IMSc Chennai, *December 18, 2023*

Alladi Ramakrishnan Centenary

December 16-18, 2023

The Institute of Mathematical Sciences Chennai

Circulant Determinants and Clifford Algebras

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Abstract

In the course of studying a higher dimensional generalization of the Pythagorean equation and its connections to the Lorentz transformation, Alladi Ramakrishnan made a conjecture on a determinant of a certain circulant matrix and published it in his paper Pythagoras to Lorentz via Fermat. In the first part of this talk we give a proof of this conjecture.

In the second part of this talk, we give an instance where Clifford algebra are used in transcendental number theory.

Pythagorean equation

$a^2 - b^2 = c^2$ $\qquad$ $\det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2 - b^2.$

$$\det \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = a^3 + b^3 + c^3 - 3abc$$

Cubic analogue of the Lorentz transformation:

$$a^3 + b^3 + c^3 - 3abc = d^3.$$

Generalization to a $n \times n$ *circulant* determinant.

Alladi Ramakrishnan, *Pythagoras to Lorentz via Fermat – spanning the interval with light and delight*, in Special Relativity, East–West Books, Madras (2003), 90–97.

Letter to Alladi Ramakrishnan, June 8, 2000

I am pleased to tell you that the conjectures you stated in your paper "Pythagoras to Lorentz" are true.

More precisely, for k a positive integer, denote by $C_k(z_1, \ldots, z_k)$ the determinant of the circulant matrix

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_{k-1} & z_k \\ z_k & z_1 & \cdots & z_{k-2} & z_{k-1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ z_2 & z_3 & \cdots & z_k & z_1 \end{pmatrix}$$

and by $P_k(z)$ the polynomial

$$C_k(z, z-1, \ldots, z-k+1).$$

Then

$$P_k(z) = k^{k-1}\left(z - \frac{k-1}{2}\right).$$

Letter to Alladi Ramakrishnan, June 8, 2000

In particular if $k = 2m+1$ is odd then

$$P_{2m+1}(m+n) = (2m+1)^{2m} n.$$

Further, for $k = 2m$ even,

$$C_{2m}(n+m, n+m-1, \dots, n+1, n-1, \dots, n-m) = c(m)n,$$

where $c(m)$ depends only on m.

M. Waldschmidt, Proof of a Conjecture of Alladi Ramakrishnan on Circulants.
In: K. Alladi, J.H. Klauder, & C.R. Rao, The legacy of Alladi Ramakrishnan in the mathematical sciences, Springer New-York (2010), 329–334.

Examples (1)

$$P_2(z) = C_2(z, z-1) = \det \begin{pmatrix} z & z-1 \\ z-1 & z \end{pmatrix}$$
$$= z^2 - (z-1)^2 = 2z - 1 = 2\left(z - \frac{1}{2}\right).$$

$$P_3(z) = C_3(z, z-1, z-2) = \det \begin{pmatrix} z & z-1 & z-2 \\ z-2 & z & z-1 \\ z-1 & z-2 & z \end{pmatrix}$$
$$= z^3 + (z-1)^3 + (z-2)^3 - 3z(z-1)(z-2)$$
$$= 9z - 9 = 3^2(z-1).$$

Examples (2)

$$C_2(n+1, n-1) = \det \begin{pmatrix} n+1 & n-1 \\ n-1 & n+1 \end{pmatrix}$$
$$= (n+1)^2 - (n-1)^2 = 4n.$$

$$C_4(n+2, n+1, n-1, n-2) =$$
$$\det \begin{pmatrix} n+2 & n+1 & n-1 & n-2 \\ n+1 & n-1 & n-2 & n+2 \\ n-1 & n-2 & n+2 & n+1 \\ n-2 & n+2 & n+1 & n-1 \end{pmatrix}$$
$$= 144n.$$

Value of $c(m)$

One can prove that

$$C_{2m}(n+m, n+m-1, \dots, n+1, n-1, \dots, n-m) = c(m)n$$

with $c(m) = 2^{2m-1} m^{m-1} (m+1)^m$.

$$\begin{aligned}
c(1) &= 2^1 1^0 2^1 = 2^2 = 4 \\
c(2) &= 2^3 2^1 3^2 = 2^4 3^2 = 144 \\
c(3) &= 2^5 3^2 4^3 = 2^{11} 3^2 = 18\,432 \\
c(4) &= 2^7 4^3 5^4 = 2^{13} 5^4 = 5\,120\,000 \\
c(5) &= 2^9 5^4 6^5 = 2^{14} 3^5 5^4 = 2\,488\,320\,000 \\
c(6) &= 2^{11} 6^5 7^6 = 2^{16} 3^5 7^6 = 1\,873\,589\,501\,952 \\
c(7) &= 2^{13} 7^6 8^7 = 2^{34} 7^6 = 2\,021\,194\,429\,628\,416 \\
c(8) &= 2^{15} 8^7 9^8 = 2^{36} 3^{16} = 2\,958\,148\,142\,320\,582\,656
\end{aligned}$$

Proof (1)

The first remark is that if $A = \left(a_{ij}\right)_{1\leq i,j\leq n}$ is a $n \times n$ square matrix, the polynomial

$$P(z) = \det(z + a_{ij})_{1\leq i,j\leq n}$$

can be written

$$P(z) = cz + \det(A)$$

with a constant c. This is easily checked by replacing each row but the first one by its difference with the first one, and then expanding with minors on the first row.

Next for $k = 2m$ consider the circulant whose determinant is

$$C_{2m}(m, m-1, \dots, 1, -1, \dots, -m+1, -m).$$

The sum of all rows (as well as the sum of all columns) is 0. Hence the determinant is 0.

Proof (2)

These two facts imply

$C_{2m}(n+m, n+m-1, \ldots, n+1, n-1, \ldots, n-m) = c(m)n.$

They also imply

$$P_k(z) = c_k \left(z - \frac{k-1}{2} \right),$$

with some constant c_k depending only on k, but we are going to reprove this result (and compute c_k) by another way.
It is well know (and easy to prove) that

$$C_k(z_1, \ldots, z_k) = \prod_{\zeta} (z_1 + \zeta z_2 + \cdots + \zeta^{k-1} z_k)$$

$$= \prod_{\zeta} \sum_{i=0}^{k-1} \zeta^i z_{i+1},$$

where ζ ranges over the k-th roots of unity.

Proof (3)

Hence

$$P_k(z) = \prod_{\zeta} \sum_{i=0}^{k-1} \zeta^i (z - i).$$

Now

$$\sum_{i=0}^{k-1} \zeta^i = \begin{cases} k & \text{for } \zeta = 1, \\ 0 & \text{for } \zeta \neq 1, \end{cases}$$

and we derive

$$P_k(z) = c_k \left(z + \frac{k-1}{2} \right)$$

with

$$c_k = k \prod_{\zeta \neq 1} \sum_{i=0}^{k-1} (-i) \zeta^i = (-1)^{k-1} k \prod_{\zeta \neq 1} \sum_{i=0}^{k-1} i \zeta^i.$$

Proof (4)

The sum

$$\sum_{i=0}^{k-1} i\zeta^i = \zeta + 2\zeta^2 + \cdots + (k-1)\zeta^{k-1}$$

is the value at the point ζ of $zf'(z)$, where f' is the derivative of the polynomial

$$f(z) = 1 + z + \cdots + z^{k-1} = \frac{z^k - 1}{z - 1}.$$

Since

$$f'(z) = \frac{kz^{k-1}}{z-1} - \frac{z^k - 1}{(z-1)^2},$$

for ζ satisfying $\zeta^k = 1$ and $\zeta \neq 1$ we have

$$\zeta f'(\zeta) = \frac{k}{\zeta - 1}.$$

Proof (5)

Now

$$\prod_{\zeta \neq 1} (\zeta - 1)$$

is nothing else than the resultant of the two polynomials $z - 1$ and $f(z)$, hence

$$\prod_{\zeta \neq 1} (\zeta - 1) = (-1)^{k-1} f(1) = (-1)^{k-1} k.$$

Therefore

$$\prod_{\zeta \neq 1} \sum_{i=0}^{k-1} i\zeta^i = \prod_{\zeta \neq 1} \frac{k}{\zeta - 1} = \frac{k^{k-1}}{(-1)^{k-1} f(1)} = (-1)^{k-1} k^{k-2}$$

and

$$c_k = k^{k-1}.$$

This completes the proof.

A further reference

Shigeru Kanemitsu

With Shigeru Kanemitsu, *Matrices of finite abelian groups, Finite Fourier Transform and codes.* 17 p. "Arithmetic in Shangrila"—Proc. the 6th China-Japan Sem. Number Theory held in Shanghai Jiao Tong University, August 15-17, 2011, ed. S. Kanemitsu, H.-Z. Li, and J.-Y. Liu. World Scientific Publishing Co, Series on Number Theory and its application, vol. **8** (2013), 90-106. arXiv:1301.1248 [math.NT].

Transcendental numbers

A complex number α is *algebraic* if there exists a nonzero polynomial $P \in \mathbb{Q}[X]$ such that $P(\alpha) = 0$.
A complex number which is not algebraic is *transcendental*.

- Examples of algebraic numbers:
  rational numbers, $\sqrt{2}$, $e^{2i\pi p/q}$.
- Examples of transcendental numbers:
  e, π, almost all numbers (for Lebesgue measure).

- Complex numbers $\alpha_1, \ldots, \alpha_n$ are *algebraically dependent* if there exists a nonzero polynomial $P \in \mathbb{Q}[X_1, \ldots, X_n]$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$.

Otherwise $\alpha_1, \ldots, \alpha_n$ are *algebraically independent*.

e and π

Charles Hermite

1822 - 1901

Ferdinand von Lindemann

1852 - 1939

e is transcendental

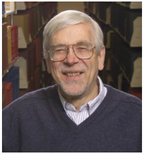
π is transcendental

If $a+b$ and ab are algebraic, then a and b are algebraic.

Hence one at least of the two numbers $\mathrm{e}+\pi$, $\mathrm{e}\pi$ is transcendental.

Conjecture. Both numbers are transcendental.
Stronger conjecture: e and π are algebraically independent.

Result MW & Dale Brownawell (1972, simultaneously and independently)



W.D. Brownawell

One at least of the two following statements is true
- e and π are algebraically independent
- e^{π^2} is a transcendental number.

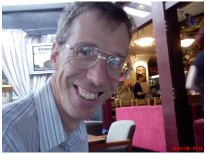
Conjecture. Both statements are true.
Stronger conjecture: e, π and e^{π^2} are algebraically independent.

Algebraic independence of logarithms

Conjecture. If $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbb{Q}$–linearly independent logarithms of algebraic numbers, then they are algebraically independent.

Remarks.
• It is not yet proved that there exist two algebraically independent logarithms of algebraic numbers.
• It is not yet proved that there is no nontrivial quadratic relation among logarithms of algebraic numbers.

Joint work with Damien Roy (1997)



Damien Roy

Theorem. *Let* $\log\alpha_1,\ldots,\log\alpha_m$ *be* $\mathbb{Q}$*–linearly independent logarithms of algebraic numbers.*
One at least of the two following statements is true:
• *At least two of the numbers* $\log\alpha_1,\ldots,\log\alpha_m$ *are algebraically independent.*
• *Let* $Q\in\mathbb{Q}[X_1,\ldots,X_m]$ *be a nonzero homogeneous polynomial of degree* 2 (a quadratic form)*. Then*

$$Q(\log\alpha_1,\ldots,\log\alpha_m)\neq 0.$$

https://mathshistory.st-andrews.ac.uk/Biographies/Clifford/

William Kingdon Clifford

1845 – 1879

Summary

William Clifford was an English mathematician who studied non-euclidean geometry arguing that energy and matter are simply different types of curvature of space. He introduced what is now called a Clifford algebra which generalises Grassmann's exterior algebra.

Clifford algebra over $\mathbb{C}$

Let $q : \mathbb{C}^m \to \mathbb{C}$ be a quadratic form. The *Clifford algebra attached to* q is a simple algebra A of dimension 2^m over $\mathbb{C}$, which contains $\mathbb{C}^m$ as a vector subspace, which is spanned by $\mathbb{C}^m$ as a $\mathbb{C}$ algebra and satisfies

$$v^2 = q(v) \cdot 1$$

for all $v \in \mathbb{C}^m$.
If $(v_1, \ldots, v_m)$ is a basis of $\mathbb{C}^m$ over $\mathbb{C}$, then the products $v_{i_1} \cdots v_{i_r}$ with $i_1 < \cdots < i_r$ are a basis of A over $\mathbb{C}$ (the empty product is 1).

Clifford algebra over $\mathbb{Q}$

Assume $q \in \mathbb{Q}[X_1, \ldots, X_m]$. Let A be the Clifford algebra attached to $q : \mathbb{C}^m \to \mathbb{C}$, A_0 the sub–$\mathbb{Q}$–algebra of A spanned by $\mathbb{Q}^m$ and $q_0 : \mathbb{Q}^m \to \mathbb{Q}$ the restriction of q. Hence A_0 is the *Clifford algebra attached to* q_0, it has dimension 2^m over $\mathbb{Q}$, and any basis of A_0 over $\mathbb{Q}$ is a basis of A over $\mathbb{C}$. Fix such a basis. For $v \in \mathbb{C}^m$ let M_v be the matrix of the linear map $L_v : A \to A$ given by the multiplication by v. Since $v^2 = q(v) \cdot 1$, we have

$$M_v^2 = q(v) \cdot I, \qquad \text{hence} \qquad \det M_v = \pm q(v)^{2^{m-1}}.$$

If $q(v) \neq 0$ then M_v is a regular matrix. If $q(v) = 0$ then M_v has rank $\leqslant 2^{m-1}$.
Define $\theta : \mathbb{C}^m \longrightarrow \mathrm{Mat}_{2^m \times 2^m}(\mathbb{C})$ by $\theta(v) = M_v$.

Clifford algebra over $\mathbb{Q}$

Let $q \in \mathbb{Q}[X_1, \ldots, X_m]$ be a nonzero quadratic form. Let $V = Z(q)$ be the set of zeros of q in $\mathbb{C}^m$. Then there is an injective linear map defined over $\mathbb{Q}$

$$\theta : \mathbb{C}^m \longrightarrow \mathrm{Mat}_{2^m \times 2^m}(\mathbb{C})$$

such that, for all $v \in \mathbb{C}^m$, the rank of $\theta(v)$ is a multiple of 2^{m-1} and such that

$$V = \{v \in \mathbb{C}^m \mid \det \theta(v) = 0\}.$$

D. Roy and M. W. *Approximation diophantienne et indépendance algébrique de logarithmes*. Annales scientifiques de l'École Normale Supérieure Sér. 4, **30** N°6 (1997), p. 753-796
MR 98f:11077 Zbl 0895.11030
The European Digital Mathematics Library (EUDML) 82449.

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