HOCHSCHILD COHOMOLOGY AND DERIVED PICARD GROUPS

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Dedicated to Idun Reiten on the occasion of her sixtieth birthday

ABSTRACT. We interpret Hochschild cohomology as the Lie algebra of the derived Picard group and deduce that it is preserved under derived equivalences.

1. Introduction

The Hochschild cohomology groups $HH^i(A, A)$ of an algebra A over a field k can be interpreted as higher extension groups of the bimodule A by itself or as morphisms from A to $A[i]$ in the derived category $\mathcal{D}(A^{op} \otimes A)$ of A-A-bimodules. This last interpretation shows that they are preserved under derived equivalences [11], i.e. if X is a complex of A-B-bimodules such that the total derived tensor product by X is an equivalence $\mathcal{D}A \to \mathcal{D}B$, then X yields a natural isomorphism from $HH^i(A, A)$ to $HH^i(B, B)$. This isomorphism is compatible with the cup product, since the cup product corresponds to the composition of morphisms in the derived category of bimodules. However, it is not clear whether the isomorphism given by X also respects the Gerstenhaber bracket on Hochschild cohomology [4]. We will show that this is indeed the case by providing an intrinsic interpretation of the Gerstenhaber bracket in terms of derived categories. The basic idea is to view Hochschild cohomology as an analogue of the Lie algebra associated with an algebraic group (more precisely, a group-valued functor). This group will be the derived Picard group [17] [13] [16] of A (more precisely, the functor which sends a commutative differential graded k-algebra R to the R-relative derived Picard group of $A \otimes_k R$). Our interpretation generalizes the fact that the first Hochschild cohomology group of a finite-dimensional algebra A is the Lie algebra of the group of outer automorphisms of A.

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2. Reminder on derived equivalences

2.1. **Derived categories and the Hochschild cohomology algebra.** Let k be a field and A a k-algebra, i.e. an associative unital k-algebra. Let $\mathsf{Mod}\, A$ denote the category of right A-modules. Let $\mathcal{D}A$ denote the (unbounded) derived category of $\mathsf{Mod}\, A$. Thus, the objects of $\mathcal{D}A$ are all complexes

$$\ldots \to M^p \xrightarrow{d} M^{p+1} \to \ldots \,,\; p \in \mathbf{Z}\,,\; d^2 = 0\,,$$

of right A-modules and its morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms, i.e. morphisms of complexes inducing isomorphisms in homology. Let us recall the most basic examples of morphisms

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in the derived category: We identify an A-module L with the complex whose 0-component is L and whose components in all other degrees vanish. Then, if L and M are A-modules, the group of morphisms in $\mathcal{D}A$ from L to M identifies with the group of A-linear maps from L to M and, more generally, we have a natural isomorphism

$$\mathsf{Hom}_{\mathcal{D}A}(L, M[i]) \xrightarrow{\sim} \mathsf{Ext}^i_A(L, M)$$

for each $i \in \mathbf{Z}$, where, for a complex K, we denote by $K[i]$ the complex with components $K[i]^p = K^{i+p}$ and differential $(-1)^i d_K$. By convention, Ext-groups vanish in negative degrees.

In particular, we can identify the Hochschild cohomology groups $HH^*(A, A)$ with groups of morphisms in the derived category of A-A-bimodules: Indeed, since A is flat over k, we have a canonical isomorphism

$$HH^i(A, A) \xrightarrow{\sim} \mathsf{Ext}^i_{A^{op} \otimes A}(A, A)$$

and thus a canonical isomorphism

$$HH^i(A, A) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}(A^{op} \otimes A)}(A, A[i]),\ i \in \mathbf{Z}.$$

Under this isomorphism, the cup product on Hochschild cohomology corresponds to the graded composition in the derived category. More precisely, the cup product of the cohomology classes corresponding to $f : A \to A[j]$ and $g : A \to A[i]$ corresponds to the composition $f[i] \circ g$.

2.2. **Derived equivalence.** Let A and B be two k-algebras. We recall one version of Rickard's Morita theorem for derived categories [11] [12].

Theorem. *The following are equivalent*

(i) *There is a triangle equivalence $F : \mathcal{D}A \xrightarrow{\sim} \mathcal{D}B$.*
(ii) *There are bimodule complexes $X \in \mathcal{D}(A^{op} \otimes B)$ and $Y \in \mathcal{D}(B^{op} \otimes A)$ and isomorphisms*

$$X \otimes^{\mathsf{L}}_B Y \xrightarrow{\sim} A \text{ in } \mathcal{D}(A^{op} \otimes A) \text{ and } Y \otimes^{\mathsf{L}}_A X \xrightarrow{\sim} B \text{ in } \mathcal{D}(B^{op} \otimes B).$$

Here the symbol $\otimes^{\mathsf{L}}$ denotes the total derived tensor functor [14]. The implication from (ii) to (i) is easy: Indeed, the functor

$$F = ? \otimes^{\mathsf{L}}_A X\ : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence whose inverse is given by $? \otimes^{\mathsf{L}}_B Y$, *cf.* [12]. The implication from (ii) to (i) is considerably more delicate. One can also show [12] that if $X \in \mathcal{D}(A^{op} \otimes B)$ is a bimodule complex such that the associated functor $? \otimes^{\mathsf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence, then (ii) holds for X and

$$Y = \mathsf{R}\,\mathsf{Hom}_B(X, B).$$

Thus the essential datum is that of X. We call such X an *invertible* bimodule complex and Y its *inverse*. Two algebras A and B are called *derived equivalent* if the conditions of the theorem hold. If we consider other variants of the derived categories (e.g. the bounded derived categories), we obtain the same equivalence relation on the class of k-algebras, *cf.* [11]. Of course, derived equivalence generalizes Morita equivalence. The following example, a particular case of Koszul duality [2], [5], shows that this generalization is non trivial.

2.3. **An example.** Let V be a vector space of dimension $n+1$, denote by S^i the i-th symmetric power of its dual space and by Λ^i the i-th exterior power of V. Let A be the algebra of upper triangular matrices

$$A = \begin{pmatrix} S^0 & S^1 & \dots & S^n \\ 0 & S^0 & \dots & S^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & S^0 \end{pmatrix}$$

and B the algebra of lower triangular matrices

$$B = \begin{pmatrix} \Lambda^0 & 0 & \dots & 0 \\ \Lambda^1 & \Lambda^0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \Lambda^n & \Lambda^{n-1} & \dots & \Lambda^0 \end{pmatrix}.$$

Then A is derived equivalent to B, but for $n > 2$, A is not Morita equivalent to B. If E_i denotes the simple A-module corresponding to the projection on the $(i+1)$-th diagonal component, $i = 0, \dots, n$, then the complex

$$\dots \to 0 \to E_n \xrightarrow{0} E_{n-1} \xrightarrow{0} \dots \xrightarrow{0} E_0 \to 0 \dots$$

is the restriction to A of a bimodule complex $X \in \mathcal{D}(A^{op} \otimes B)$, unique up to isomorphism [7], whose associated tensor functor is an equivalence $\mathcal{D}A \to \mathcal{D}B$. Historically, this example comes from geometry: Beilinson showed in [1] that the derived categories of both A and B are triangle equivalent to the derived category of the category of quasicoherent sheaves on the projective space $\mathbf{P}(V)$.

2.4. **Invariance of the algebra structure.** The following theorem is due to Dieter Happel [6] in the special case of derived equivalences coming from tilting modules and to Jeremy Rickard [12] in the general case. Let A and B be derived equivalent algebras and $X \in \mathcal{D}(A^{op} \otimes B)$ and invertible bimodule complex

Theorem. *There is a canonical algebra isomorphism*

$$\varphi_X : HH^*(A, A) \to HH^*(B, B).$$

After the preparations we have made, it is easy to construct φ_X: Indeed, let Y be the inverse of X with isomorphisms $u : Y \otimes^{\mathsf{L}}_A X \xrightarrow{\sim} B$ and $v : X \otimes^{\mathsf{L}}_B Y \xrightarrow{\sim} A$. Let $\varphi_{X,u}$ send $f : A \to A[i]$ to

$$u[i] \circ (Y \otimes^{\mathsf{L}}_A f \otimes^{\mathsf{L}}_A X) \circ u^{-1} \; : \; B \to B[i].$$

If u' is another isomorphism from $Y \otimes^{\mathsf{L}}_A X$ to B, then $u' = zu$ for an invertible central element z of B. So we have

$$\varphi_{X,u'}(f) = z[i]\, \varphi_{X,u}(f)\, z^{-1}$$

and this equals $\varphi_{X,u}(f)$ since the center of B is central in $HH^*(B, B)$. We define $\varphi_X = \varphi_{X,u}$.

3. Reminder on the Gerstenhaber bracket

Let A be an algebra. The *Hochschild complex* of A is the complex

$$C^{\cdot}(A, A) = (A \to \mathsf{Hom}_k(A, A) \to \dots \to \mathsf{Hom}_k(A^{\otimes p}, A) \to \dots)$$

with A in degree 0 whose differential maps $a \in A$ to $[a, ?]$, and, more generally, a p-cochain $c \in \mathsf{Hom}_k(A^{\otimes p}, A)$ to the $(p+1)$-cochain dc defined by

$$\begin{aligned}(dc)(a_0, \dots, a_p) &= a_0 c(a_1, \dots, a_p) - c(a_0 a_1, a_2, \dots, a_p) + \dots \\ &\quad + (-1)^{p+1} c(a_0, \dots, a_{p-1}) a_p.\end{aligned}$$

The homology in degree i of $C^{\cdot}(A,A)$ is $HH^i(A,A)$. For a p-cochain c_1, a q-cochain c_2 and an integer $0 \leq i \leq p-1$, define a $(p+q-1)$-cochain $c_1 \bullet_i c_2$ by

$$(c_1 \bullet_i c_2)(a_1, \ldots, a_{p+q-1}) = c_1(a_1, \ldots, a_i, c_2(a_{i+1}, \ldots, a_{i+q}), a_{i+q+1}, \ldots, a_{p+q-1}).$$

Then the *Gerstenhaber product* is defined by

$$(c_1 \bullet c_2) = \sum_{i=0}^{p-1} (-1)^{i(q-1)} c_1 \bullet_i c_2.$$

The Gerstenhaber product is not associative. However, the associator

$$A(c_1, c_2, c_3) = (c_1 \bullet c_2) \bullet c_3 - c_1 \bullet (c_2 \bullet c_3)$$

is super symmetric in c_2 and c_3 endowed with suitable degrees (super=$\mathbf{Z}/2\mathbf{Z}$-graded). Namely, we have

$$A(c_1, c_2, c_3) = (-1)^{(q-1)(r-1)} A(c_1, c_3, c_2)\,,$$

where c_2 is a q-cochain and c_3 an r-cochain. Therefore the super commutator of the Gerstenhaber bracket behaves like the commutator of an associative product, i.e. it satisfies the super Jacobi identity. More precisely, we have the

Lemma. [4] *Endowed with the* Gerstenhaber bracket *defined by*

$$[c_1, c_2] = c_1 \bullet c_2 - (-1)^{(p-1)(q-1)} c_2 \bullet c_1\,,\ c_1 \in C^p(A,A)\,,\ c_2 \in C^q(A,A)\,,$$

and the differential $-d$, the graded space $C^{\cdot+1}(A,A)$ becomes a differential graded Lie algebra. In particular, the homology $HH^{*+1}(A,A)$ becomes a $\mathbf{Z}$-graded super Lie algebra.

For example, let V be a vector space and A the algebra of polynomial functions on V. Then $HH^1(A,A)$ identifies with the space $Der_k(A,A)$ of k-linear derivations of A and $HH^{i+1}(A,A)$ is canonically isomorphic to the exterior power $\Lambda_A^{i+1} \mathsf{Der}_k(A,A)$. Under this isomorphism, the Gerstenhaber bracket corresponds to the Nijenhuis-Schouten-bracket, which is the natural extension of the commutator of derivations.

4. The Lie algebra of the derived Picard group

We will interpret Hochschild cohomology with the Gerstenhaber bracket as the Lie algebra of a 'generalized algebraic group', namely a group valued functor defined on a category of commutative algebras. For this, let us recall the construction of the Lie algebra of an algebraic group: Let G be an algebraic group over k considered as a group-valued functor

$$G : \{\text{commutative } k\text{-algebras}\} \to \{\text{groups}\}\,,\ R \mapsto G(R).$$

Then the Lie algebra of G is the space of tangent vectors at the origin, i.e.

$$\mathsf{Lie}(G) = \mathsf{ker}(G(k[\varepsilon]/(\varepsilon^2)) \to G(k)).$$

The bracket is induced by the commutator in G. To make this last statement more intuitive, consider the example where $G = GL_n$. We have

$$\mathsf{Lie}(GL_n) = \{1 + \varepsilon X \mid X \in M_n(k)\ \}$$

and the Lie bracket is determined by the identity

$$(1+\varepsilon_1 X_1)(1+\varepsilon_2 X_2)(1+\varepsilon_1 X_1)^{-1}(1+\varepsilon_2 X_2)^{-1} = 1 + \varepsilon_1\varepsilon_2\,[X_1, X_2]$$

in $GL_n(k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2))$.

We will now define a group valued functor DPic_A whose Lie algebra will be the Hochschild cohomology of A. Since this Lie algebra is graded, the category on which DPic_A is defined should include the category of graded commutative algebras. It turns out that a reasonable category is $\mathsf{cdg}\, k$, the category of commutative differential graded k-algebras. To define

$$\mathsf{DPic}_A : \mathsf{cdg}\, k \to \{\text{groups}\} ,$$

we need the relative derived category (*cf.* [9, Sect. 7]): Let R be a commutative differential graded algebra (for example the algebra $k[\varepsilon]/(\varepsilon^2)$, where ε has any integer degree and $d = 0$). Let E be a (typically non commutative) differential graded R-algebra. The *R-relative derived category* $\mathcal{D}_R E$ has as objects all differential graded E-modules (these are precisely the complexes of E-modules if E is concentrated in degree 0). The morphisms of $\mathcal{D}_R E$ are obtained from morphisms of differential graded E-modules by formally inverting all *R-relative quasi-isomorphisms*, i.e. all morphisms $s : L \to M$ of differential graded E-modules whose restriction to R is an homotopy equivalence. For example, the relative derived category $\mathcal{D}_k E$ equals the usual derived category $\mathcal{D}E$ of the differential graded algebra E.

Rouquier-Zimmermann [17] [13] and Yekutieli [16] have independently defined the derived Picard group of a ring. We generalize this as follows: Let R be a commutative differential graded algebra and A an algebra. A bimodule complex $U \in \mathcal{D}_R(R \otimes A^{op} \otimes A)$ is *R-semifree* if its underlying graded R-module is free; it is *invertible* if it is R-semifree[1] and there exists an R-semifree bimodule complex $V \in \mathcal{D}_R(R \otimes A^{op} \otimes A)$ such that there are isomorphisms

$$U \otimes^{\mathsf{L},\mathrm{rel}}_{A\otimes R} V \xrightarrow{\sim} R \otimes A \text{ and } V \otimes^{\mathsf{L},\mathrm{rel}}_{A\otimes R} U \xrightarrow{\sim} R \otimes A$$

in $\mathcal{D}_R(R \otimes A^{op} \otimes A)$. The *$R$-relative derived Picard group of A* is the set of isomorphism classes of invertible bimodule complexes U in $\mathcal{D}_R(R \otimes A^{op} \otimes A)$. This set is endowed with the group law induced by the derived tensor product. This group is denoted by $\mathsf{DPic}_A(R)$. It is functorial with respect to R so that we do obtain a functor

$$\mathsf{cdg}\, k \to \{\text{groups}\}.$$

If A is derived equivalent to B and $X \in \mathcal{D}(A^{op} \otimes B)$ is an invertible bimodule complex with inverse Y, then we have an isomorphism

$$\mathsf{DPic}_A(R) \xrightarrow{\sim} \mathsf{DPic}_B(R) ,\ U \mapsto Y \otimes^{\mathsf{L}}_A U \otimes^{\mathsf{L}}_A X.$$

In this sense, DPic_A is also functorial with respect to invertible bimodule complexes $X \in \mathcal{D}(A^{op} \otimes B)$.

We now define the Lie algebra of DPic_A. Fix a degree $i \in \mathbf{Z}$ and let R be the commutative differential graded algebra $k[\varepsilon]/(\varepsilon^2)$, where ε is of degree $-i$ and $d = 0$. By definition, $\mathsf{LieDPic}^i_A$ is the set of isomorphism classes U of $\mathcal{D}_R(R \otimes A^{op} \otimes A)$ such that U is free as a graded R-module and $U \otimes_R k$ is isomorphic to A in $\mathcal{D}(A^{op} \otimes A)$. The graded space $\mathsf{LieDPic}^*_A$ is endowed with a super Lie bracket defined as for algebraic groups (*cf.* 5.6). For a super Lie algebra L, we denote by L^{op} the super Lie algebra with the opposite bracket. Its appearance in the theorem below is due to the fact that we consider right modules.

Theorem. *There is a canonical isomorphism of graded super Lie algebras*

$$HH^{*+1}(A, A)^{op} \xrightarrow{\sim} \mathsf{LieDPic}^*_A$$

functorial with respect to invertible bimodule complexes $X \in \mathcal{D}(A^{op} \otimes B)$. In particular, the Gerstenhaber bracket on $HH^{*+1}(A, A)$ is preserved under derived equivalence.

The rest of the article is devoted to the proof of the theorem. In fact, we will prove it more generally for a differential graded k-algebra A.

[1] presumably, all semifreeness conditions are redundant.

5. Proof of the main theorem

5.1. **Outline of the proof.** Let A be a dg algebra. The idea is to construct an intermediate 'differential graded formal group G' whose Lie algebra is the Hochschild complex with the Gerstenhaber bracket and which acts on the relative derived category via bimodules. This group is the group G of automorphisms of the cobar construction C^+ of A, where the cobar construction is viewed as a differential graded counital (but not coaugmented) coalgebra (*cf.* 5.7). It naturally acts on the category of differential graded comodules over C^+. Via the bar-cobar-adjunction at the module level, this action translates into an action of G on the derived category via bimodules. On the other hand, the Lie algebra of G is the Lie algebra of coderivations of C^+ and, by Stasheff's interpretation [15], this Lie algebra is the Hochschild complex endowed with the Gerstenhaber bracket. This programme yields a Lie algebra morphism

$$\mathsf{Z}^0(C^{\cdot+1}(A,A)\otimes\mathfrak{m}) \longrightarrow \mathsf{LieDPic}(A,R)$$

for each augmented commutative dg algebra $R = k\oplus\mathfrak{m}$ with $\mathfrak{m}^2=0$. To check that it induces an isomorphism, we need to identify the set of deformation classes $\mathsf{LieDPic}(A,R)$ with the group

$$\mathsf{Hom}_{\mathcal{D}(A\otimes A^{op})}(A, A\otimes\mathfrak{m}[1]).$$

We will also need to know how the Lie algebra structure is reflected under this identification. This is what we study first, in the paragraphs 5.2 – 5.6 below.

5.2. **Infinitesimal deformations of modules.** Let k be a field and S a commutative dg k-algebra. Suppose that R is an augmented commutative dg S-algebra and denote by $\mathfrak{n}$ the kernel of the augmentation $R\to S$. Thus we have the decomposition $R = S\oplus\mathfrak{n}$.

Let A be a (typically noncommutative) dg S-algebra, free as a graded S-module. Then $A\otimes_S R$ is a dg R-algebra. We consider the reduction functor

$$\mathcal{D}_R(A\otimes_S R)\to\mathcal{D}_S A\ ,\ L\mapsto L\otimes_R S.$$

We will study the fibers of this functor: Let M be a dg A-module which is free as a graded S-module. Let $\mathcal{F}$ be the category whose objects are the *deformations of* M, i.e. the pairs (L,u) formed by a dg $A\otimes R$-module L, free as a graded R-module, and an isomorphism of $\mathcal{D}_S A$

$$u : L\otimes_R S \xrightarrow{\sim} M.$$

Morphisms from (L,u) to (L',u') are given by morphisms $v : L\to L'$ of $\mathcal{D}_R(A\otimes_S R)$ such that $u'\circ(v\otimes_R S) = u$. We denote by

$$\mathsf{Defo}(M, R\to S)$$

the set of isomorphism classes of $\mathcal{F}$. We denote by

$$\mathsf{Defo}'(M, R\to S)$$

the set of isomorphism classes of *weak deformations of* M, i.e. dg $A\otimes R$-modules L free as graded R-modules such that $L\otimes_R S$ is isomorphic to M. Note that we have an obvious forgetful map

$$\mathsf{Defo}(M,R\to S)\to\mathsf{Defo}'(M,R\to S).$$

The group of automorphisms of M in $\mathcal{D}_S A$ acts on $\mathsf{Defo}(M,R\to S)$ via $(L,u).f = (L, f^{-1}\circ u)$ and the forgetful map clearly induces a bijection

$$\mathsf{Defo}(M,R\to S)/\,\mathsf{Aut}_{\mathcal{D}_R A}(M)\xrightarrow{\sim}\mathsf{Defo}'(M,R\to S).$$

From now on and to the end of this section, we suppose that $\mathfrak{n}^2 = 0$. We will parametrize the connected components of $\mathcal{F}$. Let (L, u) be an object of $\mathcal{F}$. Since L is free as a graded R-module, the morphism of complexes $L \otimes_R \mathfrak{n} \to L\mathfrak{n}$ is invertible and the sequence

$$0 \to L \otimes_R \mathfrak{n} \to L \to L \otimes_R S \to 0$$

is an exact sequence of dg A-modules which splits as a sequence of dg S-modules. Thus it gives rise to a canonical triangle of $\mathcal{D}_S A$ (but not of $\mathcal{D}_R A$!)

$$L \otimes_R \mathfrak{n} \to L \to L \otimes_R S \xrightarrow{\varepsilon'} (L \otimes_R \mathfrak{n})[1] \tag{5.1}$$

Since $\mathfrak{n}^2 = 0$, we have a canonical isomorphism of dg modules

$$L \otimes_R \mathfrak{n} \xrightarrow{\sim} (L \otimes_R S) \otimes_S \mathfrak{n}.$$

Therefore, we can define a canonical morphism $\varepsilon(L, u)$ of $\mathcal{D}_S A$ by the commutative square

$$\begin{tikzcd} L \otimes_R S \arrow[r, "\varepsilon'"] \arrow[d, "u"'] & L \otimes_R \mathfrak{n}[1] \arrow[r, "\sim"] & (L \otimes_R S) \otimes_S \mathfrak{n}[1] \arrow[d, "u \otimes \mathfrak{n}[1]"] \\ M \arrow[rr, "\varepsilon(L,u)"] & & M \otimes_S \mathfrak{n}[1] \end{tikzcd}$$

Clearly the morphism $\varepsilon(L, u)$ only depends on the isomorphism class of (L, u) in the category $\mathcal{F}$.

Proposition. *The map $\Phi : (L, u) \mapsto \varepsilon(L, u)$ induces a bijection*

$$\mathsf{Defo}(M, R \to S) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}_S A}(M, M \otimes_S \mathfrak{n}[1]).$$

Clearly, the bijection of the proposition is equivariant with respect to the action of $\mathsf{Aut}_{\mathcal{D}_S A}(M)$. Therefore we have the

Corollary. *The map $\Phi : (L, u) \mapsto \varepsilon(L, u)$ induces a bijection*

$$\mathsf{Defo}'(M, R \to S) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}_S A}(M, M \otimes_S \mathfrak{n}[1]) / \mathsf{Aut}_{\mathcal{D}_S A}(M).$$

Proof. We construct a map Ψ which will turn out to be the inverse bijection. We may and will assume that M is S-relatively cofibrant in the category of dg A-modules, i.e. it satisfies the S-relative variant of property (P) of [8, 3.1]. This means [9, 7.5] that M admits an increasing filtration by dg A-submodules M_n, $n \geq 0$, such that each inclusion $M_n \subset M_{n+1}$ splits as a morphism of graded A-modules and the subquotient M_{n+1}/M_n is isomorphic to a direct summand of a module $K \otimes_S A$, where K is a dg S-module. Let a morphism

$$f : M \to M \otimes_S \mathfrak{n}[1]$$

of $\mathcal{D}_S A$ be given. Since M is S-relatively cofibrant as a dg A-module, the morphism f is realized by a map $\widetilde{f}$ of dg A-modules. We put $L = M \otimes_S R = (M \otimes_S \mathfrak{n}) \oplus M$ as a graded $A \otimes_S R$-module and we define its differential by

$$d_L = \begin{bmatrix} d_{M \otimes_S \mathfrak{n}} & f \\ 0 & d_{\mathfrak{n}} \end{bmatrix}.$$

Clearly L is free as a graded R-module (since M is free as a graded S-module) and we have an obvious isomorphism $u : L \otimes_R S \to M$. So we have constructed an object (L, φ) of $\mathcal{F}$ by choosing a representative $\widetilde{f}$ of the homotopy class $f : M \to M \otimes_S \mathfrak{n}[1]$. Let us check that the connected component of (L, φ) is independent of the choice of the representative. Let $\widetilde{f}'$ be another choice and let $h : M \to M \otimes_S \mathfrak{n}$ be a morphism of graded A-modules such that $f' = f + d \circ h + h \circ d$. Let (L', u') be the object of $\mathcal{F}$ constructed from f'. Then L' equals L as a graded $A \otimes_S R$-module and both equal $M \otimes_S \mathfrak{n} \oplus M$. The matrix

$$\begin{bmatrix} \mathbf{1}_{M\otimes_S \mathfrak{n}} & h \\ 0 & \mathbf{1}_M \end{bmatrix}$$

defines an isomorphism v of dg $A \otimes_S R$-modules from L to L' and we clearly have $u' = u \circ (v \otimes_R S)$. By definition, the map Ψ sends f to the connected component of (L, u). The easy check that $\Phi \circ \Psi$ is the identity is left to the reader. To finish the proof, it is enough to check that Ψ is surjective. For this, let an object (L', u') of $\mathcal{F}$ be given. We may and will assume that L' is R-relatively cofibrant. Put $M' = L' \otimes_R S$. Since L' is free as a graded R-module, there is an isomorphism of graded R-modules $M' \otimes_S R \xrightarrow{\sim} L'$ which lifts the identity of M'. The differential of L' then yields a differential of $M' \otimes_S R = (M' \otimes_S \mathfrak{n}) \oplus M'$ given by a matrix

$$\begin{bmatrix} d_{M'\otimes_S \mathfrak{n}} & f' \\ 0 & d_{M'} \end{bmatrix}.$$

Now define $f : M \to M \otimes_S \mathfrak{n}[1]$ by the commutative square of $\mathcal{D}_S A$

$$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, "u'"'] & M' \otimes_S \mathfrak{n}[1] \arrow[d, "u'\otimes_S \mathfrak{n}[1]"] \\ M \arrow[r, "f"] & M \otimes_S \mathfrak{n}[1] \end{tikzcd}$$

Since M' is S-relatively cofibrant, there is a morphism of dg modules $u_1 : M' \to M$ lifting u'. Moreover, $f \circ u_1$ is homotopy equivalent to $(u_1 \otimes_S \mathfrak{n}[1]) \circ f'$. Choose an homotopy h between the two. Then the matrix

$$\begin{bmatrix} u_1 \otimes \mathbf{1}_{\mathfrak{n}} & h \\ 0 & u_1 \end{bmatrix}$$

defines a map

$$\widetilde{u} : L' = (M' \otimes_S \mathfrak{n}) \oplus M' \to (M \otimes_S \mathfrak{n}) \oplus M = L.$$

This is in fact a morphism of dg $A \otimes_S R$-modules and clearly it gives a morphism of $\mathcal{F}$. Since the triangle (5.1) does not exist in the category $\mathcal{D}_R A$, it is not immediate that $\widetilde{u}$ is invertible. However, starting from an inverse of u', we can analogously construct a morphism $\widetilde{v} : L \to L'$ of $\mathcal{D}_R A$. Then the reduction of $\widetilde{u}\widetilde{v}$ modulo $\mathfrak{n}$ is homotopic to the identity. Let H be an homotopy. The homotopy H is a morphism of graded A-modules, and we can lift it to a morphism of graded $A \otimes_S R$-modules. We see that $\widetilde{u}\widetilde{v}$ is homotopic to a morphism w whose reduction modulo $\mathfrak{n}$ is the identity. Since L' and L are free over R and $\mathfrak{n}$ is nilpotent, it follows that w is invertible. So $\widetilde{u}\widetilde{v}$ is homotopic to an invertible morphism. Thus $\widetilde{u}\widetilde{v}$ is invertible in the homotopy category. Similarly, we see that $\widetilde{v}\widetilde{u}$ is invertible in the homotopy category.

We conclude that (L', u') is in the isomorphism class of $\mathcal{F}$ which is the image of f under Ψ. Hence Ψ is surjective. $\surd$

5.3. **An exact sequence.** Let T be a commutative dg algebra, S an augmented dg T-algebra and $R = S \oplus \mathfrak{n}$ an augmented dg S-algebra. Thus R becomes an augmented dg T-algebra $R = T \oplus \mathfrak{m}$ and we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{m} \arrow[r] & R \arrow[r] & T \arrow[r] & 0. \end{tikzcd}$$

Let A be a dg T-algebra free as a graded T-module and let M be a dg A-module free as a graded T-module. Then the tensor product $?\otimes_R S$ yields a natural map

$$\mathsf{Defo}(M, R\to T)\to \mathsf{Defo}(M, S\to T).$$

On the other hand, if we have a representative (L', u') of an element of $\mathsf{Defo}(M\otimes_T S, R\to S)$, we obtain an element of $\mathsf{Defo}(M, R\to T)$ by taking $L=L'$ and u the composition

$$L\otimes_S T = L'\otimes_S T \xrightarrow{u'\otimes_S T} (M\otimes_T S)\otimes_S T = M.$$

The following lemma is immediate from these definitions.

Lemma. a) *The sequence*

$$\mathsf{Defo}(M\otimes_T S, R\to S)\to \mathsf{Defo}(M, R\to T)\to \mathsf{Defo}(M, S\to T)\to *$$

is exact in the sense that the second map is surjective and its fibre over the base point is the image of the first map.

b) *The sequence*

$$*\to \mathsf{Defo}'(M\otimes_T S, R\to S)\to \mathsf{Defo}'(M, R\to T)\to \mathsf{Defo}'(M, S\to T)\to *$$

is exact in the sense that the second map is surjective, and its fiber over the base point equals the image of the first map, which is injective.

5.4. **A base change isomorphism.** Let T be a commutative dg algebra, S an augmented T-algebra and $R=S\oplus\mathfrak{n}$ an augmented S-algebra. Thus R becomes an augmented T-algebra $R=T\oplus\mathfrak{m}$. Let $R'=T\oplus\mathfrak{n}$ so that we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n} \arrow[r] & R \arrow[r] & S \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[u, equal] & R' \arrow[r] \arrow[u] & T \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Let A be a dg T-algebra free as a graded T-module and let M be a dg A-module free as a graded T-module. The tensor products $?\otimes_{R'} R$ and $?\otimes_T S$ yield a natural map

$$\mathsf{Defo}(M, R'\to T)\to \mathsf{Defo}(M\otimes_T S, R\to S).$$

Lemma. *If $\mathfrak{n}^2=0$, this map is an isomorphism.*

Proof. We have a commutative square

$$\begin{array}{ccc}
\mathsf{Defo}(M, R'\to T) & \to & \mathsf{Defo}(M\otimes_T S, R\to S) \\
\downarrow & & \downarrow \\
\mathsf{Hom}_{\mathcal{D}_T A}(M, M\otimes_T \mathfrak{n}[1]) & \to & \mathsf{Hom}_{\mathcal{D}_S A}(M\otimes_T S, M\otimes_T S\otimes_S \mathfrak{n}[1])
\end{array}$$

whose vertical maps are given by proposition (5.2) and whose horizontal maps are given by the tensor functor. So the vertical maps are bijective. And the lower horizontal arrow identifies with the adjunction isomorphism

$$\mathsf{Hom}_{\mathcal{D}_T A}(M, M\otimes_T \mathfrak{n}[1])\to \mathsf{Hom}_{\mathcal{D}_S A}(M\otimes_T S, M\otimes_T \mathfrak{n}[1])$$

where we view $M\otimes_T\mathfrak{n}[1]$ on the left hand side as the restriction to A of the $A\otimes_T S$-module $M\otimes_T \mathfrak{n}[1]$ on the right hand side. $\surd$

Corollary. *With the above notations, suppose that the map*

$$\mathsf{Defo}(M\otimes_T S, R\to S)\to \mathsf{Defo}'(M\otimes_T S, R\to S)$$

is bijective. Then we have an exact sequence

$$*\to \mathsf{Defo}(M, R'\to T)\to \mathsf{Defo}'(M, R\to T)\to \mathsf{Defo}'(M, S\to T)\to *$$

in the sense that the second map is surjective and its fibre over the base point is the image of the first map, which is injective.

This follows from the lemma and from part b) of lemma 5.3.

5.5. **Infinitesimal deformations of the bimodule A.** Let A be a dg k-algebra, T a commutative dg k-algebra and $S = T \oplus \mathfrak{n}$ an augmented commutative dg T-algebra with $\mathfrak{n}^2 = 0$.

We consider the module $M = A \otimes_k T$ over the algebra $B = A^{op} \otimes A \otimes T$.

Lemma. *Suppose that H^*T is of finite total dimension. Then the canonical map*

$$\mathsf{Defo}(M, S \to T) \to \mathsf{Defo}'(M, S \to T)$$

is bijective.

Proof. By (5.2), we have to show that the action of $\mathsf{Aut}_{\mathcal{D}_T B}(A \otimes T)$ on

$$\mathsf{Hom}_{\mathcal{D}B}(M, M \otimes_T \mathfrak{n}[1])$$

is trivial. We claim that we have an isomorphism

$$\mathbf{R}\,\mathsf{Hom}_B(A \otimes T, A \otimes T) \otimes_T \mathfrak{n}[1] \xrightarrow{\sim} \mathbf{R}\,\mathsf{Hom}_B(A \otimes T, A \otimes T \otimes_T \mathfrak{n}[1]).$$

Indeed, we have an adjunction isomorphism

$$\mathbf{R}\,\mathsf{Hom}_B(A \otimes T, A \otimes T) \otimes_T \mathfrak{n}[1] \xrightarrow{\sim} \mathbf{R}\,\mathsf{Hom}_{A^e}(A, A \otimes T) \otimes_T \mathfrak{n}[1].$$

Now since H^*T is of finite total dimension, we have an isomorphism

$$\mathbf{R}\,\mathsf{Hom}_{A^e}(A, A \otimes T) \xrightarrow{\sim} \mathbf{R}\,\mathsf{Hom}_{A^e}(A, A) \otimes T.$$

Combining the two preceding isomorphisms, we obtain an isomorphism

$$\mathbf{R}\,\mathsf{Hom}_B(A \otimes T, A \otimes T) \otimes_T \mathfrak{n}[1] \xrightarrow{\sim} \mathbf{R}\,\mathsf{Hom}_{A^e}(A, A) \otimes \mathfrak{n}[1].$$

Now we have isomorphisms

$$\mathbf{R}\,\mathsf{Hom}_{A^e}(A, A) \otimes \mathfrak{n}[1] \xrightarrow{\sim} \mathbf{R}\,\mathsf{Hom}_{A^e}(A, A \otimes \mathfrak{n}[1]) \xrightarrow{\sim} \mathbf{R}\,\mathsf{Hom}_B(A \otimes T, A \otimes T \otimes_T \mathfrak{n}[1]),$$

where we have first used that $H^*\mathfrak{n}$ is of finite total dimension and then the adjunction, as above. The claim follows. Now $\mathsf{Aut}_{\mathcal{D}_T B}(A \otimes T)$ is the group of invertible elements of the zeroth homology of

$$\mathbf{R}\,\mathsf{Hom}_B(A \otimes T, A \otimes T).$$

The claim follows since this dg algebra is commutative up to homotopy. $\surd$

5.6. **Definition of the Lie bracket.** Let A be a dg k-algebra. Let $\mathfrak{m}$ be a dg k-module whose homology is of finite total dimension. Let $R = k \oplus \mathfrak{m}$ denote the augmented commutative dg algebra with $\mathfrak{m}^2 = 0$. We consider A as an A-A-bimodule. We define

$$G(\mathfrak{m}) = \mathsf{Defo}'(A, k \oplus \mathfrak{m} \to k).$$

In other words, $G(\mathfrak{m})$ is formed by the isomorphism classes of objects in $\mathcal{D}_R(A^{op} \otimes A \otimes R)$ whose reduction modulo $\mathfrak{m}$ is isomorphic to A in $\mathcal{D}(A^{op} \otimes A)$. Note that, according to lemma 5.5, we have a canonical bijection

$$\mathsf{Defo}(A, k \oplus \mathfrak{m} \to k) \xrightarrow{\sim} \mathsf{Defo}'(A, k \oplus \mathfrak{m} \to k).$$

We will view $\mathcal{D}(A^{op} \otimes A)$ (resp. $\mathcal{D}_R(A^{op} \otimes A \otimes R)$) as a monoidal category for the derived tensor product over A (resp. for the relative derived tensor product over $A \otimes R$). The monoidal structure of $\mathcal{D}_R(A^{op} \otimes A \otimes R)$ induces a monoid structure on $G(\mathfrak{m})$ and the bijection

$$G(\mathfrak{m}) \xleftarrow{\sim} \mathsf{Defo}(A, k \oplus \mathfrak{m} \to k) \to \mathsf{Hom}_{\mathcal{D}(A^{op} \otimes A)}(A, A \otimes \mathfrak{m}[1])$$

is a monoid morphism. In particular, $G(\mathfrak{m})$ is an abelian group, functorial in $\mathfrak{m}$.

For two dg k-modules $\mathfrak{m}_1$ and $\mathfrak{m}_2$ whose homology is of finite total dimension, we define a Lie bracket

$$G(\mathfrak{m}_1) \times G(\mathfrak{m}_2) \to G(\mathfrak{m}_1 \otimes \mathfrak{m}_2)$$

as follows : Let L_1 and L_2 represent elements of $G(\mathfrak{m}_1)$ resp. $G(\mathfrak{m}_2)$. Put $R_i = k \oplus \mathfrak{m}_i$. Let U_i be the image of L_i in $\mathcal{D}_R(R \otimes A^{op} \otimes A)$ where $R = R_1 \otimes R_2$ (note that the kernel of $R \to k$ is not of square zero !). The U_i are invertible objects of a monoidal category. Let V denote the commutator of U_1 with U_2. Then V yields an element of $\mathsf{Defo}'(A, R \to k)$. We have a canonical map

$$G(\mathfrak{m}_1 \otimes \mathfrak{m}_2) \to \mathsf{Defo}'(A, R \to k)$$

and we claim that it is injective and that V lies in its image. Indeed, the image of V in $\mathsf{Defo}'(A, R_1 \to k) = G(\mathfrak{m}_1)$ vanishes since $G(\mathfrak{m}_1)$ is commutative. Thus V lies in

$$G(\mathfrak{m}_2 \oplus \mathfrak{m}_1 \otimes \mathfrak{m}_2) = \mathsf{Defo}'(A, k \oplus (\mathfrak{m}_2 \oplus \mathfrak{m}_1 \otimes \mathfrak{m}_2))$$

by (5.3). The image of V in $G(\mathfrak{m}_2) = \mathsf{Defo}'(A, k \oplus \mathfrak{m}_2 \to k)$ also vanishes since $G(\mathfrak{m}_2)$ is commutative. So again by 5.3, V lies in $\mathsf{Defo}'(A, k \oplus (\mathfrak{m}_1 \otimes \mathfrak{m}_2) \to k) = G(\mathfrak{m}_1 \otimes \mathfrak{m}_2)$.

5.7. **From coalgebra automorphisms to bimodules.** Let R be a commutative dg k-algebra and A a (typically noncommutative) dg R-algebra. Denote by SA the graded R-module with $(SA)^p = A^{p+1}$. We recall the *bar construction* of A relative to R. It is the dg R-coalgebra C^+ defined as follows: Its underlying graded space is

$$R \oplus SA \oplus (SA \otimes_R SA) \oplus \ldots \oplus (SA)^{\otimes_R p} \oplus \ldots \quad .$$

The comultiplication is defined by

$$\begin{aligned} \Delta(a_1, \ldots, a_p) \quad &= \quad 1_R \otimes (a_1, \ldots, a_p) + \sum_{i=1}^{p-1} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_p) \\ &\quad + (a_1, \ldots, a_p) \otimes 1_R \quad . \end{aligned}$$

Moreover C^+ is endowed with the counit $\eta : C^+ \to R$ given by the natural projection and the coaugmentation $\varepsilon : R \to C^+$ given by the natural inclusion. The composition of the projection $C^+ \to SA$ with the canonical morphism $s : SA \to A$ of degree $+1$ yields a homogeneous morphism $\tau : C^+ \to A$ of degree $+1$. A *coderivation* of C^+ is a homogeneous R-linear map $D : C^+ \to C^+$ such that $\Delta \circ D = D \otimes \mathbf{1} + \mathbf{1} \otimes D$. Note that this implies that $\eta \circ D = 0$. Let $\mathsf{Coder}(C^+, C^+)$ denote the graded R-module whose p-th component is formed by the coderivations of degree p. Then the composition with τ is a bijection onto the space of homogeneous R-linear morphisms from C^+ to A

$$\mathsf{Coder}(C^+, C^+) \xrightarrow{\sim} \mathsf{Hom}_R(C^+, A) \, , \; D \mapsto \tau \circ D. \tag{5.2}$$

In particular, C^+ admits a unique coderivation d_{C^+} of degree $+1$ such that $\tau \circ d_{C^+}$ restricted to $(SA)^{\otimes_R p}$ vanishes for $p \neq 1, 2$, and equals

$$\mu \circ (s \otimes s) : (SA) \otimes_R (SA) \to A$$

for $p = 2$ and $-d_A \circ s$ for $p = 1$. Here μ denotes the multiplication of A. The facts that $\mu : A \otimes A \to A$ is a morphism of complexes and that μ is associative are equivalent to the fact that $d_{C^+}^2 = 0$.

Proposition ([15])**.** *Endowed with the supercommutator and the differential $D \mapsto [d_{C^+}, D]$ the graded space $\mathsf{Coder}(C^+, C^+)$ becomes a differential graded Lie algebra which is isomorphic to the Gerstenhaber Lie algebra by the map (5.2).*

For two homogeneous R-linear morphisms $f, g : C^+ \to A$, we define $f \star g = \mu \circ (f \otimes g) \circ \Delta$. Then we have

$$\tau \star \tau = d \circ \tau + \tau \circ d \tag{5.3}$$

For a dg right A-module M and a dg left C^+-comodule N, we denote by $M \otimes_\tau N$ the dg R-module $M \otimes_R N$ with the differential defined by

$$d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes y + \sum x\, \tau(y_{(1)}) \otimes y_{(2)} \, ,$$

where x is homogeneous of degree p and $\delta(y) = \sum y_{(1)} \otimes y_{(2)}$ (Sweedler's notation). Similarly, for a left A-module M and a right C^+-comodule N, we define $N \otimes_\tau M$ to be $N \otimes_R M$ with the differential defined by

$$d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes y - \sum x_{(1)} \otimes \tau(x_{(2)})\, y \, ,$$

(note the sign in front of $\sum$). The fact that the squares of these differentials vanish follows from equation (5.3).

The dg R-module $A \otimes_\tau C^+$ inherits a right C^+-comodule structure from C^+ and a left A-module structure from A. It yields the dg A-A-bimodule

$$A \otimes_\tau C^+ \otimes_\tau A.$$

It is not hard to check that up to the signs of the differentials, this is the (sum) total dg module associated with the bar resolution of the A-A-bimodule A. In particular, we have a canonical quasi-isomorphism (which is even an homotopy equivalence of left dg A-modules or right dg A-modules)

$$A \leftarrow A \otimes_\tau C^+ \otimes_\tau A.$$

Now let $\varphi : C^+ \to C^+$ be an automorphism of the dg counital R-coalgebra C^+. Define C^+_φ to be the dg C^+-C^+-bicomodule whose left comultiplication is that of C^+ whereas the right comultiplication is $(\mathbf{1} \otimes \varphi) \circ \Delta$. We define the bimodule

$$X(\varphi) = A \otimes_\tau C^+_\varphi \otimes_\tau A.$$

Note that the underlying graded module of $X(\varphi)$ is $A \otimes_R C^+ \otimes_R A$ but the differential is twisted by φ. Now let $\psi : C^+ \to C^+$ be another automorphism. Then we have a natural morphism of dg C^+-C^+-bicomodules

$$C^+_{\psi\varphi} \to C^+_\varphi \otimes_\tau A \otimes_\tau C^+_\psi \, , \; c \mapsto \sum c_{(1)} \otimes 1_A \otimes \varphi(c_{(2)}).$$

It induces a morphism of dg A-A-bimodules

$$X(\psi\varphi) \to X(\varphi) \otimes_A X(\psi). \tag{5.4}$$

Proposition. a) *As a left dg A-module, $X(\psi)$ is relatively cofibrant. In particular, we have a canonical isomorphism*

$$X(\varphi) \otimes_A^{\mathsf{L},rel} X(\psi) \xrightarrow{\sim} X(\varphi) \otimes_A X(\psi)$$

in $\mathcal{D}_R(A^{op} \otimes A)$.

b) *The morphism (5.4) is a homotopy equivalence of dg R-modules.*

In the next section, we will deduce this from results of [10]. Note that we obtain a morphism of groups from the group

$$\mathsf{Aut}_{R-\mathrm{coalg}}(C^+)^{op}$$

to the group of autoequivalences of the relative derived category $\mathcal{D}_R A$.

Now suppose that R is an augmented dg k-algebra and $R = k \oplus \mathfrak{m}$ the corresponding decomposition. Suppose that $\varphi \otimes_R k : C^+ \otimes_R k \to C^+ \otimes_R k$ is the identity. Then clearly $X(\varphi) \otimes_R k$ is isomorphic to $X(\mathbf{1}) \otimes_R k$ as a dg A-A-bimodule and we have a canonical isomorphism $X(\mathbf{1}) \otimes_R k \to A$ in $\mathcal{D}(A^{op} \otimes A)$. So we obtain a canonical isomorphism $u_\varphi : X(\varphi) \to A$ in $\mathcal{D}(A^{op} \otimes A)$ and an object $(X(\varphi), u_\varphi)$ of the fiber category $\mathcal{F}$ associated with the reduction functor $\mathcal{D}_R(R \otimes A^{op} \otimes A) \to \mathcal{D}(A^{op} \otimes A)$ (*cf.* section 5.2).

5.8. **Modules and comodules.** Let R be a commutative dg k-algebra and A a dg R-algebra of the form $A = A' \otimes_k R$ for some dg k-algebra A'. We define A^+ to be the augmented algebra $R \oplus A$ and C^+ to be the coaugmented coalgebra defined in the previous section. We still denote by $\tau : C^+ \to A^+$ the composition of the morphism τ of the previous section with the inclusion $A \to A^+$. Denote by $\mathsf{Comod}\, C^+$ the category of dg counital right C^+-comodules.

$$\begin{tikzcd} \mathsf{Mod}\, A^+ \arrow[d, shift left, "B"] \\ \mathsf{Comod}\, C^+ \arrow[u, shift left, "\Omega"] \end{tikzcd}$$

given by

$$BM = M \otimes_\tau C^+ \ , \ \Omega N = N \otimes_\tau A^+.$$

One can check [10] that they form an adjoint pair. Let $\mathcal{W}$ denote the class of morphisms s in $\mathsf{Mod}\, A^+$ whose restriction to $\mathsf{Mod}\, R$ is an homotopy equivalence and let $\mathcal{W}'$ be the class of morphisms s of $\mathsf{Comod}\, C^+$ such that Ωs belongs to $\mathcal{W}$.

Theorem. a) *The dg A-module ΩN is relatively cofibrant for each dg C^+-comodule N.*

b) *We have $B\mathcal{W} \subset \mathcal{W}'$ and $\Omega\mathcal{W}' \subset \mathcal{W}$ and the functors B and Ω induce quasi-inverse equivalences between the localized categories*

$$(\mathsf{Mod}\, A^+)[\mathcal{W}^{-1}] \xrightarrow{\sim} (\mathsf{Comod}\, C^+)[\mathcal{W}'^{-1}].$$

Theorem. *The restriction functor $\mathsf{Mod}\, A \to \mathsf{Mod}\, A^+$ induces an equivalence from $\mathcal{D}_R(A)$ onto the full subcategory of $(\mathsf{Mod}\, A^+)[\mathcal{W}^{-1}]$ whose objects are the dg modules M such that $M \otimes_\tau C^+$ is R-relatively acyclic (i.e. its underlying dg R-module is contractible).*

These theorems are proved in [10] in the case where $R = k$ (the first one corresponds to Theorem 2.2.2.2 and the second one to Proposition 4.1.2.10 in [10]). We omit the proof in the general case since it is similar. Note however the following: If M is a dg A-module, then according to the first theorem, we have a canonical R-relative quasi-isomorphism

$$M \otimes_\tau C^+ \otimes_\tau A^+ \to M.$$

The existence of such a quasi-isomorphism is not surprising. Indeed, the decomposition $A^+ = A \oplus R$ yields an R-split short exact sequence

$$0 \to M \otimes_\tau C^+ \otimes_\tau A \to M \otimes_\tau C^+ \otimes_\tau A^+ \to M \otimes_\tau C^+ \otimes R \to 0.$$

The last term identifies with the augmented bar resolution of M. It is therefore relatively R-acyclic. The first term identifies with the bar resolution of M. It is therefore relatively R-quasi-isomorphic to M.

Now suppose that $\varphi : C^+ \to C^+$ is an automorphism of dg R-coalgebras. Then it induces a selfequivalence F_φ of $\mathsf{Comod}\, C^+$ given by twisting by φ, i.e. if N is a dg C^+-comodule, then $F_\varphi N$ is the dg comodule with the same underlying graded module and the same differential but with the new comultiplication $\delta_\varphi = (\mathbf{1} \otimes \varphi) \circ \delta$. Clearly F_φ preserves the subcategory of R-relatively acyclic comodules. Thus the composition $\Omega F_\varphi B$ preserves the image of $\mathcal{D}_R(A)$ in $(\mathsf{Mod}\, A^+)[\mathcal{W}^{-1}]$. More precisely, if M is a dg A^+-module then

$$\Omega F_\varphi BM = M \otimes_\tau C^+_\varphi \otimes_\tau A^+$$

and if M comes from a (unital) dg A-module, then the last term is R-relatively quasi-isomorphic to its submodule $M \otimes_\tau C^+_\varphi \otimes_\tau A$ since

$$M \otimes_\tau C^+_\varphi \otimes_\tau R = M \otimes_\tau C^+$$

is R-relatively acyclic. So for each $M \in \mathsf{Mod}\, A$, we have a canonical R-relative quasi-isomorphism

$$M \otimes_\tau C^+_\varphi \otimes_\tau A \to \Omega F_\varphi BM.$$

If ψ is another automorphism, by composition, we obtain a canonical R-relative quasi-isomorphism

$$M \otimes_\tau C^+_\varphi \otimes_\tau A \otimes_\tau C^+_\psi \otimes_\tau A \to \Omega F_\psi B \Omega F_\varphi BM.$$

On the other hand, the adjunction morphism $\mathbf{1} \to B\Omega$ yields a morphism

$$\Omega F_\psi B \Omega F_\varphi BM \leftarrow \Omega F_\psi F_\varphi BM = \Omega F_{\psi\varphi} BM\ ,$$

which is also an R-relative quasi-isomorphism. To prove proposition 5.7, it remains to be checked that the morphism (5.4) constructed in (5.7) makes the following square commutative

$$\begin{tikzcd} A \otimes_\tau C^+_\varphi \otimes_\tau A \otimes_\tau C^+_\psi \otimes_\tau A \arrow[r] & \Omega F_\psi B \Omega F_\varphi BA \\ A \otimes_\tau C^+_{\psi\varphi} \otimes_\tau A \arrow[r] \arrow[u] & \Omega F_{\psi\varphi} BA \arrow[u] \end{tikzcd}$$

This is left to the reader.

5.9. **Proof of the main result.** Let A be a dg k-algebra and R a commutative augmented dg k-algebra. We write $R = k \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the kernel of the augmentation.

Let C^+ be the bar construction of A relative to k (*cf.* 5.7). Then the bar construction of $A \otimes R$ relative to R identifies with $C^+ \otimes R$. We put

$$\mathsf{LieAut}(C^+, R) = \mathsf{ker}(\mathsf{Aut}_R(C^+ \otimes R) \to \mathsf{Aut}_k(C^+))$$

where Aut_R denotes the group of automorphisms of dg counital R-coalgebras.

We define $\mathsf{LieDPic}(A, R)$ to be the group of isomorphism classes (*cf.* section 4) of invertible dg bimodules $X \in \mathcal{D}_R(A^{op} \otimes A \otimes R)$ free as graded R-modules such that $X \otimes_R k$ is isomorphic to A in $\mathcal{D}(A^{op} \otimes A)$.

By section 5.7, we obtain a morphism of groups

$$\Phi : \mathsf{LieAut}(C^+, R)^{op} \to \mathsf{LieDPic}(A, R)$$

which is clearly functorial in R. As in the case of algebraic groups, one obtains canonical Lie brackets on the restrictions of these functors to the subcategory of augmented dg k-algebras $R = k \oplus \mathfrak{m}$ with $\mathfrak{m}^2 = 0$ (*cf.* section 5.6) and Φ is compatible with the bracket.

Lemma. *If $R = k \oplus \mathfrak{m}$ with $\mathfrak{m}^2 = 0$, there is natural isomorphism*

$$\mathsf{Z}^0(\mathsf{Coder}_k(C^+, C^+) \otimes_k \mathfrak{m}) \to \mathsf{LieAut}(C^+, R).$$

This is a variant of a classical result on infinitesimal deformations. The easy proof is left to the reader. As we recalled from [15] in section 5.7, we have a natural isomorphism of dg Lie algebras

$$C^{\cdot+1}(A, A) \xrightarrow{\sim} \mathsf{Coder}_k(C^+, C^+).$$

So for $\mathfrak{m}^2 = 0$, we obtain morphisms

$$\mathsf{Z}^0(C^{\cdot+1}(A, A) \otimes \mathfrak{m}) \to \mathsf{LieAut}(C^+, R) \to \mathsf{LieDPic}(A, R)^{op}.$$

compatible with the bracket. Now by 5.2, we have an isomorphism

$$\Phi : \mathsf{LieDPic}(A, R) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}(A^{op}\otimes A)}(A, A \otimes \mathfrak{m}[1]).$$

It is easy to see that the composition

$$\mathsf{Z}^0(C^{\cdot+1}(A, A) \otimes \mathfrak{m}) \to \mathsf{LieDPic}(A, R) \xrightarrow{\sim} H^0(C^{\cdot+1}(A, A) \otimes \mathfrak{m}).$$

is the canonical surjection. So we have a commutative square

$$\begin{tikzcd} \mathsf{Z}^0(C^{\cdot+1}(A, A) \otimes \mathfrak{m}) \arrow[r] \arrow[d] & \mathsf{LieDPic}(A, R) \arrow[d, "\sim"] \\ H^0(C^{\cdot+1}(A, A) \otimes \mathfrak{m}) \arrow[r, "\sim"] & \mathsf{Hom}_{\mathcal{D}(A^{op}\otimes A)}(A, A \otimes \mathfrak{m}[1]) \end{tikzcd}$$

We see that if we transport the Gerstenhaber bracket to the lower right hand corner, then the map

$$\Phi : \mathsf{LieDPic}(A, R)^{op} \to \mathsf{Hom}_{\mathcal{D}(A^{op}\otimes A)}(A, A \otimes \mathfrak{m}[1])$$

becomes an isomorphism which respects the bracket and is functorial with respect to R and with respect to invertible bimodule complexes $X \in \mathcal{D}(A^{op} \otimes B)$.

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