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### Some contributions to number theory by Chinese Mathematicians

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Hua Luogeng  
1910 – 1985



Chen Jingrun  
1933 – 1996



Wang Yuan  
1930 –

Hua Loo Keng 华罗庚 華羅庚

Hua Loo Keng and Wang Yuan 华罗庚 王元

**Mathematics Genealogy Project**

**Loo-Keng Hua (华罗庚)**

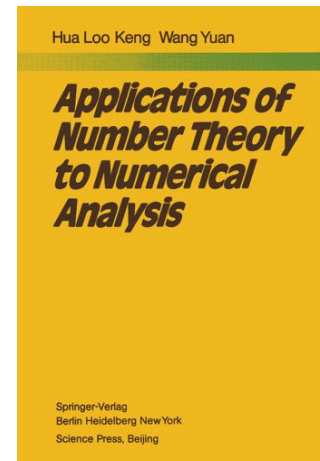
Dissertation:  
Advisor: Unknown

Students:  
Click [here](#) to see the students listed in chronological order.

Name	School	Year	Descendants
<a href="#">Ayoub, Raymond</a>	University of Illinois at Urbana-Champaign	1950	9
<a href="#">Chen (陈景润), Jingrun</a>	Chinese Academy of Sciences		3
<a href="#">Ding (丁夏畦), Xiaqi (Xian)</a>	Academia Sinica	1956	30
<a href="#">Pan (潘承洞), Chengdong</a>	Peking University	1961	39
<a href="#">Wang (王元), Yuan</a>	Academia Sinica		2
<a href="#">Wan (万哲先), Zhe-Xian</a>	Academia Sinica		17

According to our current on-line database, Loo-Keng Hua (华罗庚) has 6 [students](#) and 106 [descendants](#).  
We welcome any additional information.

<https://genealogy.math.ndsu.nodak.edu/id.php?id=4784>



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Chen's statue in Xiamen University

## Numbers

Numbers = real or complex numbers  $\mathbb{R}, \mathbb{C}$ .

Natural integers :  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

Rational integers :  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ .

## Prime numbers

Numbers with exactly two divisors.

There are 25 prime numbers less than 100 :

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,

43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

The On-Line Encyclopedia of Integer Sequences

<http://oeis.org/A000040>



Neil J. A. Sloane

## The fundamental Theorem of arithmetic

Any positive number is the product, in only one way, of prime numbers.

Prime numbers are related to multiplication, they are the building blocs of the set of integers for the product. One should multiply them, not add them !

But there is no law which would forbid to add prime numbers !!

## Sums of two primes

Let us list the numbers up to 30 that are sums of two prime numbers :  $n = p_1 + p_2$ ,  $p_1 \leq p_2$ ,  $n \leq 30$ .

$p_1 \setminus p_2$	2	3	5	7	11	13	17	19	23
2	4	5	7	9	13	15	19	21	25
3		6	8	10	14	16	20	22	26
5			10	12	16	18	22	24	28
7				14	18	20	24	26	30
11					22	24	28	30	
13						26	30		

The entries on the row with  $p_1 = 2$  after the first one are odd, all other entries are even.

## Sums of two primes

Numbers that are sums of two primes :

4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 30, ...

Numbers  $\geq 2$  that are not sums of two primes :

2, 3, 11, 17, 27, 29, ...

Numbers that are sums of at most two primes :

2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29,

Numbers  $\geq 2$  that are not sums of at most two primes :

27, ...

## Sums of three primes

Notice that 27 is sum of three primes in 7 ways :

$$\begin{aligned} 27 &= 2 + 2 + 23 = 3 + 5 + 19 = 3 + 7 + 17 \\ &= 3 + 11 + 13 = 5 + 5 + 17 = 5 + 11 + 11 = 7 + 7 + 13. \end{aligned}$$

The number of decompositions of an integer as a sum of three primes is given by the sequence <http://oeis.org/A068307>, namely

0, 0, 0, 0, 0, 1, 1, 1, 2, 1, 2, 2, 2, 1, 3, 2, 4, 2, 3, 2, 5, 2, 5, 3, 5, 3, 7, 3, 7, 2, 6, 3, 9, 2, 8, 4, 9, 4, 10, 2, 11, 3, 10, 4, 12, 3, 13, 4, 12, 5, 15, ...

## Goldbach's Conjecture



Christian Goldbach  
(1690 – 1764)



Leonhard Euler  
(1707 – 1783)

Letter of Goldbach to Euler, 1742 :  
*any integer  $\geq 6$  is sum of three primes.*

Euler : Equivalent to :

*any even integer  $\geq 4$  is sum of two primes.*

## Goldbach $\iff$ Euler

(G) : *any integer  $\geq 6$  is sum of three primes.*

(E) : *any even integer  $\geq 4$  is sum of two primes.*

Proof :

(G) $\implies$ (E)

Let  $m$  be an even number  $\geq 4$ . Assuming (G),  $m + 2$  is sum of three primes, says  $m + 2 = p_1 + p_2 + p_3$ . At least one of them is even, say  $p_3 = 2$ , and  $m = p_1 + p_2$  is sum of two primes.

(E) $\implies$ (G)

Let  $m \geq 6$ . From (E) it follows that  $m - 2$  is sum of two primes,  $m - 2 = p_1 + p_2$ , hence  $m = p_1 + p_2 + 2$  is sum of three primes.

## Number of decompositions of $2n$ into ordered sums of two odd primes.

<http://oeis.org/A002372>

0, 0, 1, 2, 3, 2, 3, 4, 4, 4, 5, 6, 5, 4, 6, 4, 7, 8, 3, 6, 8, 6, 7, 10, 8, 6,  
10, 6, 7, 12, 5, 10, 12, 4, 10, 12, 9, 10, 14, 8, 9, 16, 9, 8, 18, 8, 9, 14,  $\dots$ ,

## Circle method



Srinivasa Ramanujan  
(1887 – 1920)



G.H. Hardy  
(1877 – 1947)



J.E. Littlewood  
(1885 – 1977)

Hardy, ICM Stockholm, 1916

Hardy and Ramanujan (1918) : partitions

Hardy and Littlewood (1920 – 1928) :

Some problems in Partitio Numerorum

## Circle method

Hardy and Littlewood



Ivan Matveevich Vinogradov  
(1891 – 1983)



*Every sufficiently large odd integer is the sum of at most three primes.*

## Sums of primes

**Theorem** – I.M. Vinogradov (1937)

*Every sufficiently large odd integer is sum of three primes.*

**Theorem** – Chen Jing-Run (1966)

*Every sufficiently large even integer is sum of a prime and an integer that is either a prime or a product of two primes.*

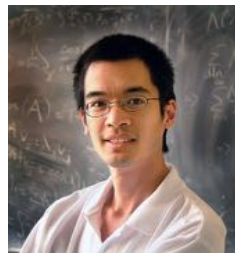


## Sums of primes

In the above proof of the equivalence between Goldbach and Euler, the prime number 2 plays a central role.

- Weak (or ternary) Goldbach Conjecture : every odd integer  $\geq 7$  is the sum of three odd primes.

- Terence Tao, February 4, 2012, arXiv:1201.6656 :  
*Every odd number greater than 1 is the sum of at most five primes.*



## Ternary Goldbach Problem

**Theorem** – Harald Helfgott (2013).

*Every odd number greater than 5 can be expressed as the sum of three primes.*

*Every odd number greater than 7 can be expressed as the sum of three odd primes.*



Harald Helfgott  
(1977- )

Earlier results due to Hardy and Littlewood (1923), Vinogradov (1937), Deshouillers et al. (1997), and more recently Ramaré, Kaniecki, Tao ...

