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A REMARK ON THE GENERALIZED SMASHING CONJECTURE

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Using one of Wodzicki's examples of H-unital algebras [14] we exhibit a ring whose derived category contains a smashing subcategory which is not generated by small objects. This disproves the generalization to arbitrary triangulated categories of a conjecture due to Ravenel [8, 1.33] and, originally, Bousfield [2, 3.4].

1. Statement of the conjecture

We refer to [7] for a nicely written analysis of the following setup: Let $\mathcal{S}$ be a triangulated category [13] admitting arbitrary (set-indexed) coproducts. An object $X \in \mathcal{S}$ is *small* if the functor $\operatorname{Hom}(X, ?)$ commutes with arbitrary coproducts. We denote the full subcategory on the small objects of $\mathcal{S}$ by $\mathcal{S}^b$. We suppose that $\mathcal{S}^b$ is equivalent to a small category. A full subcategory of $\mathcal{S}$ is *localizing* if it is a triangulated subcategory in the sense of Verdier which is closed under forming coproducts with respect to $\mathcal{S}$. We suppose that $\mathcal{S}$ is *generated* by $\mathcal{S}^b$, i.e. coincides with its smallest localizing subcategory containing $\mathcal{S}^b$.

A localizing subcategory $\mathcal{R} \subset \mathcal{S}$ is *smashing* if the inclusion $\mathcal{R} \to \mathcal{S}$ admits a right adjoint commuting with arbitrary coproducts. Suppose that $\mathcal{R}$ is generated by small objects. Since $\mathcal{S}^b$ is equivalent to a small category, the small generators of $\mathcal{R}$ may be assumed to form a set. Hence $\mathcal{R}$ is smashing by Brown's representability theorem [3]. The "generalized smashing conjecture" states the converse (which is disproved below):

> Every smashing subcategory is generated by small objects.

Remarks. a) I thank D. Ravenel for pointing out the following facts: The "generalized smashing conjecture" is not the generalization of Ravenel's Smashing Conjecture [8, 10.6], but rather of his conjecture [8, 1.33] due originally to Bousfield [2, 3.4]. This latter conjecture is now known to be false due to the failure of the telescope conjecture [8, 10.5]. The proof of this involves very hard homotopy theory (cf. [10] for an outline of the argument). More information on the conjectures of [8] is to be found in [9].

b) The quotient functor $j^* : \mathcal{S} \to \mathcal{S}/\mathcal{R}$ admits a right adjoint j_* iff the inclusion functor $i_* : \mathcal{R} \to \mathcal{S}$ admits a right adjoint $i^!$, cf. [13]. One easily checks that in this case, the functor j_* commutes with arbitrary coproducts iff the functor $i^!$ does. This leads to an equivalent formulation of the smashing conjecture where the inclusion functor is replaced by the quotient functor.

2. An example

Let A be a ring with unit and $\mathcal{D}A$ the (unbounded) derived category [13] of the category of (right, unitary) A-modules. We identify A-modules with complexes concentrated in degree 0. The unbounded derived category was studied in [12],[1],[5]. It has arbitrary coproducts. An object of $\mathcal{D}A$ is small iff it is isomorphic to a perfect complex (=finite complex of finitely generated projective modules) [11]. Moreover, $\mathcal{D}A$ is generated by the right A-module A. Hence $\mathcal{S} = \mathcal{D}A$ satisfies the above assumptions.

Let I be a two-sided ideal of A and $\mathcal{R} \subset \mathcal{D}A$ the localizing subcategory generated by the right A-module I. Suppose that

- we have $\mathrm{Tor}_i\,(A/I, A/I) = 0$ for all $i > 0$ and
- the ideal I is contained in the Jacobson radical of A.

Proposition. *The subcategory $\mathcal{R} \to \mathcal{D}A$ is smashing but $\mathcal{R}$ contains no non-zero small object of $\mathcal{D}A$.*

Note that if I satisfies both conditions and is moreover finitely generated, then we have $I = 0$, by Nakayama's lemma. In particular, no noetherian ring contains a non-trivial ideal satisfying both conditions. This is not surprising since at least for a *commutative* noetherian ring R the "generalized smashing conjecture" is true, as follows from the algebraic counterpart [6] of Hopkins–Smith's theorem on the classification of thick subcategories [4] (cf. [9] for a comprehensive account).

Now let k be a field and l an integer ≥ 2. Consider the (non-noetherian) algebra

$$B = k[t, t^{l^{-1}}, t^{l^{-2}}, \dots\,] = \bigcup_{n=0}^{\infty} k[t^{l^{-n}}]$$

and its augmentation ideal $J \subset B$, which is generated by $t, t^{l^{-1}}, t^{l^{-2}}, \dots$. This algebra is Wozicki's example 3 of [14, 4.7]. He proved in [loc. cit.] that J is H-unital. Since B is the augmented algebra obtained from J by adjoining a unit, this means that $\mathrm{Tor}_i^B\,(k, k) = 0$ for all $i > 0$ (cf. section 3 of [loc. cit.]). Now let A be the localization of B at J and let I be the maximal ideal of A. Localization preserves the vanishing of the Tor and I equals the Jacobson radical of A. Thus I satisfies both conditions.

3. Proof of the proposition

We keep the assumptions preceding the proposition. We refer to [12], [1], [5] for the definition and the basic properties of the unbounded left derived functor $\otimes_A^{\mathbf{L}}$ of the tensor product over A. In particular, this functor commutes with arbitrary coproducts. The proposition is immediate from the two following lemmas.

Lemma 1. *The functor $X \mapsto X \otimes_A^{\mathbf{L}} I$ is right adjoint to the inclusion $\mathcal{R} \to \mathcal{S} = \mathcal{D}A$.*

Proof. Let X be an object of $\mathcal{D}A$. Consider the triangle

$$X \otimes_A^{\mathbf{L}} I \to X \to X \otimes_A^{\mathbf{L}} (A/I) \to \Sigma(X \otimes_A^{\mathbf{L}} I).$$

We will show that the object $X \otimes_A^{\mathbf{L}} I$ belongs to $\mathcal{R}$ and that the object $X \otimes_A^{\mathbf{L}} (A/I)$ is $\mathcal{R}$-local, i.e. for each object $R \in \mathcal{R}$ we have $\mathrm{Hom}\,(R, X \otimes_A^{\mathbf{L}} A/I) = 0$. The assertion of the lemma is immediate from the Hom-sequence associated with the triangle.

For the generator $X = A$ of $\mathcal{D}A$, the object $A \otimes_A^{\mathbf{L}} I = I$ clearly belongs to $\mathcal{R}$. Since $? \otimes_A^{\mathbf{L}} I$ commutes with arbitrary coproducts, the object $X \otimes_A^{\mathbf{L}} I$ belongs to $\mathcal{R}$ for arbitrary $X \in \mathcal{D}A$. Now we claim that the morphism $R \otimes_A^{\mathbf{L}} I \to R$ is invertible for $R \in \mathcal{R}$. Indeed, since $? \otimes_A^{\mathbf{L}} I$ commutes with arbitrary coproducts, it is enough to check this for $X = I$. By the above triangle, we only have to show that $I \otimes_A^{\mathbf{L}} A/I = 0$. This is clear from the triangle

$$I \otimes_A^{\mathbf{L}} (A/I) \to A/I \to (A/I) \otimes_A^{\mathbf{L}} (A/I) \to \Sigma(I \otimes_A^{\mathbf{L}} (A/I))$$

since the morphism $A/I \to (A/I) \otimes_A^{\mathbf{L}} (A/I)$ is invertible by the assumption. To prove that $X \otimes_A^{\mathbf{L}} (A/I)$ is $\mathcal{R}$-local, let $R \in \mathcal{R}$ and $Y \in \mathcal{D}A$. We have $A/I \xrightarrow{\sim} (A/I) \otimes_A^{\mathbf{L}} (A/I)$ and thus the morphism $Y \otimes_A^{\mathbf{L}} (A/I) \to (Y \otimes_A^{\mathbf{L}} (A/I)) \otimes_A^{\mathbf{L}} (A/I)$ is invertible as well. Now if $f : R \to Y \otimes_A^{\mathbf{L}} (A/I)$ is a morphism, then by the diagram

$$\begin{array}{ccc} Y \otimes_A^{\mathbf{L}} (A/I) & \xrightarrow{\sim} & (Y \otimes_A^{\mathbf{L}} (A/I)) \otimes_A^{\mathbf{L}} (A/I) \\ f \uparrow & & \uparrow f \otimes_A^{\mathbf{L}} (A/I) \\ R & \to & R \otimes_A^{\mathbf{L}} (A/I) \end{array}$$

we have $f = 0$ since $R \otimes_A^{\mathbf{L}} (A/I) = 0$ by the invertibility of $R \otimes_A^{\mathbf{L}} I \to R$.

Lemma 2. *If $R \in \mathcal{D}A$ is small and belongs to $\mathcal{R}$, then $R = 0$.*

Proof. We may assume that R is a perfect complex. Since R belongs to $\mathcal{R}$, the morphism $R \otimes_A^{\mathbf{L}} I \to R$ is invertible (see the proof of lemma 1). So $R \otimes_A^{\mathbf{L}} (A/I) \xrightarrow{\sim} R \otimes_A (A/I)$ is acyclic. On the other hand, $R \otimes_A (A/I)$ is a right bounded complex of projective A/I-modules. Hence it is null-homotopic. We will deduce that R is null-homotopic. We proceed by induction on the length of R. If $R = R^0$ is concentrated in degree 0, then R^0 is a finitely generated projective A-module with $R^0 \otimes (A/I) = 0$. Hence $R^0 = 0$ by Nakayama's lemma. For general R we may assume that $R = 0$ for $i > 0$. Then $d^{-1} : R^{-1} \to R^0$ induces a split surjection $R^{-1} \otimes (A/I) \to R^0 \otimes (A/I)$. Since R^{-1} and R^0 are finitely generated projective, Nakayama's lemma implies that d^{-1} is itself a split surjection. Therefore R is homotopy equivalent to the truncated complex

$$R' = (\dots R \to R^{i+1} \to \dots \to R^{-2} \to \operatorname{Ker} d^{-1} \to 0 \to \dots).$$

By the induction hypothesis, R' is null-homotopic.

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