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On the Brahmagupta–Fermat–Pell Equation $x^2 - dy^2 = \pm 1$

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On the Brahmagupta–Fermat–Pell equation

The equation $x^2 - dy^2 = \pm 1$, where the unknowns x and y are positive integers while d is a fixed positive integer which is not a square, has been mistakenly called with the name of Pell by Euler. It was investigated by Indian mathematicians since Brahmagupta (628) who solved the case $d = 92$, next by Bhāskara II (1150) for $d = 61$ and Narayaṇa (during the 14-th Century) for $d = 103$. The smallest solution of $x^2 - dy^2 = 1$ for these values of d are respectively

$$1\,151^2 - 92 \cdot 120^2 = 1, \qquad 1\,766\,319\,049^2 - 61 \cdot 226\,153\,980^2 = 1$$

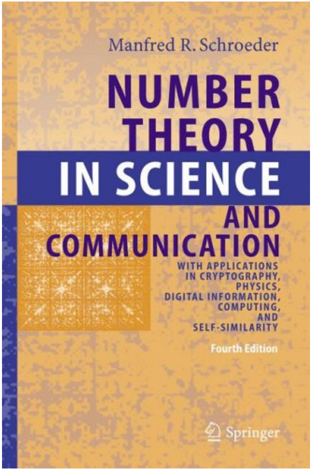
and

$$227\,528^2 - 103 \cdot 22\,419^2 = 1,$$

hence they have not been found by a brute force search !
We give a short introduction to this long story including the Archimedes Bovinum Problem.

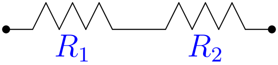
Number Theory in Science and communication

M.R. Schroeder.
Number theory in science and communication :
with applications in cryptography, physics, digital information, computing and self similarity
Springer series in information sciences **7** 1986.
4th ed. (2006) 367 p.



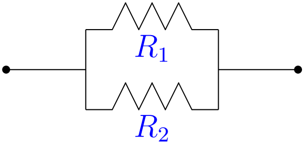
Electric networks

- The resistance of a network in series



is the sum $R_1 + R_2$.

- The resistance R of a network in parallel

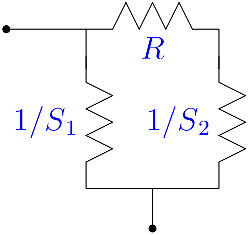


satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Electric networks and continued fractions

The resistance U of the circuit

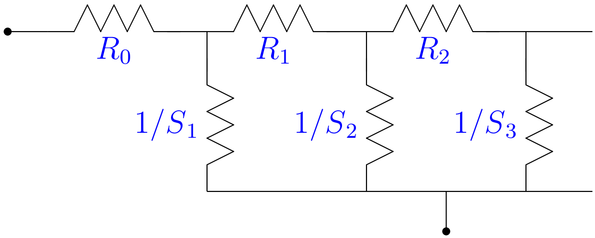


is given by

$$U = \cfrac{1}{S_1 + \cfrac{1}{R + \cfrac{1}{S_2}}} := [0, S_1, R, S_2].$$

A circuit for a continued fraction expansion

- For the network



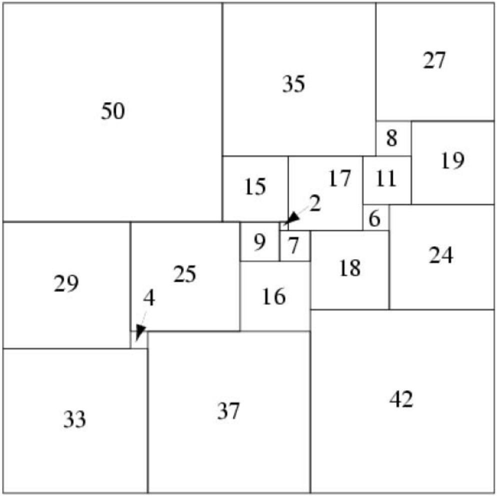
the resistance is given by a continued fraction expansion

$$R_0 + \cfrac{1}{S_1 + \cfrac{1}{R_1 + \cfrac{1}{S_2 + \cfrac{1}{\ddots}}}} := [R_0, S_1, R_1, S_2, R_2, \dots]$$

Decomposition of a square in squares

Electric networks and continued fractions have been used to find the first solution to the problem of decomposing an integer square into a disjoint union of integer squares, all of which are distinct.

Squaring the square



21-*square perfect square*

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above.

An interesting street number

The puzzle itself was about a street in the town of Louvain in Belgium, where houses are numbered consecutively. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500.

Prasanta Chandra Mahalanobis

1893 – 1972

Srinivasa Ramanujan

1887 – 1920

http://mathshistory.st-andrews.ac.uk/Biographies/Mahalanobis.html

https://www.math.auckland.ac.nz/~butcher/miniature/miniature2.pdf

Street number : examples

Examples :

- House number 6 in a street with 8 houses :

$$1+2+3+4+5=15, \quad 7+8=15.$$

- House number 35 in a street with 49 houses. To compute

$$S := 1+2+3+\cdots+32+33+34$$

write

$$S = 34+33+32+\cdots+3+2+1$$

so that $2S = 34 \times 35$:

$$1+2+3+\cdots+34 = \frac{34 \times 35}{2} = 595.$$

On the other side of the house,

$$36+37+\cdots+49 = \frac{49 \times 50}{2} - \frac{35 \times 36}{2} = 1225 - 630 = 595.$$

Other solutions to the puzzle

- House number 1 in a street with 1 house.
- House number 0 in a street with 0 house.

Ramanujan : *if no banana is distributed to no student, will each student get a banana ?*

The puzzle requests the house number between 50 and 500.

Street number

Let m be the house number and n the number of houses :

$$1 + 2 + 3 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + n.$$

$$\frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

This is $2m^2 = n(n+1)$. Complete the square on the right :

$$8m^2 = (2n+1)^2 - 1.$$

Set $x = 2n+1$, $y = 2m$. Then

$$x^2 - 2y^2 = 1.$$

Infinitely many solutions to the puzzle

Ramanujan said he has infinitely many solutions (but a single one between 50 and 500).
Sequence of balancing numbers (number of the house)
https://oeis.org/A001109

$$0, 1, 6, 35, \mathbf{204}, 1189, 6930, 40391, 235416, 1372105, 7997214 \ldots$$

This is a linear recurrence sequence $u_{n+1} = 6u_n - u_{n-1}$ with the initial conditions $u_0 = 0$, $u_1 = 1$.

The number of houses is https://oeis.org/A001108

$$0, 1, 8, 49, \mathbf{288}, 1681, 9800, 57121, 332928, 1940449, \ldots$$

OEIS

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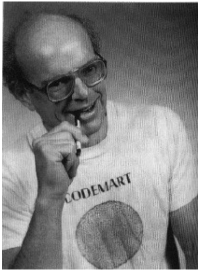
THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane



Neil J. A. Sloane's encyclopaedia

http://oeis.org/A001597

Brahmagupta (598 – 670)

Brāhmasphuṭasiddhānta (628) :

$$x^2 - 92y^2 = 1$$

Brahmagupta

The smallest solution is

$$x = 1151, \qquad y = 120.$$

Composition method : *samasa* – Brahmagupta identity

$$(a^2 - db^2)(x^2 - dy^2) = (ax + dby)^2 - d(ay + bx)^2.$$

http://mathworld.wolfram.com/BrahmaguptasProblem.html

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pell.html

Bhāskara II or Bhāskarāchārya (1114 - 1185)

Lilavati Ujjain (India)

(*Bijagaṇita*, 1150)

$$x^2 - 61y^2 = 1$$

$x = 1\,766\,319\,049, \qquad y = 226\,153\,980.$

Cyclic method (*Chakravala*) : produce a solution to Pell's equation $x^2 - dy^2 = 1$ starting from a solution to $a^2 - db^2 = k$ with a *small* k.

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pell.html

Narayaṇa Paṇḍit ∼ 1340 – ∼ 1400

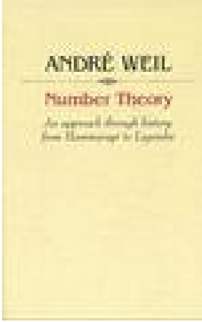
Author of *Gaṇita Kaumudi* on arithmetic in 1356.
Narayaṇa cows (*Tom Johnson*)

$$x^2 - 103y^2 = 1$$

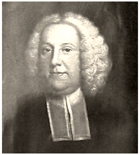
$$x = 227\,528, \qquad y = 22\,419.$$

References to Indian mathematics

André Weil
Number theory :
An approach through history. From Hammurapi to Legendre.
Birkhäuser Boston, Inc., Boston, Mass., (1984) 375 pp.
MR 85c:01004



Pell's equation $x^2 - dy^2 = \pm 1$



John Pell

1610 – 1685

It is often said that Euler mistakenly attributed Brouncker's work on this equation to Pell. However the equation appears in a book by Rahn which was certainly written with Pell's help : some say entirely written by Pell. Perhaps Euler knew what he was doing in naming the equation.

Johann Rahn (1622 - 1676) was a Swiss mathematician who was the first to use the symbol $\div$ for division.

https://mathshistory.st-andrews.ac.uk/Biographies/Pell/
https://fr.wikipedia.org/wiki/John_Pell

On the equation $x^2 - dy^2 = \pm 1$: history

Lord William Brouncker

1620–1684

Pierre de Fermat

1601–1665

Correspondence from Pierre de Fermat to Brouncker.

1657 : letter of Fermat to Frenicle de Bessy (1604–1674).

https://mathshistory.st-andrews.ac.uk/Biographies/

History (continued)

Leonard Euler

1707–1783

Joseph–Louis Lagrange

1736–1813

L. Euler : *Book of algebra in 1770 + continued fractions*

The complete theory of the equation $x^2 - dy^2 = \pm 1$ was worked out by Lagrange.

https://mathshistory.st-andrews.ac.uk/Biographies/

Solution of the equation $x^2 - dy^2 = \pm 1$

Let d be a positive integer, not a square. Then the equation $x^2 - dy^2 = \pm 1$ has infinitely many non negative solutions in integers (x, y).

There is a smallest positive *fundamental solution* (x_1, y_1) such that all non negative solutions are obtained by writing

$$x_\nu + y_\nu \sqrt{d} = (x_1 + y_1 \sqrt{d})^\nu$$

with $\nu \geq 0$.

The trivial solution $(x, y) = (1, 0)$ is obtained with $\nu = 0$.

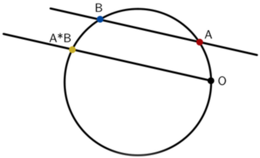
The set of solutions (x, y) in $\mathbf{Z} \times \mathbf{Z}$ is given by

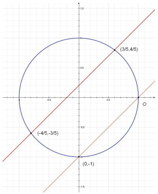
$$x_\nu + y_\nu \sqrt{d} = \pm (x_1 + y_1 \sqrt{d})^\nu$$

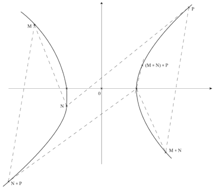
with $\nu \in \mathbf{Z}$. They form a group $\simeq \{\pm 1\} \times \mathbf{Z}$.

Group law on a conic

The curve $x^2 - dy^2 = 1$ is a conic, and on a conic there is a group law which can be described geometrically. The fact that it is associative is proved by using Pascal's Theorem.







Franz Lemmermeyer. Conics – a poor man's elliptic curves.
https://arxiv.org/pdf/math/0311306.pdf

Mahalanobis puzzle $x^2 - 2y^2 = 1$, $x = 2n + 1$, $y = 2m$

Fundamental solution : $(x_1, y_1) = (3, 2)$.
Other solutions (x_ν, y_ν) with

$$x_\nu + y_\nu\sqrt{2} = (3 + 2\sqrt{2})^\nu.$$

- $\nu = 0$, trivial solution : $x = 1$, $y = 0$, $m = n = 0$.
- $\nu = 1$, $x_1 = 3$, $y_1 = 2$, $m = n = 1$.
- $\nu = 2$, $x_2 = 17$, $y_2 = 12$, $n = 8$, $m = 6$,

$$x_2 + y_2\sqrt{2} = (3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}.$$

- $\nu = 3$, $x_3 = 99$, $y_3 = 70$, $n = 49$, $m = 35$,

$$x_3 + y_3\sqrt{2} = (3 + 2\sqrt{2})^3 = 99 + 70\sqrt{2}.$$

$+1$ or -1?

- If the fundamental solution ${x_1}^2 - d{y_1}^2 = \pm 1$ produces the $+$ sign, then the equation $x^2 - dy^2 = -1$ has no solution.

- If the fundamental solution ${x_1}^2 - d{y_1}^2 = \pm 1$ produces the $-$ sign, then the fundamental solution of the equation $x^2 - dy^2 = 1$ is (x_2, y_2) with $x_2 + y_2\sqrt{d} = (x_1 + y_1\sqrt{d})^2$, hence

$$x_2 = {x_1}^2 + d{y_1}^2, \qquad y_2 = 2x_1y_1.$$

The solutions of $x^2 - dy^2 = 1$ are the (x_ν, y_ν) with ν even, the solutions of $x^2 - dy^2 = -1$ are obtained with ν odd.

$x^2 - 2y^2 = -1$

The fundamental solution to $x^2 - 2y^2 = -1$ is $(1, 1)$.

We have $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$, which gives the fundamental solution $(3, 2)$ to $x^2 - 2y^2 = 1$.

The positive solutions (x, y) to $x^2 - 2y^2 = -1$ are given by $(1 + \sqrt{2})^\nu = x + y\sqrt{2}$ with $\nu \geq 1$ odd.

The positive solutions (x, y) to $x^2 - 2y^2 = 1$ are given by $(1 + \sqrt{2})^\nu = x + y\sqrt{2}$ with $\nu \geq 2$ even.

Algorithm for the fundamental solution

All the problem now is to find the fundamental solution.

Here is the idea. If x, y is a solution, then the equation $x^2 - dy^2 = \pm 1$, written as

$$\frac{x}{y} - \sqrt{d} = \pm \frac{1}{y(x + y\sqrt{d})},$$

shows that x/y is a good *rational approximation* to $\sqrt{d}$.

There is an algorithm for finding the *best* rational approximations of a real number : it is given by *continued fractions*.

The algorithm of continued fractions

Let $x \in \mathbf{R}$.

- Perform the Euclidean division of x by 1 :

$$x = \lfloor x \rfloor + \{x\} \quad \text{with } \lfloor x \rfloor \in \mathbf{Z} \text{ and } 0 \le \{x\} < 1.$$

- In case x is an integer, this is the end of the algorithm. If x is not an integer, then $\{x\} \neq 0$ and we set $x_1 = 1/\{x\}$, so that

$$x = \lfloor x \rfloor + \frac{1}{x_1} \quad \text{with } \lfloor x \rfloor \in \mathbf{Z} \text{ and } x_1 > 1.$$

- In the case where x_1 is an integer, this is the end of the algorithm. If x_1 is not an integer, then we set $x_2 = 1/\{x_1\}$:

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{x_2}} \quad \text{with } x_2 > 1.$$

Continued fraction expansion

Set $a_0 = \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor$ for $i \geq 1$.

- Then :

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{\lfloor x_2 \rfloor + \cfrac{1}{\ddots}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

The algorithm stops after finitely many steps if and only if x is rational.

- We shall use the notation

$$x = [a_0,\ a_1,\ a_2,\ a_3 \dots]$$

- Remark : if $a_k \geq 2$, then

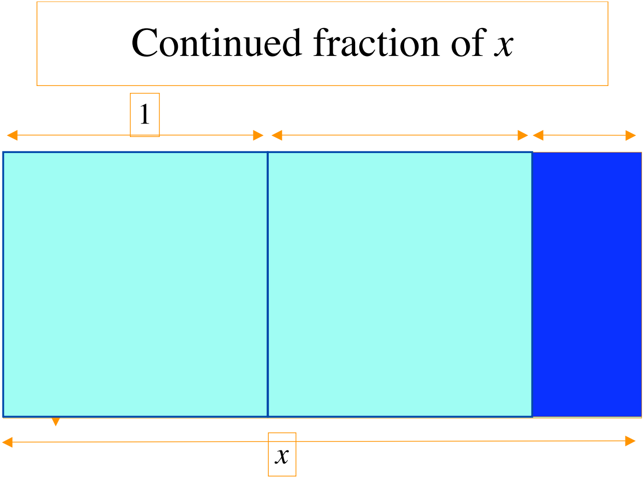
$[a_0,\ a_1,\ a_2,\ a_3, \dots, a_k] = [a_0,\ a_1,\ a_2,\ a_3, \dots, a_k - 1, 1]$.

Continued fraction expansion : geometric point of view

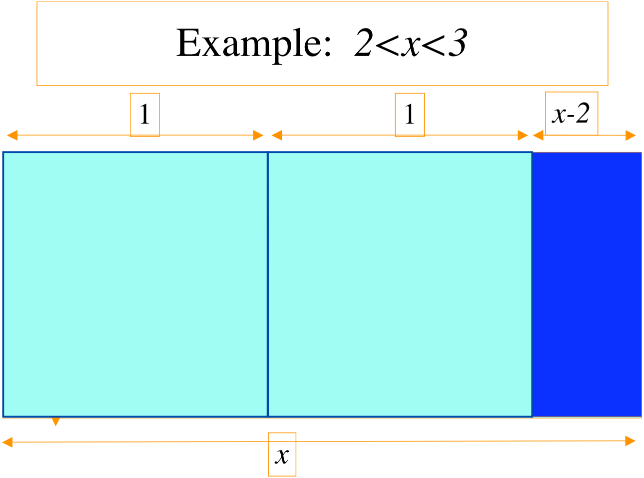
Start with a rectangle have side lengths 1 and x. The proportion is x.

Split it into $\lfloor x \rfloor$ squares with sides 1 and a smaller rectangle of sides $\{x\} = x - \lfloor x \rfloor$ and 1.

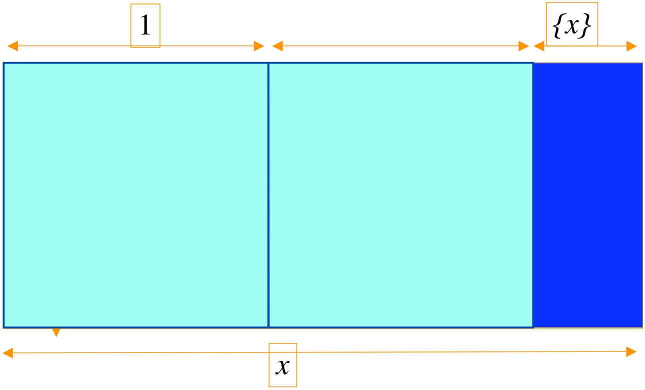
Rectangles with proportion x



Example : $2 < x < 3$



Number of squares : $a_0 = \lfloor x \rfloor$ with $x = \lfloor x \rfloor + \{x\}$



Continued fraction expansion : geometric point of view

Recall $x_1 = 1/\{x\}$

The small rectangle has side lengths in the proportion x_1.

Repeat the process : split the small rectangle into $\lfloor x_1 \rfloor$ squares and a third smaller rectangle, with sides in the proportion $x_2 = 1/\{x_1\}$.

This process produces the continued fraction expansion of x.

The sequence $a_0, a_1, \ldots$ is given by the number of squares at each step.

Example : the Golden Ratio

The Golden Ratio

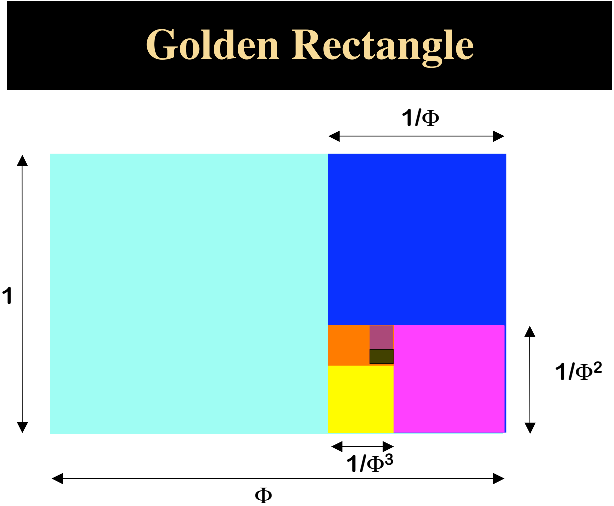
$$\Phi = \frac{1+\sqrt{5}}{2} = 1.6180339887499\ldots$$

satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

Hence if we start with a rectangle having for proportion the Golden Ratio, at each step we get one square and a remaining smaller rectangle with sides in the same proportion.

The Golden Ratio $(1+\sqrt{5})/2 = [1, 1, 1, 1 \ldots]$



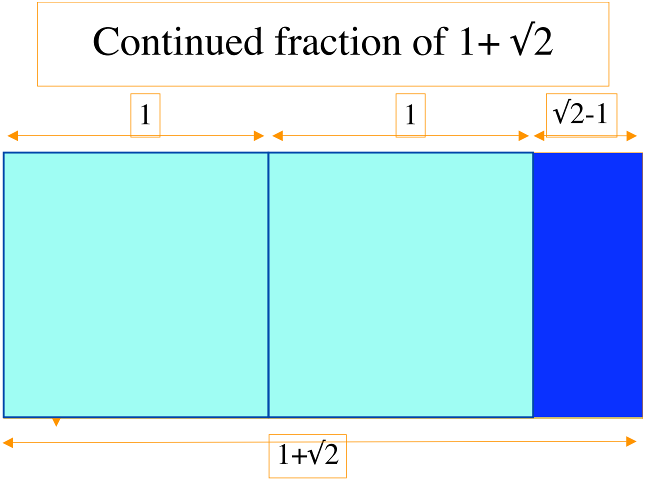
Rectangles with proportion $1+\sqrt{2}$

$$\sqrt{2} = 1.4142135623731\ldots$$

$$1+\sqrt{2} = 2 + \frac{1}{1+\sqrt{2}}$$

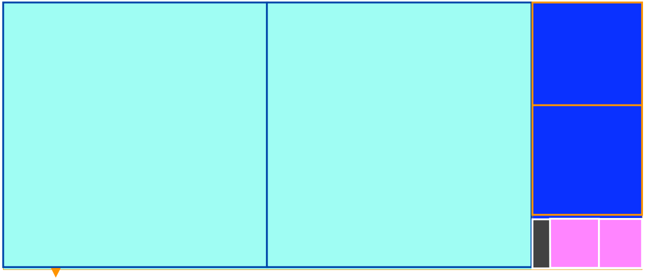
If we start with a rectangle having for proportion $1+\sqrt{2}$, at each step we get two squares and a remaining smaller rectangle with sides in the same proportion.

Rectangles with proportion $1 + \sqrt{2}$



Rectangles with proportion $1+\sqrt{2} = [2, 2, 2, 2 \dots]$

Continued fraction of $1+\sqrt{2}$



Geometric proofs of irrationality

If we start with a rectangle having integer side lengths, at each step these squares have integral side lengths, smaller and smaller. Hence this process stops after finitely many steps.

Also for a rectangle with side lengths in a rational proportion, this process stops after finitely many steps (reduce to a common denominator and scale).

For instance Φ and $1+\sqrt{2}$ are irrational numbers, hence $\sqrt{5}$ and $\sqrt{2}$ also.

Continued fractions and rational Diophantine approximation

For

$$x = [a_0,\ a_1,\ a_2, \dots, a_k, \dots],$$

the sequence of rational numbers

$$p_k/q_k = [a_0,\ a_1,\ a_2, \dots, a_k] \qquad (k = 1, 2, \dots)$$

produces rational approximations to x, and a classical result is that they are *the best possible ones* in terms of the quality of the approximation compared with the *size of the denominator.*

Continued fractions of a positive rational integer d

Recipe : let d be a positive integer which is not a square. Then the continued fraction of the number $\sqrt{d}$ is periodic.

If k is the smallest period length (that means that the length of any period is a positive integer multiple of k), this continued fraction can be written

$$\sqrt{d} = [a_0\ , \overline{a_1, a_2, \dots, a_k}],$$

with $a_k = 2a_0$ and $a_0 = \lfloor \sqrt{d} \rfloor$.
Further, $(a_1, a_2, \dots, a_{k-1})$ is a *palindrome*

$$a_j = a_{k-j} \quad \text{for} \quad 1 \leq j < k - 1.$$

Fact : the rational number given by the continued fraction $[a_0, a_1, \dots, a_{k-1}]$ is a good rational approximation to $\sqrt{d}$.

Parity of the length of the palindrome

If k is even, the fundamental solution of the equation $x^2 - dy^2 = 1$ is given by the fraction

$$[a_0, a_1, a_2, \ldots, a_{k-1}] = \frac{x_1}{y_1}.$$

In this case the equation $x^2 - dy^2 = -1$ has no solution.

Parity of the length of the palindrome

If k is odd, the fundamental solution (x_1, y_1) of the equation $x^2 - dy^2 = -1$ is given by the fraction

$$[a_0, a_1, a_2, \ldots, a_{k-1}] = \frac{x_1}{y_1}$$

and the fundamental solution (x_2, y_2) of the equation $x^2 - dy^2 = 1$ by the fraction

$$[a_0, a_1, a_2, \ldots, a_{k-1}, a_k, a_1, a_2, \ldots, a_{k-1}] = \frac{x_2}{y_2}.$$

Remark. In both cases where k is either even or odd, we obtain the sequence $(x_\nu, y_\nu)_{\nu \geq 1}$ of all solutions by repeating $\nu - 1$ times $a_1, a_2, \ldots, a_k$ followed by $a_1, a_2, \ldots, a_{k-1}$.

The simplest Pell equation $x^2 - 2y^2 = \pm 1$

Euclid of Alexandria

$\sim$325 BC - $\sim$265 BC

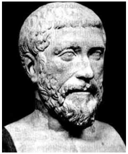
Elements, II § 10

$$17^2 - 2 \cdot 12^2 = 289 - 2 \cdot 144 = 1.$$

$$99^2 - 2 \cdot 70^2 = 9\,801 - 2 \cdot 4900 = 1.$$

$$577^2 - 2 \cdot 408^2 = 332\,929 - 2 \cdot 166\,464 = 1.$$

Pythagorean triples



Pythagoras of Samos
$\sim$569 BC – $\sim$475 BC

Which are the right angle triangles with integer sides such that the two sides of the right angle are consecutive integers ?

$$x^2 + y^2 = z^2, \qquad y = x + 1.$$

$$2x^2 + 2x + 1 = z^2$$

$$(2x+1)^2 - 2z^2 = -1$$

$$X^2 - 2Y^2 = -1$$

$x^2 - 2y^2 = \pm 1$

$$\sqrt{2} = 1,4142135623730950488016887242 \ \dots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}\cdot$$

Hence the continued fraction expansion is periodic with period length 1 :

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \dots] = [1,\ \overline{2}],$$

The fundamental solution of $x^2 - 2y^2 = -1$ is $x_1 = 1$, $y_1 = 1$

$$1^2 - 2 \cdot 1^2 = -1,$$

the continued fraction expansion of x_1/y_1 is $[1]$.

Pell's equation $x^2 - 2y^2 = 1$

The fundamental solution of

$$x^2 - 2y^2 = 1$$

is $x = 3$, $y = 2$, given by

$$[1, 2] = 1 + \frac{1}{2} = \frac{3}{2}.$$

$x^2 - 2y^2 = 1$

Fundamental solution : $x_1 = 3$, $y_1 = 2$:

$$\frac{x_1}{y_1} = \frac{3}{2} = 1 + \frac{1}{2} = [1, 2].$$

Second solution : $x_2 = 17$, $y_2 = 12$

$$\frac{x_2}{y_2} = \frac{17}{12} = 1 + \frac{5}{12}, \quad \frac{12}{5} = 2 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2},$$

hence

$$\frac{17}{12} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2}}} = [1, 2, 2, 2].$$

$x^2 - 2y^2 = 1$

Third solution of $x^2 - 2y^2 = 1$: $x_3 = 99$, $y_3 = 70$.

$$\frac{x_3}{y_3} = \frac{99}{70} = 1 + \frac{29}{70}, \quad \frac{70}{29} = 2 + \frac{12}{29}, \quad \frac{29}{12} = 2 + \frac{5}{12}$$

with

$$\frac{12}{5} = 2 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2}$$

hence

$$\frac{99}{70} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}} = [1, 2, 2, 2, 2, 2].$$

Mahalanobis puzzle (completed)

Fourth solution of $x^2 - 2y^2 = 1$

$$[1,2,2,2,2,2,2,2] = 1 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}}}} = \frac{577}{408}.$$

$$577^2 - 2 \times 408^2 = 1, \qquad 577 = 2 \times 288 + 1, \quad 408 = 2 \times 204.$$

Hence the solution to the puzzle is : *the house number is* 204 *in a street with* 288 *houses :*

$$1 + 2 + 3 + 4 + 5 + \cdots + 203 = \frac{203 \times 204}{2} = 20706,$$

$$205 + 206 + \cdots + 288 = \frac{288 \times 289}{2} - \frac{204 \times 205}{2} = 20706.$$

$x^2 - 3y^2 = 1$

The continued fraction expansion of the number

$$\sqrt{3} = 1,7320508075688772935274463415\dots$$

is

$$\sqrt{3} = [1,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ \dots] = [1,\ \overline{1,\ 2}],$$

because

$$\sqrt{3} + 1 = 2 + \cfrac{1}{1 + \cfrac{1}{\sqrt{3} + 1}}.$$

The fundamental solution of $x^2 - 3y^2 = 1$ is $x = 2$, $y = 1$, corresponding to

$$[1, 1] = 1 + \frac{1}{1} = \frac{2}{1}.$$

$x^2 - 3y^2 = 1$

The fundamental solution of $x^2 - 3y^2 = 1$ is $(x, y) = (2, 1)$:

$$(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1.$$

There is no solution to the equation $x^2 - 3y^2 = -1$.

The period of the continued fraction

$$\sqrt{3} = [1,\ \overline{1,\ 2}]$$

is $[1,\ 2]$ of even length 2.

Small values of d

$$x^2 - 2y^2 = \pm 1,\ \sqrt{2} = [1, \overline{2}],\ k = 1,\ (x_1, y_1) = (1, 1),$$
$$1^2 - 2 \cdot 1^2 = -1.$$

$$x^2 - 3y^2 = \pm 1,\ \sqrt{3} = [1, \overline{1, 2}],\ k = 2,\ (x_1, y_1) = (2, 1),$$
$$2^2 - 3 \cdot 1^2 = 1.$$

$$x^2 - 5y^2 = \pm 1,\ \sqrt{5} = [2, \overline{4}],\ k = 1,\ (x_1, y_1) = (2, 1),$$
$$2^2 - 5 \cdot 1^2 = -1.$$

$$x^2 - 6y^2 = \pm 1,\ \sqrt{6} = [2, \overline{2, 4}],\ k = 2,\ (x_1, y_1) = (5, 4),$$
$$5^2 - 6 \cdot 2^2 = 1.$$

$$x^2 - 7y^2 = \pm 1,\ \sqrt{7} = [2, \overline{1, 1, 1, 4}],\ k = 4,\ (x_1, y_1) = (8, 3),$$
$$8^2 - 7 \cdot 3^2 = 1.$$

Brahmagupta's Problem (628)

The continued fraction expansion of $\sqrt{92}$ is

$$\sqrt{92} = [9, \overline{1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1,\ 18}].$$

The fundamental solution of the equation $x^2 - 92y^2 = 1$ is given by

$$[9,\ 1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1] = \frac{1151}{120}.$$

Indeed, $1151^2 - 92 \cdot 120^2 = 1\,324\,801 - 1\,324\,800 = 1.$

Narayaṇa's equation $x^2 - 103y^2 = 1$

$$\sqrt{103} = [10, \overline{6,\ 1,\ 2,\ 1,\ 1,\ 9,\ 1,\ 1,\ 2,\ 1,\ 6,\ 20}]$$

$$[10, 6,\ 1,\ 2,\ 1,\ 1,\ 9,\ 1,\ 1,\ 2,\ 1,\ 6] = \frac{227\,528}{22\,419}$$

Fundamental solution : $x = 227\,528$, $y = 22\,419$.

$$227\,528^2 - 103 \cdot 22\,419^2 = 51\,768\,990\,784 - 51\,768\,990\,783 = 1.$$

Equation of Bhaskhara II $\quad x^2 - 61y^2 = \pm 1$

$$\sqrt{61} = [7, \overline{1,\ 4,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 4,\ 1,\ 14}]$$

$$[7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1] = \frac{29\,718}{3\,805}$$

$29\,718^2 = 883\,159\,524, \qquad 61 \cdot 3805^2 = 883\,159\,525$
is the fundamental solution of $x^2 - 61y^2 = -1$.

The fundamental solution of $x^2 - 61y^2 = 1$ is

$$[7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1] = \frac{1\,766\,319\,049}{226\,153\,980}$$

Challenge from Fermat to Brouncker

" *pour ne vous donner pas trop de peine*" (Fermat)
" *to make it not too difficult*"

$$x^2 - dy^2 = 1, \text{ with } d = 61 \text{ and } d = 109.$$

Solutions respectively :

$$(1\,766\,319\,049\ ,\ 226\,153\,980)$$
$$(158\,070\,671\,986\,249,\ 15\,140\,424\,455\,100)$$

$$158\,070\,671\,986\,249\ +\ 15\,140\,424\,455\,100\sqrt{109} = \left(\frac{261 + 25\sqrt{109}}{2}\right)^6.$$

Legendre vs Legendre

Adrien-Marie Legendre listed the fundamental solutions for $2 \leq d \leq 139$

Adrien–Marie Legendre

1752–1834

Louis Legendre

1755–1797

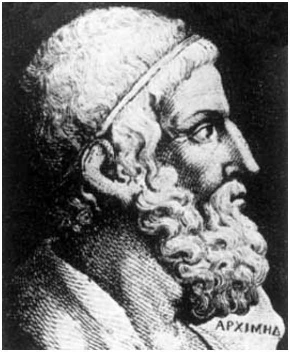
This caricature by J-L Boilly is the only known portrait of Adrien-Marie Legendre.
Louis Legendre was an active participant in the French Revolution.

https://mathshistory.st-andrews.ac.uk/Biographies/Legendre/
Peter Duren. Changing Faces : The Mistaken Portrait of Legendre.
www.ams.org/notices/200911/rtx091101440p.pdf

From 2020 to 2027

$$
\begin{aligned}
809^2 - 2020(18^2) &= 1 & \sqrt{2020} &= [44, \overline{1, 16, 1, 88}] \\
(45\,495)^2 - 2021(1012^2) &= 1 & \sqrt{2021} &= [44, \overline{1, 21, 2, 21, 1, 88}] \\
(1\,349\,495)^2 - 2022(30^2) &= 1 & \sqrt{2022} &= [44, \overline{1, 28, 1, 88}] \\
2024^2 - 2023(45^2) &= 1 & \sqrt{2023} &= [44, \overline{1, 43, 1, 88}] \\
45^2 - 2024 &= 1 & \sqrt{2024} &= [44, \overline{1, 88}] \\
2025 &= 45^2 \\
(40\,051)^2 - 2026(90^2) &= 1 & \sqrt{2026} &= [45, \overline{90}] \\
2026^2 - 2027(45^2) &= 1 & \sqrt{2027} &= [45, \overline{45, 90}].
\end{aligned}
$$

Archimedes cattle problem



The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown.

The Bovinum Problema

Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown.

The number of the black, one quarter plus one fifth the number of the spotted greater than the brown.

The number of the spotted, one sixth and one seventh the number of the white greater than the brown.

First system of equations

$B =$ white bulls, $N =$ black bulls,
$T =$ brown bulls , $X =$ spotted bulls

$$B - \left(\frac{1}{2} + \frac{1}{3}\right) N = N - \left(\frac{1}{4} + \frac{1}{5}\right) X$$
$$= X - \left(\frac{1}{6} + \frac{1}{7}\right) B = T.$$

Up to a multiplicative factor, the solution is

$$B_0 = 2226,\ N_0 = 1602,\ X_0 = 1580,\ T_0 = 891.$$

The Bovinum Problema

Among the cows, the number of white ones was one third plus one quarter of the total black cattle.

The number of the black, one quarter plus one fifth the total of the spotted cattle ;

The number of spotted, one fifth plus one sixth the total of the brown cattle ;

The number of the brown, one sixth plus one seventh the total of the white cattle.

What was the composition of the herd ?

Second system of equations

$b =$ white cows, $n =$ black cows,
$t =$ brown cows, $x =$ spotted cows

$$b = \left(\frac{1}{3} + \frac{1}{4}\right)(N + n), \quad n = \left(\frac{1}{4} + \frac{1}{5}\right)(X + x),$$
$$t = \left(\frac{1}{6} + \frac{1}{7}\right)(B + b), \quad x = \left(\frac{1}{5} + \frac{1}{6}\right)(T + t).$$

Since the solutions b, n, x, t are requested to be integers, one deduces

$$(B, N, X, T) = k \times 4657 \times (B_0, N_0, X_0, T_0).$$

Archimedes Cattle Problem

If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

The Bovinum Problema

But come, understand also all these conditions regarding the cattle of the Sun.

When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude.

Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking.

Arithmetic constraints

$$\begin{aligned} B + N &= \text{a square,} \\ T + X &= \text{a triangular number.} \end{aligned}$$

As a function of the integer k, we have $B + N = 4Ak$ with $A = 3 \cdot 11 \cdot 29 \cdot 4657$ squarefree. Hence $k = AU^2$ with U an integer. On the other side if $T + X$ is a triangular number $(= m(m+1)/2)$, then

$$8(T + X) + 1 \quad \text{is a square} \quad (2m+1)^2 = V^2.$$

Pell's equation associated with the cattle problem

Writing $T + X = Wk$ with $W = 7 \cdot 353 \cdot 4657$, we get

$$V^2 - DU^2 = 1$$

with $D = 8AW = (2 \cdot 4657)^2 \cdot 2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353$.

$$2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353 = 4\,729\,494.$$

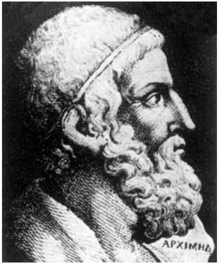
$$D = (2 \cdot 4657)^2 \cdot 4\,729\,494 = 410\,286\,423\,278\,424.$$

Cattle problem

If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

History : letter from Archimedes to Eratosthenes

Archimedes
(287 BC –212 BC)



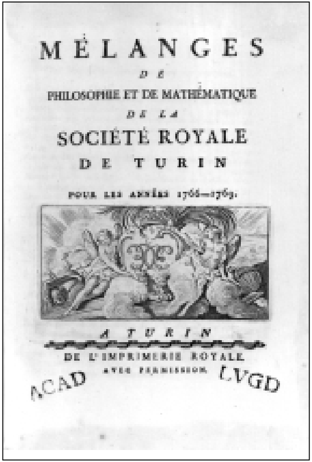
Eratosthenes of Cyrene
(276 BC - 194 BC)

History (continued)

Odyssey of Homer - the Sun God Herd

Gotthold Ephraim Lessing : 1729–1781 – Library Herzog August, Wolfenbüttel, 1773

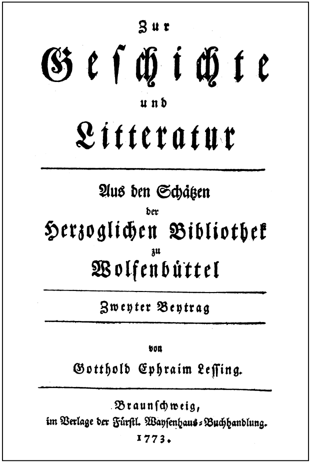
1773 : Lagrange and Lessing



MÉLANGES
DE
PHILOSOPHIE ET DE MATHÉMATIQUE
DE LA
SOCIÉTÉ ROYALE
DE TURIN
POUR LES ANNÉES 1766–1769.

A TURIN
DE L'IMPRIMERIE ROYALE
AVEC PERMISSION.

ACAD LVGD



Zur
Geschichte
und
Litteratur

Aus den Schätzen
der
Herzoglichen Bibliothek
zu
Wolfenbüttel

Zweyter Beytrag

von
Gotthold Ephraim Lessing.

Braunschweig,
im Verlage der Fürstl. Waysenhaus-Buchhandlung.
1773.

Figures 1 and 2. Title pages of two publications from 1773. The first (far left) contains Lagrange's proof of the solvability of Pell's equation, already written and submitted in 1768. The second contains Lessing's discovery of the cattle problem of Archimedes.

History (continued)

C.F. Meyer, 1867. First serious attempt to solve the problem.

A. Amthor, 1880 : the smallest solution has $206\,545$ digits, starting with 776.
B. Krumbiegel and A. Amthor, Das Problema Bovinum des Archimedes, Historisch-literarische Abteilung der Zeitschrift für Mathematik und Physik, **25** *(1880), 121–136, 153–171.*

History (continued)

A.H. Bell, The "Cattle Problem" by Archimedes 251 BC, Amer. Math. Monthly **2** (1895), 140–141.
Computation of the first 30 and last 12 decimal digits. The Hillsboro, Illinois, Mathematical Club, A.H. Bell, E. Fish, G.H. Richard – 4 years of computations.

"Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number [of cattle], it is obvious that the world will never have a complete solution"
Pre-computer-age thinking from a letter to The New York Times, January 18, 1931

History (continued)

H.C. Williams, R.A. German and C.R. Zarnke, Solution of the cattle problem of Archimedes, Math. of Computation **19** (1965), 671–674.

H.G. Nelson, A solution to Archimedes' cattle problem, J. Recreational Math. **13** (3) (1980–81), 162–176.

I. Vardi, Archimedes' Cattle Problem, Amer. Math. Monthly **105** (1998), 305-319.

H.W. Lenstra Jr, Solving the Pell Equation, Notices of the A.M.S. **49** (2) (2002) 182–192.

The solution

Equation $x^2 - 410\,286\,423\,278\,424 y^2 = 1$.

Print out of the smallest solution with $206\,545$ decimal digits : 47 pages (H.G. Nelson, 1980).

$$77602714 \star\star\star\star\star\star 37983357 \star\star\star\star\star\star 55081800$$

where each of the twelve symbols $\star$ represents $17\,210$ digits.

Large numbers

A number written with only 3 digits, but having nearly 370 millions decimal digits

The number of decimal digits of 9^{9^9} *is*

$$\left\lfloor 9^9 \frac{\log 9}{\log 10} \right\rfloor = 369\,693\,100.$$

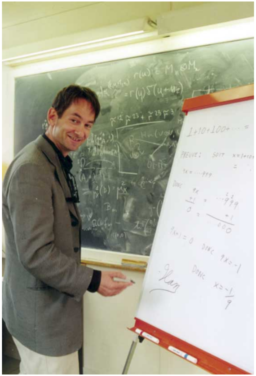
$10^{10^{10}}$ has $1 + 10^{10}$ decimal digits.

Ilan Vardi

http://www.math.nyu.edu/ crorres/Archimedes/Cattle/Solution1.html

$$\left\lfloor \tfrac{25194541}{184119152} (109931986732829734979866232821433543901088049+ \\ 50549485234315033074477819735540408986340\sqrt{4729494})^{4658} \right\rfloor$$

Archimedes' Cattle Problem, American Math. Monthly **105** (1998), 305-319.



A simple solution to Archimedes' cattle problem

Antti Nygrén, "A simple solution to Archimedes' cattle problem", University of Oulu Linnanmaa, Oulu, Finland Acta Universitatis Ouluensis Scientiae Rerum Naturalium, 2001.

50 first digits
$77602714064868182695302328332138866642323224059233$

50 last digits :
$05994630144292500354883118973723406626719455081800$

Solution of Pell's equation

H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

http://www.ams.org/notices/200202/fea-lenstra.pdf

Solution of Archimedes Problem

All solutions to the cattle problem of Archimedes

$$w = 300\,426607\,914281\,713365 \cdot \sqrt{609} + 84\,129507\,677858\,393258 \cdot \sqrt{7766}$$

$$k_j = (w^{4658 \cdot j} - w^{-4658 \cdot j})^2 / 368\,238304 \qquad (j = 1,\ 2,\ 3,\ \ldots)$$

jth solution	*bulls*	*cows*	*all cattle*
white	$10\,366482 \cdot k_j$	$7\,206360 \cdot k_j$	$17\,572842 \cdot k_j$
black	$7\,460514 \cdot k_j$	$4\,893246 \cdot k_j$	$12\,353760 \cdot k_j$
dappled	$7\,358060 \cdot k_j$	$3\,515820 \cdot k_j$	$10\,873880 \cdot k_j$
brown	$4\,149387 \cdot k_j$	$5\,439213 \cdot k_j$	$9\,588600 \cdot k_j$
all colors	$29\,334443 \cdot k_j$	$21\,054639 \cdot k_j$	$50\,389082 \cdot k_j$

Figure 4.

H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

The solution

$$x^2 - 410\,286\,423\,278\,424 y^2 = 1$$

Computation of the continued fraction of $\sqrt{410\,286\,423\,278\,424}$.

In 1867, C.F. Meyer performed the first 240 steps of the algorithm and then gave up.

The *length of the period* has now be computed : it is $203\,254$.

Solution by Amthor – Lenstra

$d = (2 \cdot 4657)^2 \cdot d' \qquad d' = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353$.

Length of the period for $\sqrt{d'}$: 92.

Fundamental unit : $u = x' + y'\sqrt{d'}$

$$u = \Big(300\,426\,607\,914\,281\,713\,365 \cdot \sqrt{609} + 84\,129\,507\,677\,858\,393\,258\sqrt{7766}\Big)^2$$

Fundamental solution of the Archimedes equation :

$$x_1 + y_1\sqrt{d} = u^{2329}.$$

$p = 4657$, $(p+1)/2 = 2329 = 17 \cdot 137$.

Size of the fundamental solution

$$2\sqrt{d} < x_1 + y_1\sqrt{d} < (4e^2 d)^{\sqrt{d}}.$$

Any method for solving the Brahmagupta–Fermat–Pell equation which requires to produce the digits of the fundamental solution has an exponential complexity.

Length L_d of the period :

$$\frac{\log 2}{2} L_d \le \log(x_1 + y_1\sqrt{d}) \le \frac{\log(4d)}{2} L_d.$$

July 23, 2020

Department of Mathematics, Bannari Amman Institute of Technology
Sathyamangalam, Erode, Tamilnadu, India ICMMSIS'20

International Web Conference on
"Mathematics for Signals, Images, and Structured Data"

On the Brahmagupta–Fermat–Pell Equation $x^2 - dy^2 = \pm 1$

Michel Waldschmidt

Institut de Mathématiques de Jussieu
Sorbonne Université (Paris)
http://webusers.imj-prg.fr/~michel.waldschmidt/