular a Frobenius category and its associated stable category $\underline{\mathcal{A}}$ is triangulated. The stable category is also called the *associated triangulated category*. If $\mathcal{A}$ is the category of complexes over an additive category, the associated triangulated category is the homotopy category.

If $\mathcal{E}$ is an exact category, it gives rise to two exact dg categories, namely the category $\mathcal{C}^b(\mathcal{E})$ of bounded complexes over $\mathcal{E}$ and its full subcategory $\mathcal{A}c^b(\mathcal{E})$ whose objects are the acyclic complexes. We will need the dg quotient category $\mathcal{C}^b(\mathcal{B})/\mathcal{A}c^b(\mathcal{E})$.

Let us recall how dg quotient categories are characterized in general: Let $\mathbf{U}$ be a universe containing an infinite set. A category is $\mathbf{U}$-*small* if the set of its morphisms belongs to $\mathbf{U}$.

We assume that the ground field k belongs to $\mathbf{U}$. The *'strict' category* $\mathcal{M}^b_{str}$ has as objects the $\mathbf{U}$-small exact dg categories. Its morphisms are the dg functors. The category $\mathcal{M}^b$ is obtained from $\mathcal{M}^b_{str}$ by localization at the class of all dg functors $F : \mathcal{A} \to \mathcal{B}$ such that F induces an equivalence in the associated triangulated categories.

Theorem. [19] *Let $\mathcal{B}$ be an exact dg category, $\mathcal{A} \subset \mathcal{B}$ a full exact dg subcategory and $I : \mathcal{A} \to \mathcal{B}$ the inclusion. Then there is a $\mathbf{U}$-small exact dg category $\mathcal{B}/\mathcal{A}$ and a morphism*

$$Q : \mathcal{B} \to \mathcal{B}/\mathcal{A}$$

of $\mathcal{M}^b$ such that $QI = 0$ and that for each morphism $F : \mathcal{B} \to \mathcal{C}$ of $\mathcal{M}^b$ such that $FI = 0$, there is a unique morphism $\overline{F} : \mathcal{B}/\mathcal{A} \to \mathcal{C}$ such that $F = \overline{F} \circ Q$.

The theorem follows by combining Theorem 4.6, Lemma 4.2 and Proposition 4.1 of [19]. A stronger statement was proved by Drinfeld in [6]. He gives a 2-universal property of the quotient (instead of a 1-universal property as in the theorem). The theorem shows that the dg quotient $\mathcal{B}/\mathcal{A}$ is indeed a quotient in the category $\mathcal{M}^b$. In particular, it is unique and functorial in this category. Let us now show that, as an object of $\mathsf{Ho}(B_\infty)$, the Hochschild complex of an object of $\mathcal{M}^b$ is well defined.

For this, we note that, by the remark following theorem 4.6, each morphism $F : \mathcal{A} \to \mathcal{B}$ of $\mathcal{M}^b_{str}$ which induces a fully faithful functor in the associated triangulated categories yields a well defined morphism

$$\varphi_F : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})$$

in $\mathsf{Ho}(B_\infty)$ and φ_F is invertible if F induces an equivalence in the triangulated categories.

Now if $F : \mathcal{A} \to \mathcal{B}$ is a morphism of the localization $\mathcal{M}^b$ inducing a fully faithful functor in the triangulated categories, then, by construction, F is the composition of functors inducing fully faithful functors in the triangulated categories and formal inverses of functors inducing equivalences in the triangulated categories. By the preceding remark, we again obtain a well-defined morphism

$$\varphi_F : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})$$

in $\mathsf{Ho}(B_\infty)$ and φ_F is invertible if F induces an equivalence in the triangulated categories.

We deduce that there is a well defined functor

$$(\mathcal{M}^b_i)^{op} \to \mathsf{Ho}(B_\infty) \, , \; \mathcal{A} \mapsto C(\mathcal{A}, \mathcal{A})$$

defined on the subcategory of $\mathcal{M}^b$ with the same objects as $\mathcal{M}^b$ and whose morphisms are those morphisms which induce fully faithful functors in the triangulated categories.

5.2. **The Hochschild complex of a small exact category.** We keep the notations of the preceding paragraph. For a $\mathbf{U}$-small k-linear exact category $\mathcal{E}$, we can now define

$$C(\mathcal{E}, \mathcal{E}) = C(\mathcal{Q}, \mathcal{Q}) \, , \quad \text{where } \mathcal{Q} = \mathcal{C}^b(\mathcal{E})/\mathcal{A}c^b(\mathcal{E}).$$

We obtain a well-defined object of $\mathsf{Ho}(B_\infty)$. If $F : \mathcal{E} \to \mathcal{E}'$ is an exact functor between exact categories which induces a fully faithful functor in the derived categories, then F yields a well-defined morphism

$$F^* : C(\mathcal{E}', \mathcal{E}') \to C(\mathcal{E}, \mathcal{E})$$

in $\mathsf{Ho}(B_\infty)$.

Theorem. *If F induces an equivalence up to factors*

$$\mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}') \, ,$$

i.e. *a fully faithful functor such that each object of $\mathcal{D}^b(\mathcal{E}')$ is a direct factor of an object in the image, then F^* is an isomorphism*

$$C(\mathcal{E}', \mathcal{E}') \xrightarrow{\sim} C(\mathcal{E}, \mathcal{E})$$

in $\mathsf{Ho}(B_\infty)$.

Proof. Let $\mathcal{Q} = \mathcal{C}^b(\mathcal{E})/\mathcal{A}c^b(\mathcal{E})$ and define $\mathcal{Q}'$ similarly. By the assumption, F induces a morphism $\mathcal{Q} \to \mathcal{Q}'$ in $\mathcal{M}^b$ which induces an equivalence up to factors in the associated triangulated categories. It follows that the induced functor

$$F^+ : \mathcal{D}\mathcal{Q} \to \mathcal{D}\mathcal{Q}'$$

induces an equivalence up to factors between the subcategories of compact objects $\mathsf{per}(\mathcal{Q}) \to \mathsf{per}(\mathcal{Q}')$. But then F^+ has to be an equivalence. So the assertion follows from part b) of theorem 4.6. $\surd$

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