Notes for an Introduction to Kontsevich's quantization theorem

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ABSTRACT. In these notes, we present Kontsevich's theorem on the deformation quantization of Poisson manifolds, his formality theorem and Tamarkin's algebraic version of the formality theorem. We also introduce the necessary material from deformation theory.

Contents

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CHAPTER 1

Presentation of the main results

In this chapter, we state the main results of M. Kontsevich's eprint [**32**]. In sections 1.1 and 3.1, we largely follow the lucid presentation of [**45**].

1. Every Poisson manifold admits a formal quantization

1.1. Deformations and Poisson structures. Let k be a commutative ring and A a k-algebra (i.e. a k-module endowed with a k-bilinear map from $A \times A$ to A). Denote by $k[[t]]$ the ring of formal power series in an indeterminate t, and by $A[[t]]$ the $k[[t]]$-module of formal power series

$$\sum_{n=0}^{\infty} a_n t^n$$

with coefficients in A. Let $*$ be a *formal deformation of the multiplication of* A, i.e. a $k[[t]]$-bilinear map

$$A[[t]] \times A[[t]] \to A[[t]]$$

such that we have

$$u * v \equiv uv \mod tA[[t]]$$

for all power series $u, v \in A[[t]]$. The product of two elements a, b of A is then of the form

$$a * b = ab + B_1(a,b)\,t + \cdots + B_n(a,b)\,t^n + \cdots$$

for a sequence of k-bilinear maps B_i, and these determine the product $*$ because it is $k[[t]]$-bilinear. We put $B_0(a,b) = ab$ and we write

$$* = \sum_{n=0}^{\infty} B_n\,t^n.$$

Let J be the group of $k[[t]]$-module automorphisms g of $A[[t]]$ such that

$$g(u) \equiv u \mod tA[[t]]$$

for all $u \in A[[t]]$. We define two formal deformations $*$ and $*'$ to be *equivalent* if there is an element $g \in J$ such that

$$g(u * v) = g(u) *' g(v)$$

for all $u, v \in A[[t]]$. Note that, for $g \in J$ and $a \in A$, we have

$$g(a) = a + g_1(a)\,t + g_2(a)\,t^2 + \cdots + g_n(a)\,t^n + \cdots$$

for certain k-linear maps $g_i : A \to A$, and that these determine g because it is $k[[t]]$-linear.

One can show (*cf.* Cor. 4.5) that if A is associative and admits a unit element 1_A, then each associative formal deformation $*$ of the multiplication of A admits a unit element 1_*. Moreover, such an associative formal deformation $*$ is always equivalent to an associative formal deformation $*'$ such that $1_{*'} = 1_A$.

Suppose that A is associative and commutative.

LEMMA 1.1. *Let $*$ be an associative (but not necessarily commutative) formal deformation of the multiplication of A. For $a, b \in A$, put $\{a, b\} = B_1(a, b) - B_1(b, a)$.*

a) *The map $\{,\}$ is a* Poisson bracket *on A, i.e. a k-bilinear map such that*
   - *the bracket $\{,\}$ is a Lie bracket and*
   - *for all $a, b, c \in A$, we have $\{a, bc\} = \{a, b\}c + b\{a, c\}$.*

b) *The bracket $\{,\}$ only depends on the equivalence class of $*$.*

PROOF. a) The map

$$(u, v) \mapsto \frac{1}{t}(u * v - v * u)$$

clearly defines a Lie bracket on $A[[t]]$. Let us denote it by $[,]$. The bracket $\{,\}$ equals the reduction modulo t of $[,]$. Therefore, it is still a Lie bracket. The second equality follows from

$$[u, vw] = [u, v]w + u[v, w]$$

for all $u, v, w \in A[[t]]$.

b) If $g \in J$ yields the equivalence of $*$ with $*'$, then we have

$$B_1(a, b) + g_1(ab) = B_1'(a, b) + g_1(a)b + ag_1(b)$$

for all $a, b \in A$. Thus the difference $B_1(a, b) - B_1'(a, b)$ is symmetric in a, b and does not contribute to $\{,\}$. $\surd$

THEOREM 1.2 (Kontsevich [**32**]). *If A is the algebra of smooth functions on a differentiable manifold M, then each Poisson bracket on A lifts to an associative formal deformation.*

In other words, the map

$$\{\text{equivalence classes of formal deformations of } A\} \longrightarrow \{\text{Poisson brackets on } A\}$$

is surjective if A is the algebra of smooth functions on a differentiable manifold M. Moreover, Kontsevich constructs a section of this map. His construction is canonical and explicit for $M = \mathbf{R}^n$; it is canonical (up to equivalence) for general manifolds M. Below, we give two simple examples of formal deformations arising from Kontsevich's construction for $M = \mathbf{R}^n$ where the Poisson bracket is given respectively by a constant and by a linear bivector field. We also give a class of examples, due to Mathieu [**41**], of finite-dimensional Poisson algebras whose brackets do not lift to formal deformations.

1.2. Example: The Moyal-Weyl product. Let $M = \mathbf{R}^2$. Consider the Poisson bracket given by

$$\{f, g\} = \mu \circ \left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right)(f \otimes g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1},$$

where μ is the multiplication of functions on M. Then Kontsevich's construction yields the associative (!) formal deformation given by

$$f * g = \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial x_1^n}\frac{\partial^n g}{\partial x_2^n}\frac{t^n}{n!}.$$

More generally, for $n \geq 2$, let $M = \mathbf{R}^n$, and let τ_{ij} be a real number, $1 \leq i < j \leq n$. Consider the Poisson bracket defined by

$$\{f, g\} = \sum_{i<j} \tau_{ij}\, \mu \circ \left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) (f \otimes g).$$

Let L be the Lie algebra of the group of translations of $\mathbf{R}^n$. Then the bracket $\{,\}$ is given by the operator

$$\overline{r} = \sum_{i<j} \tau_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

considered as an element of $\Lambda^2 L$. Let $r \in L \otimes L$ be a preimage of $\overline{r}$ and let

$$F = \exp(tr)$$

viewed as an element of $U(L) \otimes U(L)[[t]]$, where $U(L)$ is the universal enveloping algebra of L. Then the bracket $\{,\}$ comes from the associative formal deformation given by

$$f * g = \mu \circ F\,(f \otimes g).$$

The associativity of this product follows from the identity (*cf.* [**19**])

$$((\Delta \otimes \mathrm{id})(F)) \cdot (F \otimes 1) = ((\mathrm{id} \otimes \Delta)(F)) \cdot (1 \otimes F)$$

in $U(L) \otimes U(L) \otimes U(L)[[t]]$, where Δ is the comultiplication of $U(L)$. The equivalence class of $*$ is independent of the choice of r. If

$$r = \sum_{i,j=1}^{n} \sigma_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

then

$$f * g = \sum_{n=0}^{\infty} \sum_{i_1, j_1, \ldots, i_n, j_n} \sigma_{i_1 j_1} \sigma_{i_2 j_2} \ldots \sigma_{i_n j_n} \frac{\partial^n f}{\partial x_{i_1} \ldots \partial x_{i_n}} \frac{\partial^n g}{\partial x_{j_1} \ldots \partial x_{j_n}} \frac{t^n}{n!}.$$

If we choose (σ_{ij}) antisymmetric, we obtain what is known as the Moyal-Weyl product associated with $\{,\}$.

1.3. Example: The dual of a Lie algebra. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Let $\mathfrak{g}^*$ denote the dual over $\mathbf{R}$ of $\mathfrak{g}$ and A the algebra of smooth functions on $\mathfrak{g}^*$. For $f \in A$ and $x \in \mathfrak{g}^*$, we can view the differential $(df)_x$ as an element of $\mathfrak{g}$. Using this identification we define a Poisson bracket on A by

$$\{f, g\}(x) = x\left([(df)_x, (dg)_x]\right), \quad x \in \mathfrak{g}^*.$$

Kontsevich's construction yields a canonical associative product $*$ on $A[[t]]$. This product is closely linked to that of the enveloping algebra of $\mathfrak{g}$: Let $S(\mathfrak{g})$ be the symmetric algebra on $\mathfrak{g}$. We identify it with the algebra of polynomial functions on $\mathfrak{g}^*$. It is not hard to show that the subspace $B = S(\mathfrak{g})[t]$ of $A[[t]]$ formed by the polynomials in t whose coefficients are polynomial functions on $\mathfrak{g}^*$ is a subalgebra for $*$. Moreover, the inclusion $\mathfrak{g} \to B$ induces an isomorphism

$$U_{hom}(\mathfrak{g}) \xrightarrow{\sim} B$$

where $U_{hom}(\mathfrak{g})$ is the homogeneous enveloping algebra, i.e. the $\mathbf{R}[t]$-algebra generated by $\mathfrak{g}$ with relations $XY - YX - t\,[X, Y]$, $X, Y \in \mathfrak{g}$. Thus the quotient $B/(t-1)B$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$.

1.4. Mathieu's examples [41]. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra such that $\mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$ is simple and not isomorphic to $\mathfrak{sl}_n(\mathbf{C})$ for any $n \geq 2$. The bracket of $\mathfrak{g}$ uniquely extends to a Poisson bracket on the symmetric algebra $S(\mathfrak{g})$. The ideal I of $S(\mathfrak{g})$ generated by all monomials of degree 2 is a Poisson ideal (i.e. we have $\{f, I\} \subset I$ for all $f \in S(\mathfrak{g})$). So the quotient $A = S(\mathfrak{g})/I$ becomes a (finite-dimensional) Poisson algebra. Assume that its bracket comes from a formal deformation $*$. Let B be the associative algebra $A[[t]]$ endowed with $*$ and let B_L be the Lie algebra obtained by endowing B with the commutator with respect to its multiplication. Then B_L is a formal deformation of the Lie algebra $A_L = (A, \{,\})$. Now A_L is isomorphic to $\mathbf{R} \oplus \mathfrak{g}$. Since $\mathfrak{g} \otimes \mathbf{C}$ is simple, we have $H^2(A_L, A_L) = 0$ so that A_L is rigid and B_L is isomorphic to $A_L \otimes \mathbf{R}[[t]]$ (as a Lie algebra over $\mathbf{R}[[t]]$). Let K be the algebraic closure of the fraction field of $\mathbf{R}[[t]]$. By extending the scalars to K we find that $B_L \otimes_{\mathbf{R}[[t]]} K$ is isomorphic to $A_L \otimes_{\mathbf{R}[[t]]} K = K \oplus (\mathfrak{g} \otimes_{\mathbf{R}} K)$. But $B_L \otimes_{\mathbf{R}[[t]]} K$ is the Lie algebra associated with the finite-dimensional associative K-algebra $B \otimes_{\mathbf{R}[[t]]} K$. Since K is algebraically closed, this algebra is isomorphic to $M \oplus J$, where M is a product of matrix rings over K and J is nilpotent. Therefore, the only simple quotients of its associated Lie algebra are isomorphic to $\mathfrak{sl}_n(K)$ for certain $n \geq 2$. However, by assumption, $A_L \otimes_{\mathbf{R}[[t]]} K$ admits the simple quotient $\mathfrak{g}\otimes_{\mathbf{R}} K$. This contradiction shows that the bracket of A cannot come from a formal deformation.

2. Kontsevich's explicit formula

Let $d \geq 1$ be an integer and M a non empty open subset of $\mathbf{R}^d$. Let A be the algebra of smooth functions on M and $\{,\}$ a Poisson bracket on A. In this situation, Kontsevich [**32**] gives an explicit formula for a canonical formal quantization $*_K$. We describe his formula in this section.

Recall that derivations of the algebra A are given by vector fields on M. From this fact, one deduces the

Lemma 2.1. *There are unique smooth functions α^{ij}, $1 \leq i < j \leq d$, such that*

$$\{,\} = \sum_{i<j} \alpha^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \ ,$$

i.e. *for all smooth functions f and g, we have*

$$\{f, g\} = \sum_{i<j} \alpha^{ij} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right).$$

We write the quantization to be constructed in the form

$$*_K = \sum_{n=0}^{\infty} B_n t^n.$$

For smooth functions f, g, we will express $B_n(f, g)$ as a linear combination of sums of products of partial derivatives of the α^{ij} and of f and g. To describe the terms explicitly, we need a little combinatorics: A *quiver* Γ is given by

- a set Γ_0, whose elements are called the *vertices* of Γ,
- a set Γ_1, whose elements are called the *arrows* of Γ,
- two maps $s : \Gamma_1 \to \Gamma_0$ and $t : \Gamma_1 \to \Gamma_0$, which, with an arrow, associate its *source* and its *target*.

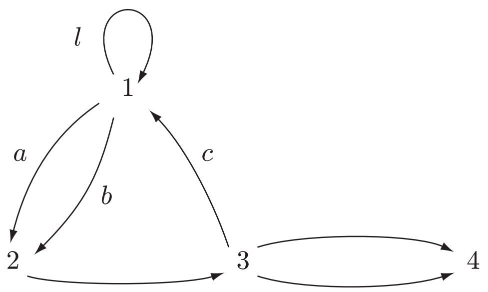

FIGURE 1. A quiver

In practice, quivers are given by drawings as in figure 1, where Γ_0 equals $\{1, 2, 3, 4\}$, Γ_1 has 7 elements and, for example, $s(c) = 3$, $t(c) = 1$. An arrow l of a quiver is a *loop* if $s(l) = t(l)$. A pair (a, b) of arrows is a *double arrow* if $s(a) = s(b)$ and $t(a) = t(b)$.

Let $n \geq 0$. We define G_n to be the set of quivers Γ such that

- $\Gamma_0 = \{1, \dots, n\} \cup \{L, R\}$, where L and R are two symbols,
- $\Gamma_1 = \{a_1, b_1, \dots, a_n, b_n\}$, where the a_i and b_i are symbols,
- for each i, we have $s(a_i) = s(b_i) = i$, and
- Γ has neither loops nor double arrows.

The unique quiver in G_0 has only the two vertices L and R and no arrow. The set G_1 contains exactly two quivers, namely

$$L \xleftarrow{a_1} 1 \xrightarrow{b_1} R \quad \text{and} \quad L \xleftarrow{b_1} 1 \xrightarrow{a_1} R\ .$$

Figure 2 shows two among the 36 quivers in G_2 and one among the 160000 quivers in G_4. In general, G_n contains $(n(n + 1))^n$ quivers. Given smooth functions f, g

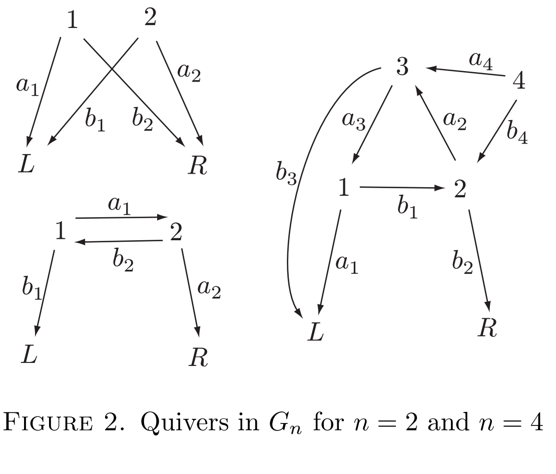

FIGURE 2. Quivers in G_n for $n = 2$ and $n = 4$

and a quiver $\Gamma \in G_n$, we will define a function $B_{\Gamma,\alpha}(f, g)$. For example, for the last quiver of figure 2, we have

$$B_{\Gamma,\alpha}(f,g) = \sum \left(\partial_{i_3}\alpha^{i_1,j_1}\right)\left(\partial_{j_1}\partial_{j_4}\alpha^{i_2,j_2}\right)\left(\partial_{i_2}\partial_{i_4}\alpha^{i_3,j_3}\right)\left(\partial_{i_1}\partial_{j_3} f\right)\left(\partial_{j_2} g\right)\ ,$$

where $i_1, j_1, \ldots, i_4, j_4$ range from 1 to d (the dimension of $M \subset \mathbf{R}^d$) and ∂_k denotes the partial derivative with respect to x_k. Note that each vertex v of the quiver corresponds to a factor and that the partial derivatives with respect to x_{i_k} or x_{j_k} correspond to the edges a_k or b_k with target v. For a general quiver Γ in G_n, we define $B_{\Gamma,\alpha}(f,g)$ to be

$$\sum \left(\prod_{i=1}^{n} \left(\prod_{a \in \Gamma(?,i)} \partial_{I(a)} \right) \alpha^{I(a_i),I(b_i)} \right) \left(\prod_{a \in \Gamma(?,L)} \partial_{I(a)} \right)(f) \left(\prod_{a \in \Gamma(?,R)} \partial_{I(a)} \right)(g) \; ,$$

where $\Gamma(?,v)$ denotes the set of arrows with target v and the sum ranges over all maps I from Γ_1 to $\{1, \ldots, d\}$.

We will define

$$B_n = \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,\alpha}$$

for certain universal constants $w_\Gamma \in \mathbf{R}$, which we now construct. For this, let $\mathcal{H}$ denote the upper half plane $\operatorname{im} z > 0$. We endow $\mathcal{H}$ with the hyperbolic metric. Its geodesics are the vertical half lines and the half circles whose center is on the real axis. For two distinct points p, q of $\mathcal{H}$, we define $l(p,q)$ to be the geodesic from p to q and we define $l(p,\infty)$ to be the vertical half line going from p to infinity. We denote by $\varphi(p,q)$ the angle from $l(p,\infty)$ to $l(p,q)$. As we see from figure 3, we have

$$\varphi(p,q) = \arg\left(\frac{q-p}{q-\overline{p}}\right) = \frac{1}{2i} \log \left(\frac{q-p}{q-\overline{p}} \cdot \frac{\overline{q}-p}{\overline{q}-\overline{p}} \right) .$$

This shows that $(p,q) \mapsto \varphi(p,q)$ is analytic. It is also clear that it admits a

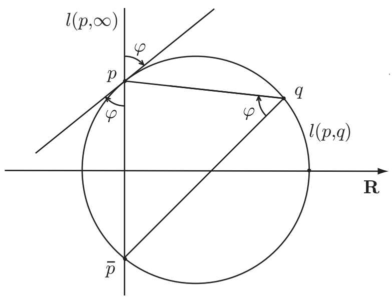

FIGURE 3. Planimetry

continuous extension to the set of pairs of complex numbers (p,q) such that $\operatorname{im} p \geq 0$, $\operatorname{im} q \geq 0$ and $p \neq q$.

Now for $n \geq 0$, let $\mathcal{H}_n$ be the set of n-tuples $(p_1, \ldots, p_n)$ of distinct points of $\mathcal{H}$. Given Γ in G_n, we interpret $\mathcal{H}_n$ geometrically as the set of all 'geodesic drawings' of Γ in the closure of $\mathcal{H}$: the vertices $1, \ldots, n$ of Γ correspond to the p_i, the vertices L and R to the points 0 and 1 of the real axis and each arrow of Γ is represented by a geodesic segment from its source point to its target point, *cf.* figure 4. With this in mind, for each arrow a of Γ, we define the function $\varphi_a : \mathcal{H}_n \to \mathbf{R}$ by

$$\varphi_a(p_1, \ldots, p_n) = \varphi(p_{s(a)}, p_{t(a)}) \; ,$$

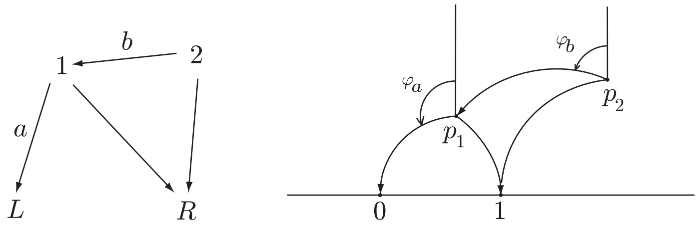

FIGURE 4. Geodesic drawing of a quiver

where we put $p_L = 0$ and $p_R = 1$. Finally, we define

$$w_\Gamma = \frac{1}{(2\pi)^n} \int_{\mathcal{H}_n} \bigwedge_{i=1}^{n} (d\varphi_{a_i} \wedge d\varphi_{b_i}).$$

Note that the integrand is a $2n$-form, and the integral is taken over a naturally oriented $2n$-dimensional manifold.

LEMMA 2.2. *The integral converges absolutely.*

For this, one shows that the integrand admits a continuous extension to a compactification of $\mathcal{H}_n$, *cf.* [**32**].

THEOREM 2.3 (Kontsevich [**32**]). *The formula*

$$f *_K g = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,\alpha}(f, g)$$

defines a formal quantization of the given Poisson bracket. Its equivalence class is independent of the choice of coordinates in M.

The essential point is that $*_K$ is associative. Kontsevich deduces this from Stokes' theorem applied to compactifications of configuration spaces $\mathcal{H}_n$. He remarks in passing that the formula has a certain physical interpretation. This claim was made precise by Cattaneo and Felder in [**4**].

3. A more precise version of Kontsevich's theorem and its link with the Duflo isomorphism

3.1. Formal quantizations. Let M be a differentiable manifold and A the algebra of smooth functions on M. Let $m \geq 1$. A *multidifferential operator on M* is a map $P : A^m \to A$ compatible with restrictions to open subsets and such that, in each system $x_1, \dots, x_n$ of local coordinates on M, we have

$$P(f_1, \dots, f_m) = \sum a_{\nu_1,\dots,\nu_m} \left(\frac{\partial^{|\nu_1|}}{\partial x_1^{\nu_1}} f_1 \right) \cdots \left(\frac{\partial^{|\nu_m|}}{\partial x_1^{\nu_m}} f_m \right) ,$$

where the ν_i are multi-indices and the $a_{\nu_1,\dots,\nu_m}$ are smooth functions which vanish for almost all $(\nu_1, \dots, \nu_m)$.

A *star product* on M is an associative formal deformation $* = \sum B_n t^n$ such that the B_n are bidifferential operators. Note that the multiplication B_0 of A is such an operator. Let J_d denote the group of $\mathbf{R}[[t]]$-module automorphisms $g = \sum g_n t^n$ of $A[[t]]$ such that g_0 is the identity and all g_n are differential operators. Two star products $*$ and $*'$ are *equivalent* if there is a $g \in J_d$ such that $g(u * v) = g(u) *' g(v)$ for all $u, v \in A[[t]]$.

As in lemma 1.1, each star product $*$ on M gives rise to a Poisson bracket $\{,\}$. We call $*$ a *formal quantization* of $\{,\}$. The Moyal-Weyl product (1.2) is an example.

THEOREM 3.1 (Kontsevich [**32**]). a) *Each Poisson bracket on A admits a formal quantization, canonical up to equivalence.*

b) *There is a bijection $[\pi] \mapsto [*_\pi]$ from the set of equivalence classes of Poisson brackets*

$$\pi = 0 + \pi_1 t + \cdots + \pi_n t^n + \cdots$$

on the commutative $\mathbf{R}[[t]]$-algebra $A[[t]]$ to the set of equivalence classes of star products on M. Moreover, if π corresponds to $*_\pi$, then the Poisson bracket on A associated with $*_\pi$ equals the coefficient π_1 of t in π.

In b), two Poisson brackets are *equivalent* if they are conjugate by an $\mathbf{R}[[t]]$-algebra automorphism belonging to J_d. The canonical quantization of a given Poisson bracket $\{,\}$ in a) is obtained by applying b) to $\pi = \{,\}\, t$.

3.2. Isomorphism of cohomology algebras. Let π be as in theorem 3.1 and let $n \geq 2$. By reduction, π defines a Poisson bracket on $A[[t]]/(t^n)$. In particular, this space becomes a Lie algebra over $\mathbf{R}[t]/(t^n)$. The associated Chevalley-Eilenberg complex (*cf.* section 5.2 of chapter 2) with coefficients in $A[[t]]/(t^n)$ admits a subcomplex, which we denote by

$$C_{Pois}(A[[t]]/(t^n), \pi) ,$$

whose pth component is formed by the cochains which are derivations in each argument (for the commutative multiplication of $A[[t]]/(t^n)$). We define

$$C_{Pois}(A[[t]], \pi) \tag{3.1}$$

to be the inverse limit of the system of complexes $C_{Pois}(A[[t]]/(t^n), \pi)$, $n \geq 2$.

Let $*$ be a formal deformation of the multiplication of A and let $n \geq 2$. By reduction, the multiplication $*$ defines an associative $\mathbf{R}[t]/(t^n)$-algebra structure on $A[[t]]/(t^n)$. We denote by

$$C_{star}(A[[t]]/(t^n), *)$$

the subcomplex of the associated Hochschild complex (*cf.* section 2 of chapter 2) with coefficients in $A[[t]]/(t^n)$ whose p-cochains have coefficients which are p-differential operators. We define

$$C_{star}(A[[t]], *) \tag{3.2}$$

to be the inverse limit of the system of complexes $C_{star}(A[[t]]/(t^n), *)$, $n \geq 2$. Let μ denote the commutative multiplication of $A[[t]]$. The complexes (3.1) and (3.2) are endowed with (associative) cup products extending the multiplications μ and $*$ on their 0th components. Their homologies

$$H^*_{Pois}(A[[t]], \pi) \text{ and } H^*_{star}(A[[t]], *)$$

become graded commutative algebras when endowed with the multiplications induced by the cup products.

THEOREM 3.2 (Kontsevich [**32**]). *Suppose that M is an open subset of $\mathbf{R}^n$. Then for each Poisson bracket π as in the preceding theorem, there is a canonical quasi-isomorphism*

$$\Psi_\pi : C_{Pois}(A[[t]], \pi) \to C_{star}(A[[t]], *_\pi)$$

which induces an algebra isomorphism

$$H^*_{Pois}(A[[t]], \pi) \to H^*_{star}(A[[t]], *_\pi).$$

The existence of the quasi-isomorphism Ψ_π follows easily from Kontsevich's formality theorem 4.1 below, *cf.* section 6 of chapter 2. In contrast, the fact that Ψ_π is compatible with the algebra structure in cohomology is highly non trivial. In the case where M is the dual of a finite-dimensional Lie algebra (*cf.* 1.3) and $\pi = t\{,\}$, it is easy to see that Ψ_π induces an algebra isomorphism

$$H^*(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} HH^*(U(\mathfrak{g}), U(\mathfrak{g})) \ ,$$

where the right hand side denotes the Hochschild cohomology algebra of $U(\mathfrak{g})$. In particular, in degree 0, we obtain an algebra isomorphism

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(U(\mathfrak{g}))$$

from the algebra of $\mathfrak{g}$-invariant polynomials on $\mathfrak{g}^*$ to the center of $U(\mathfrak{g})$. Kontsevich shows that it coincides with the Duflo isomorphism [**11**] [**12**].

4. On the proofs

4.1. Deformation theory. Let A be the algebra of smooth functions on a differentiable manifold M. The main theorem 3.1 asserts that two deformation problems are equivalent: that of deforming the zero Poisson bracket on A and that of deforming the commutative multiplication μ on A. Now 'every' deformation problem can be described in terms of a differential graded Lie algebra (=dg Lie algebra). In our case, we denote the corresponding dg Lie algebras by $L_{Pois}(M)$ and $L_{star}(M)$; their underlying complexes are

$$C_{Pois}(A, 0)[1] \quad \text{and} \quad C_{star}(A, \mu)[1] \ ,$$

where, for a complex K, we denote by $K[1]$ the *shifted complex*: $K[1]^p = K^{p+1}$, $d_{K[1]} = -d_K$.

In deformation theory, one shows that if $f : L \to L'$ is a quasi-isomorphism of differential graded Lie algebras (i.e. a morphism inducing isomorphisms in homology), then f induces an equivalence between the corresponding deformation problems. Therefore, to prove the main theorem 3.1, it is enough to show that there is a chain

$$L_{Pois}(M) \leftarrow L_1 \to \ldots \leftarrow L_n \to L_{star}(M)$$

of quasi-isomorphisms. Now in general, one can show that the existence of such a chain linking two dg Lie algebras L and L' is equivalent to that of an L_∞-*quasi-isomorphism* $\mathcal{U} : L \to L'$, i.e. a sequence of morphisms

$$\mathcal{U}_n : L^{\otimes n} \to L' \ , \ \ n \geq 1 \ ,$$

which are homogeneous of degree $1-n$, graded antisymmetric and satisfy a sequence of compatibility conditions with the brackets and the differentials of L and L'. We will review the relevant facts from deformation theory in more detail in chapter 2.

4.2. Kontsevich's proof. In the case where M is an open subset of $\mathbf{R}^n$, Kontsevich [**32**] explicitly constructed an L_∞-quasi-isomorphism

$$\mathcal{U}^M : L_{Pois}(M) \to L_{star}(M). \tag{4.1}$$

using integrals on configuration spaces. His formulas allowed him to write down an explicit star product for a given Poisson bracket on $M \subset \mathbf{R}^n$ and in particular for the canonical Poisson bracket on the dual of a finite dimensional Lie algebra (*cf.* 1.3).

Kontsevich's L_∞-morphism (4.1) is equivariant with respect to the group of affine transformations of $\mathbf{R}^n$. Using this fact and a sophisticated gluing procedure Kontsevich proved the

THEOREM 4.1 (Formality Theorem [**32**]). *For each differentiable manifold M, there is an L_∞-quasi-isomorphism $\mathcal{U}^M : L_{Pois}(M) \to L_{star}(M)$.*

4.3. Formality. A dg Lie algebra L is *formal* if it is linked to the dg Lie algebra H^*L (endowed with $d = 0$ and the bracket induced from that of L) by a chain of quasi-isomorphisms (equivalently: by an L_∞-quasi-isomorphism). The Hochschild-Kostant-Rosenberg theorem [**27**] yields that, for a differentiable manifold M, the homology of $L_{star}(M)$ is isomorphic to $L_{Pois}(M)$. Therefore, Kontsevich's formality theorem 4.1 means that for each differentiable manifold M, the dg Lie algebra $L_{star}(M)$ is formal.

4.4. Tamarkin's proof. In [**51**], D. Tamarkin gave a new proof of Kontsevich's formality theorem 4.1 for the case of $M = \mathbf{R}^n$. More precisely, he proved the following purely algebraic statement: Let k be a field of characteristic 0, let V be a finite-dimensional k-vector space and SV the symmetric algebra on V. The problem of deforming the multiplication of SV is described by the dg Lie algebra

$$L_{alg}(V^*) = C_{Hoch}(SV, \mu)[1] ,$$

i.e. the shifted Hochschild complex endowed with the Gerstenhaber bracket, *cf.* section 3 of chapter 2.

THEOREM 4.2 (Tamarkin [**51**]). *The dg Lie algebra $L_{alg}(V^*)$ is formal.*

It is easy see, *cf.* Lemma 1.2 of chapter 3, that for $k = \mathbf{R}$, the dg Lie algebra $L_{alg}(V^*)$ is linked to $L_{star}(V^*)$ by a chain of quasi-isomorphisms. Thus, Tamarkin's theorem is equivalent to the formality theorem for $k = \mathbf{R}$ and $M = \mathbf{R}^n$. We outline Tamarkin's proof in chapter 3.

5. Notes

Kontsevich's theorem 1.2 solves a conjecture which goes back to the pioneering work [**1**] by Bayen-Fronsdal-Lichnerowicz-Sternheimer. An account of the history and the motivations from physics can be found in [**8**] and [**55**]. Kontsevich's proof [**32**] of the isomorphism of cohomology algebras of Theorem 3.2 was made precise by Manchon-Torossian [**39**]. T. Mochizuki [**44**] proves that this isomorphism lifts to an A_∞-quasi-isomorphism.

The linear map underlying the Duflo isomorphism was constructed by M. Duflo in [**11**]. There he also showed that it was an algebra isomorphism for solvable and semisimple Lie algebras. Later, he showed in [**12**] that it is an isomorphism for arbitrary finite-dimensional Lie algebras.

Kontsevich deduces the formality theorem for arbitrary Poisson manifolds from the case of an open set in $\mathbf{R}^n$. The proof of this 'globalization theorem' in [**32**] is not very detailed. More details are given in the appendix to [**33**]. Alternative approaches to globalization are due to Cattaneo-Felder-Tomassini [**5**] and V. Dolgushev [**9**].

In an algebraic context, the quantization problem was studied by Kontsevich [**34**] and A. Yekutieli [**56**].

Covariant versions of the formality theorem were conjectured by B. Tsygan [**53**] and proved by B. Shoikhet [**48**].

In studying the non uniqueness of the formality morphism Kontsevich has discovered surprising links to motives and the Grothendieck-Teichmueller group, *cf.* [**33**] and [**50**].

CHAPTER 2

Deformation theory

1. Notations

Let k be a commutative ring. A *graded k-module* is a sequence $K = (K^p)$, $p \in \mathbf{Z}$, of k-modules K^p. A *morphism of degree n* between graded k-modules is a sequence $f : K \to L$ of morphisms $f^p : K^p \to L^{p+n}$. Such morphisms are composed in the natural way. A *complex* K is a graded k-module endowed with a *differential*, *i.e.* an endomorphism $d : K \to K$ of degree 1 such that $d^2 = 0$. The *suspension* or *shift* of a graded k-module K is the graded k-module denoted by SK or $K[1]$ with $(K[1])^p = K^{p+1}$, $p \in \mathbf{Z}$. If K is a complex with differential d, its suspension $SK = K[1]$ is endowed with the differential $-d$.

The *tensor product $L \otimes K$ of two graded k-modules* is the $\mathbf{Z}$-graded k-module with components

$$(L \otimes K)^n = \bigoplus_{p+q=n} L^p \otimes_k K^q.$$

The *tensor product of two morphisms f and g* is defined by

$$(f \otimes g)(x \otimes y) = (-1)^{pq} f(x) \otimes g(y)$$

where g is of degree p and x of degree q. The flip $\tau : L \otimes K \to K \otimes L$ is defined by

$$\tau(x \otimes y) = (-1)^{pq} y \otimes x \ ,$$

where x is of degree p and y of degree q. Let L be a graded k-module. A multiplication map $\mu : L \otimes L \to L$ is *graded commutative* if $\mu \circ \tau = \mu$. The *tensor coalgebra* $T^c(L)$ is the direct sum of the tensor powers $L^{\otimes n}$, $n \geq 0$. Its comultiplication is defined by

$$\Delta(x_1, \ldots, x_n) = \sum_{i=0}^{n} (x_1, \ldots, x_i) \otimes (x_{i+1}, \ldots, x_n)$$

and its counit $\eta : T^c(L) \to k$ is the canonical projection. The flips yield an action of the symmetric group S_n on the nth tensor power of L, for each n. The *symmetric coalgebra* $\mathrm{Sym}^c(L)$ is the subcoalgebra of $T^c(L)$ whose underlying graded module is the sum of the fixed point modules of S_n on $L^{\otimes n}$, $n \geq 0$.

2. R-deformations and the Hochschild complex

Let k be a field and A an associative k-algebra with multiplication μ. Let R be a commutative local k-algebra whose maximal ideal $\mathfrak{m}$ is finite-dimensional over k (and thus nilpotent). The truncated polynomial rings $k[t]/(t^n)$, $n \geq 1$, are good examples to keep in mind. An *R-deformation* of the multiplication of A is an associative R-bilinear multiplication $*$ on $A \otimes_k R$ which, modulo $\mathfrak{m}$, reduces to the multiplication μ of A, i.e. the square

$$\begin{tikzcd}
(A \otimes_k R) \otimes_R (A \otimes_k R) \arrow[r] \arrow[d, "*"'] & A \otimes_k A \arrow[d, "\mu"] \\
A \otimes_k R \arrow[r] & A
\end{tikzcd}$$

commutes. An *infinitesimal deformation* is a $k[t]/(t^2)$-deformation. Two R-deformations are *equivalent* if there is an R-module automorphism $g : A \otimes_k R \to A \otimes_k R$ which, modulo $\mathfrak{m}$, reduces to the identity of A, such that

$$g(u * v) = g(u) *' g(v)$$

for all u, v in $A \otimes_k R$. Note that, by R-bilinearity, an R-deformation is determined by the restriction of $*$ to $A \otimes_k A$ and, in fact, by its component

$$A \otimes_k A \to A \otimes_k \mathfrak{m}.$$

We *denote by* $\mathrm{Defo}(A, R)$ the set of equivalence classes of R-deformations of A. In fact, we obtain a functor, the *deformation functor associated with the associative algebra A*

$$\mathcal{R} \to \mathrm{Sets}\ , \quad R \mapsto \mathrm{Defo}(A, R)\ ,$$

where $\mathcal{R}$ denotes the *category of test algebras, i.e.* of commutative local k-algebras with finite-dimensional maximal ideal.

If $*$ is a formal deformation (in the sense of 1.1 of chapter 1) then, for each $n \geq 1$, its reduction modulo (t^n) is a $k[t]/(t^n)$-deformation. We obtain a map

$$\{\text{formal deformations}\} \longrightarrow \varprojlim \{\, k[t]/(t^n) - \text{deformations}\}\ .$$

It is not hard to see that this map is bijective and that the equivalence relations on both sides correspond to each other. Thus, the study of formal deformations reduces to that of the deformation functor.

Let us take a closer look at infinitesimal deformations: an infinitesimal deformation $*$ is determined by a k-linear map $B_1 : A \otimes_k A \to A$ such that

$$a * b = ab + B_1(a, b)t$$

for all a, b in A. The associativity of $*$ translates into

$$aB_1(b, c) - B_1(ab, c) + B_1(a, bc) - B_1(a, b)c = 0 \tag{2.1}$$

for all a, b, c in A and two infinitesimal deformations corresponding to B_1 and B_1' are equivalent iff there is a k-linear map $g_1 : A \to A$ such that

$$B_1'(a, b) - B_1(a, b) = ag_1(b) - g_1(ab) + g_1(a)b \tag{2.2}$$

for all a, b in A.

The *Hochschild complex* of A is the complex $C(A, A)$ with vanishing components in degrees $p < 0$ and whose pth component, for $p \geq 0$, is the space $\mathrm{Hom}_k(A^{\otimes p}, A)$.

By definition, the differential of a p-cochain f is the $(p+1)$-cochain defined[1] by

$$(-1)^p(df)(a_0,\dots,a_p) = a_0 f(a_1,\dots,a_p) - \sum_{i=0}^{p-1}(-1)^i f(a_0,\dots,a_ia_{i+1},\dots,a_p) + (-1)^{p-1} f(a_0,\dots,a_{p-1})a_p.$$

The *Hochschild cohomology $HH(A,A)$ of A* (with coefficients in A) is by definition the homology of the Hochschild complex.

It follows from the formulas (2.1) and (2.2) that there is a canonical bijection between the set of equivalence classes of infinitesimal deformations and the space $HH^2(A,A)$. It is also useful to note that the 1-cocyles of $C(A,A)$ are precisely the derivations of A with the 1-coboundaries corresponding to the inner derivations. Finally, the 0-coboundaries vanish and the space of 0-cocycles equals the center of A.

In order to describe non infinitesimal deformations of A, we need a finer structure on the Hochschild complex, namely the Gerstenhaber bracket.

3. The Gerstenhaber bracket

We keep the notations of the preceding section. Let f be a Hochschild p-cochain and g a q-cochain. The *Gerstenhaber product* of f by g is the $(p+q-1)$-cochain defined by

$$(f \bullet g)(a_1,\dots,a_{p+q-1}) = \sum_{i=0}^{p}(-1)^{i(q+1)} f(a_1,\dots,a_i, g(a_{i+1},\dots,a_{i+q}), a_{i+q+1},\dots,a_{p+q-1})$$

The Gerstenhaber product is not associative in general. However, its associator

$$A(f,g,h) = (f\bullet g)\bullet h - f\bullet(g\bullet h)$$

is (super) symmetric in g and h in the sense that

$$A(f,g,h) = (-1)^{(q-1)(r-1)}A(f,h,g)$$

for a q-cochain g and an r-cochain h, *cf.* figure 1. One checks that this implies that the *Gerstenhaber bracket* defined by

$$[f,g] = f\bullet g - (-1)^{(p-1)(q-1)} g\bullet f$$

satisfies the (super) Jacobi identity (3.2) below. Moreover, the Hochschild differential is expressed in terms of the Gerstenhaber bracket and the multiplication μ of A as

$$df = -[\mu, f]. \tag{3.1}$$

It follows that the shifted (*cf.* section 1) Hochschild complex

$$L_{As}(A) = C(A,A)[1]$$

endowed with the Gerstenhaber bracket is a differential graded Lie algebra in the sense of the

[1] The sign differs by a factor $(-1)^p$ from that in [**3**]. This is justified by formula (3.1) below and the relation of the Hochschild complex with the Hochschild resolution of the bimodule A.

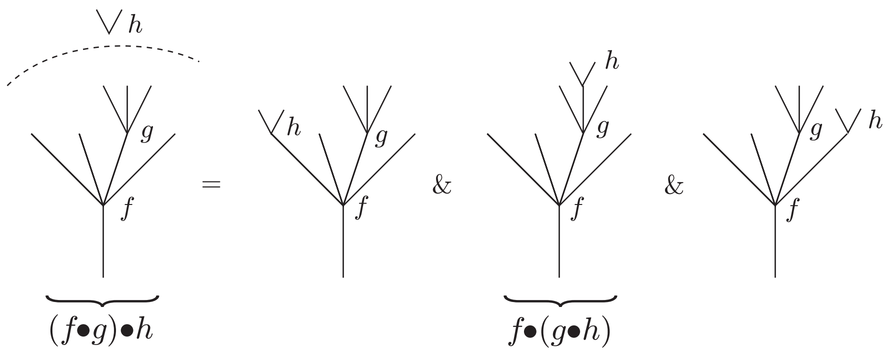

FIGURE 1. Terms occuring in $(f \bullet g) \bullet h$

DEFINITION 3.1. A **Z**-*graded Lie algebra* is a **Z**-graded vector space L endowed with a *Lie bracket*, *i.e.* a linear map

$$[,] : L \otimes L \to L$$

homogeneous of degree 0 which is

- antisymmetric, *i.e.* $[x, y] = (-1)^{pq}[y, x]$ for all $x \in L^p$ and all $y \in L^q$ and
- satisfies the Jacobi identity

$$[x, [y, z]] = [[x, y], z] + (-1)^{pq}[y, [x, z]] \tag{3.2}$$

for all $x \in L^p$, $y \in L^q$ and $z \in L$.

A *differential graded (=dg) Lie algebra* is a **Z**-graded Lie algebra L endowed with a differential d which is a derivation with respect to the bracket, *i.e.*

$$d([x, y]) = [dx, y] + (-1)^p[x, dy]$$

for all $x \in L^p$ and $y \in L$.

Note that if L is a dg Lie algebra, L^0 is an ordinary Lie algebra. For example, if $L = L_{As}(A)$, then $L^0 = \mathfrak{gl}(A)$.

Now let $R \in \mathcal{R}$ be a test algebra (section 2). Let $*$ be an R-deformation and let $B : A \otimes A \to A \otimes \mathfrak{m}$ be such that

$$a * b = ab + B(a, b)$$

for all $a, b \in A$. We view B as a homogeneous element of degree 1 of the dg Lie algebra $L_{As}(A) \otimes_k \mathfrak{m}$. Then one checks that the associativity of $*$ is expressed by the *Maurer-Cartan equation*

$$dB + \frac{1}{2}[B, B] = 0 ,$$

where we suppose that the ground field k is not of characteristic 2. We see that the solutions of this equation bijectively correspond to the R-deformations of the multiplication of A. The aim of the next section is to express the equivalence of two R-deformations in terms of the dg Lie algebra $L_{As}(A) \otimes_k \mathfrak{m}$.

4. The Maurer-Cartan equation

From now on, we suppose that the ground field k is of characteristic 0.

Let L be a differential graded Lie algebra (*cf.* section 3). Let $MC(L)$ denote the set of solutions $x \in L^1$ of the *Maurer-Cartan equation*

$$d(x) + \frac{1}{2}[x, x] = 0. \tag{4.1}$$

For $x \in MC(L)$, define $T_x MC(L)$ to be the space of vectors $X \in L^1$ such that

$$d(X) + [x, X] = 0.$$

Note that if L^1 is finite-dimensional, then $MC(L)$ is an intersection of quadrics and $T_x MC(L)$ is the (scheme-theoretic) tangent space at x of the algebraic variety $MC(L)$. For $x, y \in L$, put $(\operatorname{ad} x)(y) = [x, y]$.

Lemma 4.1. *Let $x \in MC(L)$.*

a) *The map $d_x = d + \operatorname{ad} x$ satisfies $d_x^2 = 0$.*
b) *A vector $X \in L^1$ belongs to $T_x MC(L)$ iff $d_x X = 0$.*
c) *For $X_0 \in L^0$, the map $y \mapsto d_y X_0$ yields a vector field on $MC(L)$,* i.e. *$d_y X_0 \in T_y MC(L)$ for all $y \in MC(L)$.*

Proof. a) is an easy computation, b) is immediate from a) and c) follows from b) and a). $\surd$

Now suppose that L^0 is a nilpotent Lie algebra (*i.e.* there is an $N \gg 0$ such that each composition of at least N maps $\operatorname{ad}(X)|_{L_0}$, $X \in L^0$, vanishes). Suppose moreover that the action of L^0 on L^1 is nilpotent (*i.e.* $\operatorname{ad} X_0$ induces a nilpotent endomorphism of L^1 for each $X_0 \in L^0$). Denote by $\operatorname{Aff}(L^1)$ the group of affine transformations of the vector space L^1 (the semi-direct product of the group of translations by that of linear transformations). Thanks to our nilpotency hypotheses, the Lie algebra antihomomorphism

$$L^0 \to \mathsf{Lie}(\operatorname{Aff}(L^1)) \ , \ \ X_0 \mapsto (x \mapsto d_x X_0 = dX_0 + [x, X_0])$$

integrates to a group antihomomorphism

$$\exp(L^0) \to \operatorname{Aff}(L^1)$$

so that we obtain a right action of $\exp(L^0)$ on L^1 by affine automorphisms. By point c) of the lemma, this action leaves $MC(L)$ invariant, so that we have a well defined orbit set $MC(L)/\exp(L^0)$. Notice that for $x \in MC(L)$, the 'normal space' to the orbit $x \exp(L^0)$ at x is

$$T_x MC(L)/T_x(x \exp(L^0)) = (\ker d_x)/(\operatorname{im} d_x) = H^1(L, d_x).$$

Now let $R \in \mathcal{R}$ be a test algebra (section 2) and L an arbitrary dg Lie algebra. We define

$$MC(L, R) = MC(L \otimes_k \mathfrak{m}).$$

Clearly, the dg Lie algebra $L \otimes_k \mathfrak{m}$ satisfies the nilpotency assumptions we made above. We can thus define

$$\overline{MC}(L, R) = MC(L \otimes_k \mathfrak{m})/\exp(L^0 \otimes_k \mathfrak{m}).$$

This definition is motivated by the

LEMMA 4.2. *Let A be an associative algebra. Then the dg Lie algebra $L_{As}(A)$* controls *the deformations of the multiplication of A,* i.e. *there are bijections*

$$\mathrm{Defo}(A,R) \to \overline{MC}(L_{As}(A),R)$$

functorial in $R \in \mathcal{R}$.

To prove the lemma, one checks that the bijection given at the end of section 3 is compatible with the equivalence relations.

A *morphism of dg Lie algebras* $f : L_1 \to L_2$ is a linear map homogeneous of degree 0 such that

$$f \circ d = d \circ f \text{ and } f([x,y]) = [f(x), f(y)]$$

for all $x, y \in L_1$. It is a *quasi-isomorphism of dg Lie algebras* if it induces an isomorphism in homology.

THEOREM 4.3 (Quasi-isomorphism theorem). *Let $f : L_1 \to L_2$ be a quasi-isomorphism of dg Lie algebras and let $R \in \mathcal{R}$. Then f induces a bijection*

$$\overline{MC}(L_1,R) \xrightarrow{\sim} \overline{MC}(L_2,R). \tag{4.2}$$

Note that the conclusion of the theorem concerns the solutions of systems of *quadratic* equations whereas the hypothesis that f is a quasi-isomorphism is *linear* in nature. The following proposition can be interpreted by saying that the map 4.2 induces bijections in the 'differential graded tangent spaces'. It is proved by considering the (finite!) filtrations induced by the $\mathfrak{m}^i \subset \mathfrak{m}$.

PROPOSITION 4.4. *Let $f : L_1 \to L_2$ be a quasi-isomorphism of dg Lie algebras and let $R \in \mathcal{R}$. Then for each $x \in MC(L_1,R)$, the morphism*

$$(L_1 \otimes \mathfrak{m}, d + \mathrm{ad}(x)) \longrightarrow (L_2 \otimes \mathfrak{m}, d + \mathrm{ad}(f(x))) \tag{4.3}$$

induced by f is a quasi-isomorphism.

As a simple application of the quasi-isomorphism theorem, we prove the

COROLLARY 4.5. *Let A be an associative algebra with unit* 1*. For each $R \in \mathcal{R}$, each R-deformation of the multiplication of A is equivalent to an R-deformation admitting the unit* 1*.*

PROOF. Let $L_{As,1}(A)$ be the subspace of $L_{As}(A)$ generated by all cochains f which vanish if one of their arguments equals 1. Then the subspace $L_{As,1}(A)$ is a dg Lie subalgebra of $L_{As}(A)$. For $R \in \mathcal{R}$, the subset $MC(L_{As,1}(A),R)$ of $MC(L_{As}(A),R)$ corresponds precisely to the R-deformations admitting the unit 1. Now by [**3**, Ch. IX], the inclusion of $L_{As,1}(A)$ into $L_{As}(A)$ is a quasi-isomorphism. Therefore, the claim follows from the quasi-isomorphism theorem. $\surd$

5. Deformations of star products, Lie brackets, Poisson brackets

As we have seen in Lemma 4.2, the R-deformations of an associative algebra A are controlled by the dg Lie algebra $L_{As}(A)$. Similarly, there are dg Lie algebras $L_{star}(M)$ and $L_{Pois}(M)$ which control the deformation problems appearing in Kontsevich's theorem 3.1, where M is a differentiable manifold. We will now describe these dg Lie algebras in more detail. Let A be the algebra of smooth functions on M.

5.1. Star products. Let $k = \mathbf{R}$ and $R \in \mathcal{R}$. An *R-star product* on M is an R-deformation $*$ of the multiplication of A such that the map

$$(a, b) \mapsto (\mathrm{id} \otimes \varphi)(a * b)$$

is a bidifferential operator for each linear form φ on $\mathfrak{m}$. Two R-starproducts are *equivalent* if there is an R-linear map $g : A \otimes R \to A \otimes R$ as in section 2 such that the map

$$a \mapsto (\mathrm{id} \otimes \varphi)(g(a))$$

is a differential operator for each linear form φ on $\mathfrak{m}$. Let $L_{star}(M)$ be the subspace of $L_{As}(A)$ whose p-cochains are p-differential operators. Then $L_{star}(M)$ is a dg Lie subalgebra of $L_{As}(A)$ and, for each $R \in \mathcal{R}$, there is a canonical bijection between

$$\overline{MC}(L_{star}(M), R)$$

and the set of equivalence classes of R-star products on M. In other words, $L_{star}(M)$ controls the problem of deforming the commutative multiplication of M into a star product. Following [**32**] we use the notation

$$D_{poly}(M) = L_{star}(M).$$

5.2. Lie brackets and the Chevalley-Eilenberg complex. Suppose that k is a field of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra over k. For $R \in \mathcal{R}$, the set of equivalence classes of R-deformations of the Lie bracket of $\mathfrak{g}$ is defined in analogy with the case of an associative multiplication (*cf.* section 2). The *Chevalley-Eilenberg complex* $C_{CE}(\mathfrak{g}, \mathfrak{g})$ of $\mathfrak{g}$ has the components $\mathrm{Hom}_k(\Lambda^p \mathfrak{g}, \mathfrak{g})$ in degrees $p \geq 0$ and vanishing components in negative degrees. Its differential is defined[2] by

$$(-1)^p (df)(X_0, \ldots, X_p) = \sum_{i<j} (-1)^{i+j+1} f([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p) - \sum_i (-1)^i [X_i, f(X_0, \ldots, \widehat{X_i}, \ldots, X_p)].$$

As usual, the symbol $\widehat{X}$ indicates that X is to be omitted. The *Richardson-Nijenhuis product* of a p-cochain f by a q-cochain g is the $(p+q-1)$-cochain defined by

$$(f \bullet g) = \sum \mathrm{sign}(\sigma) f(g(X_{\sigma(1)}, \ldots, X_{\sigma(q)}), X_{\sigma(q+1)}, \ldots, X_{\sigma(p+q-1)}) \,,$$

where σ runs through the permutations which are increasing on $\{1, \ldots, q\}$ and $\{q+1, \ldots, p+q-1\}$. The *Richardson-Nijenhuis bracket* of a p-cochain f by a q-cochain g is the commutator

$$[f, g] = f \bullet g - (-1)^{(p-1)(q-1)} g \bullet f.$$

If we let β denote the Lie bracket of $\mathfrak{g}$, we have

$$df = -[\beta, f].$$

As in section 3, one checks that the shifted complex

$$L_{Lie}(\mathfrak{g}) = C_{CE}(\mathfrak{g}, \mathfrak{g})[1]$$

endowed with the Richardson-Nijenhuis bracket is a dg Lie algebra. It controls the R-deformations of the Lie bracket of $\mathfrak{g}$.

[2] with the same modification of the sign as for the Hochschild differential in section 2

5.3. Poisson brackets. Suppose that k is a field of characteristic 0 and A a Poisson algebra over k. For $R \in \mathcal{R}$, a *Poisson R-deformation of the bracket of A* is an R-linear Poisson bracket π on the commutative algebra $A \otimes_k R$ which, modulo $\mathfrak{m}$, reduces to the bracket of A. Note that we deform only the bracket, the commutative multiplication of A remains unchanged. Two Poisson R-deformations are *equivalent* if they are conjugate by an automorphism of the commutative algebra $A \otimes_k R$ which, modulo $\mathfrak{m}$, induces the identity on A.

Let $L_{Pois}(A)$ be the subspace of $L_{Lie}(A)$ formed by the cochains which are derivations in each argument. Then $L_{Pois}(A)$ is a differential graded Lie subalgebra of $L_{Lie}(A)$. The dg Lie algebra $L_{Pois}(A)$ controls the Poisson deformations of the bracket of A.

Now suppose that $k = \mathbf{R}$ and that A is the algebra of smooth functions on a Poisson manifold M. Let $T_{poly}(M)$ be the graded space with vanishing components in degrees < -1 and whose pth component is the space of *$(p+1)$-polyvector fields on M*, *i.e.* the space $\Gamma(M, \Lambda^{(p+1)}TM)$ of global sections of the $(p+1)$th exterior power of the tangent bundle TM of M. For vector fields ξ_i and functions f_j, we define

$$(\xi_1 \wedge \ldots \wedge \xi_p)(f_1 \wedge \ldots \wedge f_p) = \frac{1}{p!} \det(\xi_i(f_j)).$$

This yields a canonical isomorphism of $L^p_{Pois}(A)$ with $T_{poly}(M)^p$. Thus we obtain a dg Lie algebra structure on the graded space $T_{poly}(M)$. Its bracket is uniquely determined by the following conditions:

$$T^0_{poly}(M) \text{ is the Lie algebra of vector fields on } M, \tag{5.1}$$

$$[\xi, f] = \xi(f) \ , \tag{5.2}$$

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(p-1)q} \beta \wedge [\alpha, \gamma] \ , \tag{5.3}$$

where ξ is a vector field, f a smooth function, and α, β, γ are polyvector fields of degree p, q and r, respectively.

6. Quasi-isomorphisms and L_∞-morphisms of dg Lie algebras

We will sketch a conceptual approach to the notions and results of this section in section 7.

Let L_1 and L_2 be two dg Lie algebras. By definition, an *L_∞-morphism* $f : L_1 \to L_2$ is given by a sequence of maps

$$f_n : L_1^{\otimes n} \to L_2 \ , \quad n \geq 1 \ ,$$

homogeneous of degree $1 - n$ and such that the following conditions are satisfied:

- The morphism f_n is graded antisymmetric, i.e. we have
$$f_n(x_1, \ldots, x_i, x_{i+1}, \ldots x_n) = -(-1)^{|x_i|\,|x_{i+1}|} f_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$
for all homogeneous $x_1, \ldots, x_n$ of L_1.
- We have $f_1 \circ d = d \circ f_1$, *i.e.* the map f_1 is a morphism of complexes.
- We have
$$f_1([x_1, x_2]) = [f_1(x_1), f_1(x_2)] + d(f_2(x_1, x_2)) + f_2(d(x_1), x_2) + (-1)^{|x_1|} f_2(x_1, d(x_2))$$
for all homogeneous x_1, x_2 in L. This means that f_1 is compatible with the brackets up to a homotopy given by f_2. In particular, f_1 induces a morphism of graded Lie algebras from H^*L_1 to H^*L_2.

- More generally, for each $n \geq 1$ and all homogeneous elements $x_1, \ldots, x_n$ of L_1, we have

$$
\begin{aligned}
(6.1) \quad & (-1)^n \sum_{i<j} (-1)^s f_{n-1}([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n) \\
&= \frac{1}{2} \sum_{p+q=n} \sum_{\sigma} (-1)^{pn+t} [f_p(x_{\sigma(1)}, \ldots, x_{\sigma(p)}), f_q(x_{\sigma(p+1)}, \ldots, x_{\sigma(n)})] + \\
&\qquad d(f_n(x_1, \ldots, x_n)) - (-1)^{n-1} \sum_{i=1}^{n} (-1)^u f_n(x_1, \ldots, d(x_i), \ldots, x_n).
\end{aligned}
$$

  Here, σ runs through all (p, q)-*shuffles*, *i.e.* all permutations of $\{1, \ldots, n\}$ which are increasing on $\{1, \ldots, p\}$ and on $\{p+1, \ldots, p+q\}$; the exponents s, t and u are respectively the numbers of transpositions of odd elements in passing from $(x_1, \ldots, x_n)$ to $(x_i, x_j, x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n)$, from $(f_p, f_q, x_1, \ldots, x_n)$ to

$$(f_p, x_{\sigma(1)}, \ldots, x_{\sigma(p)}, f_q, x_{\sigma(p+1)}, \ldots, x_{\sigma(n)})$$

  and from $(d, x_1, \ldots, x_n)$ to $(x_1, \ldots, d, x_i, \ldots, x_n)$.

Roughly speaking, an L_∞-morphism is a map between dg Lie algebras which is compatible with the brackets up to a given coherent system of higher homotopies. An L_∞-*quasi-isomorphism* is an L_∞-morphism whose first component is a quasi-isomorphism. The importance of this notion is apparent from the

THEOREM 6.1. *The following are equivalent*

(i) *There is an L_∞-quasi-isomorphism $L_1 \to L_2$.*

(ii) *There is a diagram of two quasi-isomorphisms of dg Lie algebras*

$$L_1 \leftarrow L_3 \rightarrow L_2.$$

(iii) *There is a chain of quasi-isomorphisms of dg Lie algebras*

$$L_1 \leftarrow L_3 \rightarrow L_4 \leftarrow \cdots \rightarrow L_n \leftarrow L_2$$

The dg Lie algebras L_1 and L_2 are *homotopy equivalent* if they satisfy the conditions of the theorem. A dg Lie algebra L is *formal* if it is homotopy equivalent to its homology H^*L (viewed as a dg Lie algebra with vanishing differential). The quasi-isomorphism theorem 4.3 implies that homotopy equivalent dg Lie algebras yield equivalent deformation problems. More precisely, we have the

THEOREM 6.2 (L_∞-quasi-isomorphism theorem). *Let $f : L_1 \to L_2$ be an L_∞-quasi-isomorphism of dg Lie algebras. Then, for each test algebra $R \in \mathcal{R}$ (section 2), the map*

$$x \mapsto \sum_{n \geq 1} \frac{1}{n!} f_n(x, \ldots, x)$$

induces a bijection

$$\overline{MC}(L_1, R) \xrightarrow{\sim} \overline{MC}(L_2, R).$$

Here and below, the R-multilinear extension of $f_n : L_1^{\otimes n} \to L_2$ to the nth tensor power over R of $L_1 \otimes_k \mathfrak{m}$ is still denoted by f_n. In analogy with proposition 4.4, we also have quasi-isomorphisms in the 'differential graded tangent spaces':

PROPOSITION 6.3. *Let $f : L_1 \to L_2$ be an L_∞-quasi-isomorphism of dg Lie algebras and let $R \in \mathcal{R}$. Then, for each $x \in MC(L_1, R)$, the map*

$$y \mapsto \sum_{n \geq 1} \frac{1}{(n-1)!} f_n(x, \ldots, x, y)$$

is a quasi-isomorphism

$$(L_1 \otimes \mathfrak{m}, d + \mathrm{ad}(x)) \longrightarrow (L_2 \otimes \mathfrak{m}, d + \mathrm{ad}(f(x))).$$

7. Formal deformation theory via Quillen's equivalence

In this section, we describe the framework for formal deformation theory provided by Quillen's equivalence between the homotopy category of dg Lie algebras and that of (certain) dg cocommutative coalgebras. In this way, we will obtain a better understanding of L_∞-morphisms and L_∞-algebras, *i.e.* a formal manifolds [**32**]. We follow V. Hinich's article [**24**].

7.1. From test algebras to test coalgebras. Let k be a field of caracteristic 0 and A an associative k-algebra. Let $R = k \oplus \mathfrak{m}$ be a test algebra (*cf.* section 2). Since R is finite-dimensional, its dual space $C = \mathrm{Hom}_k(R, k)$ is naturally a coalgebra. We have a natural isomorphism of vector spaces

$$A \otimes_k R \xrightarrow{\sim} \mathrm{Hom}_k(C, A)$$

and the canonical multiplication on the left corresponds to the *convolution product* defined by

$$f \cdot g = \mu \circ (f \otimes g) \circ \Delta$$

for all f, g linear maps $C \to A$, where μ is the multiplication of A and Δ the comultiplication on C. The R-deformations of A then correspond to the associative $\mathrm{Hom}_k(C, k)$-bilinear multiplications on $\mathrm{Hom}_k(C, A)$ which induce the multiplication of A after passage to the quotient

$$\mathrm{Hom}_k(C, A) \to \mathrm{Hom}_k(k, A) = A.$$

This description has the advantage that it naturally generalizes to certain infinite dimensional coalgebras. For example, if C is the coalgebra $k[T]$ with

$$\Delta(f(T)) = f(T \otimes 1 + 1 \otimes T)\ ,$$

then $\mathrm{Hom}_k(C, A)$ identifies with the power series algebra $A[[t]]$ and our description yields precisely the associative formal deformations of the multiplication of A (in the sense of 1.1 of chapter 1). The appropriate class of *test coalgebras* to consider is that of *cocommutative cocomplete augmented coalgebras*, *i.e.* cocommutative coassociative coalgebras C endowed with a counit $\eta : C \to k$ and an augmentation $\varepsilon : k \to C$ such that, if $\overline{C} = C/\varepsilon(C)$ is the *reduction of* C, then each element of $\overline{C}$ is annihilated by a sufficiently high iterate $\overline{C} \to \overline{C}^{\otimes i}$ of the comultiplication induced by Δ. Note that the dual of a test coalgebra C is a complete local ring with maximal ideal $\mathrm{Hom}_k(\overline{C}, k)$.

7.2. The Maurer-Cartan equation. Let L be a dg Lie algebra. For a test algebra $R = k \oplus \mathfrak{m}$, the set

$$\overline{MC}(R, L) = \overline{MC}(L \otimes_k \mathfrak{m})$$

of equivalence classes of solutions of the Maurer-Cartan equation (4.1) only depends on the 'piece'

$$L^0 \to L^1 \to L^2$$

of the dg Lie algebra L. In order to capture the whole information given by L, we have to allow R to have components in several degrees and to have a non zero differential. If we combine this observation with the remarks of the preceding paragraph, we arrive at the notion of a *dg test coalgebra*, *i.e.* a test coalgebra endowed with a $\mathbf{Z}$-grading and a coalgebra differential $d : C \to C$ of degree 1. This means that $d^2 = 0$ and that d is a *coderivation*, *i.e.*

$$\Delta \circ d = (\mathrm{id} \otimes d + d \otimes \mathrm{id}) \circ \Delta.$$

For a dg test coalgebra C, the graded space $\mathrm{Hom}_k(C, L)$, whose n-th component consists of the homogeneous k-linear maps of degree n, becomes a dg Lie algebra for the differential

$$d(f) = d \circ f - (-1)^{|f|} f \circ d$$

and the convolution bracket

$$[f, g] = [,]_L \circ (f \otimes g) \circ \Delta.$$

We define the set of *twisting cochains* $C \to L$ to be the set

$$\mathrm{Tw}(C, L) = MC(\mathrm{Hom}_k(C, L))$$

of solutions of the Maurer-Cartan equation (4.1). For example, if we take $C = k[T]$ (as in 7.1) concentrated in degree 0 and with $d = 0$ and if $L = L_{As}(A)$ for an associative algebra A (as in section 3), then $\mathrm{Tw}(C, L)$ naturally identifies with the associative formal deformations of the multiplication of A (as in paragraph 1.1 of chapter 1).

7.3. The bar and the cobar constructions. Let Lie be the category of dg Lie algebras and Cog that of dg test coalgebras.

LEMMA AND DEFINITION 7.1. a) For $L \in$ Lie, the functor

$$\mathrm{Tw}(?, L) : \mathrm{Cog}^{op} \to \mathrm{Sets}$$

is representable. We denote a representative by BL.

b) For $C \in$ Com, the functor

$$\mathrm{Tw}(C, ?) : \mathrm{Lie} \to \mathrm{Sets}$$

is representable. We denote a representative by ΩC.

c) We have canonical bijections

$$\mathrm{Hom}_{\mathrm{Lie}}(\Omega C, L) = \mathrm{Tw}(C, L) = \mathrm{Hom}_{\mathrm{Cog}}(C, \Omega L).$$

In particular, B and Ω are a pair of adjoint functors between Lie and Cog.

Note that c) is a reformulation of a) and b). Explicitly, part a) of the Lemma claims that there is a dg test coalgebra BL and a twisting cochain $\tau : BL \to L$ which is *universal*, *i.e.* for each twisting cochain $\tau' : C \to L$ there is a unique morphism of dg test coalgebras $f : BL \to C$ such that $\tau' = \tau \circ f$. Concretely, BL is given by the *bar construction on* L, *i.e.* the graded symmetric (section 1) coalgebra $\mathrm{Sym}^c(L[1])$ on the suspension $L[1]$ of L endowed with the unique coderivation d such that the evident morphism $\tau : \mathrm{Sym}^c(L[1]) \to L$ of degree 1 becomes a twisting cochain:

$$d_L \circ \tau + \tau \circ d - \frac{1}{2}[\tau, \tau] = 0.$$

Note that if L is concentrated in degree 0, the underlying complex of BL is the homological Chevalley-Eilenberg complex which computes $H^*(L, k)$.

Dually, part b) claims that there is a dg Lie algebra ΩC and a twisting cochain $\tau : C \to \Omega C$ which is co-universal. Explicitly, ΩC is given by the free graded (section 1) Lie algebra on $C[-1]$ endowed with the unique derivation d such that the evident morphism $\tau : C \to \Omega C$ of degree 1 becomes a twisting cochain.

7.4. Quillen's equivalence. A morphism $f : C \to C'$ of Cog is a *weak equivalence* if Ωf is a quasi-isomorphism. The following lemma is not hard to show:

Lemma 7.2. a) *The functor* $B : \mathrm{Lie} \to \mathrm{Cog}$ *takes quasi-isomorphisms to weak equivalences.*

b) *For each* $L \in \mathrm{Lie}$*, the adjunction morphism* $\Omega BL \to L$ *is a quasi-isomorphism, and for each* $C \in \mathrm{Cog}$*, the adjunction morphism* $C \to B\Omega C$ *is a weak equivalence.*

Let Ho(Lie) be the *localization* of the category Lie at the class of quasi-isomorphisms, *i.e.* the category whose objects are the same as those of Lie and whose morphisms are obtained from those of Lie by formally inverting all quasi-isomorphisms. Analogously, let Ho(Cog) be the localization of Cog at the class of weak equivalences. We refer to these localizations as *homotopy categories*. From the lemma, we immediately obtain the

Theorem 7.3 ([**46**],[**24**]). *The functors* B *and* Ω *induce quasi-inverse equivalences between homotopy categories* Ho(Lie) *and* Ho(Cog).

Quillen's equivalence is the equivalence $B : \mathrm{Ho}(\mathrm{Lie}) \to \mathrm{Ho}(\mathrm{Cog})$.

7.5. Morphisms in the homotopy categories. Let L', L be dg Lie algebras. In general, the map

$$\mathrm{Hom}_{\mathrm{Lie}}(L', L) \to \mathrm{Hom}_{\mathrm{Ho(Lie)}}(L', L)$$

will not be surjective. However, if $L' = \Omega C$ for some $C \in \mathrm{Cog}$, it is surjective and we can describe the image in terms of equivalence classes of solutions of the Maurer-Cartan equation: Put

$$\overline{\mathrm{Tw}}(C, L) = \overline{MC}(\mathrm{Hom}_k(C, L)).$$

Theorem 7.4 ([**24**]). *The maps*

$$\mathrm{Hom}_{\mathrm{Lie}}(\Omega C, L) \to \mathrm{Hom}_{\mathrm{Ho(Lie)}}(\Omega C, L) \text{ and } \mathrm{Hom}_{\mathrm{Cog}}(C, BL) \to \mathrm{Hom}_{\mathrm{Ho(Cog)}}(C, BL)$$

are surjective and we have bijections

$$\mathrm{Hom}_{\mathrm{Ho(Lie)}}(\Omega C, L) = \overline{\mathrm{Tw}}(C, L) = \mathrm{Hom}_{\mathrm{Ho(Cog)}}(C, BL).$$

7.6. Formal deformation problems. A *formal deformation problem* is a representable functor

$$F = \mathrm{Hom}_{\mathrm{Ho(Cog)}}(?, C) : \mathrm{Ho(Cog)}^{op} \to \mathrm{Sets}\,.$$

If L is a dg Lie algebra, the problem of deforming the zero solution of the Maurer-Cartan equation in L is the functor

$$\overline{\mathrm{Tw}}(?, L) : \mathrm{Ho(Cog)}^{op} \to \mathrm{Sets}\,.$$

It is a formal deformation problem since it is represented by ΩL, by theorem 7.4. Conversely, this theorem yields that for each formal deformation problem F there is a dg Lie algebra L, unique up to isomorphism in Ho(Lie), such that

$$F \xrightarrow{\sim} \overline{\mathrm{Tw}}(?, L).$$

In summary, we have bijections between isomorphism classes of formal deformation problems, homotopy types of dg Lie algebras and homotopy types of dg cocomplete cocommutative coalgebras.

7.7. Link with L_∞-morphisms. Let L and L' be dg Lie algebras. In section 6, we have defined the notion of L_∞-morphism from L to L'.

LEMMA 7.5. *There is a canonical bijection between the set of L_∞-morphisms $L \to L'$ and the set of morphisms of dg coalgebras $BL \to BL'$. Under this bijection, the L_∞-quasi-isomorphisms correspond to the weak equivalences. The dg Lie algebras L and L' are homotopy equivalent iff they are isomorphic in the homotopy category* Ho(Lie).

Let us deduce the L_∞-quasi-isomorphism theorem 6.2: If R is a test algebra, then we have a bijection (*cf.* paragraph 7.1)

$$\overline{MC}(L, R) = \overline{\mathrm{Tw}}(DR, L) = \mathrm{Hom}_{\mathrm{Ho(Cog)}}(DR, BL)\,,$$

where $DR = \mathrm{Hom}_k(R, k)$ is the coalgebra dual to the (finite-dimensional) algebra R. The lemma shows that the right hand side is preserved under L_∞-quasi-isomorphisms. It is easy to check the explicit formula in theorem 6.2.

7.8. L_∞-algebras and fibrant coalgebras. Let $C \in$ Cog.

PROPOSITION 7.6. *The following are equivalent*

(i) *There is a graded vector space L such that the underlying graded augmented coalgebra of C is isomorphic to* $\mathrm{Sym}^c(L[1])$.
(ii) *For each morphism $i : D \to E$ of* Cog *such that i is injective (on the underlying vector spaces) and $\Omega(i)$ is a quasi-isomorphism, and for each morphism $f : D \to C$ of* Cog, *there is a morphism $i : E \to C$ such that $h \circ i = f$.*

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "i"'] & C \\ E \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

Suppose that C is *fibrant*, *i.e.* it satisfies the properties of the proposition. Then the graded space $L[1]$ of (i) is isomorphic to the space $\mathrm{Prim}(C)$ of primitive elements of C (*i.e.* the kernel of the map $\overline{C} \to \overline{C} \times \overline{C}$ induced by the comultiplication). Note however that there is no canonical isomorphism between C and $\mathrm{Sym}^c(\mathrm{Prim}(C))$. The differential d of C yields a sequence of graded maps

$$Q_n : L^{\otimes n} \to L \; , \quad n \geq 1 \; ,$$

which are homogeneous of degree $2-n$, graded antisymmetric and satisfy quadratic equations which express the fact that $d^2 = 0$. The first two of these equation imply that $Q_1^2 = 0$ (so (L, Q_1) is a complex) and that $Q_2 : L \otimes L \to L$ is a map of complexes which induces a graded Lie bracket in homology. By definition, the space L endowed with the Q_n becomes an *L_∞-algebra*. Each dg Lie algebra is naturally an L_∞-algebra but there are many other examples. By definition, *morphisms between L_∞-algebras* correspond bijectively to morphisms between the corresponding objects of Cog. Thus we have a fully faithful functor

$$B_\infty : \{L_\infty\text{-algebras}\} \to \mathrm{Cog} \; ,$$

which extends the bar construction $B : \mathrm{Lie} \to \mathrm{Cog}$. It is easy to see that the category of L_∞-algebras admits products and that the product of L_1 with L_2 is $L_1 \oplus L_2$ with the natural maps Q_n. An L_∞-algebra L is *linear contractible* if $Q_n = 0$ for $n \geq 2$ and the complex (L, Q_1) is contractible. It is *minimal* if Q_1 vanishes.

PROPOSITION 7.7. *Each L_∞-algebra L is isomorphic to the product $M \oplus C$ of a minimal L_∞-algebra M and a linear contractible L_∞-algebra C.*

It follows from the proposition that an L_∞-algebra C is contractible iff $B_\infty C$ is isomorphic to zero in Ho(Cog). Moreover, an L_∞-algebra M is minimal iff we have

$$f : M \to M \text{ is invertible} \Leftrightarrow B_\infty(f) \text{ becomes invertible in } \mathrm{Ho(Cog)}.$$

7.9. Formal manifolds. A formal (graded) manifold is a graded cocomplete coalgebra C which is isomorphic to the symmetric coalgebra $\mathrm{Sym}^c(V)$ of some graded vector space V. However, the isomorphism $C \xrightarrow{\sim} \mathrm{Sym}^c(V)$ is not part of the structure. For a formal manifold P, a P-point of C is a morphism of coaugmented coalgebras $P \to C$. The formal manifold C comes with a distinguished point $k \to C$ given by the coaugmentation. Its *tangent space at the distinguished point* is

$$T_0 C = \mathrm{Prim}(C) = \ker(\overline{C} \to \overline{C} \otimes \overline{C}).$$

Let $d : C \to C$ be a coalgebra differential, homogeneous of degree 1. Geometrically, we view d as a vector field of degree 1 on the formal manifold C satisfying $[d, d] = 0$. Let us call the datum of C with d a *Q-manifold*. If L is a graded vector space such that $C \cong \mathrm{Sym}^c(L[1])$, then d corresponds to a structure of L_∞-algebra on L. By lemma 7.5, morphisms of Q-manifolds are in bijection with L_∞-morphisms. One defines the notion of homotopy between morphisms of Q-manifolds using polynomial families of morphisms. Then one can show [**24**] that the homotopy category of Q-manifolds is equivalent to the homotopy categories Ho(Cog) and Ho(Lie).

8. Notes

According to Goldman and Millson [**22**], the philosophy of controlling deformation problems by differential graded Lie algebras is due to Schlessinger-Stasheff [**47**] and P. Deligne. The quasi-isomorphism theorem 4.3 is stated and proved in this generality in [**32**]. The material presented in section 7 is due to Quillen [**46**], Hinich-Schechtman [**26**], Kontsevich [**32**], Hinich [**24**], ...

CHAPTER 3

On Tamarkin's approach

1. Tamarkin's theorem

Let k be a field of characteristic 0, let V be a finite-dimensional k-vector space and SV the symmetric algebra on V. The problem of deforming the multiplication of SV is described by the dg Lie algebra

$$L_{As}(SV) = C(SV, SV)[1] ,$$

i.e. the shifted Hochschild complex endowed with the Gerstenhaber bracket (*cf.* section 3 of chapter 2).

THEOREM 1.1 (Tamarkin [**51**]). *The dg Lie algebra $L_{As}(SV)$ is formal.*

1.1. Kontsevich's formality theorem follows for $M = \mathbf{R}^n$. We keep the above notations and let $k = \mathbf{R}$. We consider the dual $M = V^*$ as a Poisson manifold with vanishing bracket. Let $L_{As,md}(SV)$ be the subcomplex of $L_{As}(SV)$ whose components are formed by the cochains which are multidifferential operators with polynomial coefficients. The following lemma results from suitable variants of the Hochschild-Kostant-Rosenberg theorem [**27**].

LEMMA 1.2. *$L_{As,md}(SV)$ is a dg Lie subalgebra of $L_{As}(SV)$ and of $L_{star}(V^*)$. Moreover, both inclusions are quasi-isomorphisms.*

It now follows from Tamarkin's theorem that $L_{star}(V^*)$ is linked to its homology by a chain of quasi-isomorphisms of dg Lie algebras, so that we obtain Kontsevich's formality theorem 4.1 of chapter 1 using the quasi-isomorphism theorem 4.3 of chapter 2.

1.2. Outline of Tamarkin's approach. We essentially follow Kontsevich's presentation [**33**]. The basic idea (which, according to [**51**], goes back to B. Tsygan) consists in using the additional structure present on the Hochschild complex in the form of the cup product. We use the notations introduced in section 1 of chapter 2.

A *Gerstenhaber algebra* is given by a $\mathbf{Z}$-graded vector space G, a graded commutative associative multiplication on G and a Lie bracket on $G[1]$ such that for each $x \in G^p$, the bracket $[x, ?]$ is a derivation of degree $p+1$ of the associative algebra G. A *dg Gerstenhaber algebra* is a Gerstenhaber algebra endowed with a differential which is a derivation for both operations, the multiplication and the bracket.

If A is an associative algebra, the *cup product* of a Hochschild p-cochain f by a q-cochain g is the $(p+q)$-cochain $f \cup g$ defined by

$$(f \cup g)(a_1, \ldots, a_{p+q}) = (-1)^{pq} f(a_1, \ldots, a_p) g(a_{p+1}, \ldots, a_{p+q}).$$

LEMMA 1.3 (Gerstenhaber [**15**]). a) *Endowed with the cup product and the Hochschild differential the Hochschild complex becomes a dg associative algebra.*

b) *Hochschild cohomology endowed with the cup product and the Gerstenhaber bracket is a Gerstenhaber algebra.*

REMARK 1.4. It is important to note that the Hochschild complex itself is *not*, in general, a Gerstenhaber algebra for the cup product and the Gerstenhaber bracket. For example, the cup product of cochains is not commutative in general.

Let us examine the Gerstenhaber algebra structure on the Hochschild cohomology of a *commutative* k-algebra A. It is easy to see that we have an isomorphism of Lie algebras

$$\mathrm{Der}_k(A,A) \xrightarrow{\sim} HH^1(A,A)\ ,$$

where $\mathrm{Der}_k(A,A)$ denotes the space of k-linear derivations from A to itself. The bracket on $\mathrm{Der}_k(A,A)$ admits a unique extension which makes the exterior algebra $\Lambda_A \mathrm{Der}_k(A,A)$ into a Gerstenhaber algebra (where the elements of $\mathrm{Der}_k(A,A)$ are in degree 1). The above isomorphism uniquely extends to a morphism of Gerstenhaber algebras

$$\Lambda_A \mathrm{Der}_k(A,A) \to HH^*(A,A).$$

By the Hochschild-Kostant-Rosenberg theorem [**27**], this is an isomorphism if k is perfect and A is the algebra of polynomial functions on a smooth affine variety over k. It is also invertible if k is an arbitrary field and $A = SV$ the symmetric algebra on a finite-dimensional vector space V (use the Koszul resolution as in [**36**, 3.3.3]). Thus we have an isomorphism of Gerstenhaber algebras

$$\Lambda_{SV} \mathrm{Der}_k(SV,SV) \xrightarrow{\sim} HH^*(SV,SV).$$

A *quasi-isomorphism* of dg Gerstenhaber algebras is a morphism of dg Gerstenhaber algebras which induces isomorphisms in homology. A dg Gerstenhaber algebra G is *formal* if it is linked to its homology H^*G (considered as a dg Gerstenhaber algebra with vanishing differential) by a sequence of quasi-isomorphisms of dg Gerstenhaber algebras.

PROPOSITION 1.5. *Let V be a finite-dimensional vector space and G a dg Gerstenhaber algebra such that H^*G is isomorphic to $HH^*(SV,SV)$. Then G is formal.*

Though non-trivial, the proposition is not deep. Proofs can be found in section 3 of [**51**], in [**25**], or in [**20**]. The deep part of Tamarkin's contribution is contained in the following theorem.

THEOREM 1.6. *For each associative (not necessarily commutative) k-algebra A, there is a dg Gerstenhaber algebra $\tilde{G}$ such that*

a) *$H^*\tilde{G}$ is isomorphic to $HH^*(A,A)$ as a Gerstenhaber algebra and*

b) *$\tilde{G}$ is linked to $L_{As}(A)$ by a sequence of quasi-isomorphisms of dg Lie algebras.*

Together, the proposition and the theorem imply Tamarkin's formality theorem 1.1: Indeed, it follows from a) and the proposition that for $A = SV$, the dg Gerstenhaber algebra $\tilde{G}$ is formal. In particular, it is formal as dg Lie algebra. So we obtain sequences of quasi-isomorphisms of dg Lie algebras

$$H^*(\tilde{G})[1] \rightsquigarrow \tilde{G}[1] \rightsquigarrow L_{As}(SV).$$

Since we have isomorphisms of dg Lie algebras

$$H^*L_{As}(SV) = HH^*(SV, SV)[1] = H^*(\tilde{G})[1] \ ,$$

it follows that $L_{As}(SV)$ is formal as a dg Lie algebra, as claimed by theorem 1.1.

Tamarkin's proof of theorem 1.6 uses the language of operads [**42**]. Its two main ingredients are the following theorems

1) Deligne's question [**7**] has a positive answer: There is a homotopy action of the (normalized singular chain operad of) the little squares operad on the Hochschild cochain complex of any associative algebra.
2) The little squares operad is formal.

In the sequel, we will succinctly introduce the language of operads, present these two theorems and show how they imply theorem 1.6.

2. Operads

2.1. A first example: The associative operad. Let k be a field, $\mathrm{Vec}\, k$ the category of k-vector spaces and $\mathcal{A}$ the category of associative (non unital) k-algebras. For $n \geq 1$, we consider the functor

$$T_n : \mathcal{A} \to \mathrm{Vec}\, k \ , \quad A \mapsto A^{\otimes n}.$$

A *natural n-ary operation* is a morphism of functors $T_n \to T_1$, *i.e.* a morphism of vector spaces

$$A^{\otimes n} \to A$$

which is functorial in the algebra A. For example, the multiplication of A yields a natural binary operation and the identical map a natural unary operation. Denote by $\mathrm{As}(n)$ the space of natural n-ary operations. It is a right Σ_n-module: If $\lambda \in \mathrm{As}(n)$ and σ is a permutation of $\{1, \ldots, n\}$, then the operation $\lambda\sigma$ is defined by

$$a_1 \otimes \ldots \otimes a_n \mapsto \lambda(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}).$$

If we have natural operations $\lambda \in \mathrm{As}(n)$ and $\mu_i \in \mathrm{As}(k_i)$, $1 \leq i \leq n$, then the composition $\lambda(\mu_1, \ldots, \mu_n)$ defined by

$$a_1 \otimes \ldots \otimes a_N \mapsto \lambda(\mu_1(a_1, \ldots, a_{k_1}), \ldots, \mu_n(a_{N-k_n+1}, \ldots, a_N)) \ , \quad N = \sum k_i \ ,$$

belongs to $\mathrm{As}(k_1 + \ldots + k_n)$. Thus we obtain a composition map

$$\gamma : \mathrm{As}(n) \otimes \mathrm{As}(k_1) \otimes \ldots \otimes \mathrm{As}(k_n) \to \mathrm{As}(k_1 + \ldots + k_n).$$

It is clear that composition is compatible in a suitable way with the actions of the symmetric groups and that the identity of A yields a 'neutral element' for the composition. The spaces $\mathrm{As}(n)$, $n \geq 1$, together with the actions of the symmetric groups, the composition and the identity morphism form the *associative operad*. It is not hard to show that $\mathrm{As}(n)$ is in fact a free $k[\Sigma_n]$-module and to describe the compositions explicitly, *cf.* for example [**2**, 0.10].

2.2. Operads and their algebras. More generally, an *operad of vector spaces* is given by a sequence $O(n)$, $n \geq 1$, of vector spaces, a right action of Σ_n on $O(n)$ for each $n \geq 1$, composition maps

$$\gamma : O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) \to O(k_1 + \ldots + k_n)$$

for all integers $n, k_1, \ldots, k_n \geq 1$, and a distinguished element $1 \in O(1)$. One imposes natural conditions to the effect that

- composition is compatible with the actions of the symmetric groups,
- composition is associative,
- composition admits 1 as a neutral element.

(The reader can find the complete definition in [**21**], for example). *Morphisms of operads* are defined in the natural way.

If V is a vector space, the *endomorphism operad* $\mathrm{Endop}(V)$ has the components $\mathrm{Hom}(V^{\otimes n}, V)$ with the natural action of Σ_n, $n \geq 1$, and the natural composition. If O is an operad, an *algebra over O (=O-algebra)* is a vector space A together with a morphism of operads

$$\rho : O \to \mathrm{Endop}(A).$$

For example, one can check that the algebras over the associative operad are precisely the associative algebras. Similarly, there is the *commutative operad* Com with $Com(n) = k$ (the trivial module) for all $n \geq 1$ whose algebras are precisely the commutative k-algebras. Another example is the *Lie operad* Lie whose algebras are precisely the Lie algebras over k.

If O is an operad and $\mathrm{Alg}(O)$ the category of algebras over O, the forgetful functor

$$\mathrm{Alg}(O) \to \mathrm{Vec}\, k$$

admits a left adjoint: the free algebra functor, which takes a vector space V to

$$F(O, V) = \sum_{n \geq 1} O(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

This shows that the Σ_n-module $O(n)$ can be recovered from the free algebra $F(O, k^n)$ as the $(1, \ldots, 1)$-component of the natural $\mathbf{N}^n$-grading. For example, we thus obtain a description of $\mathrm{Lie}(n)$, $n \geq 1$, as the $(1, \ldots, 1)$-component of the free Lie algebra on n generators.

The definition of an operad still makes sense if we replace the category of vector spaces by that of topological spaces and the tensor product by the cartesian product. This yields the notion of a *topological operad* and of an algebra over such an operad. More generally, we may replace the category of vector spaces by any symmetric monoidal category (*cf.* [**38**]). We thus obtain the notion of *graded operad* and *differential graded (=dg) operad*. The *Gerstenhaber operad* Gerst is the graded operad whose algebras are the Gerstenhaber algebras (1.2). We have a natural morphism of graded operads Com $\to$ Gerst. The restriction of a Gerstenhaber algebra along this morphism is its underlying commutative algebra. Similarly, we have a canonical morphism $\Sigma\,\mathrm{Lie} \to \mathrm{Gerst}$, where $\Sigma\,\mathrm{Lie}$ denotes the graded operad whose algebras are the suspensions $L[1]$, where L is a (graded) Lie algebra. More generally, for any graded operad O, the *suspended operad* ΣO whose algebras are the suspensions $A[1]$ of O-algebras A, is given by

$$(\Sigma O)(n) \xrightarrow{\sim} O(n)[n-1] \otimes \mathrm{sgn}_n\ ,$$

where sgn_n is the sign representation of the symmetric group Σ_n. We have

$$\mathrm{Gerst}(2) = \mathrm{Lie}(2)[1] \otimes \mathrm{sgn}_2 \oplus \mathrm{Com}(2).$$

If O is a topological operad, then, thanks to the Künneth theorem, the (singular) homologies $H_*(O(n), k)$ naturally form a graded operad. More subtly, the *normalized singular chain complexes* $N_*(O(n), k)$ (the quotients of the complexes of singular chains by all degenerate chains) form a dg operad, thanks to the Eilenberg-Zilber theorem (*cf.* [**37**, VIII.8]).

2.3. Little squares. The *little squares operad* $\mathcal{C}_2$ is an example of a topological operad. It is defined as follows: Let J be the open unit interval $]0,1[$. A *little square* is an affine embedding with parallel oriented axes of J^2 into itself. In other words, it is a map

$$c : J^2 \to J^2 \ , \ \ (t_1, t_2) \to ((1-t_1)x_1 + t_1 y_1, (1-t_2)x_2 + t_2 y_2) \ ,$$

where $0 \leq x_i < y_i \leq 1$. The nth component of the *little squares operad* $\mathcal{C}_2$ is the set of all n-tuples $(c_1, \ldots, c_n)$ of little squares with disjoint images. We identify this set with a subspace of the space of maps (with the compact open topology) from the disjoint union of n copies of J^2 to J^2. The group Σ_n acts on $\mathcal{C}_2(n)$ by permuting the squares:

$$(c_1, \ldots, c_n)\,\sigma = (c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(n)}).$$

For $c \in \mathcal{C}_2(n)$ and $d_i \in \mathcal{C}_2(k_i)$, $1 \leq i \leq n$, the composition $\gamma(c, d_1, \ldots, d_n)$ is defined via the composition of maps

$$\coprod_{i=1}^n \coprod_{k_i} J^2 \xrightarrow{(d_1,\ldots,d_n)} \coprod_n J^2 \xrightarrow{\quad c \quad} J^2$$

Finally, the unit element is the identity map $\mathrm{id} \in \mathcal{C}_2(1)$. By sending each square to its center we obtain a homotopy equivalence between $\mathcal{C}_2(n)$ and the space of n-tuples of distinct points in J^2. In particular, we see that $\mathcal{C}_2(2)$ is homotopy equivalent to the circle.

Let Y be a topological space with a base point $*$. Define its second loop space $\Omega^2 Y$ to be the space of all continuous maps $f : [0,1]^2 \to Y$ which send the boundary of the unit square to the base point $*$. We see that $\Omega^2 Y$ carries a natural structure of $\mathcal{C}_2$-algebra. Conversely, if X is a connected topological space which is a $\mathcal{C}_2$-algebra, then X is weakly equivalent to $\Omega^2 Y$ for some topological space with base point Y, *cf.* [**42**].

THEOREM 2.1 (Cohen [**6**]). *Let k be a field of characteristic* 0. *The homology operad $H_*(\mathcal{C}_2, k)$ of the little squares operad is isomorphic to the Gerstenhaber operad* Gerst.

As a simple instance of the theorem, note that

$$H_*(\mathcal{C}_2(2), k) \xrightarrow{\sim} H_*(S^1, k) \xrightarrow{\sim} k\mu \oplus k\lambda = \mathrm{Gerst}(2) \ ,$$

where μ corresponds to the commutative multiplication (of degree 0) and λ to the bracket (of cohomological degree -1) of a Gerstenhaber algebra.

3. Application in Tamarkin's proof

3.1. Homotopy action of the Gerstenhaber operad on Hochschild cochains. Let O, P be dg operads. By definition, a *quasi-isomorphism* $O \to P$ is a morphism of dg operads such that $O(n) \to P(n)$ is a quasi-isomorphism of complexes for each $n \geq 1$. The *homotopy category of operads* $\mathrm{Ho}(Op)$ is obtained from the category of dg operads by formally inverting all quasi-isomorphisms. A dg operad O is *formal* if, in $\mathrm{Ho}(Op)$, O is isomorphic to H^*O viewed as a dg operad with vanishing differential. An *homotopy action* of the dg operad O on a dg vector space C is by definition a morphism

$$\rho : O \to \mathrm{Endop}(C)$$

in $\mathrm{Ho}(Op)$. Such a morphism induces a structure of $H^*(O)$-algebra on the homology $H^*(C)$. In general, it is a highly non trivial problem to determine whether a given $H^*(O)$-action on $H^*(C)$ lifts to a homotopy action of O on C.

Let A be an associative algebra. We know that the Lie algebra structure on $HH^*(A,A)[1]$ extends to a Gerstenhaber algebra structure. This means that we have a commutative triangle of graded operads

$$\begin{tikzcd} \Sigma\,\mathrm{Lie} \arrow[r, "\lambda"] \arrow[d] & \mathrm{Endop}(HH^*(A,A)) \\ \mathrm{Gerst} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We also know that the Lie structure on $HH^*(A,A)[1]$ comes from a Lie algebra structure on the shifted Hochschild complex $C(A,A)[1]$ itself. So we have a canonical lift of λ to morphism of operads

$$\Sigma\,\mathrm{Lie} \xrightarrow{\Lambda} \mathrm{Endop}(C(A,A)).$$

THEOREM 3.1 (Tamarkin). *The morphism φ lifts to a morphism*

$$\Phi : \mathrm{Gerst} \to \mathrm{Endop}(C(A,A))$$

of $\mathrm{Ho}(Op)$ *such that the triangle*

$$\begin{tikzcd} \Sigma\,\mathrm{Lie} \arrow[r, "\Lambda"] \arrow[d] & \mathrm{Endop}(C(A,A)) \\ \mathrm{Gerst} \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes in $\mathrm{Ho}(Op)$.

Theorem 1.6 follows from this theorem: for $\tilde{G}$, one takes the 'restriction along Φ' of the algebra $C(A,A)$. Since Φ is not a morphism of operads but only a morphism in $\mathrm{Ho}(Op)$, some work is required to define the restriction. The main point is the

THEOREM 3.2 (Hinich [**23**, 4.7.4]). *If $\alpha : O \to O'$ is a quasi-isomorphism of dg operads over a field of characteristic* 0*, then the restriction along α is an equivalence*

$$\mathrm{Ho}(\mathrm{Alg}(O')) \to \mathrm{Ho}(\mathrm{Alg}(O)) ,$$

where $\mathrm{Ho}(\mathrm{Alg}(O))$ *is the localization of the category of dg O-algebras with respect to the class of quasi-isomorphisms.*

3.2. On the proof of Theorem 3.1. Let us fix an isomorphism

$$\alpha : \mathrm{Gerst} \to H^*(\mathcal{C}_2, k)$$

as in Cohen's theorem 2.1. The morphism Φ of theorem 3.1 is constructed as a composition of two morphisms of $\mathrm{Ho}(Op)$:

$$\mathrm{Gerst} \xrightarrow{\Phi_1} N_*(\mathcal{C}_2, k) \xrightarrow{\Phi_2} \mathrm{Endop}(C(A,A)).$$

Here, N_* denotes the normalized singular cochain complex defined at the end of section 2.2 and Φ_1 is an isomorphism inducing α in homology. The existence of Φ_1 is immediate from the

THEOREM 3.3. *If k is a field of characteristic 0, the normalized singular chains operad $N_*(\mathcal{C}_2, k)$ of the little squares operad is formal.*

The existence of a suitable morphism Φ_2 follows from the

THEOREM 3.4. *There is a morphism $\Phi_2 : N_*(\mathcal{C}_2, k) \to \mathrm{Endop}(C(A,A))$ of $\mathrm{Ho}(Op)$ such that the triangle*

$$\begin{tikzcd} & \mathrm{Endop}(HH^*(A,A)) \\ \mathrm{Gerst} \arrow[ur, "\varphi"] \arrow[r, "\alpha"'] & N_*(\mathcal{C}_2) \arrow[u, "H^*(\Phi_2)"'] \end{tikzcd}$$

commutes.

The morphism Φ is defined as the composition $\Phi_2 \circ \Phi_1$. It is then clear that Φ lifts φ and it only remains to prove the commutativity of the triangle in theorem 3.1. For this, one needs a slightly more precise version of theorem 3.4: Let P be the dg suboperad of $\mathrm{Endop}\, C(A,A)$ generated by the cup-product and by the brace operations: If $f_0, \ldots, f_p$ are Hochschild cochains, the brace operation is given by an expression of the form (*cf.* [**43**])

$$f_0\{f_1, \ldots, f_p\} = \sum \pm f_0 \circ (\mathrm{id}^{\otimes i_0} \otimes f_1 \otimes \mathrm{id}^{\otimes i_1} \otimes \ldots \otimes \mathrm{id}^{\otimes i_{p-1}} \otimes f_p \otimes \mathrm{id}^{\otimes i_p})$$

where the sequence $i_0, \ldots, i_p$ ranges over all possibilities such that the composition with f_0 makes sense. Note that if f_i is of degree r_i, then the degree r of the resulting cochain satisfies

$$r - 1 = \sum_i (r_i - 1).$$

It follows that the complexes $(\Sigma^{-1}P)(n)$ are all concentrated in degrees ≥ 0. The Gerstenhaber bracket is expressed in terms of brace operations so that Λ factors as the composition of the inclusion of P with a morphism $\overline{\Lambda}$. Similarly, φ factors as the composition of the map $H^*(P) \to \mathrm{Endop}(HH^*(A,A))$ with a morphism $\overline{\varphi}$. And even a very superficial inspection of the proofs [**43**] [**2**] [**35**] of theorem 3.4 yields the

PORISM 3.5. The morphism Φ_2 of Theorem 3.4 factors through the morphism

$$P \to \mathrm{Endop}(C(A,A)).$$

It remains to prove that the following diagram is commutative

$$\begin{tikzcd} \mathrm{Lie} \arrow[r, "\overline{\Lambda}"] \arrow[d] & \Sigma^{-1}P \arrow[r] & \Sigma^{-1}\,\mathrm{Endop}(C(A,A)) \\ \Sigma^{-1}\,\mathrm{Gerst} \arrow[ur, "\overline{\Phi}"'] & & \end{tikzcd}$$

Clearly, it induces a commutative diagram in homology. Since the $\Sigma^{-1}P(n)$ are all concentrated in degrees ≥ 0, the proof is completed by the following easy

LEMMA 3.6. *Let O and O' be dg operads such that $O(n)$ is concentrated in degree 0 and $O'(n)$ in degrees ≥ 0 for all $n \geq 1$. Then the map*

$$\mathrm{Hom}_{\mathrm{Ho}(Op)}(O, O') \to \mathrm{Hom}_{Op}(H^*O, H^*O')$$

is bijective.

4. Notes

Tamarkin's proof [**51**] of his theorem relies on Kazhdan-Etingof's biquantization theory [**13**], and thus, ultimately, on the existence of a rational Drinfeld associator [**10**]. The proof was streamlined in [**49**] and presented with more details in [**25**]. We have essentially followed Kontsevich's interpretation [**33**], where the use of Drinfeld associators becomes more transparent: they appear naturally in the construction of the formality isomorphism Φ_1 in [**52**].

A comprehensive reference on operads is [**40**]. The first sections of [**21**] also offer a nice introduction to the subject.

Theorem 3.3 on the formality of the little squares operad was first announced by Getzler-Jones in [**18**]. However, the proof contained an error. A correct proof was given by Tamarkin in [**52**] using work by Fiedorowicz [**14**] and the existence of a rational Drinfeld associator [**10**]. Later, Kontsevich [**33**] gave a different proof and also proved that, more generally, the little d-cubes operad is formal for all $d \geq 0$.

Theorem 3.4 on the homotopy action of the little squares operad on the Hochschild complex goes back to a question formulated by Deligne in [**7**]. It was proved in [**54**] by correcting a method proposed in [**18**]. It also results by combining [**51**] with [**52**]. A geometric proof was given in [**35**] and purely topologico-combinatorial proofs in [**43**] and [**2**]. In [**31**], a conceptual approach was proposed and a closely related statement proved in a 'non-linear' context. A proof involving an operad related to Connes-Kreimer's renormalization Hopf algebra was given in [**30**]. A generalization of the theorem to little cubes was announced in [**33**] and proved in [**28**].

The brace operations on two arguments were introduced in [**15**] and on an arbitrary number of arguments in [**29**] and [**17**]. Their action on the Hochschild complex was systematized in [**16**].

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