On the construction of triangle equivalences

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Summary. We give a self-contained account of an alternative proof of J. Rickard's Morita-theorem for derived categories [32] and his theorem on the realization of derived equivalences as derived functors [33]. To this end, we first review the basic facts on unbounded derived categories (complexes unbounded to the right and to the left) and on derived functors between such categories (cf. [34], [2]). We then extend the formalism of derived categories to differential graded algebras (cf. [14]). This allows us to write down a formula for a bimodule complex given a tilting complex. We then deduce J. Rickard's results.

As a second application of the differential graded algebra techniques, we prove a structure theorem for stable categories admitting infinite sums and a small generator. This yields a natural construction of D. Happel's equivalence [11] between the derived category of a finite-dimensional algebra and the stable category of the associated repetitive algebra.

Finally, we use differential graded algebras to show that cyclic homology is preserved by derived equivalences (following [23]).

1. Unbounded derived categories

In the terminology we use, *unbounded* derived categories have as objects all complexes of modules without any restriction on the vanishing of components in large or small degrees. We consider unbounded derived categories because they are *easier* to handle than their bounded subcategories. For instance, we will see that *both* total derived functors

$$? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B \ , \ \mathbf{R}\mathcal{H}om_B\,(X, ?) : \mathcal{D}B \to \mathcal{D}A$$

are defined and form an adjoint pair *without any hypotheses* on the rings A, B, or the complex X of A-B-bimodules. These functors may or may not give rise to functors between the bounded categories, but in any case, the question about the existence of functors between the bounded categories becomes much simpler to study once the phenomena at the unbounded level are understood.

The situation is similar to that encountered in studying a variety over a field. It is natural to pass to the algebraic closure in a first step, and to study the problem of descent in a second step.

1.1 Unbounded resolutions. Let A be a ring (associative, with one) and denote by $\mathcal{H}A$ the homotopy category of (unbounded) complexes of (possibly infinitely generated, right) A-modules. We will define homotopically projective complexes, and analyze their structure and their relationship to arbitrary complexes. Homotopically projective complexes in the unbounded homotopy category will play the rôle of right bounded complexes with projective components in the homotopy category of right bounded complexes.

A complex K is *homotopically projective* [34] if we have

$$\mathrm{Hom}_{\mathcal{H}A}(K, N) = 0$$

for each acyclic complex N. Dually, it is *homotopically injective* if we have

$$\mathrm{Hom}_{\mathcal{H}A}(N, K) = 0$$

for each acyclic complex N. Denote by $\mathcal{H}_pA$ (resp. $\mathcal{H}_iA$) the full subcategories of homotopically projective (resp. homotopically injective) complexes of $\mathcal{H}A$. For example, if K is right bounded with projective components, it is homotopically projective. On the other hand, if K is any complex with projective components and *vanishing* differential, then it is homotopically projective. Indeed, it is then the countable sum of complexes having at most one non-vanishing component, which moreover is projective, and clearly (arbitrary) sums of homotopically projective complexes are homotopically projective. We get even more unbounded homotopically projective complexes by observing that homotopically projective complexes form a full triangulated subcategory of $\mathcal{H}A$. Now suppose that K is the direct limit in the category of complexes of a direct system

$$P_0 \xrightarrow{i_0} P_1 \xrightarrow{i_1} P_2 \to \ldots \to P_q \xrightarrow{i_q} P_{q+1} \to \ldots\,,\ q \in \mathbf{N}\,, \tag{1}$$

where each morphism i_q has split injective components and all the subquotients P_{q+1}/P_q have projective components and vanishing differential. By induction, we see that each P_q is homotopically projective. We claim that this continues to hold for $K = \varinjlim P_q$. To prove our claim, we invoke *Milnor's triangle* [28]. It is constructed as follows: Let

$$\Phi : \bigoplus_{p\in\mathrm{N}} P_p \to \bigoplus_{q\in\mathrm{N}} P_q$$

be the morphism whose components are the

$$P_p \xrightarrow{[1,-i_p]^t} P_p \oplus P_{p+1} \xrightarrow{\mathrm{can}} \bigoplus_{q\in\mathrm{N}} P_q.$$

Then we clearly have an exact sequence

$$0 \to \bigoplus_{p\in\mathrm{N}} P_p \xrightarrow{\Phi} \bigoplus_{q\in\mathrm{N}} P_q \to \varinjlim P_q \to 0$$

where the second morphism is given by the canonical morphisms $P_q \to \varinjlim P_q$. This sequence is in fact split exact in each component. So it gives rise to a triangle (where S=suspension functor=translation functor)

$$\bigoplus_{p\in\mathrm{N}} P_p \xrightarrow{\overline{\Phi}} \bigoplus_{q\in\mathrm{N}} P_q \to \varinjlim P_q \to S\bigoplus_{p\in\mathrm{N}} P_p \tag{2}$$

in the homotopy category. This is Milnor's triangle [28]. To see that $\varinjlim P$ is indeed homotopically projective, we apply the cohomological functor $\mathrm{Hom}_{\mathcal{H}A}(?, N)$ to Milnor's triangle and use the fact that

$$\mathrm{Hom}_{\mathcal{H}A}(\textstyle\bigoplus_{p\in\mathrm{N}} P_p, N) \xrightarrow{\sim} \displaystyle\prod_{p\in\mathrm{N}} \mathrm{Hom}_{\mathcal{H}A}(P_p, N) = 0.$$

Variants of the following theorem can be found in [34], [21], [2],[8, Ch. 12].

Theorem.

a) *A complex K is homotopically projective if and only if it isomorphic in $\mathcal{H}A$ to the direct limit in the category of complexes of a system (1).*

b) *For each complex K, there is a triangle*

$$\boldsymbol{p}K \to K \to \boldsymbol{a}K \to S\boldsymbol{p}K$$

where $\boldsymbol{p}K$ is homotopically projective and $\boldsymbol{a}K$ is acyclic. Any triangle (P, K, A) with homotopically projective P and acyclic A is isomorphic to $(\boldsymbol{p}K, K, \boldsymbol{a}K)$ and there is a unique such isomorphism extending the identity of K.

a') *A complex K is homotopically injective if and only it is isomorphic in $\mathcal{H}A$ to the inverse limit in the category of complexes of a system*

$$I_0 \xleftarrow{p_0} I_1 \leftarrow \ldots \leftarrow I_j \xleftarrow{p_j} I_{j+1} \leftarrow \ldots\,,\ j \in \mathbf{N}$$

where each p_j is split surjective in each component and $\mathrm{Ker}\, p_j$ has injective components and vanishing differential for each j.

b') For each complex K, there is a triangle

$$\boldsymbol{a}'K \to K \to \boldsymbol{i}K \to S\boldsymbol{a}'K\,,$$

where $\boldsymbol{a}'K$ is acyclic and $\boldsymbol{i}K$ is homotopically injective ...

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The theorem is proved in the appendix. The first part of the theorem shows that the above construction actually yields *all* homotopically projective complexes.

The second part of the theorem states in particular that each complex is quasi-isomorphic to a homotopically projective complex. We therefore call $\boldsymbol{p}K$ a *homotopically projective resolution* of the complex K. It is well-known that the projective resolution of an ordinary module is *functorial* in the module if we regard the resolution as a functor from modules to the *homotopy* category. Similarly, $\boldsymbol{p}K$ and $\boldsymbol{a}K$ underlie functors $\mathcal{H}A \to \mathcal{H}A$ and in fact *triangle functors* (by [18, 6.7]). More precisely, $\boldsymbol{p}$ yields a right adjoint to the inclusion of the full triangulated subcategory of homotopically projective complexes, and $\boldsymbol{a}$ yields a left adjoint of the inclusion of the full triangulated subcategory of acyclic complexes. This makes it clear that $\boldsymbol{p}$ vanishes on acyclic complexes and $\boldsymbol{a}$ on homotopically projective complexes.

It will be crucial for us that *the functors $\boldsymbol{p}$ and $\boldsymbol{a}$ commute with infinite direct sums.* This results from the unicity of the triangle and the fact that infinite direct sums preserve the classes of homotopically projective resp. of acyclic complexes.

1.2 Unbounded derived categories. Keep the hypotheses of (1.1). Denote by Σ the class of (homotopy classes of) quasi-isomorphisms. The *unbounded derived category* is by definition the localization [36],[9]

$$\mathcal{D}A = (\mathcal{H}A)[\Sigma^{-1}].$$

The category $\mathcal{D}A$ has infinite direct sums and these are given by direct sums of complexes (because infinite sums of quasi-isomorphisms are quasi-isomorphisms).

Variants of the following theorem can be found in [34], [21], [2].

Theorem. *The quotient functor $\mathcal{H}A \to \mathcal{D}A$ induces equivalences $\mathcal{H}_pA \xrightarrow{\sim} \mathcal{D}A$ and $\mathcal{H}_iA \xrightarrow{\sim} \mathcal{D}A$. The quasi-inverse functors are induced by $\boldsymbol{p} : \mathcal{D}A \xrightarrow{\sim} \mathcal{H}_pA$ and $\boldsymbol{i} : \mathcal{D}A \xrightarrow{\sim} \mathcal{H}_iA$. More precisely, $\boldsymbol{p}$ induces a fully faithful left adjoint to the quotient functor and $\boldsymbol{i}$ a fully faithful right adjoint.*

Note that $\boldsymbol{p}$ and $\boldsymbol{i}$ induce indeed well-defined functors $\mathcal{D}A \to \mathcal{H}A$, since they vanish on acyclic complexes. The theorem follows from (1.1) by the classical arguments. It implies that if L and M are complexes, we have the usual formulae

$$\mathrm{Hom}_{\mathcal{H}A}(L, \boldsymbol{i}M) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(\boldsymbol{p}L, M). \tag{3}$$

1.3 Infinite dévissage. The principle of (finite) dévissage states that by 'unscrewing' (dévissage), any finite complex may be reduced to complexes concentrated in one degree. This is the basis for inductive arguments as the one of Beilinson's lemma [1].

The following principle of 'infinite dévissage' states that by 'infinite unscrewing', an arbitrary complex may be reduced to complexes isomorphic to shifted copies of the free module of rank one. More precisely, under the hypotheses of (1.1), we have the following statement:

Proposition. *A full triangulated subcategory of $\mathcal{D}A$ equals $\mathcal{D}A$ if (and only if) it contains A_A and is closed under forming infinite direct sums.*

Proof. Since we have an equivalence $\mathcal{H}_pA \xrightarrow{\sim} \mathcal{D}A$, it is enough to prove the corresponding statement for $\mathcal{H}_pA$. We have to show that the smallest full triangulated subcategory $\mathcal{U}$ of $\mathcal{H}_pA$ which contains A_A and is closed under forming infinite direct sums equals $\mathcal{H}_pA$. Clearly, $\mathcal{U}$ contains each free module F and each finite complex of free modules. Let us prove that $\mathcal{U}$ contains all projective modules. So let e be an idempotent endomorphism of a free module F. We have the (split) resolution

$$\ldots \to F \xrightarrow{1-e} F \xrightarrow{e} F \xrightarrow{1-e} F \to e(F) \to 0\,,$$

So $e(F)$ is isomorphic in $\mathcal{H}A$ to the complex

$$\ldots F \overset{1-e}{\to} F \overset{e}{\to} F \overset{1-e}{\to} F \to 0.$$

This complex belongs to $\mathcal{U}$: just look at at the Milnor triangle associated with the sequence of (stupidly) truncated subcomplexes which have zero components to the far left. By looking at Milnor's triangle, we see that $\mathcal{H}_p A$ coincides with the smallest full triangulated subcategory of $\mathcal{H}A$ which contains all projective modules and is closed under forming infinite direct sums. Thus $\mathcal{U}$ equals $\mathcal{H}_p A$ by Milnor's triangle.

1.4 Derived equivalences. If $\mathcal{C}$ is a category and $F : \mathcal{H}A \to \mathcal{C}$ is a functor, one defines [5] the *total left derived functor* $\mathbf{L}F$ as the composition

$$F \circ \boldsymbol{p} : \mathcal{D}A \to \mathcal{C}\ ,\ K \mapsto F(\boldsymbol{p}K).$$

Similarly, the *total right derived functor* $\mathbf{R}F$ is the composition $F \circ \boldsymbol{i} : \mathcal{D}A \to \mathcal{C}$.

For example, let B be another ring and X a complex of B-A-bimodules. Recall the two functors of complexes associated with X : If L is a complex of A-modules, the complex of B-modules $L \otimes_A X$ is defined by

$$(L \otimes_A X)^n = \bigoplus_{p+q=n} L^p \otimes_A X^q\ ,\ d(l \otimes x) = (dl) \otimes x + (-1)^p l \otimes dx\ ,\ l \in L^p\ ,\ x \in X^q \quad .$$

If M is a complex of B-modules, the complex of A-modules $\mathcal{H}om_B\,(X, M)$ is defined by

$$\mathcal{H}om_B\,(X, M)^n = \prod_{-p+q=n} \mathrm{Hom}_B(X^p, M^q),\ (df)(x) = d(f(x)) - (-1)^n f(dx),\ f \in \mathcal{H}om_B\,(X, M)^n.$$

Note that $\mathrm{Hom}_{\mathcal{D}B}(X, M)$ is only an abelian group whereas $\mathcal{H}om_B\,(X, M)$ is a complex (of A-modules). The functors $F := ? \otimes_A X$ and $G := \mathcal{H}om_B\,(X, ?)$ induce functors between $\mathcal{H}A$ and $\mathcal{H}B$ which will be denoted by the same symbols. They form an adoint pair:

$$\begin{array}{ccc} & \mathcal{H}A & \\ F = ? \otimes_A X & \downarrow\uparrow & \mathcal{H}om_B\,(X, ?) = G \\ & \mathcal{H}B & \end{array}$$

So we have the total derived functors $\mathbf{L}F$ and $\mathbf{R}G$ between $\mathcal{D}A$ and $\mathcal{D}B$. These form again an adjoint pair. Indeed by the formulae for the morphisms in the derived category and the adjointness of F and G at the level of homotopy categories, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}B}(\mathbf{L}F\,L, M) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}B}(F\,(\boldsymbol{p}L), \boldsymbol{i}M) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(\boldsymbol{p}L, G\,(\boldsymbol{i}M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(L, \mathbf{R}G\,M) \end{aligned}$$

Now suppose that X viewed as a complex of B-modules is quasi-isomorphic to a *perfect complex*, i.e. a bounded complex of finitely generated projective B-modules. Denote by $\mathrm{per}\,A$ the full subcategory of $\mathcal{D}A$ formed by complexes quasi-isomorphic to perfect complexes. The following proposition is a special case of [21, 6.1].

Proposition. *The following are equivalent*

i) The functor $\mathbf{L}F : \mathcal{D}A \to \mathcal{D}B$ *is an equivalence.*

ii) The functor $\mathbf{L}F$ *induces an equivalence* $\mathrm{per}\,A \to \mathrm{per}\,B$.

iii) The object $T = \mathbf{L}F\,A$ *satisfies*

a) The map

$$A \to \mathrm{Hom}_{\mathcal{D}B}(T, T)$$

is bijective and $\mathrm{Hom}_{\mathcal{D}B}(T, T[n]) = 0$ *for* $n \neq 0$.

b) $T \in \mathrm{per}\, B$.

c) *The smallest full triangulated subcategory of* $\mathcal{D}B$ *containing* T *and closed under forming direct summands equals* $\mathrm{per}\, B$.

Remark. An equivalence $\mathcal{D}A \to \mathcal{D}B$ isomorphic to a functor $? \otimes_A^{\mathbf{L}} X$ is called a *standard derived equivalence* [33]. No example of a 'non-standard' derived equivalence is known.

Proof. i) implies ii) thanks to the following intrinsic characterization of $\mathrm{per}\, A$ due to J. Rickard

Lemma. [32, 6.3] *A complex* $K \in \mathcal{D}A$ *belongs to* $\mathrm{per}\, A$ *if and only if is is* compact, *i.e. the functor* $\mathrm{Hom}_{\mathcal{D}A}(K, ?)$ *commutes with infinite direct sums.*

Let us show that ii) implies iii). Indeed, condition a) follows from the full faithfulness of $\mathbf{L}F$. Condition b) holds since $\mathbf{L}F$ carries $\mathrm{per}\, A$ to $\mathrm{per}\, B$, and condition c) holds because the essential image of $\mathrm{per}\, A$ under $\mathbf{L}F$ equals $\mathrm{per}\, B$.

We show that iii) implies i). In a first step, we show that $\mathbf{L}F$ is fully faithful. Since $\mathbf{R}G$ is right adjoint to $\mathbf{L}F$, it is enough to show that the adjunction morphism

$$\varphi M : M \to \mathbf{R}G\, \mathbf{L}F\, M$$

is invertible for each $M \in \mathcal{D}A$. Let $\mathcal{U}$ be the full subcategory on the objects M of $\mathcal{D}A$ for which φM is invertible. We will use the principle of infinite dévissage to show that $\mathcal{U}$ equals $\mathcal{D}A$. Indeed, $\mathcal{U}$ contains A_A, for, if we take the n-th homology group of

$$\varphi A : A \to \mathbf{R}\mathcal{H}om_B\,(X, A \otimes_A^{\mathbf{L}} X) = \mathbf{R}\mathcal{H}om_B\,(T, T)$$

we find the isomorphism $A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(T, T)$ resp. $0 \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(T, T)$ of condition a). Moreover $\mathcal{U}$ is the subcategory of objects on which a morphism of triangle functors is invertible. Hence it is a triangulated subcategory. Finally, we have to show that $\mathcal{U}$ is closed under infinite direct sums. For this it is enough to show that $\mathbf{R}G\mathbf{L}F$ commutes with infinite direct sums. Since $\mathbf{L}F$ is a left adjoint, it commutes with infinite direct sums. To see that $\mathbf{R}G = \mathbf{R}\mathcal{H}om_B\,(T, ?)$ commutes with direct sums, it suffices to prove that $\mathrm{H}^n\mathbf{R}G = \mathrm{Hom}_{\mathcal{D}B}(T, ?[n])$ commutes with direct sums for each n. This follows from condition b) by the above lemma.

Finally, we note that $\mathbf{L}F$ is essentially surjective by infinite dévissage applied to the essential image of $\mathbf{L}F$.

Remark. a) The lemma also yields intrinsic characterizations of $\mathcal{D}^-\mathrm{Mod}\, A$, $\mathcal{D}^+\mathrm{Mod}\, A$ and $\mathcal{D}^b\mathrm{Mod}\, A$. For example, an object $K \in \mathcal{D}A$ belongs to $\mathcal{D}^-\mathrm{Mod}\, A$ iff for each $P \in \mathrm{per}\, A$, there is an $N \gg 0$ such that $\mathrm{Hom}_{\mathcal{D}A}(P, S^n K)$ vanishes for all $n > N$.

b) The lemma admits a surprising generalization (cf. 1.5) to the class of triangulated categories sharing the following two essential features with $\mathcal{D}A$:

1) The category $\mathcal{D}A$ admits infinite direct sums.
2) The category $\mathcal{D}A$ admits a set of compact generators.

Indeed, by infinite dévissage, the free A-module A_A generates $\mathcal{D}A$ in the sense that $\mathcal{D}A$ coincides with its smallest triangulated subcategory containing A_A and closed under forming infinite direct sums. Moreover, the generator A_A is compact. Indeed, we have

$$\mathrm{Hom}_{\mathcal{D}A}(A, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(\boldsymbol{p}A, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(A, M) \xrightarrow{\sim} \mathrm{H}^0 M$$

and the homology functor clearly commutes with direct sums.

1.5 Triangulated categories with infinite sums and a set of compact generators. Let $\mathcal{S}$ be a triangulated category which admits infinite sums and a set $\mathcal{X}$ of compact objects which *generate* $\mathcal{S}$, i.e. such that we have

Infinite dévissage. *A full triangulated subcategory of* $\mathcal{S}$ *equals* $\mathcal{S}$ *if it contains* $\mathcal{X}$ *and is closed under forming infinite direct sums.*

In such a category, the following remarkable theorems hold :

Brown's representability theorem [3] *A cohomological functor $F : \mathcal{S} \to (\mathcal{A}b)^{\mathrm{op}}$ is representable iff it commutes with infinite sums.*

Note that infinite sums in $\mathcal{A}b^{\mathrm{op}}$ correspond to infinite products in $\mathcal{A}b$. We refer to [21, 5.2] for the (easy) proof.

Characterization of compact objects (Ravenel–Neeman [30] [31]) *An object of $\mathcal{S}$ is compact iff it is contained in the smallest full triangulated subcategory of $\mathcal{S}$ containing $\mathcal{X}$ and closed under forming direct summands.*

The proof is tricky but elementary (cf. [21, 5.3]). In fact, the theorem is a special case of the following important result: Denote by $\mathcal{S}^c$ the full subcategory of compact objects of $\mathcal{S}$. According to the caracterization it is the smallest full triangulated subcategory of $\mathcal{S}$ containing $\mathcal{X}$ and closed under forming direct summands.

Localization theorem (Ravenel–Neeman [30] [31]) *Suppose that $\mathcal{R}$ is a thick subcategory of $\mathcal{S}$ generated by a set of compact objects of $\mathcal{S}$ and $\mathcal{T} = \mathcal{S}/\mathcal{R}$. Then the inclusion functor $\mathcal{R} \to \mathcal{S}$ maps $\mathcal{R}^b$ to $\mathcal{S}^b$, the quotient functor $\mathcal{S} \to \mathcal{T}$ maps $\mathcal{S}^b$ to $\mathcal{T}^b$, the induced functor $\mathcal{S}^b/\mathcal{R}^b \to \mathcal{T}^b$ is fully faithful and the closure of its image under forming direct summands equals $\mathcal{T}^b$.*

2. Differential graded algebras and their derived categories

In section 1 we assumed that a bimodule complex (for example a two-sided tilting complex) was given. We will show in section 3 how to construct such bimodule complexes from complexes of modules endowed with a second action which is well defined only 'up to homotopy'. Our technical tool is the generalization of the formalism of section 1 to the situation of differential graded algebras. The main reference for this section is [21]. Positively graded DG algebras, their modules and their derived categories were previously considered by Illusie [14]. Differential graded algebras (and co-algebras) and their (co-)modules have also been used by topologists for many years (cf. for example Moore's report [29]).

2.1 DG algebras. Let k be a commutative ring and A a *differential graded k-algebra* (=DG algebra), i.e. a $\mathbf{Z}$-graded associative k-algebra

$$A = \bigoplus_{p \in \mathbf{Z}} A^p$$

endowed with a k-linear differential $d : A \to A$ which is homogeneous of degree 1 (i.e. $dA^p \subset A^{p+1}$ for each p) and satisfies the graded Leibniz rule

$$d(ab) = (da)\, b + (-1)^p a\, db\ ,\ \forall\, a \in A^p\ ,\ \forall\, b \in A.$$

It turns out to be convenient *not* to impose any a priori finiteness conditions on A.

Examples. a) If B is an 'ordinary' k-algebra, it gives rise to a DG algebra A defined by

$$A^p = \begin{cases} B & p = 0 \\ 0 & p \neq 0. \end{cases}$$

Conversely, any DG algebra A which is *concentrated in degree* 0 (i.e. A^p=0 for all $p \neq 0$) is obtained in this way from an 'ordinary' algebra.

b) If B is a k-algebra and

$$M = (\ldots \to M^i \xrightarrow{d} M^{i+1} \to \ldots)\ ,\ i \in \mathbf{Z}\ ,\ dd = 0$$

a complex of right B-modules, the complex $A = \mathcal{H}om_B(M, M)$ (cf. 1.4) has a natural structure of DG algebra. Note that even if $M^i = 0$ for all $i \gg 0$, there will in general be non-vanishing components of A in arbitrarily small *and* arbitrarily large degrees.

2.2 DG modules. A *differential graded module over* A (=DG A-module) is a Z-graded right A-module

$$M = \bigoplus_{p \in \mathbf{Z}} M^p$$

endowed with a k-linear differential $d : M \to M$ which is homogeneous of degree 1 and satisfies the graded Leibniz rule

$$d(ma) = (dm)\, a + (-1)^p m\, da\ ,\ \forall\, m \in M^p\ ,\ \forall\, a \in A.$$

Differential graded left A-modules are defined similarly. A *morphism* of DG A-modules $f : M \to N$ is a morphism of the underlying graded A-modules which is homogeneous of degree 0 and commutes with the differential. For example, for each DG A-module M, we have a canonical isomorphism

$$\mathrm{Hom}\,(A_A, M) \xrightarrow{\sim} \mathrm{Z}^0\, M\ ,\ f \mapsto f(1)\ ,$$

where $\mathrm{Hom}\,(,)$ denotes morphisms of DG modules.

Examples. a) In the situation of example 2.1 a), the category of DG A-modules identifies with the category of differential complexes of right B-modules.

b) In the situation of example 2.1 b), each complex N of right B-modules gives rise to a DG A-module $\mathcal{H}om_B\,(M, N)$ endowed with the A-action $(g^j)(f^i) = (g^{i+p} \circ f^i)$, where $(g^j) \in \mathrm{Hom}_B(M, N)^q$ and $(f^i) \in A^p$. On the other hand, M becomes itself a DG left A-module for the action $(f^i)(m^j) = (f^i(m^i))$.

2.3 The homotopy category. Let $f : M \to N$ be a morphism of DG A-modules. We say that f *is null-homotopic* if we have $f = dr + rd$, where $r : M \to N$ is a morphism of the underlying graded A-modules which is homogeneous of degree -1. The *homotopy category* $\mathcal{H}A$ has the DG A-modules as *objects*. Its *morphisms* are classes $\overline{f}$ of morphisms f of DG A-modules modulo null-homotopic morphisms. For example, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, M) \xrightarrow{\sim} \mathrm{H}^n M \tag{4}$$

for each DG A-module M.

Define the *suspension functor* $S : \mathcal{H}A \to \mathcal{H}A$ by

$$(SM)^p = M^{p+1}\ ,\ d_{SM} = -d_M\ ,\ \mu_{SM}(m, a) = \mu_M(m, a)\ ,$$

for $m \in M$ and $a \in A$, where μ_M and μ_{SM} are the multiplication maps of the respective modules. Define a *standard triangle of* $\mathcal{H}A$ to be a sequence

$$L \xrightarrow{\overline{f}} M \xrightarrow{\overline{g}} Cf \xrightarrow{\overline{h}} SL\ ,$$

where $f : L \to M$ is a morphism of DG modules, $Cf = M \oplus SL$ as a graded k-module,

$$d_{Cf} = \begin{bmatrix} d_M & f \\ 0 & d_{SL} \end{bmatrix}\ ,\ \mu_{Cf}(\begin{bmatrix} m \\ l \end{bmatrix}, a) = \begin{bmatrix} ma \\ la \end{bmatrix}\ ,$$

for $m \in M\ ,\ l \in L^p$, the morphism g is the canonical injection $M \to Cf$, and $-h$ (note the sign) is the canonical projection. As usual, Cf is called the *mapping cone* over f.

Lemma. *Endowed with the suspension functor S and the triangles isomorphic to standard triangles, the category $\mathcal{H}A$ becomes a triangulated category in the sense of Verdier.*

To prove the lemma, we endow the category of DG A-modules with the exact structure given by the sequences which split as sequences of graded A-modules. One checks that one obtains a Frobenius category and that the associated stable category is $\mathcal{H}A$. This implies the claim by [11]. In the situation of example 2.1 a), the category $\mathcal{H}A$ identifies with the homotopy category of complexes of right B-modules.

2.4 Resolutions. In analogy with (1.1), we define a DG module K to be *homotopically projective* (resp. *injective*) if

$$\mathrm{Hom}_{\mathcal{H}A}(K,N)=0 \qquad (\text{resp. } \mathrm{Hom}_{\mathcal{H}A}(N,K)=0\ )$$

for all acyclic DG A-modules N. For example, A_A is homotopically projective (clear by formula (4)). The structure of a general homotopically projective DG module admits a description similar to that in the case of an ordinary algebra. Moreover, *statements b) and b') of theorem (1.1) carry over literally to the case of DG algebras.*

2.5 The derived category. A morphism of DG A-modules $s : M \to M'$ is a *quasi-isomorphism* if the induced morphism in homology $\mathrm{H}^*s : \mathrm{H}^*M \to \mathrm{H}^*M'$ is invertible. By definition, the *derived category of* A is the localization

$$\mathcal{D}A := (\mathcal{H}A)[\Sigma^{-1}]\,,$$

where Σ denotes the class of all (homotopy classes of) quasi-isomorphisms. In the situation of example 2.1 a), the category $\mathcal{D}A$ identifies with the unbounded derived category of the category of right B-modules. *Theorem (1.2), formula (3) for the morphisms in the derived category and the infinite dévissage principle all continue to hold for DG algebras.* Moreover, we have the formula

$$\mathrm{Hom}_{\mathcal{D}A}(A,M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(\boldsymbol{p}A,M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(A,M) \xrightarrow{\sim} \mathrm{H}^0M\,,$$

and hence $\mathcal{D}A$ is a triangulated category with infinite sums and a compact generator.

2.6 Derived equivalences. One defines total derived functors by the same formulas as for ordinary algebras (1.4). For example, let B be another DG algebra, and ${}_AX_B$ a DG A-B-*bimodule*, i.e.

$$X=\bigoplus_{p\in\mathbb{Z}} X^p$$

is simultaneously a graded left A-module and a graded right B-module, the two actions commute and coincide on k, and X is endowed with a homogeneous k-linear differential d of degree 1 satisfying

$$d(axb)=(da)\,xb+(-1)^p a\,(dx)\,b+(-1)^{p+q}ax\,(db)$$

for all $a\in A^p$, $x\in X^q$, $b\in B$. We define the DG *algebra* $A^{\mathrm{op}}\otimes B$ by

$$(A^{\mathrm{op}}\otimes B)^n=\bigoplus_{p+q=n} A^p\otimes B^q\,,\quad \begin{array}{l} d(a\otimes b)=(da)\otimes b+(-1)^p a\otimes(db)\,,\\ (a\otimes b)(a'\otimes b')=(-1)^{qp'}aa'\otimes bb' \end{array}$$

for all $a\in A^p$, $b\in B^q$, $a'\in A^{p'}$, $b'\in B$. We may then view X as a (right) DG $A^{\mathrm{op}}\otimes B$-module via

$$x.(a\otimes b)=(-1)^{rp}axb\,,\ \forall\, x\in X^r\,,\ \forall\, a\in A^p\,,\ \forall\, b\in B.$$

Let M be a DG A-module. We define $M\otimes_k X$ to be the DG B-module with the action of B on X and with the DG structure

$$(M\otimes_k X)^m=\bigoplus_{p+q=m} M^p\otimes_k X^q\,,\ d(m\otimes x)=(dm)\otimes x+(-1)^p m\otimes(dx)\,,$$

for all $m\in M^p$, $x\in X$. The k-submodule generated by all differences $ma\otimes x-m\otimes ax$ is stable under d and under multiplication by elements of B. So $M\otimes_A X$, the quotient modulo this submodule, is a well defined DG B-module, which is moreover functorial in M and X. Let N be a DG B-module. By definition, $\mathcal{H}om_B(X,N)^n$ is formed by the graded homogeneous B-module homomorphisms $f : X_B\to N$ of degree n. The differential of $\mathcal{H}om_B(X,N)$ is defined by $(df)(x)=d(f(x))-(-1)^n f(dx)$ for f of degree n and $x\in X$. We make $\mathcal{H}om_B(X,N)$ into a right DG A-module by $(fa)(x)=f(ax)$.

As in the case of ordinary algebras, the derived functors $? \otimes^{\mathbf{L}}_A X$ and $\mathbf{R}\mathrm{Hom}_B(X, ?)$ form a pair of adjoint functors between the derived categories $\mathcal{D}A$ and $\mathcal{D}B$.

We define the subcategory of perfect DG modules per A to be the smallest full triangulated subcategory of $\mathcal{H}A$ containing A_A and closed under forming direct summands. By Ravenel-Neeman's characterization of compact objects (1.5), an object of $\mathcal{D}A$ is compact iff it is quasi-isomorphic to a perfect DG module. Suppose that X_B is a quasi-isomorphic to a perfect DG B-module. *Theorem 1.4 and its proof carry over to DG algebras.*

3. Applications

We apply the techniques of the preceding section to the construction of bimodule complexes from tilting complexes. We deduce J. Rickard's Morita theorem for derived categories in a form where the equivalence appears as a derived functor.

We point out that the first construction of bimodule complexes is due to J. Rickard [33] in the case of a tilting complex. Whereas he uses the existence of a triangle equivalence to construct the bimodule complex, we give an a priori construction, which can then be used to define a standard derived equivalence.

As a second application, we prove a structure theorem for stable categories analogous to the characterization of module categories by Freyd [6] and Gabriel [7].

Finally, we show that cyclic homology is preserved by derived equivalences. The analogous statement for Hochschild homology is due to D. Happel [12] and J. Rickard [33]. The case of cyclic homology seemed harder to prove because cyclic homology does not admit an intrinsic homological interpretation analogous to that of Hochschild homology as Tor-groups of bimodules.

The references for this section are [19], [20], [19], [23], [22].

3.1 Construction of bimodule complexes. Let B be a flat k-algebra and T a homotopically projective complex over B (for example a right bounded complex with projective components). Suppose that B satisfies the *Toda condition* [35]

$$\mathrm{Hom}_{\mathcal{H}B}(T, T[n]) = 0$$

for all $n < 0$. Let

$$A := \mathrm{Hom}_{\mathcal{H}B}(T, T).$$

We would like to construct a complex of A-B-bimodules X such that the functor $? \otimes^{\mathbf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$ sends A_A to T. So X_B has to be quasi-isomorphic to T and the left 'action up to homotopy' of A on T should correspond to the (strict) left action of A on X.

More precisely, we will construct a complex of A-B-bimodules X and a quasi-isomorphism $\varphi : T \to X_B$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & X_B \\ a \downarrow & & \downarrow \lambda(a) \\ T & \xrightarrow{\varphi} & X_B \end{array}$$

commutes in $\mathcal{H}B$ for all $a \in A$, where $\lambda(a)$ is left multiplication by a.

We proceed as follows: Let $C = \mathcal{H}om_B(T, T)$. So C is a differential graded k-algebra in the sense of (2.1). Let C_- be the differential graded subalgebra (!) of C whose underlying complex is

$$\ldots \to C^{-2} \to C^{-1} \to \mathrm{Z}^0 C \to 0 \to 0 \to \ldots$$

Note that the 0-component of C_- is canonically isomorphic to the group of morphisms of complexes $T \to T$ so that A viewed as a DG algebra concentrated in degree 0 becomes a quotient of C. So we have morphisms of DG algebras

$$A \leftarrow C_- \subset C.$$

The complex T is canonically a DG C–B–bimodule in the sense of (2.6). By restriction, it becomes a DG C_-–B–bimodule. We put

$$X := A \otimes_{C_-} \boldsymbol{p}T$$

where the homotopically projective resolution $\boldsymbol{p}T$ is taken over $B \otimes C_-^{\mathrm{op}}$.

Note that we have used none of the hypotheses we made. However, we need them to construct the morphism $\varphi : T \to X$. Note first that the Toda condition means precisely that the morphism $C_- \to A$ is a quasi-isomorphism. We claim that the induced map

$$A \otimes_{C_-} \boldsymbol{p}T \leftarrow C_- \otimes_{C_-} \boldsymbol{p}T$$

is still a quasi-isomorphism, or equivalently, that its mapping cone is acyclic. Indeed, by infinite dévissage, it is enough to check that the functor $? \otimes_{C_-} (C_- \otimes_k B)$ preserves acyclicity; this results from the flatness assumption on B. Clearly we have an isomorphism

$$C_- \otimes_{C_-} \boldsymbol{p}T \xrightarrow{\sim} \boldsymbol{p}T.$$

Thus, at least in $\mathcal{D}B$, there is a unique morphism φ making the following diagram commutative

$$\begin{array}{ccc} A \otimes_{C_-} \boldsymbol{p}T & \xleftarrow{\varphi} & T \\ \uparrow & & \uparrow \\ C_- \otimes_{C_-} \boldsymbol{p}T & \xrightarrow{\sim} & \boldsymbol{p}T \end{array}$$

Since T is homotopically projective, φ is realized by a genuine morphism of complexes.

3.2 A Morita theorem. Let k be a commutative ring and A, B two k-algebras such that B is flat over k. The following theorem is due to J. Rickard [32], [33] (cf. also [17], [21]).

Theorem. *The following are equivalent*

i) There is a complex of A-B-bimodules such that the functor $? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence.

ii) There is a triangle equivalence $F : \mathcal{D}A \to \mathcal{D}B$

iii) There is a triangle equivalence $\mathrm{per}\, A \to \mathrm{per}\, B$.

iv) There is an object $T \in \mathcal{D}B$ such that

a) There is an isomorphism

$$A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(T, T)$$

and we have $\mathrm{Hom}_{\mathcal{D}B}(T, T[n]) = 0$ *for* $n \neq 0$.

b) $T \in \mathrm{per}\, B$.

c) The smallest full triangulated subcategory of $\mathcal{D}B$ containing T and closed under forming direct summands equals $\mathrm{per}\, B$.

Proof. Clearly, i) implies ii). The implications from ii) to iii) and from iii) to iv) are proved as in proposition (1.4). To prove that iv) implies i), we may assume that T is homotopically projective. We then apply construction (3.1) to T. The claim then follows from proposition (1.4).

3.3 Stable categories and DG algebras. Let $\mathcal{E}$ be a k-linear Frobenius category [11]. Suppose that $\mathcal{E}$ has infinite direct sums. Since $\mathcal{E}$ has enough injectives, sums of admissible short exact sequences are automatically admissible short exact. Moreover, infinite direct sums of projective-injectives are projective-injective. The stable category $\underline{\mathcal{E}}$ then becomes a triangulated category with infinite direct sums. Suppose that it admits a compact generator X.

Theorem.

a) There is a DG algebra A and a triangle equivalence $F : \mathcal{D}A \xrightarrow{\sim} \underline{\mathcal{E}}$ with $FA \xrightarrow{\sim} X$. In particular, we have

$$\mathrm{H}^n A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(A, S^n A) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathcal{E}}}(X, S^n X).$$

b) If we have $\mathrm{Hom}_{\underline{\mathcal{E}}}(X, S^n X) = 0$ *for all* $n \neq 0$*, then there is a* k*-algebra* A *and a triangle equivalence* $F : \mathcal{D}A \xrightarrow{\sim} \underline{\mathcal{E}}$ *with* $FA \xrightarrow{\sim} X$*.*

Remark. The theorem yields for example a construction of Happel's equivalence between the derived category of a finite-dimensional algebra of finite global dimension and the stable category of the associated repetitive algebra [11]. It also yields a construction of Beilinson's equivalence [1].

Proof. Let $\widetilde{\mathcal{E}}$ be the category of acyclic differential complexes

$$P = (\ldots \to P^n \xrightarrow{d} P^{n-1} \to \ldots)\ ,\ n \in \mathbf{Z}$$

with projective components $P^n \in \mathcal{E}$. Endow $\widetilde{\mathcal{E}}$ with the componentwise split short exact sequences. Then $\widetilde{\mathcal{E}}$ is a Frobenius category and it is easy to see that the functor $P \mapsto \mathrm{Z}^0 P$ induces an equivalence

$$G_1 : \underline{\widetilde{\mathcal{E}}} \to \underline{\mathcal{E}}.$$

Now choose $\widetilde{X} \in \widetilde{\mathcal{E}}$ with $\mathrm{Z}^0\,\widetilde{X} \xrightarrow{\sim} X$. Define the DG algebra A by $A = \mathcal{H}om_{\mathcal{E}}\,(\widetilde{X}, \widetilde{X})$. Note that

$$\mathrm{Hom}_{\underline{\widetilde{\mathcal{E}}}}(X, S^n X) \xrightarrow{\sim} \mathrm{H}^n \mathcal{H}om_{\mathcal{E}}\,(X, X) = \mathrm{H}^n A.$$

It is clear that the composition of the exact functor

$$\widetilde{\mathcal{E}} \to \mathcal{C}A\ ,\ P \mapsto \mathcal{H}om_{\mathcal{E}}\,(\widetilde{X}, P)$$

with the canonical projection $\mathcal{C}A \to \mathcal{D}A$ vanishes on projectives of $\widetilde{\mathcal{E}}$ (=null-homotopic complexes in $\widetilde{\mathcal{E}}$) and hence induces an triangle functor

$$G_2 : \underline{\widetilde{\mathcal{E}}} \to \mathcal{D}A.$$

The module $G_2\widetilde{X}$ is isomorphic to A, the free module. If P_i, $i \in I$, is a family in $\widetilde{\mathcal{E}}$, the nth homology of the morphism

$$\bigoplus \mathcal{H}om_{\mathcal{E}}\,(\widetilde{X}, P_i) \to \mathcal{H}om_{\mathcal{E}}\,(\widetilde{X}, \bigoplus P_i)$$

identifies with

$$\bigoplus \mathrm{Hom}_{\underline{\widetilde{\mathcal{E}}}}(\widetilde{X}, S^n P_i) \to \mathrm{Hom}_{\underline{\widetilde{\mathcal{E}}}}(\widetilde{X}, \bigoplus S^n P_i)\ ,$$

which is bijective since $\widetilde{X}$ is small in $\widetilde{\mathcal{E}}$. Hence G_2 commutes with direct sums. We have already seen that G_2 induces bijections

$$\mathrm{Hom}_{\underline{\widetilde{\mathcal{E}}}}(\widetilde{X}, S^n \widetilde{X}) \xrightarrow{\sim} \mathrm{H}^n \mathcal{H}om_{\mathcal{E}}\,(\widetilde{X}, \widetilde{X}) \xrightarrow{\sim} \mathrm{H}^n A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(G_2\widetilde{X}, S^n G_2 \widetilde{X})\ ,\ n \in \mathbf{Z}.$$

By infinite dévissage, we conclude that G_2 is fully faithful. The essential image of G_2 contains the generator A, of $\mathcal{D}A$. So G_2 is essentially surjective. We let F be the composition of G_2 with a quasi-inverse of G_1. Part b) follows from part a) by a truncating argument as in (3.1).

3.4 Invariance of cyclic homology under derived equivalence. We refer to [27], [25], [26] for the definition and the basic properties of cyclic homology.

Let k be a commutative ring and A and B two k-algebras which are projective as k-modules. Suppose that they are derived equivalent and that X is a complex of A-B-bimodules such that the functor $? \otimes^{\mathbf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence.

Theorem. *There is an isomorphism* $\mathrm{HC}_* X : \mathrm{HC}_* A \xrightarrow{\sim} \mathrm{HC}_* B$*.*

Remarks. The theorem also holds if we only suppose A and B to be flat over k, cf. [22].

The morphism $\mathrm{HC}_* X$ is functorial in the following sense: Consider the full subcategory rep (A, B) of the derived category of A-B-bimodules formed by the bimodule complexes X which when restricted to B become quasi-isomorphic to perfect complexes (compare with [15], [16]). One can show [22] that each such complex X gives rise to a morphism in cyclic homology

$$\mathrm{HC}_*\,(X) : \mathrm{HC}_*(A) \to \mathrm{HC}_*(B).$$

This morphism is functorial in the sense that if we view A as an A-A-bimodule complex, then $\mathrm{HC}_*(A) = \mathbf{1}$ and if $Y \in \mathrm{rep}\,(B, C)$ then $\mathrm{HC}_*(X \otimes_B^{\mathbf{L}} Y) = \mathrm{HC}_*(Y) \circ \mathrm{HC}_*(X)$.

Moreover, one can show [22] that $\mathrm{HC}_*(X)$ only depends on the class of X in the Grothendieck group of the triangulated category rep (A, B). These Grothendieck groups are naturally viewed as the morphism spaces of a category whose objects are all algebras. A *K-theoretic equivalence* is an isomorphism of this category. Thus, cyclic homology is invariant under K-theoretic equivalence. For example, a finite-dimensional algebra over an algebraically closed field is K-theoretically equivalent to its largest semi-simple quotient (cf. [22]). Thus, if k is an algebraically closed field, the cyclic homology of a finite-dimensional algebra A of finite global dimension only depends on the number of isomorphism classes of simple A-modules. This yields the 'no loops conjecture' in the algebraically closed case, which was first proved by H. Lenzing [24]. We refer to K. Igusa's article [13] for a proof under more general hypotheses.

Proof. Consider the object $T = A \otimes_A^{\mathbf{L}} X$ of $\mathcal{D}B$. By proposition 1.4, it is quasi-isomorphic to a perfect complex over B. In particular, X lies in $\mathcal{D}^-(A^{\mathrm{op}} \otimes B)$. So there is a right bounded complex of projective $A^{\mathrm{op}} \otimes B$-modules P and a quasi-isomorphism $P \to X$ over $A^{\mathrm{op}} \otimes B$. Since A is projective over k, the components of P are projective over B. Thus, for any $n \in \mathbf{Z}$, the canonical map

$$\mathrm{Hom}_{\mathcal{H}B}(P, P[n]) \to \mathrm{Hom}_{\mathcal{D}B}(P, P[n])$$

is bijective. By the faithfulness of $? \otimes_A^{\mathbf{L}} X$, we have isomorphisms

$$\begin{aligned} 0 &= \mathrm{Hom}_{\mathcal{H}B}(P, P[n]), \quad \text{for } n \neq 0 \\ A &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}B}(P, P). \end{aligned} \tag{5}$$

Now consider the endomorphism complex $C = \mathcal{H}om_B\,(P, P)$. Recall from section 1.4 that C is a complex of k-modules whose nth component is formed by homogeneous maps of degree n

$$f : \bigoplus_{p \in \mathbf{Z}} P^p \to \bigoplus_{q \in \mathbf{Z}} P^q$$

of $\mathbf{Z}$-graded B-modules. It is endowed with the differential defined by

$$df = d \circ f - (-1)^n f \circ d$$

where $f \in C^n$. The composition of graded maps makes C into a differential graded k-algebra (cf. section 2.1). We claim that C is flat as a DG k-module, i.e. that $C \otimes N$ is acyclic for each acyclic DG k-module N. Indeed, P is homotopy equivalent to a finite complex of finitely generated projective B-modules P'. Since $\mathcal{H}om_B\,(?, ?)$ induces a functor

$$(\mathcal{H}B)^{\mathrm{op}} \times (\mathcal{H}B) \to \mathcal{H}k,$$

this implies that C is homotopy equivalent to $\mathcal{H}om_B\,(P', P')$. In turn, $\mathcal{H}om_B\,(P', P')$ is obtained from $\mathcal{H}om_B\,(B, B) = B$ by forming shifts, mapping cones and direct summands. Since B is projective over k, we conclude that $B \otimes N$ and hence $C \otimes N$ are acyclic whenever N is acyclic.

Let us now consider the Hochschild complex $H(C)$ associated with C. It is the simple complex associated with the double complex whose n-th column is $C \otimes C^{\otimes n}$ (vanishing columns for $n < 0$) and whose horizontal differential is given by

$$\begin{aligned} d(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= \sum_{i=0}^{n-1} (-1)^i c_0 \otimes \ldots \otimes c_{i-1} \otimes c_i c_{i+1} \otimes c_{i+2} \otimes \ldots \otimes c_n \\ &\quad + (-1)^n c_n c_0 \otimes c_1 \otimes \ldots \otimes c_{n-1}. \end{aligned}$$

The column filtration of the double complex is complete and its subquotients are shifted copies of the complexes $C \otimes C^{\otimes n}$. Now consider the map

$$\lambda : A \to C = \mathcal{H}om_B\,(P, P)$$

given by the left action of A on the components of P. It is well known (and easy to check) that we have canonical isomorphisms

$$\mathcal{H}om_{\mathcal{H}B}\,(P, P[n]) \xrightarrow{\sim} \mathrm{H}^n C.$$

The formulae (5) therefore imply that λ is a quasi-isomorphism. Since $A\otimes ?$ and $C\otimes ?$ both preserve acyclicity, it follows that λ induces quasi-isomorphisms in all tensor powers $A\otimes A^{\otimes n} \to C\otimes C^{\otimes n}$ and hence a quasi-isomorphism between the Hochschild complexes associated with A and C. Now the cyclic complexes admit complete filtrations whose subquotients are homotopy equivalent to shifted copies of the Hochschild complexes. Thus λ also induces isomorphisms in the cyclic complexes.

We will now compare C to B. Consider the morphisms of DG algebras

$$\mathcal{H}om_B\,(P, P) \xrightarrow{\alpha} \mathcal{H}om_B\,(B \oplus P, B \oplus P) \xleftarrow{\beta} \mathcal{H}om_B\,(B, B) = B$$

given by

$$\alpha\,(f) = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}, \quad \beta\,(b) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}.$$

We claim that α and β induce isomorphisms in Hochschild homology (and hence in cyclic homology). Indeed, as a complex of B-modules, P is homotopy equivalent to a tilting complex. So B and P may be obtained from one another by forming shifts, mapping cones and direct summands. Our claim is therefore a consequence of the following

Lemma. *Let P and Q be complexes of B-modules which are homotopy equivalent to perfect complexes. Suppose that Q may be obtained from P by forming shifts, mapping cones and direct summands. Then the canonical map*

$$\mathcal{H}om_B\,(P, P) \to \mathcal{H}om_B\,(P \oplus Q, P \oplus Q)\,,\ f \mapsto \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$$

induces isomorphisms in Hochschild homology and cyclic homology.

Proof. Put $C = \mathcal{H}om_B\,(P, P)$ and $D = \mathcal{H}om_B\,(P \oplus Q, P \oplus Q)$ and let

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in D.$$

The Hochschild resolution M associated with D is the total complex associated with the double complex whose n-th column is $D \otimes D^{\otimes n} \otimes D$ (vanishing columns for $n < 0$) and whose horizontal differential is given by

$$d(x_0 \otimes x_1 \otimes \ldots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \ldots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \ldots \otimes x_{n+1}.$$

We view it as a differential graded D-D-bimodule, respectively as a DG module over $D^e = D\otimes D^{\mathrm{op}}$, in the sense of section 2.2. It is easy to check that $M \otimes_{D^e} D$ is isomorphic to the Hochschild complex associated with D. Consider the DG submodule $M' \subset M$ given by the total complex of the double complex whose n-th column is $De \otimes C^{\otimes n} \otimes eD$ (vanishing columns for $n < 0$). Clearly M' is a D^e-submodule of M. It is easy to check that $M' \otimes_{D^e} D$ is isomorphic to the Hochschild complex $H(C)$ and that the inclusion $M' \subset M$ induces the same morphism $H(C) \to H(D)$ as the map $C \to D$. To prove that $H(C) \to H(D)$ is an homotopy equivalence, we will prove that $M' \subset M$ is an homotopy equivalence of differential graded D^e-modules. Now both, M' and M are filtered by the column filtration with subquotients of the form $D \otimes L \otimes D$, where L is homotopy equivalent to a complex of projective k-modules. Moreover, the column filtrations split when considered as filtrations of graded D^e-modules. Thus M and M' are homotopically projective as D^e-modules in the sense of section 2.2. To prove that the inclusion $M' \subset M$ is an homotopy equivalence, it is therefore enough to prove that it is a quasi-isomorphism. Since the augmentation $\varepsilon : M \to D$ is a quasi-isomorphism (indeed, it is a homotopy equivalence of DG right D-modules), it is enough to show that the restriction of ε to M' induces a quasi-isomorphism $M' \to D$. For this we introduce the complex $R(X,Y)$, where X and Y are arbitrary complexes of B-modules. The complex $R(X,Y)$ is the total complex of the double complex whose n-th column is

$$\mathcal{H}om_B(P,Y) \otimes C^{\otimes n} \otimes \mathcal{H}om_B(X,P)$$

for $n \geq 0$ and whose column of index -1 is $\mathcal{H}om_B(X,Y)$. The other columns vanish. The differential is defined in analogy with that of the Hochschild resolution in degrees > 0 and via the augmentation in degree 0. Then $R(P \oplus Q, P \oplus Q)$ identifies with the mapping cone over $\varepsilon : M' \to D$ and we have to prove that $R(P \oplus Q, P \oplus Q)$ is acyclic. On the other hand, $R(P,P)$ is null-homotopic (it is the mapping cone over the augmentation of the Hochschild resolution for C). Now if we view R as a triangle functor

$$(\mathcal{H}B)^{\mathrm{op}} \times (\mathcal{H}B) \to \mathcal{H}k\,,$$

then this means that (P,P) is in the kernel of R. By the hypothesis on Q, we see that the objects $(P, P \oplus Q)$ and $(P \oplus Q, P \oplus Q)$ belong to the kernel as well. This means that $R(P \oplus Q, P \oplus Q)$ is null-homotopic (as a DG k-module) and hence acyclic.

4. Appendix: Proof of theorem 1.1

We first prove the existence of the triangle in b). Let

$$\underline{A} = (\ldots \to P_q \to P_{q-1} \to \ldots \to P_0 \to K \to 0) \tag{6}$$

be a projective resolution of the complex K in the sense of Cartan–Eilenberg [4, Ch. XVII]. This means that (6) is a sequence of complexes such that for each p the sequences of modules

$$\begin{array}{llll}
\ldots \to P_q^p \to P_{q-1}^p \to \ldots & \to P_0^p \to & K & \to 0 \\
\ldots \to \mathrm{Z}^p\, P_q \to \mathrm{Z}^p\, P_{q-1} \to \ldots & \to \mathrm{Z}^p\, P_0 \to & \mathrm{Z}^p\, K & \to 0 \\
\ldots \to \mathrm{B}^p\, P_q \to \mathrm{B}^p\, P_{q-1} \to \ldots & \to \mathrm{B}^p\, P_0 \to & \mathrm{B}^p\, K & \to 0 \\
\ldots \to \mathrm{H}^p P_q \to \mathrm{H}^p P_{q-1} \to \ldots & \to \mathrm{H}^p P_0 \to & \mathrm{H}^p K & \to 0
\end{array}$$

are projective resolutions. We identify $\underline{A}$ with a double complex (with commuting differentials) whose columns are K and the P_q. Now consider the sequence of double complexes

$$\begin{array}{llllllll}
\underline{P} & = (\ldots \to & P_2 & \to & P_1 & \to & P_0 & \to 0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\underline{K} & = (\ldots \to & 0 & \to & 0 & \to & K & \to 0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\underline{A} & = (\ldots \to & P_1 & \to & P_0 & \to & K & \to 0)
\end{array}$$

Let $P \to K \to A$ be the associated sequence of simple complexes. Clearly, A is isomorphic to the mapping cone over $P \to K$. Thus, we have a triangle

$$P \to K \to A \to P[1]$$

in $\mathcal{H}A$. Let us prove that P is homotopically projective. Indeed, let $n \in \mathbf{N}$ and consider the subcomplex

$$\underline{P}_{\leq n} = (\ldots 0 \to 0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to 0)$$

of $\underline{P}$. Denote by $P_{\leq n}$ the associated simple complex. Then P has the filtration

$$P_{\leq 0} \subset P_{\leq 1} \subset \ldots \subset P_{\leq q-1} \subset P_{\leq q} \subset \ldots \subset P\,,$$

where each inclusion $P_{\leq q-1} \subset P_q$ has split injective components. Using first induction and then Milnor's triangle (2), we see that it is enough to prove that each subquotient $P_{\leq q}/P_{\leq q-1}$ is homotopically projective. Now clearly $P_{\leq q}/P_{\leq q-1}$ is isomorphic to a shifted copy of P_q, the q-th component of the resolution we started with. Now by definition, P_q has projective homology modules and projective boundary modules. It is therefore isomorphic to the direct sum of a null-homotopic complex and a complex with vanishing differential, both having projective components. So P_q is clearly homotopically projective.

Let us prove that A is acyclic. For each complex L and each $n \in \mathbf{Z}$ let $\tau_{\leq n} L$ be the subcomplex

$$\ldots \to L^{n-2} \to L^{n-1} \to \mathrm{Z}^{n-1} L \to 0 \to 0 \to \ldots .$$

Let $n \in \mathbf{Z}$ and consider the double complex

$$\underline{F}_n = \ldots \to \tau_{\leq n} P_q \to \tau_{\leq n} P_{q-1} \to \ldots \to \tau_{\leq n} P_1 \to \tau_{\leq n} P_0 \to \tau_{\leq n} K \to 0$$

and let F_n be the associated simple complex. Clearly, we have $A = \varinjlim F_n$. Since homology commutes with filtered direct limits, it is enough to prove that F_n is acyclic. Since F_n is the total complex of a third quadrant double complex, it is enough to check that the rows of F are acyclic. But the only (possibly) non-vanishing rows of F_n are of the types

$$\begin{array}{llll} \ldots \to \mathrm{Z}^n P_q \to \mathrm{Z}^n P_{q-1} \to \ldots & \to \mathrm{Z}^n P_0 \to & \mathrm{Z}^n K & \to 0 \\ \ldots \to P_q^p \to P_{q-1}^p \to \ldots & \to P_0^p \to & K & \to 0 \end{array}$$

and these are acyclic by construction.

This construction yields a homotopically projective complex P which is almost of the type required in a). Namely, P is the direct limit of the $P_{\leq q}$ but the subquotients $P_{\leq q}/P_{\leq q-1}$ are only *homotopy equivalent* to complexes with projective components and vanishing differential. We will now correct this. We will inductively construct a new sequence

$$P'_{\leq 0} \to P'_{\leq 1} \to \ldots \to P'_{\leq q-1} \to P'_{\leq q} \to \ldots$$

and a commutative diagram (in the category of complexes)

$$\begin{array}{ccccccc} P'_{\leq 0} & \to & P'_{\leq 1} & \to \ldots \to & P'_{\leq q-1} & \to & P'_{\leq q} \to \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_{\leq 0} & \to & P_{\leq 1} & \to \ldots \to & P_{\leq q-1} & \to & P_{\leq q} \to \ldots \end{array}$$

where the morphisms $P'_{\leq q} \to P_{\leq q}$ are homotopy equivalences and the subquotients $P'_{\leq q}/P'_{\leq q-1}$ have projective components and vanishing differentials. If we put $P' = \varinjlim P'_{\leq q}$, it will follow from Milnor's triangle that the induced morphism $P' \to P$ is a homotopy equivalence. Let $P'_{\leq 0}$ be a complex with projective components and vanishing differential such that there is a homotopy equivalence $P'_{\leq 0} \to P_{\leq 0}$. Suppose that we have already constructed $P'_{\leq q-1}$ and the homotopy equivalence $P'_{\leq q-1} \to P_{\leq q-1}$. Choose a homotopy equivalence $P'_q \to P_{\leq q}/P_{\leq q-1}$, where P'_q has projective components and vanishing differential. We define $P'_{\leq q}$ by the pullback diagram

$$\begin{array}{ccc} P'_{\leq q} & \to & P'_q \\ \downarrow & & \downarrow \\ P_{\leq q} & \to & P_{\leq q}/P_{\leq q-1} \end{array}$$

Then it is easy to check that we have a commutative diagram

$$\begin{array}{ccccc} 0 \to P'_{\leq q-1} & \to & P'_{\leq q} & \to & P'_q \to 0 \\ \| & & \downarrow & & \downarrow \\ 0 \to P'_{\leq q-1} & \to & P_{\leq q} & \to & P_{\leq q}/P_{\leq q-1} \to 0 \end{array}$$

whose rows are exact with split exact components and whose vertical morphisms are homotopy equivalences.

We have thus proved that there is a triangle

$$\boldsymbol{p}K \to K \to \boldsymbol{a}K \to S\boldsymbol{p}K ,$$

where $\boldsymbol{p}K$ is isomorphic to the direct limit in the category of complexes of a direct system (1). Suppose that $P \to K \to A \to SP$ is another such triangle. Consider the sequence

$$(P, \boldsymbol{a}K[-1]) \to (P, \boldsymbol{p}K) \to (P, K) \to (P, \boldsymbol{a}K)\,,$$

where $(,)$ stands for $\mathrm{Hom}_{\mathcal{H}A}(,)$. Its outer terms vanish since P is homotopically projective and $\boldsymbol{a}K$ is acyclic. So we have an isomorphism

$$\mathrm{Hom}_{\mathcal{H}A}(P, \boldsymbol{p}K) \to \mathrm{Hom}_{\mathcal{H}A}(P, K) \tag{7}$$

and there is a unique morphism $P \to \boldsymbol{p}K$ of $\mathcal{H}A$ such that the diagram

$$\begin{array}{ccc} P & \to & K \\ \downarrow & & \| \\ \boldsymbol{p}K & \to & K \end{array}$$

commutes. Since $\mathcal{H}A$ is triangulated, this diagram embeds into a morphism of triangles

$$\begin{array}{ccccccc} P & \to & K & \to & A & \to & SP \\ \downarrow & & \| & & \downarrow & & \downarrow \\ \boldsymbol{p}K & \to & K & \to & \boldsymbol{a}K & \to & S\boldsymbol{p}K. \end{array}$$

The morphism $A \to \boldsymbol{a}K$ is unique by the sequence

$$(SP, \boldsymbol{a}k) \to (A, \boldsymbol{a}K) \to (K, \boldsymbol{a}K) \to (P, \boldsymbol{a}K)$$

whose outer terms vanish.

Let us prove a). For any complex K, the morphism $\boldsymbol{p}K \to K$ is universal among the morphisms $P \to K$ of $\mathcal{H}A$ with homotopically projective P. This follows from the isomorphism (7). If K is itself homotopically projective, the identity $K \to K$ is universal as well. Thus the canonical morphism $\boldsymbol{p}K \to K$ is a homotopy equivalence. But we have seen above that $\boldsymbol{p}K$ may be chosen as a direct limit in the category of complexes of a system (1).

This ends the proof of a) and b). The proof of a') and b') is analogous with a slight extra difficulty due to the fact that homology does not commute with (filtered) inverse limits in general. We therefore sketch the proof of the existence of the triangle of b') :

Let

$$\underline{A} = (0 \to K \to I^0 \to I^1 \to \ldots \to I^q \to I^{q+1} \to \ldots)$$

be an injective resolution of the complex K in the sense of Cartan–Eilenberg [loc. cit.]. Consider the sequence of double complexes

$$\begin{array}{llllllll} \underline{A} & = (0 \to & K & \to & I^0 & \to & I^1 & \to \ldots) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \underline{K} & = (0 \to & K & \to & 0 & \to & 0 & \to \ldots) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \underline{I} & = (0 \to & I^0 & \to & I^1 & \to & I^2 & \to \ldots) \end{array}$$

Denote by $A \to K \to I$ the associated sequence of *completed* simple complexes. Recall that the completed simple complex L associated with a double complex $\underline{L}$ has the components

$$L^n = \prod_{p+q=n} \underline{L}^{p,q}.$$

Clearly, I is homotopy equivalent to the mapping cone over $A \to K$. Thus we have a triangle

$$A \to K \to I \to SA$$

in $\mathcal{H}A$. The complex I is homotopically injective by an argument dual to the one we used in the proof of b). Let us prove that A is acyclic. For each complex L and each $m \geq 0$, we denote by $\sigma_m L$ the quotient complex

$$\sigma_m L = (\ldots \to 0 \to 0 \to \mathrm{B}^{-m} L \to L^{-m+1} \to \ldots \to L^{m-1} \to L^m \to 0 \to 0 \to \ldots).$$

Let

$$\underline{F}_m = (0 \to \sigma_m K \to \sigma_m I^0 \to \sigma_m I_1 \to \dots)$$

and let F_m be the completed simple complex associated with $\underline{F}_m$. Clearly we have $A = \varprojlim F_m$. Moreover, each F_m is acyclic. Indeed, it is the total complex of a double complex with finitely many non-vanishing acyclic rows. The canonical morphism

$$F_{m+1} \to F_m$$

is surjective in each component. Hence it also induces surjections in the boundary modules and hence in the cycle modules (since the F_m are acyclic). By the Mittag-Leffler criterion [10, 0_{III}, 13.1], we can conclude that $A = \varprojlim F_m$ is acyclic.

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