The square root of 2, the Golden Ratio and the Fibonacci sequence

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Abstract

The square root of 2,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\ldots,$$

and the Golden ratio

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.618\,033\,988\,749\,894\ldots$$

are two irrational numbers with many remarkable properties. The Fibonacci sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233\ldots$$

occurs in many situations, in mathematics as well as in the real life. We review some of these properties.

Tablet YBC 7289 : 1800 – 1600 BC

Babylonian clay tablet, accurate sexagesimal approximation to $\sqrt{2}$ to the equivalent of six decimal digits.

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.\mathbf{414\,212}\,962\,962\,962\ldots$$

$$\sqrt{2} = 1.\mathbf{414\,213}\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\ldots$$

https://en.wikipedia.org/wiki/YBC_7289

A4 format 21×29.7

$$\frac{297}{210} = \frac{99}{70} = 1.414\,285\,714\,285\,714\,285\,714\,285\,714\ldots$$

A4 format

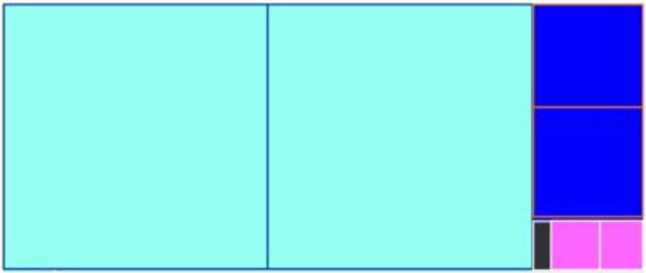
The number $\sqrt{2}$ is twice its inverse : $\sqrt{2} = 2/\sqrt{2}$.
Folding a rectangular piece of paper with sides in proportion $\sqrt{2}$ yields a new rectangular piece of paper with sides in proportion $\sqrt{2}$ again.

Paper format A0, A1, A2,... in cm

$$x_1 = 100\sqrt[4]{2} = 118.9, \qquad x_2 = \frac{100}{\sqrt[4]{2}} = 84.1.$$

$$\begin{array}{llll}
A0: & x_1 = 118.9 & x_2 = 84.1 \\
A1: & x_2 = 84.1 & \frac{x_1}{2} = 59.4 \\
A2: & \frac{x_1}{2} = 59.4 & \frac{x_2}{2} = 42 \\
A3: & \frac{x_2}{2} = 42 & \frac{x_1}{4} = 29.7 \\
A4: & \frac{x_1}{4} = 29.7 & \frac{x_2}{4} = 21 \\
A5: & \frac{x_2}{4} = 21 & \frac{x_1}{8} = 14.8
\end{array}$$

Rectangles with proportion $1 + \sqrt{2}$



Irrationality of $\sqrt{2}$: geometric proof

- Start with a rectangle have sides lengths 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $\sqrt{2}-1$ and 1.
- This second small rectangle has sides lengths in the proportion

$$\frac{1}{\sqrt{2}-1} = 1+\sqrt{2},$$

which is the same as for the large one.
- Hence the second small rectangle can be split into two squares and a third smaller rectangle with the same proportion $1+\sqrt{2}$.
- This process does not end.

Irrationality of $\sqrt{2}$: geometric proof

If we start with a rectangle having a rational proportion, say $297/210 = 99/70$, using an appropriate unit the sides lengths are integers. For instance 99 and 70.

The successive squares have decreasing integer sides lengths, say 70, 29, 12, 5, 2, 1 :
$99 = 70 + 29$, $\quad 70 = 2 \times 29 + 12$, $\quad 29 = 2 \times 12 + 5$,
$12 = 2 \times 5 + 2$, $\quad 5 = 2 \times 2 + 1$.

Hence this process stops after finitely may steps.

Hence $1 + \sqrt{2}$ is an irrational number, and $\sqrt{2}$ also.

Continued fraction of $\sqrt{2}$

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,20\ \ldots$$

satisfies

$$\boxed{\sqrt{2}} = 1 + \cfrac{1}{1 + \boxed{\sqrt{2}}}.$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \boxed{\sqrt{2}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1,\ \overline{2}].$$

A4 format

$$\frac{297}{210} = 1 + \frac{29}{70},$$
$$\frac{70}{29} = 2 + \frac{12}{29},$$
$$\frac{29}{12} = 2 + \frac{5}{12},$$
$$\frac{12}{5} = 2 + \frac{2}{5},$$
$$\frac{5}{2} = 2 + \frac{1}{2}.$$

Hence

$$\frac{297}{210} = [1, 2, 2, 2, 2, 2].$$

First decimals of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.41421356237309504880168872420969807856967187537694807317667973
7990732478462107038850387534327641572735013846230912297024924836
0558507372126441214970999358314132226659275055927557999505011527
8206057147010955997160597027453459686201472851741864088919860955
2329230484308714321450839762603627995251407989687253396546331808
8296406206152583523950547457502877599617298355752203375318570113
5437460340849884716038689997069900481503054402779031645424782306
8492936918621580578463111596668713013015618568987237235288509264
8612494977154218334204285686060146824720771435854874155657069677
6537202264854470158588016207584749226572260020855844665214583988
9394437092659180031138824646815708263010059485870400318648034219
4897278290641045072636881313739855256117322040245091227700226941
1275736272804957381089675040183698683684507257993647290607629969
4138047565482372899718032680247442062926912485905218100445984215
0591120249441341728531478105803603371077309182869314710171111683
91658172688941975871658215212822951848847 ...

First binary digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.0110101000001001111001100110011111110011101111001100100100000
1000101100101111101100010011011001101110101010010101011111010 0
1111100011101011011110110000010111010100010010011101110101000 0
1001100111011010001011110101100100001011000001100110011100110 0
1000101010100101011111100100000110000010000111010101110001010 0
0101100001110101000101100011111111001101111111011100100000111 10
1101100111001000011110111010010101000010111100100001110011100 0
1111011010010100111100000000010010000111001101100011110111111 01
0001001110110100011010010001000000010111010000111010000101010 1
1110001111101001110010100110000010110011100011000000000100011 01
1110000110011011110111100101010110001101111001001000100010110 1
0001000010001011000101001000110000010101011110001110010001011 1
1011111000100111000110011110001101101010110101000101000111000 1
0111011011111101001110111001100101100101010011000110100001100 1
1000111110011110010000100110111110101001011110001001000001111 1
0000011011011100101100000101110111010101010010010100000010001 00
1100100000010000000110010100100101010000000100111001010010 10 . . .

Computation of decimals of $\sqrt{2}$

$1\,542$ decimals computed by hand by Horace Uhler in 1951

$14\,000$ decimals computed in 1967

$1\,000\,000$ decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of π.

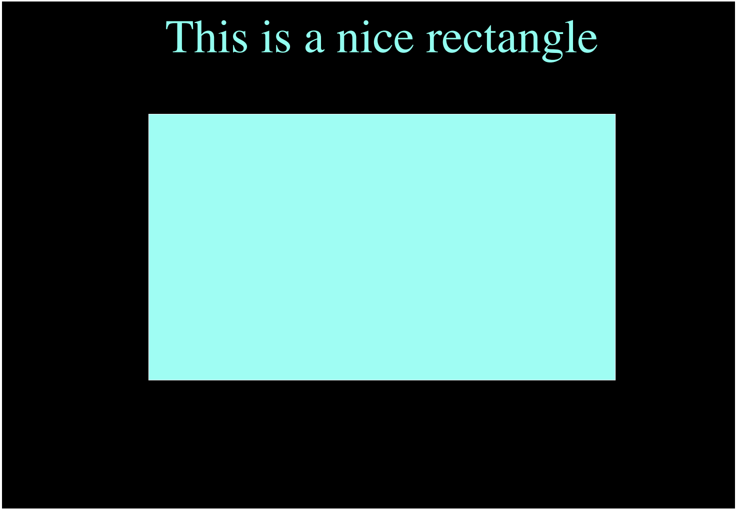
Émile Borel (1871–1956)

• *Les probabilités dénombrables et leurs applications arithmétiques,*
Palermo Rend. **27**, 247-271 (1909).
Jahrbuch Database JFM 40.0283.01
http://www.emis.de/MATH/JFM/JFM.html

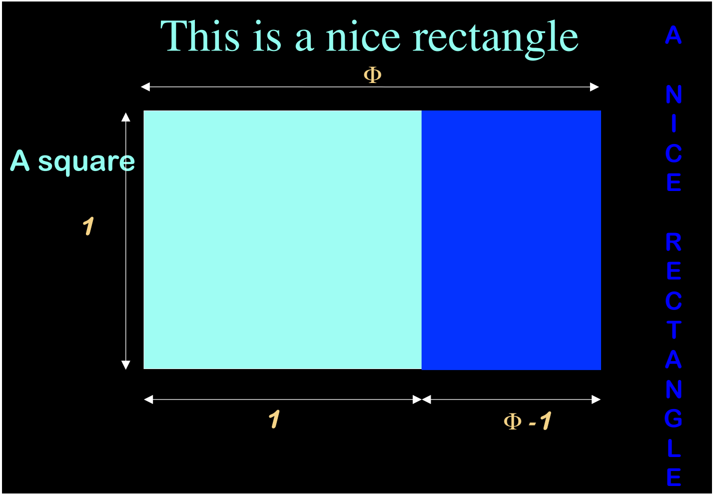
• *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
C. R. Acad. Sci., Paris **230**, 591-593 (1950).
Zbl 0035.08302

Émile Borel : 1950

Let $g \geq 2$ be an integer and x a real irrational algebraic number. *The expansion in base g of x should satisfy some of the laws which are valid for almost all real numbers (with respect to Lebesgue's measure).*


This is a nice rectangle

Golden rectangle



$$\frac{\Phi}{1} = \frac{1}{\Phi - 1}.$$

Irrationality of Φ and of $\sqrt{5}$

The number

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.618\,033\,988\,749\,894\dots$$

satisfies

$$\boxed{\Phi} = 1 + \frac{1}{\boxed{\Phi}}\cdot$$

Hence

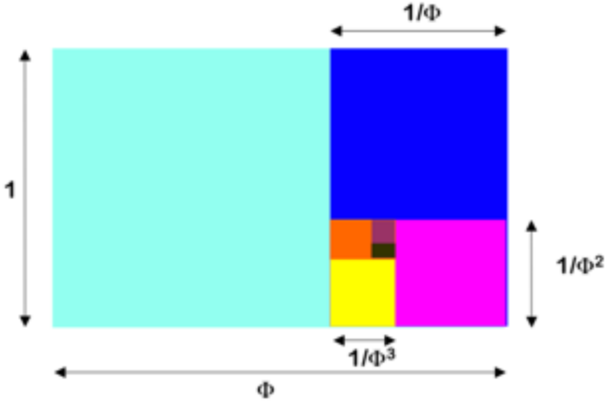
$$\Phi = 1 + \frac{1}{1 + \frac{1}{\boxed{\Phi}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$$

If we start from a rectangle with the Golden ratio as proportion of sides lengths, at each step we get a square and a smaller rectangle with the same proportion for the sides lengths.
http://oeis.org/A001622

The Golden Ratio

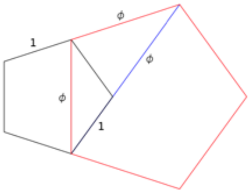
$$(1+\sqrt{5})/2 = 1.618\,033\,988\,749\,894\ldots$$

Golden Rectangle

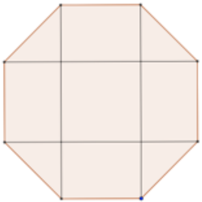


The diagonal of the pentagon and the diagonal of the octogon

The diagonal of the pentagon is Φ



The diagonal of the octogon is $1+\sqrt{2}$



Nested roots

$$\Phi^2 = 1 + \Phi.$$

$$\begin{aligned}
\Phi &= \sqrt{1+\Phi} \\
&= \sqrt{1+\sqrt{1+\Phi}} \\
&= \sqrt{1+\sqrt{1+\sqrt{1+\Phi}}} \\
&= \dots \\
&= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}
\end{aligned}$$

Nested roots

Journal of the Indian Mathematical Society (1912) – problems solved by Ramanujan

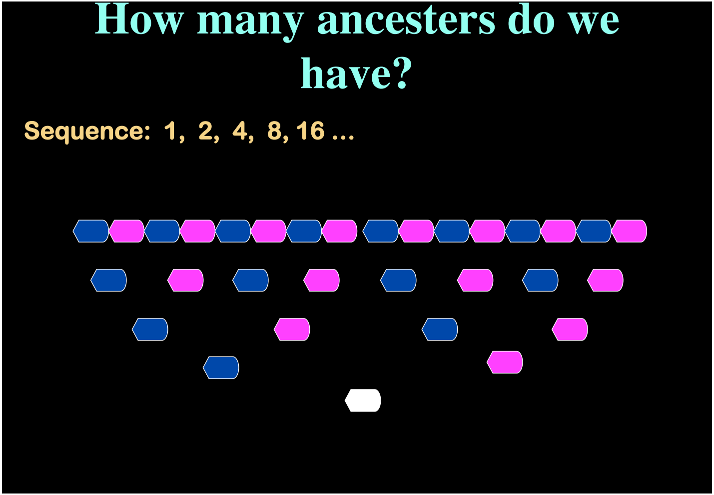
$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}=3$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}}=4$$

Srinivasa Ramanujan

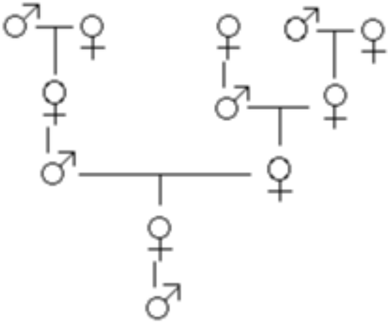
1887 – 1920

Geometric series $\quad u_0 = 1, \quad u_{n+1} = 2u_n$



Bees genealogy

Male honeybees are born from unfertilized eggs. Female honeybees are born from fertilized eggs. Therefore males have only a mother, but females have both a mother and a father.



Genealogy of a male bee (bottom – up)

Number of bees :

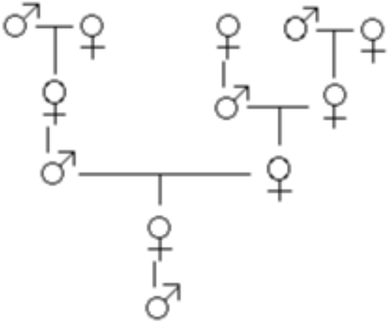
$$1,\ 1,\ 2,\ 3,\ 5\ldots$$

Number of females :

$$0,\ 1,\ 1,\ 2,\ 3\ldots$$

Rule :

$$u_{n+2} = u_{n+1} + u_n.$$



Bees genealogy $u_1 = 1,\ u_2 = 1,\ u_{n+2} = u_{n+1} + u_n$

Number of females at a given level =
total population at the previous level

Number of males at a given level=
number of females at the previous level

3 + 5 = 8

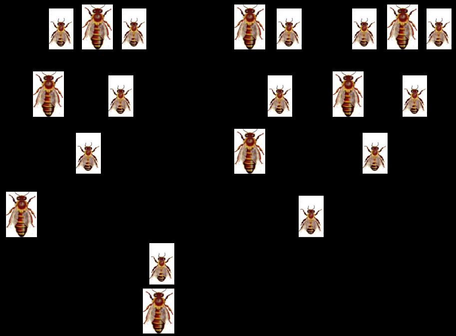
2 + 3 = 5

1 + 2 = 3

1 + 1 = 2

0 + 1 = 1

1 + 0 = 1



The Lamé Series

Gabriel Lamé

1795 – 1870

Edouard Lucas

1842 - 1891

In 1844 the sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233, \ldots$$

was referred to as the Lamé series, because Gabriel Lamé used it to give an upper bound for the number of steps in the Euclidean algorithm for the gcd.
On a trip to Italy in 1876 Edouard Lucas found them in a copy of the Liber Abbaci of Leonardo da Pisa.

Leonardo Pisano (Fibonacci)

The Fibonacci sequence $(F_n)_{n\geq 0}$,

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233,\ldots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 0.$$

http://oeis.org/A000045

Leonardo Pisano (Fibonacci)
(1170–1250)

Leonardo Pisano (Fibonacci)

Guglielmo Bonacci : filius Bonacci or Fibonacci

travels around the mediterranean,

learns the techniques of Al-Khwarizmi

Liber Abbaci (1202)

https://commons.wikimedia.org/w/index.php?curid=720501

Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, ...

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

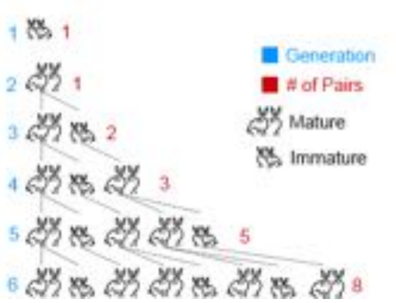
Neil J. A. Sloane

http://oeis.org/A000045

Fibonacci rabbits

Fibonacci considered the growth of a rabbit population.

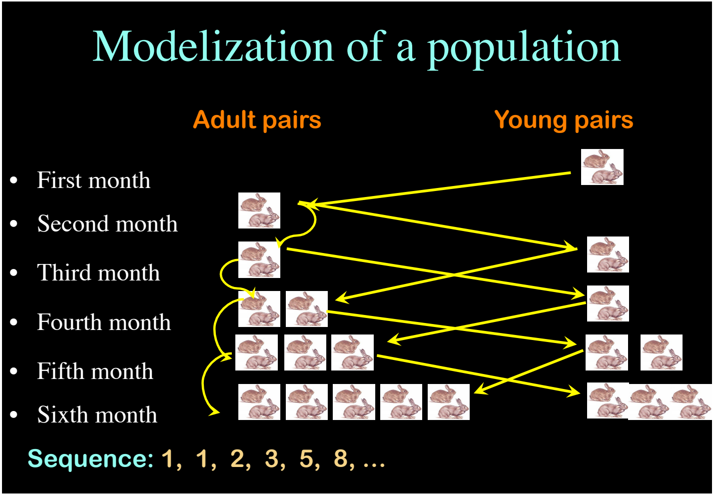
A newly born pair of rabbits, a male and a female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits ; rabbits never die and a mating pair always produces

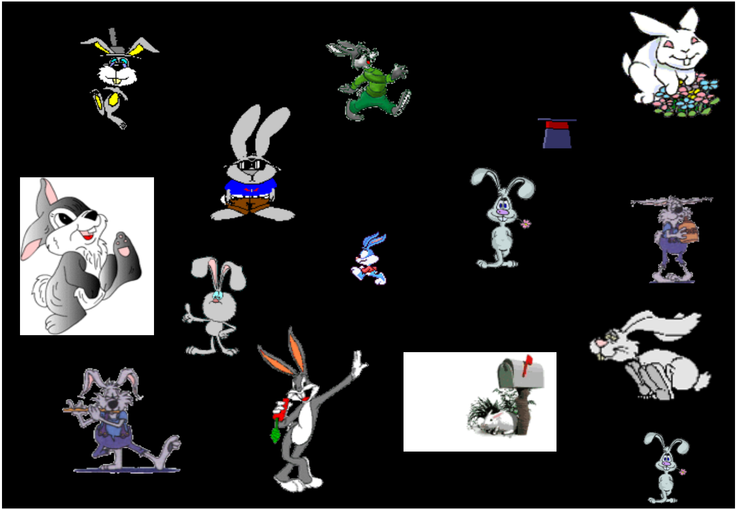


one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was : how many pairs will there be in one year ?

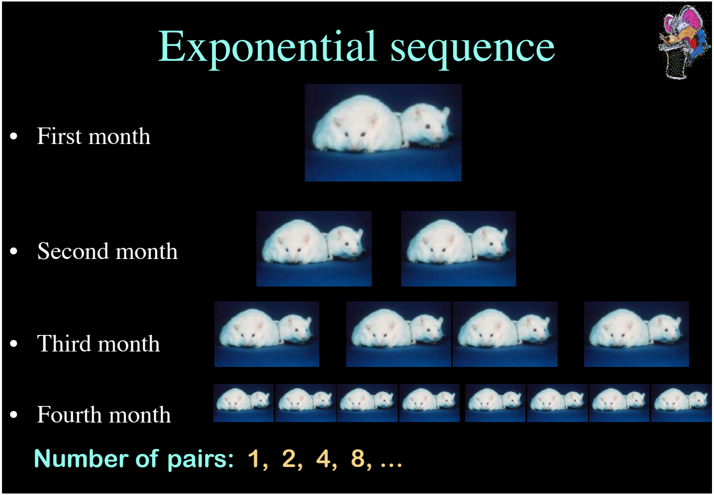
Answer : $F_{12} = 144$.

Fibonacci's rabbits





Modelization of a population of mice



Is-it a realistic model ?

The genealogy of the ancestors of a human being is not a mathematical tree :
30 generations would give 2^{30} ancestors, more than a billion people, three to four times more than the total population on earth one thousand years ago.

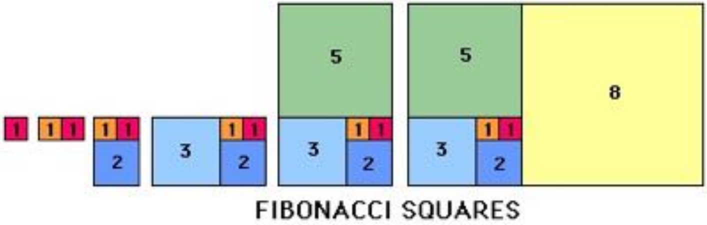
Even worse for the genealogy of bees :

In every bee hive there is one female queen bee which lays all the eggs. If an egg is not fertilised it eventually hatches into a male bee, called a drone. If an egg is fertilised by a male bee, then the egg produces a female worker bee, which doesn't lay any eggs herself.

Alfred Lotka : arctic trees

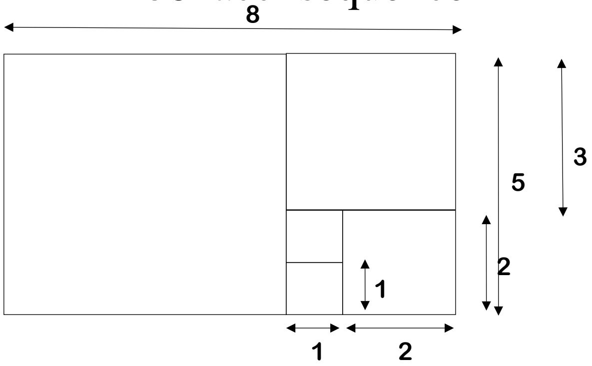
In cold countries, each branch of some trees gives rise to another one after the second year of existence only.

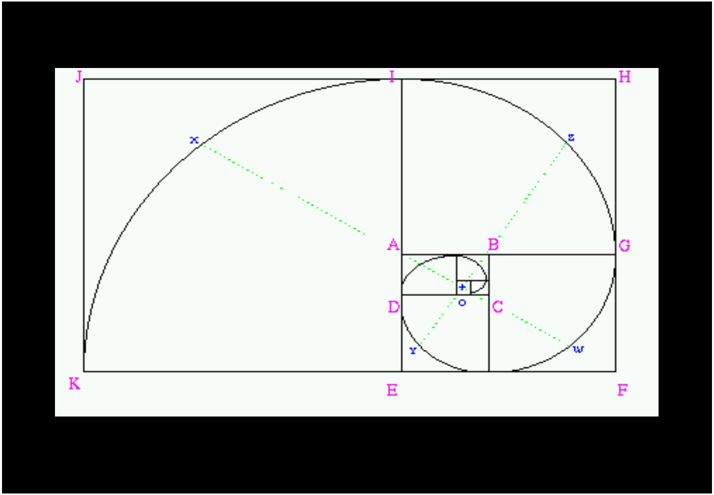
Fibonacci squares



http://mathforum.org/dr.math/faq/faq.golden.ratio.html

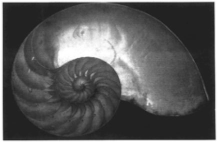
Geometric construction of the Fibonacci sequence

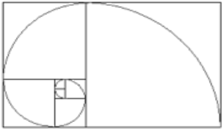


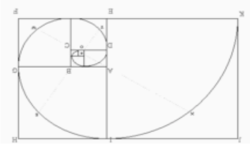

J
I
H
x
z
A
B
G
D
o
C
y
w
K
E
F

The Fibonacci numbers in nature

Ammonite (Nautilus shape)



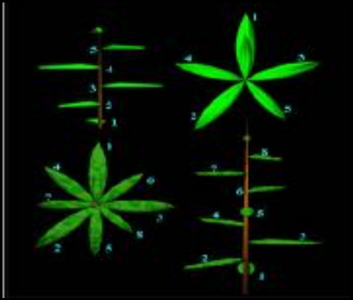




Phyllotaxy

- Study of the position of leaves on a stem and the reason for them
- Number of petals of flowers: daisies, sunflowers, aster, chicory, asteraceae,…
- Spiral patern to permit optimal exposure to sunlight
- Pine-cone, pineapple, Romanesco cawliflower, cactus

Leaf arrangements



- Université de Nice,
  Laboratoire Environnement Marin Littoral,
  Equipe d'Accueil "Gestion de la Biodiversité"

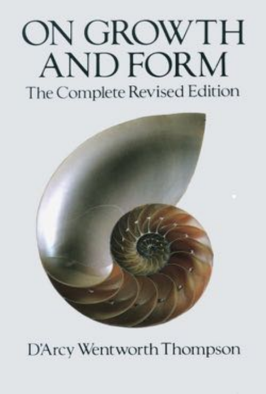
http://www.unice.fr/LEML/coursJDV/tp/tp3.htm

Phyllotaxy

Phyllotaxy

- J. Kepler (1611) uses the Fibonacci sequence in his study of the dodecahedron and the icosaedron, and then of the symmetry of order 5 of the flowers.
- Stéphane Douady and Yves Couder
  Les spirales végétales
  La Recherche *250* (Jan. 1993) vol. **24.**


PATTERNS
IN NATURE
PETER S.
STEVENS


ON GROWTH
AND FORM
The Complete Revised Edition
D'Arcy Wentworth Thompson

Why are there so many occurrences of the Fibonacci numbers and of the Golden ratio in the nature ?

According to Leonid Levin, objects with a small algorithmic Kolmogorov complexity (generated by a short program) occur more often than others.

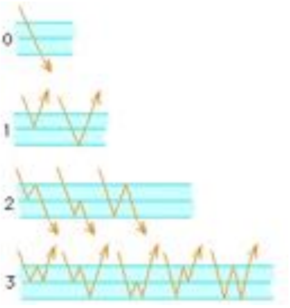
Another example is given by Sierpinski triangles.

Reference : J-P. Delahaye.
http://cristal.univ-lille.fr/~jdelahay/pls/

Reflections of a ray of light

Consider three parallel sheets of glass and a ray of light which crosses the first sheet. Each time it touches one of the sheets, it can cross it or reflect on it.

Denote by p_n the number of different paths with the ray going out of the system after n reflections.



$p_0 = 1$,

$p_1 = 2$,

$p_2 = 3$,

$p_3 = 5$.

In general, $p_n = F_{n+2}$.

Sequences of 0, 1 and 2

Denote by u_n the number of sequences of n elements, each of them is 0, 1 or 2, starting with 0, and obeying the following rule : the sequence is alternatively increasing and decreasing.

For $n = 1$ we have $u_1 = 1$ since there is just the sequence (0).

For $n = 2$ we have $u_2 = 2$ since there are two sequences, namely $(0, 1)$ and $(0, 2)$.

For $n = 3$ we have $u_3 = 3$ since there are three sequences, namely $(0, 1, 0)$, $(0, 2, 1)$ and $(0, 2, 0)$.

For $n = 4$ we have $u_4 = 5$ since there are five sequences, namely

$$(0, 1, 0, 1), \quad (0, 1, 0, 2), \quad (0, 2, 1, 2), \quad (0, 2, 0, 1), \quad (0, 2, 1, 2).$$

Sequences of 0, 1 and 2

We found $u_n = F_{n+1}$ for $n = 1, 2, 3, 4$. Let us check this formula for $n \geq 5$ as well, by induction on n.

For n odd, an admissible sequence ends with 0 or 1. For n even, it ends with 1 or 2.

Denote by v_n the number of sequences of length n ending with 0 or 2 :

$$v_1 = 1, \quad v_2 = 1, \quad v_3 = 2, \quad v_4 = 3.$$

Sequences of 0, 1 and 2

For n even, we obtain all sequences of length n ending with 2 as follows :
• we consider the sequences of length $n-1$ ending with 0 and we complete with 2
• we consider the sequences of length $n-1$ ending with 1 and we complete with 2

The number of sequences of length n ending with 02 is v_{n-1}.

A sequence ending with 12 ends with 212. The number of sequences of length n ending with 212 is v_{n-2}.

This gives $v_n = v_{n-1} + v_{n-2}$ for n even.

The same proof gives the result also for n odd.

Hence $v_n = F_n$ for $n \geq 1$.

Sequences of 0, 1 and 2

Denote by w_n the number of sequences of length n ending with 1 :

$$w_1 = 0, \quad w_2 = 1, \quad w_3 = 1, \quad w_4 = 2.$$

A sequence of length n ending with 1 ends with 21 if n is odd, with 01 if n is even. Hence $w_n = v_{n-1}$.

Therefore $w_n = F_{n-1}$ for $n \geq 1$.

Sequences of 0, 1 and 2

Finally we have $u_n = v_n + w_n$.
Hence

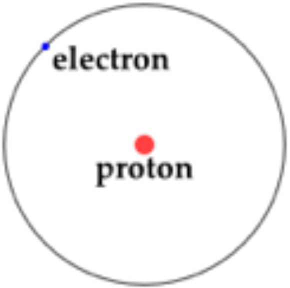
$$u_n = F_n + F_{n-1} = F_{n+1}.$$

Reflection of the ray of light

Give a label to the three glasses : 0, 1 and 2.
To each path associate the sequence of 0, 1 and 2 starting with 0 followed by the labels of the glasses where the ray reflects.
One deduces $p_n = u_{n+1}$. Hence $p_n = F_{n+2}$ for $n \geq 0$.

Levels of energy of an electron of an atom of hydrogen

An atom of hydrogen can have three levels of energy, 0 at the ground level when it does not move, 1 or 2. At each step, it **alternatively** gains and looses some level of energy, either 1 or 2, without going sub 0 nor above 2. Let ℓ_n be the number of different possible scenarios for this electron after n steps.



In general, $\ell_n = F_{n+2}$.

We have $\ell_0 = 1$ (initial state level 0)

$\ell_1 = 2$: state 1 or 2, scenarios (ending with gain) 01 or 02.

$\ell_2 = 3$: scenarios (ending with loss) 010, 021 or 020.

$\ell_3 = 5$: scenarios (ending with gain) 0101, 0102, 0212, 0201 or 0202.

Electron of the atom of hydrogen

Recall that u_n denotes the number of sequences of n elements, each of them is 0, 1 or 2, starting with 0, and obeying the following rule : the sequence is alternatively increasing and decreasing.

From the definition of u_n we deduce $\ell_n = u_{n+1}$.

Hence $\ell_n = F_{n+2}$.

Rhythmic patterns

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody. Several Indian scholars, Pingala (200 BC), Virahanka (c. 700 AD), Gopāla (c. 1135), and the Jain scholar Hemachandra (c. 1150). studied rhythmic patterns that are formed by concatenating one beat notes ● and double beat notes ■■.

one-beat note ● : short syllabe (`ti` in Morse Alphabet)

double beat note ■■ : long syllabe (`ta ta` in Morse)

1 beat, 1 pattern : ●

2 beats, 2 patterns : ● ● and ■■

3 beats, 3 patterns : ● ● ●, ● ■■ and ■■ ●

4 beats, 5 patterns :

● ● ● ●, ■■ ● ●, ● ■■ ●, ● ● ■■, ■■ ■■

n beats, F_{n+1} patterns.

Fibonacci sequence and Golden Ratio

The developments

$$[1], \quad [1,1], \quad [1,1,1], \quad [1,1,1,1], \quad [1,1,1,1,1], \quad [1,1,1,1,1,1], \dots$$

are the quotients

$$\begin{array}{cccccccc} \frac{F_2}{F_1} & \frac{F_3}{F_2} & \frac{F_4}{F_3} & \frac{F_5}{F_4} & \frac{F_6}{F_5} & \frac{F_7}{F_6} & \\ \| & \| & \| & \| & \| & \| & \cdots \\ \frac{1}{1} & \frac{2}{1} & \frac{3}{2} & \frac{5}{3} & \frac{8}{5} & \frac{13}{8} & \end{array}$$

of consecutive Fibonacci numbers.

The development $[1,1,1,1,1,\dots]$ is the continued fraction expansion of the *Golden Ratio*

$$\Phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1.618\,033\,988\,749\,894\dots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

The Fibonacci sequence and the Golden ratio

For $n \geq 0$, the Fibonacci number F_n is the nearest integer to

$$\frac{1}{\sqrt{5}}\Phi^n,$$

where Φ is the *Golden Ratio* :

$$\Phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

Binet's formula

For $n \geq 0$,

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},$$

Jacques Philippe Marie Binet
(1843)

$$\Phi = \frac{1+\sqrt{5}}{2}, \quad -\Phi^{-1} = \frac{1-\sqrt{5}}{2},$$

$$X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1}).$$

The so–called Binet Formula

Formula of A. De Moivre (1718, 1730), Daniel Bernoulli (1726), L. Euler (1728, 1765), J.P.M. Binet (1843) : for $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Abraham de Moivre (1667–1754)

Daniel Bernoulli (1700–1782)

Leonhard Euler (1707–1783)

Jacques P.M. Binet (1786–1856)

Generating series

A single series encodes all the Fibonacci sequence :

$$\sum_{n\geq 0} F_n X^n = X + X^2 + 2X^3 + 3X^4 + 5X^5 + \cdots + F_n X^n + \cdots$$

Fact : this series is the Taylor expansion of a rational fraction :

$$\sum_{n\geq 0} F_n X^n = \frac{X}{1 - X - X^2}.$$

`Proof` : the product

$$(X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots)(1 - X - X^2)$$

is a telescoping series

$$\begin{aligned} &X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots \\ &\quad -X^2 - X^3 - 2X^4 - 3X^5 - 5X^6 - \cdots \\ &\qquad -X^3 - X^4 - 2X^5 - 3X^6 - \cdots \\ &= X. \end{aligned}$$

Generating series of the Fibonacci sequence

Remark. The denominator $1 - X - X^2$ in the right hand side of

$$X + X^2 + 2X^3 + 3X^4 + \cdots + F_n X^n + \cdots = \frac{X}{1 - X - X^2}$$

is $X^2 f(X^{-1})$, where $f(X) = X^2 - X - 1$ is the irreducible polynomial of the Golden ratio Φ.

Homogeneous linear differential equation

Consider the homogeneous linear differential equation

$$y'' - y' - y = 0.$$

If $y = \mathrm{e}^{\lambda x}$ is a solution, from $y' = \lambda y$ and $y'' = \lambda^2 y$ we deduce

$$\lambda^2 - \lambda - 1 = 0.$$

The two roots of the polynomial $X^2 - X - 1$ are Φ (the Golden ratio) and Φ' with

$$\Phi' = 1 - \Phi = -\frac{1}{\Phi}.$$

A basis of the space of solutions is given by the two functions $\mathrm{e}^{\Phi x}$ and $\mathrm{e}^{\Phi' x}$. Since (Binet's formula)

$$\sum_{n\geq 0} F_n \frac{x^n}{n!} = \frac{1}{\sqrt{5}}\left(\mathrm{e}^{\Phi x} - \mathrm{e}^{\Phi' x}\right),$$

this exponential generating series of the Fibonacci sequence is a solution of the differential equation.

Fibonacci and powers of matrices

The Fibonacci linear recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ can be written

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}.$$

By induction one deduces, for $n \geq 0$,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An equivalent formula is, for $n \geq 1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ -1 & X - 1 \end{pmatrix} = X^2 - X - 1,$$

which is the irreducible polynomial of the Golden ratio Φ.

The Fibonacci sequence and the Golden ratio (continued)

For $n \geq 1$, $\Phi^n \in \mathbb{Z}[\Phi] = \mathbb{Z} + \mathbb{Z}\Phi$ is a linear combination of 1 and Φ with integer coefficients, namely

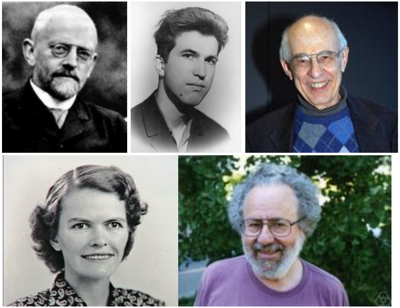
$$\Phi^n = F_{n-1} + F_n \Phi.$$

$$\begin{aligned}
\Phi &= 0 + \Phi \\
\Phi^2 &= 1 + \Phi \\
\Phi^3 &= 1 + 2\Phi \\
\Phi^4 &= 2 + 3\Phi \\
\Phi^5 &= 3 + 5\Phi \\
\Phi^6 &= 5 + 8\Phi \\
\Phi^7 &= 8 + 13\Phi \\
&\vdots
\end{aligned}$$

The Fibonacci sequence and Hilbert's 10th problem

Yuri Matiyasevich (1970) showed that there is a polynomial P in n, m, and a number of other variables $x, y, z, \ldots$ having the property that $n = F_{2m}$ iff there exist integers $x, y, z, \ldots$ such that $P(n, m, x, y, z, \ldots) = 0$.

This completed the proof of the impossibility of the tenth of Hilbert's problems (*does there exist a general method for solving Diophantine equations ?*) thanks to the previous work of Hilary Putnam, Julia Robinson and Martin Davis.



The Fibonacci Quarterly

The Fibonacci sequence satisfies a lot of very interesting properties. Four times a year, the *Fibonacci Quarterly* publishes an issue with new properties which have been discovered.


The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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VOLUME 50 AUGUST 2012 NUMBER 3

Narayana's cows

Narayana was an Indian mathematician in the 14th century who proposed the following problem :
A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year. How many calves are there altogether after, for example, 17 *years ?*

Narayana sequence https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.
It starts with

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277, \ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\dfrac{29+3\sqrt{93}}{2}} + \sqrt[3]{\dfrac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

Generating series and power of matrices

$$2 + 3X + 4X^2 + 6X^3 + \cdots + C_n X^n + \cdots = \frac{2 + X + X^2}{1 - X - X^3}.$$

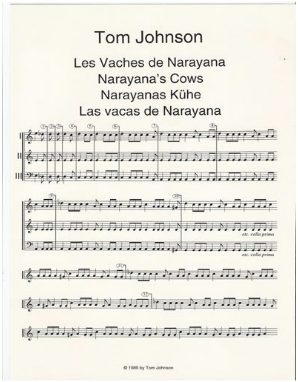
Differential equation : $y''' - y'' - y = 0$;
initial conditions : $y(0) = 2$, $y'(0) = 3$, $y''(0) = 4$.

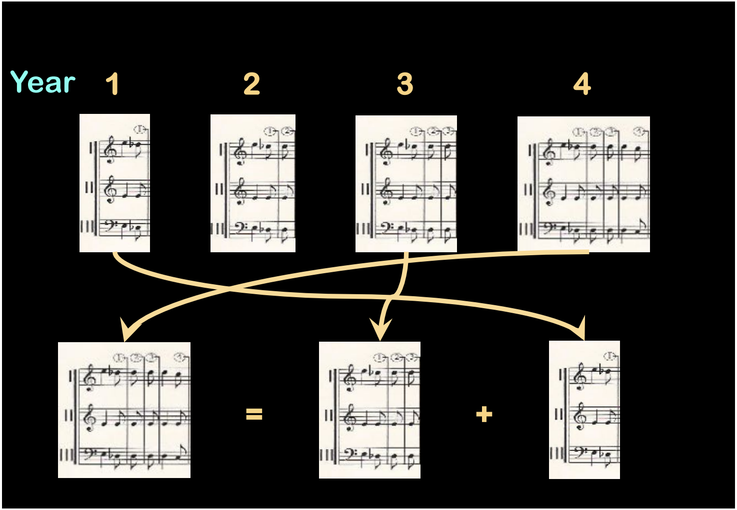
For $n \geq 0$,

$$\begin{pmatrix} C_n \\ C_{n+1} \\ C_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Music : http://www.pogus.com/21033.html

In working this out, Tom Johnson found a way to translate this into a composition called *Narayana's Cows.*
Music : Tom Johnson
Saxophones : Daniel Kientzy




Year
1
2
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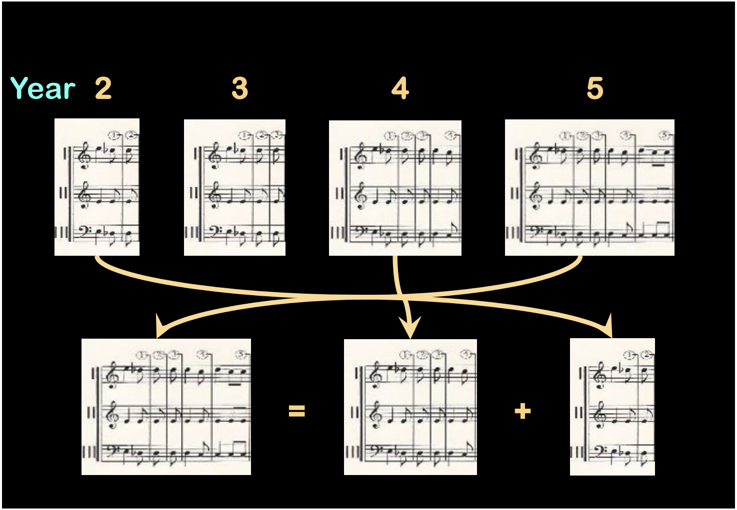
+


Year 2
3
4
5
=
+

Narayana's cows

http://www.math.jussieu.fr/~michel.waldschmidt/

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Original Cow	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Second generation	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Third generation	0	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105
Fourth generation	0	0	0	0	0	0	1	4	10	20	35	56	84	120	165	220	286
Fifth generation	0	0	0	0	0	0	0	0	0	1	5	15	35	70	126	210	330
Sixth generation	0	0	0	0	0	0	0	0	0	0	0	0	1	6	21	56	126
Seventh generation	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	7
Total	**2**	**3**	**4**	**6**	**9**	**13**	**19**	**28**	**41**	**60**	**88**	**129**	**189**	**277**	**406**	**595**	**872**


17th year: 872 cows

Jean-Paul Allouche and Tom Johnson

http://www.math.jussieu.fr/~jean-paul.allouche/
bibliorecente.html
http://www.math.jussieu.fr/~allouche/johnson1.pdf

Cows, music and morphisms

Jean-Paul Allouche and Tom Johnson

- Narayana's Cows and Delayed Morphisms
In 3èmes Journées d'Informatique Musicale (JIM '96), Ile de Tatihou, Les Cahiers du GREYC (1996 no. 4), pages 2-7, May 1996.
http://kalvos.org/johness1.html

- Finite automata and morphisms in assisted musical composition,
Journal of New Music Research, no. 24 (1995), 97 – 108.
http://www.tandfonline.com/doi/abs/10.1080/09298219508570676
http://web.archive.org/web/19990128092059/www.swets.nl/jnmr/vol24_2.html

Music and the Fibonacci sequence

- Dufay, XVème siècle
- Roland de Lassus
- Debussy, Bartok, Ravel, Webern
- Stockhausen
- Xenakis
- **Tom Johnson** *Automatic Music for six percussionists*

Fibonacci numbers with odd indices

The sequence of Fibonacci numbers with odd indices is

$$F_1 = 1,\ F_3 = 2,\ F_5 = 5,\ F_7 = 13,\ F_9 = 34,\ F_{11} = 89,\ \dots$$

$$1,\ 2,\ 5,\ 13,\ 34,\ 89,\ 233,\ 610,\ 1597, 4181,\ 10946,\ 28657,\ \dots$$

They produce solutions of a special case of the Markoff equation

$$x^2 + y^2 + 1 = 3xy.$$

with $x = F_{m-1}$ and $y = F_{m+1}$:

$1^2 + 2^2 + 1 = 3 \cdot 1 \cdot 2$,
$2^2 + 5^2 + 1 = 3 \cdot 2 \cdot 5$,
$5^2 + 13^2 + 1 = 3 \cdot 5 \cdot 13$, ...

The sequence of Markoff numbers

A *Markoff number* is a positive integer z such that there exist two positive integers x and y satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$

For instance 1 is a Markoff number, since $(x, y, z) = (1, 1, 1)$ is a solution.

Andrei Andreyevich Markoff
(1856–1922)

Photos :
http://www-history.mcs.st-andrews.ac.uk/history/

The On-Line Encyclopedia of Integer Sequences

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461, 37666, 43261, 51641, 62210, 75025, 96557, 135137, 195025, 196418, 294685, ...

The sequence of Markoff numbers is available on the web

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

Neil J. A. Sloane

http://oeis.org/A002559

Integer points on a surface

Given a Markoff number z, there exist infinitely many pairs of positive integers x and y satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$

This is a cubic equation in the 3 variables (x, y, z), of which we know a solution $(1, 1, 1)$.

There is an algorithm producing all integer solutions.

Markoff's cubic variety

The surface defined by Markoff's equation

$$x^2 + y^2 + z^2 = 3xyz.$$

is an algebraic variety with many automorphisms : permutations of the variables, changes of signs and

$$(x, y, z) \mapsto (3yz - x, y, z).$$

A.A. Markoff (1856–1922)

Algorithm producing all solutions

Let (m, m_1, m_2) be a solution of Markoff's equation :

$$m^2 + {m_1}^2 + {m_2}^2 = 3mm_1m_2.$$

Fix two coordinates of this solution, say m_1 and m_2. We get a quadratic equation in the third coordinate m, of which we know a solution, hence, the equation

$$x^2 + {m_1}^2 + {m_2}^2 = 3xm_1m_2.$$

has two solutions, $x = m$ and, say, $x = m'$, with $m + m' = 3m_1m_2$ and $mm' = {m_1}^2 + {m_2}^2$. This is the *cord and tangente process.*

Hence, *another* solution is (m', m_1, m_2) with $m' = 3m_1m_2 - m$.

Three solutions derived from one

Starting with one solution (m, m_1, m_2), we derive three *new* solutions :

$$(m', m_1, m_2), \quad (m, m_1', m_2), \quad (m, m_1, m_2').$$

If the solution we start with is $(1, 1, 1)$, we produce only one new solution, $(2, 1, 1)$ (up to permutation).

If we start from $(2, 1, 1)$, we produce only two *new* solutions, $(1, 1, 1)$ and $(5, 2, 1)$ (up to permutation).

A *new* solution means *distinct from the one we start with.*

New solutions

We shall see that any solution different from $(1,1,1)$ and from $(2,1,1)$ yields three new different solutions – and we shall see also that, in each other solution, the three numbers m, m_1 and m_2 are pairwise distinct.

Two solutions are called *neighbors* if they share two components.

For instance

- $(1,1,1)$ has a single neighbor, namely $(2,1,1)$,
- $(2,1,1)$ has two neighbors : $(1,1,1)$ et $(5,2,1)$,
- any other solution has exactly three neighbors.

Markoff's tree

Assume we start with (m, m_1, m_2) satisfying $m > m_1 > m_2$. We shall check

$$m_2' > m_1' > m > m'.$$

We order the solution according to the largest coordinate. Then two of the neighbors of (m, m_1, m_2) are larger than the initial solution, the third one is smaller.

Hence, if we start from $(1, 1, 1)$, we produce infinitely many solutions, which we organize in a tree : this is *Markoff's tree*.

This algorithm yields all the solutions

Conversely, starting from any solution other than $(1, 1, 1)$, the algorithm produces a *smaller* solution.

Hence, by induction, we get a sequence of smaller and smaller solutions, until we reach $(1, 1, 1)$.

Therefore the solution we started from was in Markoff's tree.

First branches of Markoff's tree

Markoff's tree starting from $(2, 5, 29)$

Markoff's tree up to $100\,000$

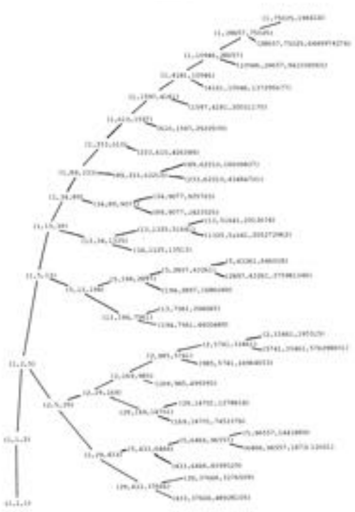


FIGURE 2
Markoff triples (p, q, r) *with* $\max(p, q) < 100000$

Don Zagier,
On the number of Markoff numbers below a given bound.
Mathematics of Computation,
39 160 (1982), 709–723.

Markoff's Conjecture

The previous algorithm produces the sequence of Markoff numbers. Each Markoff number occurs infinitely often in the tree as one of the components of the solution.

According to the definition, for a Markoff number $m > 2$, there exist a pair (m_1, m_2) of positive integers with $m > m_1 > m_2$ such that $m^2 + {m_1}^2 + {m_2}^2 = 3mm_1m_2$.

Question : *Given* m*, is such a pair* (m_1, m_2) *unique ?*

The answer is yes, as long as $m \leq 10^{105}$.

The Fibonacci sequence and the Markoff equation

The smallest Markoff number is 1. When we impose $z = 1$ in the Markoff equation $x^2 + y^2 + z^2 = 3xyz$, we obtain the equation

$$x^2 + y^2 + 1 = 3xy.$$

Going along the Markoff's tree starting from $(1, 1, 1)$, we obtain the subsequence of Markoff numbers

$$1,\ 2,\ 5,\ 13,\ 34,\ 89,\ 233,\ 610,\ 1597, 4181,\ 10946,\ 28657,\ \dots$$

which is the sequence of Fibonacci numbers with odd indices

$$F_1 = 1,\ F_3 = 2,\ F_5 = 5,\ F_7 = 13,\ F_9 = 34,\ F_{11} = 89,\ \dots$$

Fibonacci numbers with odd indices

Fibonacci numbers with odd indices are Markoff's numbers :

$$F_{m+3}F_{m-1} - F_{m+1}^2 = (-1)^m \quad \text{for} \quad m \geq 1$$

and

$$F_{m+3} + F_{m-1} = 3F_{m+1} \quad \text{for} \quad m \geq 1.$$

Set $y = F_{m+1}$, $x = F_{m-1}$, $x' = F_{m+3}$, so that, for even m,

$$x + x' = 3y, \quad xx' = y^2 + 1$$

and

$$X^2 - 3yX + y^2 + 1 = (X - x)(X - x').$$

A computer should not be a Black Box

Computers will play an increasing role everywhere.
You need to understand fully all what they are doing.

IBM Releases “Black Box” Breaker on IBM Cloud

https://www.cbronline.com/news/ai-bias-ibm

$$u_0 = 1,\ u_1 = (1-\sqrt{5})/2, \qquad u_n = u_{n-1} + u_{n-2}$$

Question : compute u_{100}.

Pierre Arnoux

$$\frac{1-\sqrt{5}}{2} = -0.61803398874989484820458683436 5\ldots$$

https://oeis.org/A001622

http://iml.univ-mrs.fr/~arnoux/

Excel file Column A : n Column B : u_n

$u_0 = 1$, $u_1 = (1-\sqrt{5})/2$, $u_n = u_{n-1} + u_{n-2}$

	A	B
1	0	1
2	1	`=(1-RACINE(5))/2`

	A	B
1	0	1
2	1	-0.618034

	A	B
1	0	1
2	1	-0.618034
3	=1+A2	=B1+B2

	A	B
1	0	1
2	1	-0.618034
3	2	0.38196601

Copy $A3$ $B3$ down

Excel file : u_1 to u_{39}

1	-0,61803399
2	0,381966011
3	-0,23606798
4	0,145898034
5	-0,09016994
6	0,05572809
7	-0,03444185
8	0,021286236
9	-0,01315562
10	0,008130619
11	-0,005025
12	0,00310562
13	-0,00191938
14	0,001186241
15	-0,00073314
16	0,000453104
17	-0,00028003
18	0,00017307
19	-0,00010696
20	6,6107E-05
21	-4,0856E-05
22	2,52506E-05
23	-1,5606E-05
24	9,64487E-06
25	-5,9609E-06
26	3,68401E-06
27	-2,2769E-06
28	1,40715E-06
29	-8,6971E-07
30	5,37445E-07
31	-3,3226E-07
32	2,05185E-07
33	-1,2708E-07
34	7,8109E-08
35	-4,8967E-08
36	2,91423E-08
37	-1,9824E-08
38	9,31784E-09
39	-1,0507E-08

Excel file : u_1 to u_{39}

1	-0,61803399
2	0,381966011
3	-0,23606798
4	0,145898034
5	-0,09016994
6	0,05572809
7	-0,03444185
8	0,021286236
9	-0,01315562
10	0,008130619
11	-0,005025
12	0,00310562
13	-0,00191938
14	0,001186241
15	-0,00073314
16	0,000453104
17	-0,00028003
18	0,00017307
19	-0,00010696

20	6,6107E-05
21	-4,0856E-05
22	2,52506E-05
23	-1,5606E-05
24	9,64487E-06
25	-5,9609E-06
26	3,68401E-06
27	-2,2769E-06
28	1,40715E-06
29	-8,6971E-07
30	5,37445E-07
31	-3,3226E-07
32	2,05185E-07
33	-1,2708E-07
34	7,8109E-08
35	-4,8967E-08
36	2,91423E-08
37	-1,9824E-08
38	9,31784E-09
39	-1,0507E-08

Exact value of u_n

Observations : The signs of u_n alternate, the absolute value is decreasing.

Set $\widetilde{\Phi} = (1 - \sqrt{5})/2$. Notice that $\widetilde{\Phi}$ is a root of $X^2 - X - 1$, the other root is $\Phi = (1 + \sqrt{5})/2$, the golden ratio.

From $\widetilde{\Phi}^n = \widetilde{\Phi}^{n-1} + \widetilde{\Phi}^{n-2}$ with $u_0 = 1$, $u_1 = \widetilde{\Phi}$, we deduce by induction $u_n = \widetilde{\Phi}^n$.

Exact value of u_{39}

Numerical values :

$$\widetilde{\Phi} = -0.618\,033\,988\,749\,894\ldots$$

$$\log|\widetilde{\Phi}| = -0.481\,211\,825\,059\,603\,4\ldots$$

$$u_{39} = -\widetilde{\Phi}^{39} = -\mathrm{e}^{-18.767\,261\,177\,324,453\ldots} = -7.071\,019\ldots 10^{-9}.$$

PARI GP : https://pari.math.u-bordeaux.fr/

Comparing the excel values with the exact values

	excel value	exact value
30	5,37445E-07	5,3749E-07
31	-3,32261E-07	-3,32187E-07
32	2,05185E-07	2,05303E-07
33	-1,27076E-07	-1,26884E-07
34	7,8109E-08	7,84188E-08
35	-4,89667E-08	-4,84655E-08
36	2,91423E-08	2,99533E-08
37	-1,98244E-08	-1,85122E-08
38	9,31784E-09	1,14411E-08
39	-1,05066E-08	-7,07102E-09
40	-1,18878E-09	4,37013E-09
41	-1,16954E-08	-2,70089E-09

Exact value of u_{100}

The answer to initial question is

$$u_{100} = \widetilde{\Phi}^{100}$$

$$\widetilde{\Phi} = -0.618\,033\,988\,749\,894\ldots, \; \log|\widetilde{\Phi}| = -0.481\,211\,825\,059\,603\,4\ldots$$

$$\widetilde{\Phi}^{100} = \mathrm{e}^{-48.121\,182\,505\,960\,34\ldots} = 1.262\,513\,338\,064\ldots 10^{-21}.$$

Excel (continued)

$$u_{100} = -19\,241.901\,833\,167\ldots$$

38	9,31784E-09
39	-1,05066E-08
40	-1,18878E-09
41	-1,16954E-08
42	-1,28842E-08
43	-2,45796E-08
44	-3,74637E-08
45	-6,20433E-08
46	-9,9507E-08
47	-1,6155E-07
48	-2,61057E-07
49	-4,22608E-07
50	-6,83665E-07
51	-1,10627E-06
52	-1,78994E-06

85	-14,10695857
86	-22,82553845
87	-36,93249702
88	-59,75803546
89	-96,69053248
90	-156,4485679
91	-253,1391004
92	-409,5876684
93	-662,7267688
94	-1072,314437
95	-1735,041206
96	-2807,355643
97	-4542,396849
98	-7349,752492
99	-11892,14934
100	-19241,90183

The linear recurrence sequence $u_n = u_{n-1} + u_{n-2}$

From the two solutions Φ^n and $\widetilde{\Phi}^n$ one deduces that any solution is of the form $u_n = a\Phi^n + b\widetilde{\Phi}^n$.

Since $|\Phi| > 1$, the term Φ^n tends to ∞.

Since $|\widetilde{\Phi}| < 1$, the term $b\widetilde{\Phi}^n$ tends to 0.

If $a \neq 0$, then $|u_n|$ tends to infinity like $a\Phi^n$.

If $a = 0$, then $u_n = b\widetilde{\Phi}^n$ tends to 0.

If two consecutive terms are of the same sign, then all the next ones have the same sign and $|u_n|$ tends to infinity.

Two computers may give different answers

One of the objectives of the *Aric* project (Arithmetic and Computing)

http://www.ens-lyon.fr/LIP/AriC/

is to build correctly rounded mathematical function programs.

The IEEE 754-2008 standard

https://en.wikipedia.org/wiki/IEEE_754

specifies the behavior of floating-point arithmetic. This standard defines rounding rules : properties to be satisfied when rounding numbers during arithmetic and conversions.

Institute of Electrical and Electronics Engineers (IEEE).

Decimal expansion of real numbers

A real number has a decimal expansion

$a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0 + b_1 10^{-1} + b_2 10^{-2} + \cdots$

where the digits a_i and b_j belong to $\{0, 1, \ldots 9\}$.
Any sequence of digits defines a real number, but some numbers have two decimal expansions, namely the rational numbers with denominator a power of 10.
From the relation

$$1 + a + a^2 + a^3 + \cdots + a^m + \cdots = \frac{1}{1-a}$$

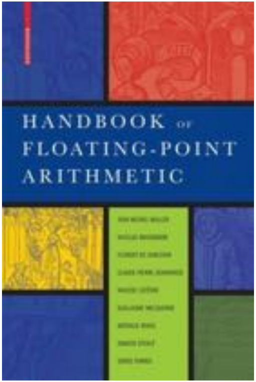
which is valid for $-1 < a < 1$ we deduce

$$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots \frac{1}{10^m} + \cdots = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9},$$

hence

$$0.999\,999\,999 \cdots = 1.$$

Handbook of floating-point arithmetic



Jean-Michel Muller, Nicolas Brisebarre, Florent de Dinechin, Claude-Pierre Jeannerod, Vincent Lefèvre, Guillaume Melquiond, Nathalie Revol, Damien Stehlé, Serge Torres. *Handbook of floating-point arithmetic.* Birkhäuser Basel, 2010.

Y. V. Nesterenko and M. Waldschmidt. On the approximation of the values of exponential function and logarithm by algebraic numbers (in Russian). Mat. Zapiski, 2 :23–42, 1996. Available in English at http://www.math.jussieu.fr/~miw/articles/ps/Nesterenko.ps

Connection with Diophantine approximation

Many functions considered in the IEEE 754-2008 standard are transcendental, including the exponentials, logarithms, trigonometric functions, and inverse trigonometric functions.

The Table Maker's Dilemma.
Accurate rounding of transcendental mathematical functions is difficult because the number of extra digits that need to be calculated to resolve whether to round up or down cannot be known in advance.
https://en.wikipedia.org/wiki/Rounding

The Table Maker's Dilemma for the exponential function

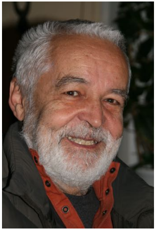
Let α be a precision-p floating-point number in $[1,2]$. The exact value $\exp(\alpha)$ belongs to the interval $[\mathrm{e},\mathrm{e}^2)$. We now use the theorem of Nesterenko and Waldschmidt with $E = \mathrm{e} = 2.7182818\ldots$ and $\theta = \alpha'$, where α' is any precision-p floating-point number in $[1,6)$. We obtain the following :

$$|\mathrm{e}^{\alpha'} - \alpha| \geq 2^{-688p^2 - 992p\log(p+1) - 67514p - 71824\log(p+1) - 1283614}.$$

Reference : *Handbook of floating-point arithmetic*, § 12.4. Solving the Table Maker's Dilemma for Arbitrary Functions, p. 431.

$|e^b - a|$ for a and b rational integers

Kurt Mahler
(1903 – 1988)



Maurice Mignotte

Franck Wielonsky

http://www-history.mcs.st-and.ac.uk/Biographies/Mahler.html
https://www.i2m.univ-amu.fr/perso/franck.wielonsky/

$|e^b - a|$ for a and b rational integers

K. Mahler noticed that an integer power of e is never an integer, since e is transcendental. Hence when a and b are rational integers, we have $e^b \neq a$.

Mahler obtained a lower bound for $|e^b - a|$ in 1953 and 1967. His estimates were improved by M. Mignotte (1974), and later by F. Wielonsky (1997). The sharpest known estimate is

$$|e^b - a| > b^{-20b}.$$

$|e^b - a|$ for a and b rational integers

Mahler asked whether there exists an absolute constant $c > 0$ such that, for a and b positive integers,

$$|e^b - a| > a^{-c}?$$

This is not yet solved. He also noticed that the inequality

$$|b - \log a| < \frac{1}{a}$$

has infinitely many solutions in positive integers a and b. Indeed, if a denotes the integral part of e^b, then we have

$$0 < e^b - a < 1, \qquad 0 < a(b - \log a) < e^b - a < e^b(b - \log a),$$

hence

$$0 < b - \log a < \frac{e^b - a}{a} < \frac{1}{a}\cdot$$

Mahler's conjecture

Mahler's conjecture arises by considering the numbers $\log a - b_a$ for $a = 1, \ldots, A$, where b_a is the nearest integer to $\log a$, for growing values of A, and assuming that these numbers are more or less evenly distributed in the interval $(-1/2, 1/2)$.

Mahler's conjecture is equivalent to the existence of a constant $c > 0$ such that, for a and b positive integers,

$$|\mathrm{e}^b - a| > \mathrm{e}^{-cb}.$$

Stronger conjecture

I suggest that the numbers $\mathrm{e}^b - a_b$ for $b = 1, \ldots, B$, for growing values of B, are evenly distributed in the interval $(-1/2, 1/2)$, where a_b is the nearest integer to e^b. This amounts to suggest the stronger conjecture that there exists a constant $c > 0$ for which

$$|\mathrm{e}^b - a| > b^{-c}.$$

This conjecture is equivalent to the existence of a constant $c > 0$ for which

$$|\mathrm{e}^b - a| > \frac{1}{a(\log a)^c}.$$

$|e^b - a|$ for a and b rational numbers

Define $H(p/q) = \max\{|p|, q\}$.

Then for a and b in $\mathbb{Q}$ with $b \neq 0$, the estimate is

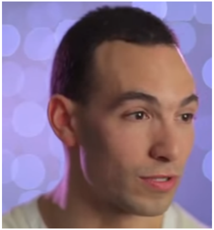
$$|e^b - a| \geq \exp\{-1,3 \cdot 10^5 (\log A)(\log B)\}$$

where $A = \max\{H(a),\ A_0\}$, $B = \max\{H(b),\ 2\}$.

Yu. V. Nesterenko & M. Waldschmidt – *On the approximation of the values of exponential function and logarithm by algebraic numbers*. (In russian) Mat. Zapiski, **2** *Diophantine approximations, Proceedings of papers dedicated to the memory of Prof. N. I. Feldman*, ed. Yu. V. Nesterenko, Centre for applied research under Mech.-Math. Faculty of MSU, Moscow (1996), 23–42.
http://fr.arXiv.org/abs/math/0002047

$|e^b - a|$ for a and b rational numbers

A refinement of our estimate has been obtained in
SAMY KHÉMIRA & PAUL VOUTIER.
Diophantine approximation and Hermite-Padé approximants of type I of exponential functions.
Ann. Sci. Math. Québec 35 (2011), no. 1, 85–116.

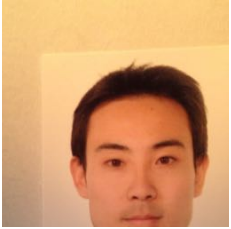


Samy Khemira

Paul Voutier

https://www.youtube.com/watch?v=1WnoyYPu65g
Parlons Passion : Samy donne des cours aux enfants hospitalisés

$|e^b - a|$ for a and b rational numbers



Makoto Kawashima

Makoto Kawashima,
Linear independence of values of logarithms revisited,
April 3, 2019
https://arxiv.org/abs/1904.01737

New lower bound for linear form in

$$1, \log(1+\alpha), \ldots, \log^{m-1}(1+\alpha)$$

with algebraic integer coefficients in both complex and p–adic case. Refinement of the result of Nesterenko-Waldschmidt on the lower bound of linear form in certain values of power of logarithms.

Further applications of Diophantine Approximation

Hua Loo Keng & Wang Yuan – *Application of number theory to numerical analysis*, Springer Verlag (1981).

Hua Loo Keng
(1910 – 1985)

Wang Yuan

Further applications of Diophantine Approximation include equidistribution modulo 1, discrepancy, numerical integration, interpolation, approximate solutions to integral and differential equations.

http://www-history.mcs.st-and.ac.uk/Biographies/Hua.html

http://www-history.mcs.st-and.ac.uk/PictDisplay/Wang_Yuan.html

The square root of 2, the Golden Ratio and the Fibonacci sequence

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/