A course on interpolation

Third Course :
Several Points
Poritsky, Gontcharoff

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Abstract

Given two sequences $(\sigma_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ of complex numbers and a sequence $(\tau_n)_{n\geq 0}$ of nonnegative integers, the interpolation problem asks for the existence and unicity of an entire function f satisfying

$$f^{(\tau_n)}(\sigma_n) = a_n$$

for all $n \geq 0$.

We consider special cases.

Two interpolation problems

We are going to consider the following interpolation problems:

- (Poritsky): For $m \geq 2$ and $\sigma_0, \ldots, \sigma_{m-1}$ in $\mathbb{C}$,

  $$f^{(mn)}(\sigma_j) = a_{nj} \quad \text{for} \quad n \geq 0 \quad \text{and} \quad j = 0, 1, \ldots, m-1.$$

- (Gontcharoff): For $(\sigma_n)_{n\geq 0}$ a sequence of complex numbers,

  $$f^{(n)}(\sigma_n) = a_n \quad \text{for} \quad n \geq 0.$$

  Periodic sequence:

  $$f^{(mn+j)}(\sigma_j) = a_{mn+j} \quad \text{for} \quad n \geq 0 \quad \text{and} \quad j = 0, 1, \ldots, m-1.$$

Recall : Lidstone vs Whittaker

Let us display horizontally the points and vertically the derivatives.

- $\bullet$ interpolation values $\circ$ no condition

Lidstone interpolation

$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	$\circ$	$\circ$
$f^{(2n)}$	$\bullet$	$\bullet$
$\vdots$	$\vdots$	$\vdots$
f''	$\bullet$	$\bullet$
f'	$\circ$	$\circ$
f	$\bullet$	$\bullet$
	s_0	s_1

Whittaker interpolation

$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	$\bullet$	$\circ$
$f^{(2n)}$	$\circ$	$\bullet$
$\vdots$	$\vdots$	$\vdots$
f''	$\circ$	$\bullet$
f'	$\bullet$	$\circ$
f	$\circ$	$\bullet$
	s_0	s_1

Interpolation with 3 points

Poritsky

	s_0	s_1	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	○
$f^{(3n+1)}$	○	○	○
$f^{(3n)}$	●	●	●
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\mathrm{iv})}$	○	○	○
f'''	●	●	●
f''	○	○	○
f'	○	○	○
f	●	●	●

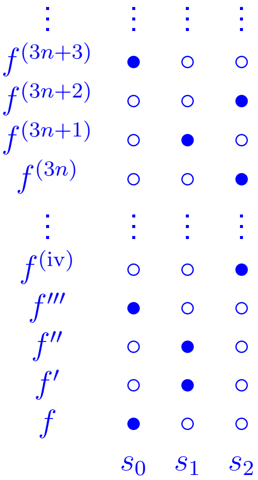
Gontcharoff periodic

	s_0	s_1	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	●
$f^{(3n+1)}$	○	●	○
$f^{(3n)}$	●	○	○
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\mathrm{iv})}$	○	●	○
f'''	●	○	○
f''	○	○	●
f'	○	●	○
f	●	○	○

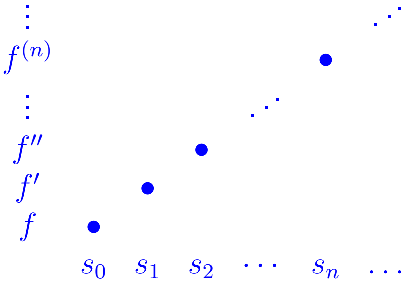
Gontcharoff – Abel interpolation

The set of points may not be finite (or may not be distinct)

Gontcharoff



Abel



Poritsky interpolation: unicity

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem [H. Poritsky, 1932]. If

$$f^{(mn)}(s_0) = f^{(mn)}(s_1) = \cdots = f^{(mn)}(s_{m-1}) = 0$$

for all sufficiently large n, then f is a polynomial.

For $m = 2$, $s_0 = 0$, $s_1 = 1$, this reduces Poritsky's result on Lidstone expansion (up to the exact bound on the exponential type).

Gontcharoff interpolation: unicity

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem [W. Gontcharoff 1930, A. J. Macintyre 1954]. If

$$f^{(n)}(s_0)f^{(n)}(s_1)\cdots f^{(n)}(s_{m-1}) = 0$$

for all sufficiently large n, then f is a polynomial.

For $m = 2$, $s_0 = 0$, $s_1 = 1$, this implies Whittaker's result for $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ (up to the exact bound on the exponential type).

Periodic sequences

Let $s_0, s_1, \ldots, s_{m-1}$ be complex numbers, not necessarily distinct. We write $\mathbf{s}$ for the tuple $(s_0, s_1, \ldots, s_{m-1})$. Let $r_0, \ldots, r_{m-1}$ be m nonnegative integers satisfying
$0 \leq r_0 \leq r_1 \leq \cdots \leq r_{m-1} \leq m-1$.
We investigate the interpolation problem for the values

$$f^{(mn+r_j)}(s_j) \quad (n \geq 0,\ j = 0, \ldots, m-1).$$

Examples
(1) Poritsky:

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

(2) Gontcharoff periodic:

$$r_j = j \qquad \text{for} \quad j = 0, 1, \ldots, m-1.$$

Periodic sequences: $f^{(mn+r_j)}(s_j)$

Let $\mathbb{C}[z]_{\leq m-1}$ be the space of polynomials of degree $\leq m-1$.

If there is a nonzero polynomial $f \in \mathbb{C}[z]_{\leq m-1}$ such that

$$f^{(r_j)}(s_j) = 0 \quad (j = 0, \dots, m-1),$$

then there is no unicity. So we assume that the linear map

$$\begin{array}{ccc} \mathbb{C}[z]_{\leq m-1} & \longrightarrow & \mathbb{C}^m \\ f(z) & \longmapsto & \left(f^{(r_j)}(s_j)\right)_{0\leq j\leq m-1} \end{array}$$

is an isomorphism of $\mathbb{C}$–vector spaces.

The determinant $\mathrm{D}(\mathbf{s})$

In other words we assume that the determinant

$$\mathrm{D}(\mathbf{s}) = \det\left(\frac{k!}{(k-r_j)!}{s_j}^{k-r_j}\right)_{0\le j,k\le m-1}$$

does not vanish.
It follows that $r_j \le j$ for all $j = 0, 1, \ldots, m-1$.

Proposition.

Assume $\mathrm{D}(\mathbf{s}) \ne 0$. Then there exists a unique family of polynomials $(\Lambda_{nj}(z))_{n\ge 0, 0\le j\le m-1}$ satisfying

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}, \quad \textit{for} \quad n, k \ge 0 \quad \textit{and} \quad 0 \le j, \ell \le m-1.$$

For $n \ge 0$ and $0 \le j \le m-1$ the polynomial Λ_{nj} has degree $\le mn + m - 1$.

Recurrence relations

Under the assumption $\mathrm{D}(\mathbf{s}) \neq 0$, the polynomials $\Lambda_{nj}(z)$, ($n \geq 0$, $j = 0, \ldots, m-1$), are the unique solution of the recurrence relations $\Lambda_{nj}^{(m)} = \Lambda_{n-1,j}$ with initial conditions

$$\begin{cases} \Lambda_{nj}^{(r_\ell)}(s_\ell) = 0 \quad \text{for} \quad n \geq 1, \\ \Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \quad \text{for} \quad 0 \leq j, \ell \leq m-1, \end{cases}$$

with Λ_{nj} of degree $\leq mn + m - 1$.

Expansion of polynomials into interpolation series

Proposition.

Assume $\mathrm{D}(\mathbf{s}) \neq 0$. *Then any polynomial* f *has a finite expansion*

$$f(z) = \sum_{j=0}^{m-1} \sum_{n \geq 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z),$$

where only finitely many terms on the right hand side are nonzero.

Goal: to extend this expansion to entire functions of sufficiently small exponential type.

Expansion of entire functions

We will produce a number $\tau > 0$ such that the following holds.

Theorem.
Assume $\mathrm{D}(\mathbf{s}) \neq 0$. Then any entire function f of exponential type $< \tau$ has an expansion of the form

$$f(z) = \sum_{j=0}^{m-1} \sum_{n \geq 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z),$$

where the series in the right hand side is absolutely and uniformly convergent for z on any compact space in $\mathbb{C}$.

Unicity of the expansion

Corollary.

Assume $\mathrm{D}(\mathbf{s}) \neq 0$. *If an entire function* f *has exponential type* $< \tau$ *and satisfies*

$$f^{(mn+r_j)}(s_j) = 0$$

for $j = 0, \ldots, m-1$ *and all sufficiently large* n, *then* f *is a polynomial.*

The main examples

Examples

(1) Lidstone polynomials: $\tau = \pi$,

$$m = 2,\ s_0 = 0,\ s_1 = 1,\ r_0 = r_1 = 0,$$
$$\Lambda_{n0}(z) = \Lambda_n(1 - z),\ \Lambda_{n1}(z) = \Lambda_n(z).$$

(2) Whittaker polynomials: $\tau = \pi/2$,

$$m = 2,\ s_0 = 1,\ s_1 = 0,\ r_0 = 0,\ r_1 = 1,$$
$$\Lambda_{n0}(z) = M_n(z),\ \Lambda_{n1}(z) = M'_{n+1}(z - 1).$$

(3) Poritsky: assuming $s_0, s_1, \ldots, s_{m-1}$ are pairwise distinct,

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

(4) Gontcharoff periodic:

$$r_j = j \quad \text{for} \quad j = 0, 1, \ldots, m - 1.$$

Strategy of proof of the Theorem

Goal: for f of exponential type $< \tau$,

$$f(z) = \sum_{j=0}^{m-1} \sum_{n \geq 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z),$$

First prove it for e^{tz} for sufficiently small $|t|$, say $|t| < \tau$, next use Laplace transform to deduce it for f entire of type $< \tau$.

Special case $f_t(z) = \mathrm{e}^{tz}$, ${f_t}^{(mn+r_j)}(s_j) = t^{mn+r_j} \mathrm{e}^{ts_j}$

$$\mathrm{e}^{tz} = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \sum_{n \geq 0} t^{mn+r_j} \Lambda_{nj}(z).$$

Exponential sums

For $j = 0, 1, \ldots, m-1$ and $z \in \mathbb{C}$, consider the power series $\varphi_j(t, z) \in \mathbb{C}[[t]]$ defined by

$$\varphi_j(t, z) := \sum_{n \geq 0} t^{mn + r_j} \Lambda_{nj}(z).$$

Goal: there exists $\Theta > 0$ such that, for $|t| < 1/\Theta$ and $z \in \mathbb{C}$, we have

$$\mathrm{e}^{tz} = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t, z).$$

Examples

(1) Lidstone : $m = 2$, $s_0 = 0$, $s_1 = 1$, $r_0 = r_1 = 0$,

$$\varphi_0(t,z) = \frac{\sinh((1-z)t)}{\sinh(t)}, \quad \varphi_1(t,z) = \frac{\sinh(tz)}{\sinh(t)}.$$

(2) Whittaker : $m = 2$, $s_0 = 1$, $s_1 = 0$, $r_0 = 0$, $r_1 = 1$,

$$\varphi_0(t,z) = \frac{\cosh(tz)}{\cosh(t)}, \quad \varphi_1(t,z) = \frac{\sinh((z-1)t)}{\cosh(t)}.$$

Upper bound for the interpolation polynomials

Given complex numbers $a_0, a_1, \cdots$ and non negative real numbers $c_0, c_1, \ldots,$ we write

$$\sum_{i\geq 0} a_i z^i \preceq_z \sum_{i\geq 0} c_i z^i$$

if $|a_i| \leq c_i$ for all $i \geq 0$.

Lemma (D. Roy).

There exists a constant $\Theta > 0$ *such that*

$$\Lambda_{nj}(z) \preceq_z \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i$$

for all $n \geq 0$ *and* $j = 0, 1, \ldots, m-1$.

Expansion of e^{tz} for $|t| < 1/\Theta$

The functions

$$\varphi_j(t, z) := \sum_{n \geq 0} t^{mn+r_j} \Lambda_{nj}(z)$$

are analytic in the disc $|t| < 1/\Theta$.
Let us prove, for $|t| < 1/\Theta$ and $z \in \mathbb{C}$,

$$\mathrm{e}^{tz} = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t, z).$$

Proof. Define, for $|t| < 1/\Theta$ and $z \in \mathbb{C}$,

$$F(t, z) = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t, z) - \mathrm{e}^{tz}.$$

Proof of the expansion of e^{tz}

We have

$$\begin{aligned} F(t,z) &= \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t,z) - \mathrm{e}^{tz}. \\ &= \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}(z) - \mathrm{e}^{tz} \\ &= \sum_{n\geq 0} a_n(z) t^n, \end{aligned}$$

where $a_n(z) \in \mathbb{C}[z]_{\leq n+m-1}$ for all $n \geq 0$.

Proof of the expansion of e^{tz}

$$F(t,z) = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}(z) - \mathrm{e}^{tz} = \sum_{n\geq 0} a_n(z) t^n.$$

We obtain, for all $k \geq 0$ and $\ell = 0, 1, \ldots, m-1$,

$$\left(\frac{\partial}{\partial z}\right)^{mk+r_\ell} F(t,z)|_{z=s_\ell} = \sum_{n\geq 0} a_n^{(mk+r_\ell)}(s_\ell) t^n$$

$$= \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}^{(mk+r_\ell)}(s_\ell) - t^{mk+r_\ell} \mathrm{e}^{ts_\ell} = 0.$$

Therefore $a_n^{(mk+r_\ell)}(s_\ell) = 0$ for all $k \geq 0$, $n \geq 0$ and $\ell = 0, 1, \ldots, m-1$. We conclude $a_n(z) = 0$ for all $n \geq 0$.

Differential equations for $\varphi_j(t, z)$

Recall, for $|t| < 1/\Theta$,

$$\varphi_j(t, z) := \sum_{n \geq 0} t^{mn+r_j} \Lambda_{nj}(z)$$

and $\Lambda_{nj}^{(m)} = \Lambda_{n-1,j}$, with initial conditions

$$\Lambda_{nj}^{(r_\ell)}(s_\ell) = 0 \quad \text{for} \quad n \geq 1, \quad \Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \quad \text{for} \quad 0 \leq j, \ell \leq m-1.$$

Hence the functions $\varphi_0(t, z), \varphi_1(t, z), \ldots, \varphi_{m-1}(t, z)$ satisfy the differential equation

$$\left(\frac{\partial}{\partial z}\right)^m \varphi_j(t, z) = t^m \varphi_j(t, z) \quad \text{for} \quad j = 0, \ldots, m-1$$

with the initial conditions

$$\left(\frac{\partial}{\partial z}\right)^{r_\ell} \varphi_j(t, s_\ell) = t^{r_\ell} \delta_{j\ell} \quad \text{for} \quad 0 \leq j, \ell \leq m-1.$$

Differential equations

Let ζ be a primitive m-th root of unity. For $t \neq 0$, the general solution of the differential equation

$$f^{(m)}(z) = t^m f(z)$$

is a linear combination of the functions

$$\mathrm{e}^{\zeta^k t z} \qquad (k = 0, 1, \ldots, m-1)$$

with coefficients depending on t.

Hence for $0 < |t| < 1/\Theta$ there exist complex numbers $c_{jk}(t)$ $(j, k = 0, 1 \ldots, m-1)$ such that

$$\varphi_j(t, z) = \sum_{k=0}^{m-1} c_{jk}(t) \mathrm{e}^{\zeta^k t z}.$$

The coefficients $c_{jk}(t)$

Recall

$$\varphi_j(t,z) = \sum_{n\geq 0} t^{mn+r_j}\Lambda_{nj}(z) = \sum_{k=0}^{m-1} c_{jk}(t)\mathrm{e}^{\zeta^k tz}$$

and

$$\left(\frac{\partial}{\partial z}\right)^{r_\ell}\varphi_j(t,s_\ell) = t^{r_\ell}\delta_{j\ell} \quad \text{for} \quad 0\leq j,\ell\leq m-1.$$

Hence, for $0\leq j,\ell\leq m-1$ and $0<|t|<1/\Theta$, we have

$$\sum_{k=0}^{m-1} c_{jk}(t)\zeta^{kr_\ell}\mathrm{e}^{\zeta^k ts_\ell} = \delta_{j\ell}.$$

Product of matrices

For $0 \le j, \ell \le m-1$ and $0 < |t| < 1/\Theta$, we have

$$\sum_{k=0}^{m-1} c_{jk}(t) \zeta^{kr_\ell} \mathrm{e}^{\zeta^k t s_\ell} = \delta_{j\ell}.$$

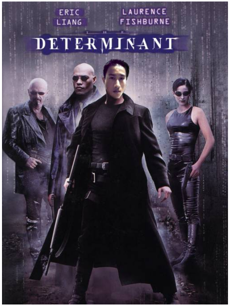
This means that the product

$$(c_{jk}(t))_{0 \le j,k \le m-1} \left(\zeta^{kr_\ell} \mathrm{e}^{\zeta^k t s_\ell} \right)_{0 \le k,\ell \le m-1}$$

is the identity $m \times m$ matrix.

A matrix and its determinant

When you have a matrix, you consider its determinant



The determinant $\Delta(t)$

For $t \in \mathbb{C}$, consider the $m \times m$ matrix

$$M(t) = \left(\zeta^{kr_\ell} \mathrm{e}^{\zeta^k t s_\ell}\right)_{0 \le k, \ell \le m-1}$$

and its determinant $\Delta(t) =$

$$\det \begin{pmatrix} \mathrm{e}^{ts_0} & \mathrm{e}^{ts_1} & \cdots & \mathrm{e}^{ts_{m-1}} \\ \zeta^{r_0}\mathrm{e}^{\zeta ts_0} & \zeta^{r_1}\mathrm{e}^{\zeta ts_1} & \cdots & \zeta^{r_{m-1}}\mathrm{e}^{\zeta ts_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{(m-1)r_0}\mathrm{e}^{\zeta^{m-1}ts_0} & \zeta^{(m-1)r_1}\mathrm{e}^{\zeta^{m-1}ts_1} & \cdots & \zeta^{(m-1)r_{m-1}}\mathrm{e}^{\zeta^{m-1}ts_{m-1}} \end{pmatrix}$$

Therefore the determinant $\Delta(t)$ does not vanish for $0 < |t| < 1/\Theta$.

The value of τ

Let τ be the least positive number such that $\Delta(t)$ does not vanish for $0 < |t| < \tau$.

For $|t| < 1/\Theta$ the matrix $\big(c_{jk}(t)\big)_{0\le j,k\le m-1}$ is the inverse of the matrix $M(t)$. We deduce that the functions $c_{jk}(t)$ are analytic in the domain $0 < |t| < \tau$.

The functions $\varphi_j(t,z)$ are now defined by

$$\varphi_j(t,z) = \sum_{k=0}^{m-1} c_{jk}(t)\mathrm{e}^{\zeta^k tz}$$

for all $z \in \mathbb{C}$ and for all t with $\Delta(t) \neq 0$. In particular the function of two variables $(t,z) \mapsto \varphi_j(t,z)$ is analytic in the domain $|t| < \tau$, $z \in \mathbb{C}$, and the equations

$$\varphi_j(t,z) = \sum_{n\ge 0} t^{mn+r_j}\Lambda_{nj}(z).$$

are valid in this domain.

Poritsky interpolation

$r_0 = r_1 = \cdots = r_{m-1} = 0$.

The condition $\mathrm{D}(\mathbf{s}) = 0$ means that $s_0, s_1, \ldots, s_{m-1}$ are pairwise distinct.

The function $\Delta(t)$ has a zero at the origin of multiplicity $m(m-1)/2$. The coefficient of $t^{m(m-1)/2}$ in the Taylor expansion at the origin of $\Delta(t)$ is given by a product of two Vandermonde determinants.

Gontcharoff interpolation (periodic)

$r_j = j$ for $j = 0, 1, \ldots, m-1$.

In this case $\Delta(0)$ is the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^{m-1} \\ 1 & \zeta^2 & \cdots & \zeta^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{m-1} & \cdots & \zeta^{(m-1)^2} \end{pmatrix},$$

and hence is not zero.

Recall Laplace transform

Let

$$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$. The Laplace transform of f,

$$F(t) = \sum_{n \geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t| > \tau(f)$. For $\varrho > \tau(f)$ we have

$$f(z) = \frac{1}{2\pi i} \int_{|t|=\varrho} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Hence

$$f^{(mn+r_j)}(z) = \frac{1}{2\pi i} \int_{|t|=\varrho} t^{mn+r_j} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

End of the proof

Assume $\tau(f) < \tau$. Let ϱ satisfy $\tau(f) < \varrho < \tau$. For $|t| = \varrho$, we have

$$\mathrm{e}^{tz} = \sum_{n\geq 0} \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} t^{mn+r_j} \Lambda_{nj}(z),$$

hence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|t|=\varrho} \mathrm{e}^{tz} F(t) \mathrm{d}t \\ &= \sum_{n\geq 0} \sum_{j=0}^{m-1} \left(\frac{1}{2\pi i} \int_{|t|=\varrho} t^{mn+r_j} \mathrm{e}^{ts_j} F(t) \mathrm{d}t \right) \Lambda_{nj}(z) \\ &= \sum_{n\geq 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z). \end{aligned}$$

The last series is absolutely and uniformly convergent for z on any compact space in $\mathbb{C}$.

Abel – Gontcharoff interpolation

Let $\mathbf{w} = (w_n)_{n\geq 0}$ be a sequence of complex numbers. There exists a sequence of polynomials $(\Omega_{n;\mathbf{w}})_{n\geq 0}$ in $\mathbb{C}[z]$ such that any polynomial f can be written as a finite sum

$$f(z) = \sum_{n\geq 0} f^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z).$$

We define $\Omega_{n;\mathbf{w}} = \Omega_{w_0,w_1,\ldots,w_{n-1}} \in \mathbb{C}[z]$ by induction on n so that

$$\Omega_{n;\mathbf{w}}^{(k)}(w_k) = \delta_{kn} \quad \text{for} \quad n \geq 0 \quad \text{and} \quad k \geq 0.$$

We set $\Omega_{0,\mathbf{w}} = \Omega_{\emptyset} = 1$, $\Omega_{1,\mathbf{w}} = \Omega_{w_0}(z) = z - w_0$.
For $n \geq 1$, we define $\Omega_{w_0,w_1,w_2,\ldots,w_n}(z)$ as the polynomial of degree $n+1$ which is the primitive of $\Omega_{w_1,w_2,\ldots,w_n}$ vanishing at w_0.

The polynomials $\Omega_{n;\mathbf{w}} = \Omega_{w_0,w_1,\ldots,w_{n-1}}$

For $n \geq 0$, $\Omega_{n;\mathbf{w}}$ is a polynomial of degree n which depends only on the first n terms of the sequence $\mathbf{w}$.
The leading term of $\Omega_{n;\mathbf{w}}$ is $(1/n!)z^n$.
For $N \geq 0$ we have

$$\frac{z^N}{N!} = \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\mathbf{w}}(z).$$

This gives an inductive formula defining $\Omega_{N;\mathbf{w}}$: for $N \geq 0$,

$$\Omega_{N;\mathbf{w}}(z) = \frac{z^N}{N!} - \sum_{n=0}^{N-1} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\mathbf{w}}(z).$$

The polynomials $\Omega_{n;\mathbf{w}}$

From the definition we deduce the following formula, involving iterated integrals

$$\Omega_{w_0,w_1,\ldots,w_{n-1}}(z) = \int_{w_0}^{z} \mathrm{d}t_1 \int_{w_1}^{t_1} \mathrm{d}t_2 \cdots \int_{w_{n-1}}^{t_{n-1}} \mathrm{d}t_n.$$

Examples: since

$$\Omega_{w_0,w_1,\ldots,w_n}(z) = \Omega_{0,w_1-w_0,w_2-w_0,\ldots,w_n-w_0}(z-w_0),$$

it suffices to consider the case $w_0 = 0$.

$$\begin{aligned} 2!\Omega_{0,w_1}(z) &= (z-w_1)^2 - {w_1}^2, \\ 3!\Omega_{0,w_1,w_2}(z) &= (z-w_2)^3 - 3(w_1-w_2)^2 z + {w_2}^3, \\ 4!\Omega_{0,w_1,w_2,w_3}(z) &= (z-w_3)^4 - 6(w_2-w_3)^2(z-w_1)^2 \\ &\quad - 4(w_1-w_3)^3 z + 6{w_1}^2(w_2-w_3)^2 - {w_3}^4. \end{aligned}$$

Gontcharoff determinant for $\Omega_{w_0,w_1,\dots,w_{n-1}}(z)$

$$\Omega_{w_0,w_1,\dots,w_{n-1}}(z) = (-1)^n \begin{vmatrix} 1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{n-1}}{(n-1)!} & \frac{z^n}{n!} \\ 1 & \frac{w_0}{1!} & \frac{{w_0}^2}{2!} & \cdots & \frac{{w_0}^{n-1}}{(n-1)!} & \frac{{w_0}^n}{n!} \\ 0 & 1 & \frac{w_1}{1!} & \cdots & \frac{{w_1}^{n-2}}{(n-2)!} & \frac{{w_1}^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & \cdots & \frac{{w_2}^{n-3}}{(n-3)!} & \frac{{w_2}^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{w_{n-1}}{1!} \end{vmatrix}.$$

Two examples

- For $\mathbf{w} = (1, 0, 1, 0, \ldots, 0, 1, \ldots)$, we recover the Whittaker polynomials

$$\Omega_{2n;\mathbf{w}}(z) = M_n(z), \quad \Omega_{2n+1,\mathbf{w}}(z) = M'_{n+1}(z-1).$$

- For the arithmetic progression $\mathbf{w} = (a + nt)_{n\geq 0}$ with a in $\mathbb{C}$ and t in $\mathbb{C} \setminus \{0\}$, we obtain Abel's polynomials

$$\Omega_{n;\mathbf{w}}(z) = \frac{1}{n!}(z-a)(z-a-nt)^{n-1}$$

for $n \geq 1$, which satisfy

$$\Omega'_{n;\mathbf{w}}(z) = \Omega_{n-1;\mathbf{w}}(z-t).$$

Estimate for $|\Omega_{n;\mathbf{w}}|$ when $\sup_{n\geq 0}|w_n| < \infty$

Assume that the sequence $(|w_n|)_{n\geq 0}$ is bounded. Let $A > \sup_{n\geq 0}|w_n|$.

Proposition.

Let $\kappa > 1/\log 2$. *For* n *sufficiently large, we have, for all* $r \geq |A|$,

$$|\Omega_{n;\mathbf{w}}|_r \leq (\kappa r)^n.$$

Expansion in a disc containing $|z| \le A$

Recall $\sup_{n\ge 0} |w_n| < A$.

Proposition.

Let f *be an entire function of exponential type* $\tau(f)$ *satisfying* $\tau(f) < \log 2/A$. *Then*

$$f(z) = \sum_{n\ge 0} f^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z),$$

where the series on the right hand side is absolutely and uniformly convergent in any disk $|z| \le r$ *with* $r < \log 2/\tau(f)$.

Two examples

Corollary.

If an entire function f of exponential type $\tau(f) < \log 2/A$ satisfies $f^{(n)}(w_n) = 0$ for all sufficiently large n, then f is a polynomial.

Special case where the set $\{w_0, w_1, w_2, \dots\}$ is finite, say $\{s_0, s_1, \dots, s_{m-1}\}$, with

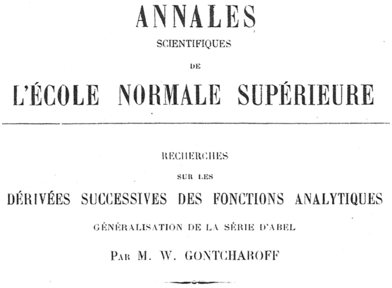
$$\max\{|s_0|, |s_1|, \dots, |s_{m-1}|\} < A.$$

Corollary.

If an entire function f of exponential type $\tau(f) < \log 2/A$ satisfies

$$\prod_{j=0}^{m-1} f^{(n)}(s_j) = 0$$

for all sufficiently large n, then f is a polynomial.


ANNALES
SCIENTIFIQUES
DE
L'ÉCOLE NORMALE SUPÉRIEURE

RECHERCHES
SUR LES
DÉRIVÉES SUCCESSIVES DES FONCTIONS ANALYTIQUES
GÉNÉRALISATION DE LA SÉRIE D'ABEL
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Interpolation problem for

$$f^{(n)}(\sigma_n), \quad n \geq 0.$$

Example:

$$f^{(nm+j)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m-1.$$

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Interpolation problem for

$$f^{(nm)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m-1.$$

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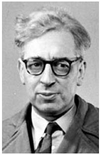

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Interpolation problem for

$$f^{(nm+b_j)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m-1.$$

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A course on interpolation

Third Course :
Several Points
Poritsky, Gontcharoff

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Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/

09/12/2020