# Lecture I

1. The theories of model categories and of derivators are part of the homotopical or homological algebra, homological algebra being concerned with the additive or commutative part of homotopical algebra. Typically, homotopical algebra studies categories like the homotopy category of CW-complexes, or the derived categories of categories of modules or more generally of abelian or exact categories. One main feature of these categories is that, except in some very trivial cases, they do not admit limits or colimits other than products and coproducts. What replace these limits and colimits are homotopy limits and homotopy colimits. But homotopy limits and homotopy colimits can not be defined in terms of these categories. More structure is needed: a model or more canonically a derivator.

Terminological remark

| Notation | Topologists | Category theorists | Bourbaki |
| --- | --- | --- | --- |
| $\varinjlim$ = colim | colimit | direct limit | inductive limit |
| $\varprojlim$ = lim | limit | inverse ~~people~~ limit | projective limit |

In this course (and hopefully in the others) we will use the notation $\varinjlim$ and $\varprojlim$ and the terminology of topologists. For the homotopy case, the notation will be $\underrightarrow{\mathrm{holim}}$ and $\underleftarrow{\mathrm{holim}}$ and the terminology homotopy colimit and homotopy limit.

We will now remind the precise definition of the homotopy category of spaces and the derived categories.

The objects of the homotopy category of CW-complexes are the CW-complexes and the morphisms are the homotopy classes of continuous maps. It is known that this category is equivalent to the category Hot obtained from the category Top of <u>all</u> topological spaces and continuous maps by formally inverting weak equivalences by a universal construction that we will soon recall. A continuous map of topological spaces $f : X \to Y$ is a weak equivalence if

0) $f$ induces a bijection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ of the sets of arcwise connected components;

1) for every $i \geq 1$ and every point $x$ of $X$, $f$ induces an isomorphism $\pi_i(f;x) : \pi_i(X;x) \to \pi_i(Y; f(x))$ of the homotopy groups.

The definition of derived categories is somehow similar. Let $\mathcal{A}$ be an abelian category and $C(\mathcal{A})$ the category of cochain complexes

$$X = \cdots \to X^{n-1} \xrightarrow{d} X^n \xrightarrow{d} X^{n+1} \longrightarrow \cdots , \quad d \circ d = 0$$

The derived category $\mathrm{Der}(\mathcal{A})$ of $\mathcal{A}$ is the category obtained from $C(\mathcal{A})$ by formally inverting quasi-isomorphisms

A map $f : X \to Y$ of complexes is a quasi-isomorphism if for every $i \in \mathbb{Z}$, $f$ induces an isomorphism $H^i(f) : H^i(X) \to H^i(Y)$ of the cohomology groups.

2. So we observe that all these categories are obtained from a category $C$ by formally inverting the arrows of a class of arrows in $C$. Such a datum is called a localizer. A <u>localizer</u> is just a pair $(C, W)$ where $C$ is a category and $W$ is a class of arrows of $C$ (and we give this name to such a pair if we have in mind to study the category obtained by formally inverting the arrows in $W$). Usually the arrows in $W$ are called <u>weak equivalences</u>. Given an arbitrary localizer $(C, W)$, if we neglect the set theoretic questions of size, there exists always a category $W^{-1}C$ and a functor $P : C \to W^{-1}C$ that takes arrows in $W$ to isomorphisms of $W^{-1}C$ satisfying the following universal property. For every functor $F : C \to \mathcal{D}$ that takes arrows in $W$ to isomorphisms of $\mathcal{D}$, there exists a unique functor $\overline{F} : W^{-1}C \to \mathcal{D}$ such that $F = \overline{F} \circ P$.

$$\begin{tikzcd} C \arrow[d, "P"'] \arrow[dr, "F"] & \\ W^{-1}C \arrow[r, "\overline{F}"'] & \mathcal{D} \end{tikzcd}$$

A quick construction of the category $W^{-1}C$ and the functor $P$ is obtained as a pushout. Let

$$I = \{0 \xrightarrow{s} 1\}, \quad J = \{0 \underset{s}{\overset{s^{-1}}{\rightleftarrows}} 1\}$$

be the categories with two objects $0$ and $1$ and respectively with exactly one non identical arrow $s: 0 \to 1$ and exactly two non identical arrows $s: 0 \to 1$, $s^{-1}: 1 \to 0$ mutually inverses. Consider the inclusion $I \hookrightarrow J$. The functor $C \to W^{-1}C$ can be defined as the pushout of the inclusion $\coprod_W I \to \coprod_W J$ by the obvious functor $\coprod_W I \to C$. In other words the square

$$\begin{tikzcd}
\coprod_W I \arrow[r] \arrow[d, hook] & C \arrow[d, "P"] \\
\coprod_W J \arrow[r] & W^{-1}C
\end{tikzcd}$$

is cocartesian. (This does <u>not</u> imply that $P$ is faithfull, which is the case only in very particular situations).

The category $W^{-1}C$ is called the <u>localization</u> of $C$ by $W$ and the functor $P$ the <u>localization functor</u> or the <u>canonical</u> <u>functor</u>

As colimits of categories are "2-universal", the description of the localization functor as a pushout implies a stronger universal property ("2-universal"), which is very important:
Let $\mathcal{D}$ a category and denote by $\underline{\mathrm{Hom}}_W(C, \mathcal{D})$ the full subcategory of the category $\underline{\mathrm{Hom}}(C, \mathcal{D})$ of functors from $C$ to $\mathcal{D}$ and natural transformations whose objects are functors taking arrows in $W$ to isomorphisms of $\mathcal{D}$. The precise universal

property says that the functor

$$\underline{\mathrm{Hom}}\,(W^{-1}C, \mathcal{D}) \longrightarrow \underline{\mathrm{Hom}}_W\,(C, \mathcal{D})$$

$$G \longmapsto G \circ P$$

is an <u>isomorphism</u> of categories.

A more concrete description of the category $W^{-1}C$ is obtained as follows. The objects of $W^{-1}C$ are the same as those of $C$. The arrows of $W^{-1}C$ are equivalence classes of composable zigzags of arrows of $C$, the arrows going in the "wrong" direction belonging to $W$

Example of such a zigzag from $X$ to $Y$

$$X \xrightarrow{f} T_1 \xleftarrow{s} T_2 \xrightarrow{g} T_3 \xleftarrow{t} T_4 \xleftarrow{w} T_5 \xrightarrow{h} T_6 \xrightarrow{k} Y$$

$$f, g, h, k \in \mathrm{Ar}\, C \quad , \quad s, t, w \in W$$

The equivalence relation is the equivalence relation compatible with composition of zigzags generated by the elementary relations

$$\xrightarrow{v}\xrightarrow{u} \;\sim\; \xrightarrow{uv} \qquad v, u \in \mathrm{Ar}\, C$$

$$\xrightarrow{s}\xleftarrow{s} \;\sim\; \xrightarrow{1} \qquad s \in W$$

$$\xleftarrow{s}\xrightarrow{s} \;\sim\; \xrightarrow{1} \qquad s \in W$$

$$X \xrightarrow{1} X \;\sim\; X \qquad \text{(zigzag of length 0 from } X \text{ to } X\text{)}$$

Example of deduced relation : $s, t \in W$, $st \in W$

$$X \xleftarrow{s} Y \xleftarrow{t} Z \sim X \xleftarrow{s} Y \xleftarrow{t} Z \xrightarrow{1_Z} Z \sim$$

$$\sim X \xleftarrow{s} Y \xleftarrow{t} Z \xrightarrow{st} X \xleftarrow{st} Z \sim X \xleftarrow{s} Y \xleftarrow{t} Z \xrightarrow{t} Y \xrightarrow{s} X \xleftarrow{st} Z \sim$$

$$\sim X \xleftarrow{s} Y \xrightarrow{1_Y} Y \xrightarrow{s} X \xleftarrow{st} Z \sim X \xleftarrow{s} Y \xrightarrow{s} X \xleftarrow{st} Z \sim$$

$$\sim X \xrightarrow{1_X} X \xleftarrow{st} Z \sim X \xleftarrow{st} Z .$$

This description of $W^{-1}C$ is due to Gabriel and Zisman and is known as the Gabriel-Zisman localization. An equivalent description can be given using the "hammock"(*) localization of Dwyer and Kan. Let $(C, W)$ a localizer such that $W$ contains the identities and is stable by composition, and $X, Y$ two objects of $C$. We define the *hammock simplicial set* from $X$ to $Y$ as the simplicial set whose $p$-simplices are commutative diagrams in $C$ of the forme

$$\begin{tikzcd}
 & T_{0,1} \arrow[r, dash] \arrow[d] & T_{0,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & T_{0,n-1} \arrow[d] \arrow[ddr, dash] & \\
 & T_{1,1} \arrow[r, dash] \arrow[d] & T_{1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & T_{1,n-1} \arrow[d] \arrow[dr, dash] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[ddr, dash] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \\
 & & & & & \\
 & T_{p,1} \arrow[r, dash] & T_{p,2} \arrow[r, dash] & \cdots \arrow[r, dash] & T_{p,n-1} \arrow[uur, dash] &
\end{tikzcd} , \quad n \geq 0,$$

in which

a) all vertical maps are in $W$;

b) in each column, all maps (horizontal or oblique) go in the same direction; if they go to the left, then they are in $W$;

c) the maps (horizontal or oblique) in adjacent

(*) Hanging bed of canvas or netting suspended by cords at ends.

columns go in different directions;

d) no column contains only identity maps.

It can be proven that the set of arrows from $X$ to $Y$ in $W^{-1}C$ is in canonical bijection with the set of connected components of this simplicial set.

The hammock construction defines a simplicial category whose objects are the objects of $C$ and such that for every pair of objects $X, Y$ the map simplicial set from $X$ to $Y$ is the hammock simplicial set from $X$ to $Y$. This simplicial category is known as the hammock Dwyer-Kan localization of $(C, W)$ and is denoted by $L^H(C, W)$. In this course, all localizations considered will be Gabriel-Zisman localizations and <u>not</u> Dwyer-Kan localizations. As it was mentionned before, the relation between them is that Gabriel-Zisman localization is the "$\pi_0$ of the Dwyer-Kan localization":

$$W^{-1}C = \pi_0 \, L^H(C, W).$$

3. Let $(\mathcal{C}, W)$ an <u>arbitrary</u> localizer and $F : \mathcal{C} \to \mathcal{D}$ a functor. By the universal property of the localization if $F$ takes arrows in $W$ to isomorphisms of $\mathcal{D}$, there exists a unique functor $\overline{F} : W^{-1}\mathcal{C} \to \mathcal{D}$ such that $F = \overline{F} \circ P$ where $P : \mathcal{C} \to W^{-1}\mathcal{C}$ is the localization functor. The aim of the theory of <u>derived</u> functors is to define a functor $W^{-1}\mathcal{C} \to \mathcal{D}$ "approximating" this situation in some cases in which $F$ does not satisfy this condition.

Let $(\mathcal{C}, W)$ be a localizer, $P : \mathcal{C} \to W^{-1}\mathcal{C}$ the localization functor and $F : \mathcal{C} \to \mathcal{D}$ an arbitrary functor. A <u>right</u> <u>derived</u> <u>functor</u> of $F$ is a pair $(RF, \alpha_F)$ (although $W$ is not in the notation this <u>do</u> depend on $W$) where $RF : W^{-1}\mathcal{C} \to \mathcal{D}$ is a functor and $\alpha_F : F \to RF \circ P$ a natural transformation, satisfying the following universal property. For every functor $G : W^{-1}\mathcal{C} \to \mathcal{D}$, and every natural transformation $\alpha : F \to G \circ P$, there is a unique natural transformation $\gamma : RF \to G$ such that $\alpha = (\gamma * P)\alpha_F$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "P"'] \arrow[dr, "F"] & \\ W^{-1}\mathcal{C} \arrow[r, "RF"'] \arrow[ur, phantom, "\Uparrow \alpha_F"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[d, "\alpha_F"'] \arrow[dr, "\alpha"] & \\ RF \circ P \arrow[r, "\gamma * P"'] & G \circ P \end{tikzcd}$$

This condition means exactly that the functor $RF$ (together with the natural transformation $\alpha$) is a <u>left</u> Kan extension of $F$ along the localization functor $P$, but I don't want to insist on this fact in order to avoid confusion between left and right. The pair $(RF, \alpha_F)$ is an <u>absolute right derived functor</u> of $F$ if for every functor $H : D \to D'$ the pair $(H \circ RF, H * \alpha_F)$ is a right derived functor of $H \circ F$. An absolute right derived functor of $F$ is in particular a right derived functor of $F$ (take $H = 1_D$).

The notions of left derived functor $(LF, \beta_F)$ and absolute left derived functor are defined dually. If $F : C \to D$ takes arrows in $W$ to isomorphisms of $D$ then $(\overline{F}, 1_F)$ is a left and right absolute derived functor of $F$. One of the goals of the theory of model categories is to prove the existence of derived functors in less trivial situations, and <u>all</u> derived functors produced by this theory are <u>absolute</u> derived functors.

Let $(C, W)$, $(C', W')$ be two localizers and $\rho : C \to W^{-1}C$, $\rho' : C' \to W'^{-1}C'$ the localization functors. A <u>morphism of localizers</u> $F : (C, W) \to (C', W')$ is a functor $F : C \to C'$ such that $F(W) \subset W'$. The functor $\rho' \circ F : C \to W'^{-1}C'$ takes then arrows of $W$ to isomorphisms of $W'^{-1}C'$, and by the universal property of the localization there exists a unique functor $\overline{F} : W^{-1}C \to W'^{-1}C'$ such that $\rho' \circ F = \overline{F} \circ \rho$

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\rho"'] & C' \arrow[d, "\rho'"] \\ W^{-1}C \arrow[r, "\overline{F}"'] & W'^{-1}C' \end{tikzcd}$$

The aim of the theory of <u>total</u> derived functors is to approximate this situation in some cases when the functor $F : C \to C'$ is <u>not</u> a morphism of localizers.

Let $(C, W)$ and $(C', W')$ be two localizers, $p : C \to W^{-1}C$ and $p' : C' \to W'^{-1}C'$ the localization functors, and $F : C \to C'$ an arbitrary functor. A <u>total right derived functor</u> (resp. an <u>absolute total right derived functor</u>) of $F$ is a pair $(\underline{R}F, \alpha)$, where $\underline{R}F : W^{-1}C \to W'^{-1}C'$ is a functor and $\alpha : p' \circ F \to \underline{R}F \circ p$ a natural transformation, which is a right derived functor (resp. an absolute right derived functor) of $p' \circ F$.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "p"'] \arrow[dr, phantom, "\swArrow\alpha"] & C' \arrow[d, "p'"] \\ W^{-1}C \arrow[r, "\underline{R}F"'] & W'^{-1}C' \end{tikzcd}$$

The notions of <u>total left derived functor</u> and <u>absolute total left derived functor</u> are defined dually. If $F$ is a morphism of localizers $(\overline{F}, 1_{p'F})$ is an absolute total left and right derived functor of $F$. The theory of model categories produce many other, non trivial, examples.

(Abstract adjunction theorem)
<u>Theorem</u> Let $(C, W)$ and $(C', W')$ be two localizers

$$F : C \longrightarrow C' \ , \quad G : C' \longrightarrow C$$

a pair of adjoint functors ($F$ left adjoint, $G$ right adjoint). Suppose that the functor $F$ (resp. $G$) has an absolute total left (resp. right) derived functor $(\underline{L}F, \alpha)$ (resp. $(\underline{R}G, \beta)$). Then the pair of functors

$$\underline{L}F : W^{-1}C \to W'^{-1}C' \ , \quad \underline{R}G : W'^{-1}C' \longrightarrow W^{-1}C$$

is a pair of adjoint functors.

<u>Remark</u> Let $(C, W)$, $(C', W')$ and $(C'', W'')$ be three localizers, $P : C \longrightarrow W^{-1}C$, $P' : C' \longrightarrow W'^{-1}C'$ and $P'' : C'' \longrightarrow W''^{-1}C''$ the localization functors and $F : C \to C'$, $G : C' \longrightarrow C''$ two functors. Suppose that $F$ (resp. $G$) admits an (absolute) total right derived functor $(\underline{R}F, \alpha)$ (resp. $(\underline{R}G, \beta)$). In general the composed functor $H = G \circ F$ need not admit a total right derived functor, and even if it does and if $(\underline{R}H, \gamma)$ is such a derived functor, the natural transformation $\underline{R}H \to \underline{R}G \circ \underline{R}F$ deduced by the universal property of derived functors need not be an isomorphism. Model category theory gives sufficient conditions for the total derived functor of $H$ to exist and for the natural transformation $\underline{R}H \to \underline{R}G \circ \underline{R}F$ to be an isomorphism

4. Fix a category $C$. For every small category $I$ denote $C^I := \underline{\mathrm{Hom}}(I, C)$ the category of functors from $I$ to $C$, and

$$\Delta_I : C \longrightarrow C^I$$

the "diagonal" functor

$$X \longmapsto \text{constant functor } I \to C \text{ of value } X.$$

Recall that $C$ admits "colimits of type $I$" if and only if this functor has a left adjoint, and then the pair of functors

$$C^I \xrightarrow{\varinjlim_I} C \quad , \quad C \xrightarrow{\Delta_I} C^I$$

$$F \longmapsto \varinjlim F$$

is an adjoint pair. More precisely, if $F : I \longrightarrow C$ is a functor, $\varinjlim F$ exists in $C$ if and only if the left adjoint of $\Delta_I$ is "defined" at $F$, and then its value is $\varinjlim F$

Fix now a localizer $(C, W)$. For every small category $I$ denote $W_I$ the class of arrows of $C^I$ that are pointwise in $W$

$$W_I = \{\alpha : F \to G \mid \forall i \in \mathrm{Ob}\, I \ \ \alpha_i : F(i) \to G(i) \in W\}$$

The functor $\Delta_I : C \to C^I$ is a morphism of localizers $(C, W) \longrightarrow (C^I, W_I)$ and induces a functor

$$\overline{\Delta_I} : W^{-1}C \longrightarrow W_I^{-1} C^I$$

Let $F$ be an object of $W_I^{-1} C^I$, i.e a functor $F : I \longrightarrow C$. We will say that $F$ has a <u>homotopy colimit</u> if the left adjoint of $\overline{\Delta}_I$ is "defined" at $F$ in $W^{-1}C$. This means that there exists an object $\underrightarrow{\mathrm{holim}}\, F$ of $W^{-1}C$, called the <u>homotopy colimit</u> of $F$, and a bijection

$$\mathrm{Hom}_{W^{-1}C}(\underrightarrow{\mathrm{holim}}\, F, T) \xrightarrow{\sim} \mathrm{Hom}_{W_I^{-1}C^I}(F, \overline{\Delta}_I(T))$$

natural in $T$. We will say that the localizer $(C, W)$ <u>admits homotopy colimits of type $I$</u> if the functor $\overline{\Delta}_I$ has a left adjoint $\underrightarrow{\mathrm{holim}}_I$

$$W_I^{-1}C^I \xrightarrow{\underrightarrow{\mathrm{holim}}_I} W^{-1}C \qquad W^{-1}C \xrightarrow{\overline{\Delta}_I} W_I^{-1}C^I$$

$$F \longmapsto \underrightarrow{\mathrm{holim}}\, F$$

Observe that these notions can <u>not</u> be defined in terms of the sole localized category $W^{-1}C$.

The following proposition is an immediate consequence of the abstract adjunction theorem.

<u>Proposition</u> Let $(C, W)$ be a localizer, $I$ a small category, and suppose that the category $C$ admits colimits of type $I$ and that the functor $\underrightarrow{\lim}_I : C^I \longrightarrow C$ has an absolute total left derived functor $(\underline{L}\, \underrightarrow{\lim}_I, \ )$. Then the localizer $(C, W)$ admits homotopy colimits of type $I$ and $\underrightarrow{\mathrm{holim}}_I \simeq \underline{L}\, \underrightarrow{\lim}_I$

We will say that the localizer $(\mathcal{C}, W)$ admits homotopy colimits if it admits homotopy colimits of type $I$ for every small category $I$.

An important particular case of homotopy colimit is the <u>homotopy amalgamated sum</u> or <u>homotopy pushout</u> of a diagram of the form

$$\begin{array}{ccc} X_{00} & \longrightarrow & X_{01} \\ \downarrow & & \\ X_{10} & & \end{array}$$

Such a diagram can be considered as an object $X$ of the category $W_\Gamma^{-1}\mathcal{C}^\Gamma$ where $\Gamma$ is the category:

$$\begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ \downarrow & & \\ (1,0) & & \end{array}$$

The homotopy pushout of the given diagram is the homotopy colimit $\underrightarrow{\mathrm{holim}}_\Gamma X$

A commutative square

$$\begin{array}{ccc} X_{00} & \longrightarrow & X_{01} \\ \downarrow & & \downarrow \\ X_{10} & \longrightarrow & X_{11} \end{array}$$

is called <u>homotopy pushout</u> or <u>homotopy cocartesian</u> if it "identifies" $X_{11}$ to the homotopy colimit of its upper left corner, which means exactly that for every object $T$ of $W^{-1}\mathcal{C}$ the map

$$\mathrm{Hom}_{W^{-1}\mathcal{C}}(X_{11}, T) \longrightarrow \mathrm{Hom}_{W_\Gamma^{-1}\mathcal{C}^\Gamma}\left(\begin{array}{ccc} X_{00} & \to & X_{01} \\ \downarrow & & \\ X_{10} & & \end{array}, \begin{array}{ccc} T & \xrightarrow{=} & T \\ \Vert & & \\ T & & \end{array}\right)$$

deduced from the arrows of the square is bijective

The notions of homotopy limit, homotopy fiber product, homotopy cartesian square etc. are defined dually

Let $C$ be a cocomplete category and $u : I \to J$ a functor between small categories. It is well known that the functor

$$u^* : C^J \longrightarrow C^I \, , \quad F \longmapsto F \circ u$$

has a left adjoint

$$u_! : C^I \longrightarrow C^J$$

and that this adjoint satisfies the relation

$$u_!(F)(j) = \varinjlim_{\substack{I/j \\ (i,\, u(i) \to j)}} F(i) \; , \quad F : I \to C \, , \; j \in Ob(J)$$

where $I/j$ is the "comma category" whose objects are the pairs $(i, u(i) \xrightarrow{l} j)$, $i \in Ob\, I$, $l \in Ar(J)$ with morphisms

$$(i, u(i) \xrightarrow{l} j) \longrightarrow (i', u(i') \xrightarrow{l'} j)$$

$k : i \to i' \in Ar(I)$

$$\begin{tikzcd} u(i) \arrow[rr, "u(k)"] \arrow[dr, "l"'] & & u(i') \arrow[dl, "l'"] \\ & j & \end{tikzcd}$$

$l = l' u(k)$

If $J$ is the point category, that we will denote by $e$, and $u = p = p_I : I \to e$ the unique functor then

$$p^* = \Delta_I \quad \text{and} \quad p_! = \varinjlim_I$$

Let's come back to a general $J$. If we denote by

$$\varepsilon : u_! u^* \longrightarrow 1_{C^J} \quad , \quad 1_{C^I} \longrightarrow u^* u_!$$

the counit and unit of the adjunction, it is easily verified that for every functor $F : I \to C$, the pair $(u_! F, \eta_F)$ is a left

$$\begin{tikzcd} I \arrow[d, "u"'] \arrow[dr, "F"] & \\ J \arrow[r, "u_! F"'] \arrow[ur, phantom, "\Uparrow \eta_F"] & C \end{tikzcd} \qquad F \xrightarrow{\eta_F} u^* u_! F = u_! F \circ u$$

Kan extension of $F$ along the functor $u$:

$$f \longmapsto u^*(f)\, \eta_F = (f * u)\, \eta_F$$

$$\mathrm{Hom}_{C^J}(u_! F, G) \xrightarrow{\sim} \mathrm{Hom}_{C^I}(F, u^* G) = \mathrm{Hom}_{C^I}(F, Gu) \quad , \quad G : J \to C.$$

For this reason $u_!$ is called the left Kan extension functor

Fix now a localizer $(\mathcal{C}, W)$ and let $u : I \to J$ be a functor between small categories. The functor $u^* : \mathcal{C}^J \longrightarrow \mathcal{C}^I$ is a morphism of localizers $(\mathcal{C}^J, W_J) \longrightarrow (\mathcal{C}^I, W_I)$ and induces a functor

$$\overline{u^*} : W_J^{-1}\mathcal{C}^J \longrightarrow W_I^{-1}\mathcal{C}^I$$

In contrast to the classical situation, in order for $\overline{u^*}$ to have a left adjoint it is <u>not</u> enough to suppose that $(\mathcal{C}, W)$ admits homotopy colimits

We will say that the localizer $(\mathcal{C}, W)$ <u>admits homotopy left Kan extensions</u> if for every functor $u : I \to J$ between small categories $\overline{u^*}$ admits a left adjoint $u_{!R}$.
In particular, such a localizer admits homotopy colimits which as in the classical case correspond to $J$ being the point category. Nevertheless, again in contrast to the classical situation, even though for every object $F : I \to \mathcal{C}$ of $W_I^{-1}\mathcal{C}^I$ and every object $j$ of $J$ there is a natural canonical map in $W^{-1}\mathcal{C}$

$$(*) \qquad u_! \,d(F)(j) \longleftarrow \underrightarrow{\mathrm{holim}}\, F \,|\, (I/j)$$

this map need not be an isomorphism. We will say that the localizer $(\mathcal{C}, W)$ is <u>homotopically cocomplete</u> if $(\mathcal{C}, W)$ admits homotopy left Kan extensions and if $(*)$ is an isomorphism of $W^{-1}\mathcal{C}$ for every $u$, $F$, $j$.