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# Two invariants related to two conjectures due to Nagata

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# Abstract

Seshadri's constant is related to a conjecture due to Nagata. Another conjecture, also due to Nagata and solved by Bombieri in 1970, is related with algebraic values of meromorphic functions. The main argument of Bombieri's proof leads to a Schwarz Lemma in several variables, the proof of which gives rise to another invariant associated with symbolic powers of the ideal of functions vanishing on a finite set of points. This invariant is an asymptotic measure of the least degree of a polynomial in several variables with given order of vanishing on a finite set of points. Recent works on the resurgence of ideals of points and the containment problem compare powers and symbolic powers of ideals.

# Schneider – Lang Theorem (1949, 1966)

Theodor Schneider
(1911 – 1988)

Serge Lang
(1927 – 2005)

*Let $f_1, \ldots, f_m$ be meromorphic functions in $\mathbb{C}$. Assume $f_1$ and $f_2$ are algebraically independent and of finite order. Let $\mathbb{K}$ be a number field. Assume $f_j'$ belongs to $\mathbb{K}[f_1, \ldots, f_m]$ for $j = 1, \ldots, m$. Then the set*

$$S = \{w \in \mathbb{C} \mid w \text{ not pole of } f_j,\ f_j(w) \in \mathbb{K}\ (j = 1, \ldots, m)\}$$

*is finite.*

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lang.html

# Hermite – Lindemann Theorem (1882)

Charles Hermite
(1822 – 1901)

Carl Louis Ferdinand von Lindemann
(1852 – 1939)

**Corollary**. *If $w$ is a non zero complex number, one at least of the two numbers $w$, $e^w$ is transcendental.*

**Consequence** : transcendence of $e$, $\pi$, $\log \alpha$, $e^\beta$, for algebraic $\alpha$ and $\beta$ assuming $\alpha \neq 0$, $\beta \neq 0$, $\log \beta \neq 1$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hermite.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lindemann.html

# Gel'fond – Schneider Theorem (1934)

Aleksandr Osipovich Gelfond
(1906 – 1968)

Theodor Schneider
(1911 – 1988)

**Corollary (Hilbert's seventh problem)**. *If $\beta$ is an irrational algebraic number and $w$ a non zero complex number, one at least of the two numbers $e^w$, $e^{\beta w}$ is transcendental.*

**Consequence** : transcendence of $e^\pi$, $2^{\sqrt{2}}$, $\alpha^\beta$, $\log\alpha_1/\log\alpha_2$, for algebraic $\alpha$, $\alpha_1$, $\alpha_2$ and $\beta$ assuming $\alpha \neq 0$, $\log\alpha \neq 0$, $\beta \notin \mathbb{Q}$, $\log\alpha_1/\log\alpha_2 \notin \mathbb{Q}$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Gelfond.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html

# Proofs of the corollaries

Hermite – Lindemann. Let $\mathbb{K} = \mathbb{Q}(w, e^w)$. The two functions $f_1(z) = z$, $f_2(z) = e^z$ are algebraically independent, of finite order, and satisfy the differential equations $f_1' = 1$, $f_2' = f_2$. The set $S$ contains $\{\ell w \mid \ell \in \mathbb{Z}\}$. Since $w \neq 0$, this set is infinite ; it follows that $\mathbb{K}$ is not a number field. $\square$

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# Schneider's Theorems on elliptic functions (1937)

**Corollary (Schneider).** *Let $\wp$ be an elliptic function of Weierstrass with algebraic invariants $g_2$, $g_3$. Let $w$ be a complex number, not pole of $\wp$. Then one at least of the two numbers $w$, $\wp(w)$ is transcendental.*

Proof. Let $\mathbb{K} = \mathbb{Q}(g_2, w, \wp(w), \wp'(w))$. The two functions $f_1(z) = z$, $f_2(z) = \wp(z)$ are algebraically independent, of finite order. Set $f_3(z) = \wp'(z)$. From $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ one deduces

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6f_2^2 - (g_2/2).$$

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# The transcendence machinery

The prototype of transcendence methods is Hermite's proof of the transcendence of $e$.

The proof of the Schneider – Lang Theorem follows the following scheme :

**Step 1** Construct an auxiliary function $f$ with many zeroes.

**Step 2** Find a point $z_0$ where $f(z_0) \neq 0$.

**Step 3** Give a lower bound for $|f(z_0)|$ using arithmetic arguments.

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# Schwarz Lemma in one variable

Hermann Amandus Schwarz
(1843 – 1921)

Let $f$ be an analytic function in a disc $|z| \leq R$ of $\mathbb{C}$, with at least $N$ zeroes in a disc $|z| \leq r$ with $r < R$. Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^N |f|_R.$$

We use the notation

$$|f|_r = \sup_{|z|=r} |f(z)|.$$

When $R > 3r$, this improves the maximum modulus bound $|f|_r \leq |f|_R$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schwarz.html

# Schwarz Lemma in one variable : proof

Let $a_1, \ldots, a_N$ be zeroes of $f$ in the disc $|z| \le r$, counted with multiplicities. The function

$$g(z) = f(z) \prod_{j=1}^{N} (z - a_j)^{-1}$$

is analytic in the disc $|z| \le R$. Using the maximum modulus principle, from $r \le R$ we deduce $|g|_r \le |g|_R$. Now we have

$$|f|_r \le (2r)^N |g|_r \quad \text{and} \quad |g|_R \le (R-r)^{-N} |f|_R.$$

Finally, assuming (wlog) $R > 3r$,

$$\frac{2r}{R-r} \le \frac{3r}{R}.$$

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# Blaschke factor

Wilhelm Johann Eugen Blaschke
(1885 – 1962)

Let $R > 0$ and let $a \in \mathbb{C}$ satisfy $|a| \leq R$. The Blaschke factor is defined in $|z| \leq R$ by

$$B_a(z) = \frac{z - a}{R^2 - \overline{a}z},$$

where $\overline{a}$ is the complex conjugate of $a$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Blaschke.html

http://mathworld.wolfram.com/BlaschkeFactor.html

# Estimating a Blaschke factor

**Lemma.** *Let* $|a| < R$*. The function*

$$B_a(z) = \frac{z-a}{R^2 - \overline{a}z} \quad (|z| \le R)$$

*satisfies*

$$|B_a(z)| = \frac{1}{R} \quad \text{for} \quad |z| = R.$$

*Moreover, for* $r$ *in the range* $|a| \le r < R$*, we have*

$$\sup_{|z|=r} |B_a(z)| = |B_a(-ar/|a|)| = \frac{r+|a|}{R^2 + r|a|} \le \frac{2r}{R^2+r^2}.$$

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# Schwarz Lemma with a Blaschke product

**Refinement of Schwarz Lemma in one variable.**
*Let* $f$ *be an analytic function in a disc* $|z| \leq R$ *of* $\mathbb{C}$*, with at least* $N$ *zeroes in a disc* $|z| \leq r$ *with* $r < R$*. Then*

$$|f|_r \leq \left(\frac{2rR}{R^2 + r^2}\right)^N |f|_R.$$

Proof. The function

$$g(z) = f(z) \prod_{j=1}^{N} \frac{R^2 - \bar{a}_j z}{z - a_j}$$

is analytic in the disc $|z| \leq R$. □

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# Schneider – Lang Theorem in several variables : cartesian products (1941, 1966)

*Let* $f_1,\ldots,f_m$ *be meromorphic functions in* $\mathbb{C}^n$ *with* $m\geq n+1$. *Assume* $f_1,\ldots,f_{n+1}$ *are algebraically independent of finite order. Let* $\mathbb{K}$ *be a number field. Assume* $(\partial/\partial z_i)f'_j$ *belongs to* $\mathbb{K}[f_1,\ldots,f_m]$ *for* $j=1,\ldots,m$ *and* $i=1,\ldots,n$. *If* $e_1,\ldots,e_n$ *is a basis of* $\mathbb{C}^n$, *then the set*

$$S=\{w\in\mathbb{C}^n \mid w \text{ not pole of } f_j,\ f_j(w)\in\mathbb{K}\ (j=1,\ldots,m)\}$$

*does not contain a cartesian product*

$$\{s_1e_1+\cdots+s_ne_n \mid (s_1,\ldots,s_n)\in S_1\times\cdots\times S_n\}$$

*where each* $S_i$ *is infinite.*

# Schneider's Theorem on Euler's Beta function

Leonhard Euler
(1707 – 1783)

*Let $a$, $b$ be rational numbers, not integers. Then the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

*is transcendental.*

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Euler.html

Further results by Th. Schneider and S. Lang on abelian functions and algebraic groups.

# Schwarz lemma in several variables : cartesian products

*Let* $f$ *be an analytic function in a ball* $|z| \leq R$ *of* $\mathbb{C}^n$*. Assume* $f$ *vanishes with multiplicity at least* $t$ *on a set* $S_1 \times \cdots \times S_n$ *where each* $S_i$ *is contained in a disc* $|z| \leq r$ *with* $r < R$ *and has at least* $s$ *elements.*
*Then*

$$|f|_r \leq \left(\frac{3r}{R}\right)^{st} |f|_R.$$

# Cartesian products

Schwarz Lemma for Cartesian products can be proved by induction.

§4.3 of M.W.. *Diophantine Approximation on Linear Algebraic Groups.* Grund. Math. Wiss. **326** Springer-Verlag (2000).

Another proof, based on integral formulae, yields a weaker result : for $R > 3r$,

$$|f|_r \leq \left(\frac{R-3r}{2r}\right)^n \left(\frac{3r}{R}\right)^{st} |f|_R.$$

The conclusion follows from a homogeneity argument : replace $f$ by $f^N$ (and $t$ by $Nt$) and let $N \to \infty$.

Chap. 7 of M.W.. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70** (1979).

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# Landau's trick (Pólya – Szegő)

Edmund Georg Hermann Landau (1877 – 1938)

George Pólya (1887 – 1985)

Gábor Szegő (1895 – 1985)

G. Pólya and G. Szegő. *Problems and theorems in analysis. Vol. II. Theory of functions, zeros, polynomials, determinants, number theory, geometry.* Grundlehren der Mathematischen Wissenschaften, Band **216**. Springer-Verlag, New York-Heidelberg, 1976.

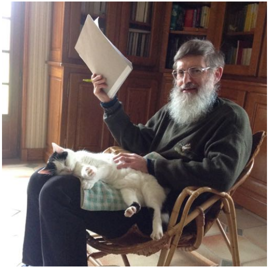
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Landau.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Polya.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Szego.html

# Philippon's redundant variables

Recall step 1 of the transcendence machinery :
**Step 1** Construct an auxiliary function $f$ with many zeroes.



For completing proofs of algebraic independence suggested by G.V. Chudnovsky, P. Philippon introduces several variables and derives the conclusion by letting the number of variables tend to infinity.

# Nagata's suggestion (1966)

Masayoshi Nagata
(1927 – 2008)

In the conclusion of the Schneider – Lang Theorem, replace the fact that $S$ does not contain a cartesian product $S_1 \times \cdots \times S_n$ where each $S_i$ is infinite by the fact that $S$ is contained in an algebraic hypersurface.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Nagata.html

# Bombieri's Theorem (1970)

*Let $f_1, \ldots, f_m$ be meromorphic functions in $\mathbb{C}^n$ with $m \geq n+1$. Assume $f_1, \ldots, f_{n+1}$ are algebraically independent and of finite order. Let $\mathbb{K}$ be a number field. Assume $(\partial/\partial z_i) f'_j$ belongs to $\mathbb{K}[f_1, \ldots, f_m]$ for $j = 1, \ldots, m$ and $i = 1, \ldots, n$.*

Enrico Bombieri

*Then the set*

$$S = \{w \in \mathbb{C}^n \mid w \text{ not pole of } f_j, \; f_j(w) \in \mathbb{K} \; (j = 1, \ldots, m)\}$$

*is contained in an algebraic hypersurface.*

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Bombieri.html

# Bombieri – Lang (1970)

*Let* $f$ *be an analytic function in a ball* $|z| \leq R$ *of* $\mathbb{C}^n$. *Assume* $f$ *vanishes at* $N$ *points* $z_i$ *(counting multiplicities) in a ball* $|z| \leq r$ *with* $r < R$. *Assume* $\min_{z_i \neq z_k} |z_i - z_k| \geq \delta$.

*Then*

$$|f|_r \leq \left(\frac{3r}{R}\right)^M |f|_R$$

*with*

$$M = N\left(\frac{\delta}{6r}\right)^{2n-2}.$$

# Lelong number

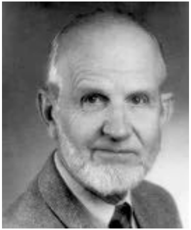
E. Bombieri. *Algebraic values of meromorphic maps.* Invent. Math. **10** (1970), 267–287.
E. Bombieri and S. Lang. *Analytic subgroups of group varieties.* Invent. Math. **11** (1970), 1–14.

Pierre Lelong
(1912 – 2011)

P. Lelong. *Intégration sur un ensemble analytique complexe*, Bulletin S.M.F. **85** (1957), 239–262,

`https://fr.wikipedia.org/wiki/Pierre_Lelong`

# $L^2$- estimates of Hörmander



Lars Hörmander
(1931 – 2012)

Existence theorems for the $\overline{\partial}$ operator.

E. Bombieri. *Let $\varphi$ be a plurisubharmonic function in $\mathbb{C}^n$ and $z_0 \in \mathbb{C}^n$ be such that $e^{-\varphi}$ is integrable near $z_0$. Then there exists a nonzero entire function $F$ such that*

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)} (1 + |z|^2)^{-3n} d\lambda(z) < \infty.$$

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hormander.html

# Towards a Schwarz lemma in several variables

Let $S$ be a finite subset of $\mathbb{C}^n$ and $t$ a positive integer. Let $M$ be a positive number with the following property.
*There exists a real number $r$ such that for $R > r$, if $f$ is an analytic function in the ball $|z| \leq R$ of $\mathbb{C}^n$ which vanishes with multiplicity at least $t$ at each point of $S$, then*

$$|f|_r \leq \left(\frac{c(n)r}{R}\right)^M |f|_R,$$

*where $c(n)$ depends only on the dimension $n$.*

Question: what is the largest possible value for $M$ ?

Answer: use the property for $f$ a nonzero polynomial vanishing on $S$ with multiplicity $t$. We deduce that $f$ has degree at least $M$.

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# Degree of hypersurfaces

Let $S$ be a finite set of $\mathbb{C}^n$ and $t$ a positive integer.

Denote by $\omega_t(S)$ the smallest degree of a polynomial vanishing at each point of $S$ with multiplicity $\geq t$.

M.W. *Propriétés arithmétiques de fonctions de plusieurs variables* (II). Sém. P. Lelong (Analyse), 16è année, 1975/76 ; Lecture Notes in Math., **578** (1977), 274–292.

M.W. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70**. Société Mathématique de France, Paris, 1979.

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$$|f|_r \leq \left(\frac{e^n r}{R}\right)^{\omega_t(S)} |f|_R.$$

This is a refined asymptotic version due to Jean-Charles Moreau.

The exponent $\omega_t(S)$ cannot be improved : take for $f$ a non–zero polynomial of degree $\omega_t(S)$.

# Homogeneous ideals of $\mathbb{K}[X_0, \ldots, X_n]$

For $p = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n(\mathbb{K})$, denote by $I(p)$ the homogeneous ideal generated by the polynomials $\alpha_i X_j - \alpha_j X_i$ $(0 \leq i, j \leq n)$ in the polynomial ring $R = \mathbb{K}[X_0, \ldots, X_n]$.

For $S = \{p_1, \ldots, p_s\} \subset \mathbb{P}^n(\mathbb{K})$, set

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# Initial degree

Generally, when $J$ is a nonzero homogeneous ideal of $R$, define $\omega(J)$ as the least degree of a polynomial in $J$.

Since $J$ is homogeneous,

$$J = \bigoplus_{m \geq 0} J_m$$

we have

$$\omega(J) = \min\{m \geq 0 \mid J_m \neq 0\}$$

and $\omega(J)$ is also called the *initial degree* of $J$.

Since $J_1 J_2$ is generated by the products $P_1 P_2$ with $P_i \in J_i$, it is plain that $\omega(J_1 J_2) = \omega(J_1) + \omega(J_2)$, hence

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# Symbolic powers

For $t \geq 1$, define the symbolic power $I^{(t)}(S)$ by

$$I^{(t)}(S) = I(p_1)^t \cap \cdots \cap I(p_s)^t.$$

This is the ideal of forms vanishing on $S$ with multiplicities $\geq t$. Hence

$$\omega(I^{(t)}(S)) = \omega_t(S).$$

We have $I(S)^t \subset I(p_i)^t$ for all $i$, hence

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# Dirichlet's box principle

Johann Peter Gustav Lejeune Dirichlet
(1805 – 1859)

Given a finite subset $S$ of $\mathbb{K}^n$ and a positive integer $t$, if $D$ is a positive integer such that

$$|S|\binom{t+n-1}{n} < \binom{D+n}{n},$$

then

$$\omega_t(S) \leq D.$$

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Dirichlet.html

# Properties of $\omega_t(S)$

Consequence of Dirichlet's box principle :

$$\omega_t(S) \leq (t+n-1)|S|^{1/n}.$$

Subadditivity :

$$\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S).$$

For a Cartesian product $S = S_1 \times \cdots \times S_n$ in $\mathbb{K}^n$,

$$\omega_t(S) = t \min_{1 \leq i \leq n} |S_i|.$$

In particular for $n = 1$ we have $\omega_t(S) = t|S|$.

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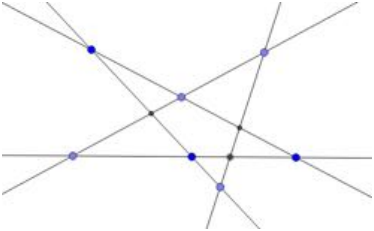
# Complete intersections of hyperplanes

Let $H_1, \dots, H_N$ be $N$ hyperplanes in general position in $\mathbb{K}^n$ with $N \geq n$ and $S$ the set of $\binom{N}{n}$ intersection points of any $n$ of them. Then, for $t \geq 1$,

$$\omega_{nt}(S) = Nt.$$



$n = 2$, $N = 5$, $|S| = 10$.

# An asymptotic invariant

**Theorem**. *The sequence*

$$\left(\frac{1}{t}\omega_t(S)\right)_{t\geq 1}$$

*has a limit* $\Omega(S)$ *as* $t\to\infty$, *and*

$$\frac{1}{n}\omega_1(S)-2\leq\Omega(S)\leq\omega_1(S).$$

*Further, for all* $t\geq 1$ *we have*

$$\Omega(S)\leq\frac{\omega_t(S)}{t}.$$

Remark : $\Omega(S)\leq|S|^{1/n}$.

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# Improvement of $L^2$ estimate by Henri Skoda

*Let $\varphi$ be a plurisubharmonic function in $\mathbb{C}^n$ and $z_0 \in \mathbb{C}^n$ be such that $e^{-\varphi}$ is integrable near $z_0$. For any $\epsilon > 0$ there exists a nonzero entire function $F$ such that*

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)} (1 + |z|^2)^{-n-\epsilon} d\lambda(z) < \infty.$$

**Corollary** :

$$\frac{1}{n}\omega_1(S) \le \Omega(S) \le \omega_1(S).$$

H. Skoda. *Estimations $L^2$ pour l'opérateur $\overline{\partial}$ et applications arithmétiques.* Springer Lecture Notes in Math., **578** (1977), 314–323.

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# Comparing $\omega_{t_1}(S)$ and $\omega_{t_2}(S)$

`Idea:` Let $P$ be a polynomial of degree $\omega_{t_1}(S)$ vanishing on $S$ with multiplicity $\geq t_1$. If the function $P^{t_2/t_1}$ were an entire function, it would be a polynomial of degree $t_2/t_1$ vanishing on $S$ with multiplicity $\geq t_2$, which would yield $\omega_{t_2}(S) \leq \frac{t_2}{t_1}\omega_{t_1}(S)$.

$P^{t_2/t_1}$ is usually not an entire function but $\varphi = \frac{t_2}{t_1} \log P$ is a plurisubharmonic function. By the $L^2$–estimates of Hörmander – Bombieri – Skoda, $e^{\varphi}$ is well approximated by a nonzero entire function. This function is a polynomial vanishing on $S$ with multiplicity $\geq t_2$.

**Theorem.** *For all* $t \geq 1$*,*

$$\frac{\omega_t}{t+n-1} \leq \Omega(S) \leq \frac{\omega_t}{t}.$$

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# Connection with C.S. Seshadri constant

Conjeevaram Srirangachari Seshadri

For a generic set $S$ of $s$ points in $\mathbb{P}^n$, Seshadri's constant $\epsilon(S)$ is related to $\Omega(S)$ by

$$\epsilon(S)^{n-1} = \frac{\Omega(S)}{s}.$$

# C.S. Seshadri's criterion

*Let $X$ be a smooth projective varitety and $L$ a line bundle on $X$. Then $L$ is ample if and only if there exists a positive number $\epsilon$ such that for all points $x$ on $X$ and all irreducible curves $C$ passing through $x$ one has*

$$L \cdot C \leq \epsilon \, \mathrm{mult}_x C.$$

R. Hartshorne. *Ample subvarieties of algebraic varieties.* Springer Lecture Notes in Math., vol. **156**, Springer (1970).

# C.S. Seshadri constant at a point

Let $X$ be a smooth projective variety and $L$ a nef line bundle on $X$. For a fixed point $x \in X$ the real number

$$\epsilon(X, L; x) := \inf \frac{L \cdot C}{\mathrm{mult}_x C}$$

is the Seshadri constant of $L$ at $x$.
The infimum is over all irreducible curves passing through $x$.

J.-P. Demailly. *Singular Hermitian metrics on positive line bundles*. Complex algebraic varieties (Bayreuth, 1990), Lecture Notes Math. **1507**, Springer-Verlag, (1992) 87–104.

# C.S. Seshadri constant at a subscheme

For an arbitrary subscheme $Z \subset X$, let $f : Y \to X$ be the blow-up of $X$ along $Z$ with the exceptional divisor $E$. The Seshadri constant of $L$ at $Z$ is the real number

$$\epsilon(X, L; Z) := \sup\{\lambda : f^*L - \lambda E \text{ is ample on } Y\}.$$

T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek, and T. Szemberg. *A primer on Seshadri constants.* Contemp. Math., **496** (2009), 33–70.

# C.S. Seshadri

Conjeevaram Srirangachari Seshadri

C.S. Seshadri FRS (born 29 February 1932) is an eminent Indian mathematician.

He is Director-Emeritus of the Chennai Mathematical Institute, and is known for his work in algebraic geometry. The Seshadri constant is named after him.
He is a recipient of the Padma Bhushan in 2009, the third highest civilian honor in the country. In 2013 he received a Doctorate Honoris Causa from Université P. et M. Curie (Paris 6).

https://en.wikipedia.org/wiki/C._S._Seshadri

# Zero estimates

Recall step 2 of the transcendence machinery :

**Step 2** Find a point $z_0$ where $f(z_0) \neq 0$.

`Context` : zero estimates, multiplicity estimates, interpolation estimates on an algebraic group.

Results of C Hermite, C.L. Siegel, Th. Schneider, K. Mahler, A.O. Gel'fond, R. Tijdeman, W.D. Brownawell, D.W. Masser, G. Wüstholz, P. Philippon, J-C. Moreau, D. Roy, M. Nakamaye, N. Rattazzi, S. Fischler. . .

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# Michael Nakamaye and Nicolas Ratazzi

M. Nakamaye and N. Ratazzi. *Lemmes de multiplicités et constante de Seshadri.* Math. Z. 259, No. 4, 915-933 (2008).

http://www.math.unm.edu/research/faculty_hp.php?d_id=96

http://www.math.u-psud.fr/~ratazzi/

# Stéphane Fischler and Michael Nakamaye

S. Fischler and M. Nakamaye. *Seshadri constants and interpolation on commutative algebraic groups.* Ann. Inst. Fourier **64**, No. 3, 1269-1289 (2014).

http://www.math.u-psud.fr/~fischler/

http://www.math.unm.edu/research/faculty_hp.php?d_id=96

# S. David, M. Nakamaye, P. Philippon

  

*Bornes uniformes pour le nombre de points rationnels de certaines courbes*, Diophantine geometry, 143–164, CRM Series, 4, Ed. Norm., Pisa, 2007.

`http://www.math.unm.edu/~nakamaye/Pisa.pdf`

# S. David, M. Nakamaye, P. Philippon

*Au Professeur C. S. Seshadri à l'occasion de son 75ème anniversaire.*

Nous commençons par une étude indépendante des jets des sections de fibrés amples sur une surface lisse, puis sur le carré d'une courbe elliptique, utile pour le Théorème 4.2. Ceci nous permet en particulier d'introduire dans le présent contexte les constantes de Seshadri, dont l'utilisation en géométrie diophantienne nous semble devoir être positivement stimulée.

http://www.math.unm.edu/~nakamaye/Pisa.pdf

# Hilbert's 14th problem

David Hilbert
(1862 – 1943)

*Let $k$ be a field and $K$ a subfield of $k(X_1, \dots, X_n)$ containing $k$. Is the $k$–algebra*

$$K \cap k[X_1, \dots, X_n]$$

*finitely generated ?*

Oscar Zariski (1954) : true for $n = 1$ and $n = 2$.
Counterexample by Masayoshi Nagata in 1959.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hilbert.html

http://www.clarku.edu/~djoyce/hilbert/

# Hilbert's 14th problem : restricted case

Masayoshi Nagata
(1927 – 2008)

Original 14th problem :
*Let $G$ be a subgroup of the full linear group of the polynomial ring in indeterminate $X_1, \ldots, X_n$ over a field $k$, and let $\mathfrak{o}$ be the set of elements of $k[X_1, \ldots, X_n]$ which are invariant under $G$. Is $\mathfrak{o}$ finitely generated ?*

M. Nagata. *On the 14-th Problem of Hilbert*. Amer. J. Math **81** (1959), 766–772.
http://www.jstor.org/stable/2372927

# Fundamental Lemma of Nagata

Given $16$ independent generic points of the projective plane over a prime field and a positive integer $t$, there is no curve of degree $4t$ which goes through each $p_i$ with multiplicity at least $t$.

In other words for $|S| = 16$ generic in $\mathbb{K}^2$, we have $\omega_t(S) > 4t$.

Reference: M. Nagata. *On the fourteenth problem of Hilbert.* Proc. Internat. Congress Math. 1958, Cambridge University Press, pp. 459–462.

http://www.mathunion.org/ICM/ICM1958/Main/icm1958.0459.0462.ocr.pdf

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`http://www.mathunion.org/ICM/ICM1958/Main/icm1958.0459.0462.ocr.pdf`

# Nagata' contribution

Masayoshi Nagata
(1927 – 2008)

**Proposition**. *Let $p_1, \ldots, p_r$ be independent generic points of the projective plane over the prime field. Let $C$ be a curve of degree $d$ passing through the $p_i$'s with multiplicities $\geq m_i$. Then*

$$m_1 + \cdots + m_r < d\sqrt{r}$$

*for $r = s^2$, $s \geq 4$.*

It is not known if $r > 9$, is sufficient to ensure the inequality of the Proposition.

M. Nagata. *Lectures on the fourteenth problem of Hilbert.* Tata Institute of Fundamental Research Lectures on Mathematics **31**, (1965), Bombay.

http://www.math.tifr.res.in/~publ/ln/tifr31.pdf

# Reformulation of Nagata's Conjecture

By considering $\sum_\sigma C_\sigma$ where $\sigma$ runs over the cyclic permutations of $\{1, \ldots, r\}$, it is sufficient to consider the case $m_1 = \cdots = m_r$.

**Conjecture**. *Let $S$ be a finite generic subset of the projective plane over the prime field with $|S| \geq 10$. Then*

$$\omega_t(S) > t\sqrt{|S|}.$$

Nagata :

- True for $|S|$ a square.
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# The Nagata – Biran Conjecture

Masayoshi Nagata

Paul Biran

Let $X$ be a smooth algebraic surface and $L$ an ample line bundle on $X$ of degree $d$. For sufficiently large $r$, the Seshadri constant of a generic set $Z = \{p_1, \ldots, p_r\}$ satisfies

$$\epsilon(X, L; Z) = \frac{d}{\sqrt{r}}.$$

The Oberwolfach Photo Collection http://owpdb.mfo.de/

# $|S| \leq 9$ in $\mathbb{K}^2$

Nagata : for $|S| \leq 9$ in $\mathbb{K}^2$ we have $\dfrac{\omega_t(S)}{t} \leq \sqrt{|S|}$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\|S\|$ | $=$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ |
| $\omega_1(S)$ | $=$ | $1$ | $1$ | $2$ | $2$ | $2$ | $3$ | $3$ | $3$ | $3$ |
| $t$ | $=$ | $1$ | $1$ | $2$ | $1$ | $1$ | $5$ | $8$ | $17$ | $1$ |
| $\omega_t(S)$ | $=$ | $1$ | $1$ | $3$ | $2$ | $2$ | $12$ | $21$ | $48$ | $3$ |
| $\dfrac{\omega_t(S)}{t}$ | $=$ | $1$ | $1$ | $\dfrac{3}{2}$ | $2$ | $2$ | $\dfrac{12}{5}$ | $\dfrac{21}{8}$ | $\dfrac{48}{17}$ | $3$ |
| $\sqrt{\|S\|}$ | $=$ | $1$ | $\sqrt{2}$ | $\sqrt{3}$ | $2$ | $\sqrt{5}$ | $\sqrt{6}$ | $\sqrt{7}$ | $\sqrt{8}$ | $3$ |

# $|S| = 1$ or 2 in $\mathbb{K}^2$

$|S| = 1$ : $S = \{(0,0)\}$, $P_t(X,Y) = X^t$,
$\omega_t(S) = t$, $\Omega(S) = 1$.

$|S| = 2$ : $S = \{(0,0),(1,0)\}$, $P_t(X,Y) = Y^t$,
$\omega_t(S) = t$, $\Omega(S) = 1$.

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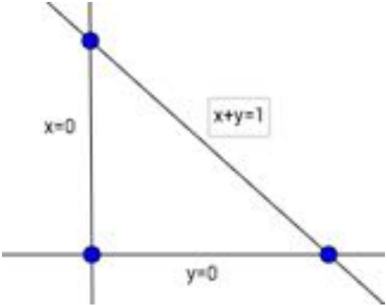
# Generic $S \subset \mathbb{K}^2$ with $|S| = 3$




$S = \{(0,0),(0,1),(1,0)\}$.

$P_1(X,Y) = XY$

$P_2(X,Y) = XY(X+Y-1)$

$\omega_1(S) = 2, \quad \omega_2(S) = 3.$

With

$P_{2m-1} = X^m Y^m (X+Y-1)^{m-1}, \; P_{2m} = X^m Y^m (X+Y-1)^m,$

we deduce

$$\omega_{2m-1}(S) = 3m-1, \quad \omega_{2m}(S) = 3m.$$

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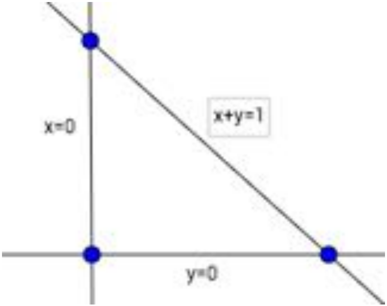


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# Generic $S$ with $|S| = 3$ in $\mathbb{K}^2$

Given a set $S$ of $3$ points in $\mathbb{K}^2$, not on a straight line, we have

$$\omega_t(S) = \begin{cases} \dfrac{3t+1}{2} & \text{for } t \text{ odd}, \\ \\ \dfrac{3t}{2} & \text{for } t \text{ even}, \end{cases}$$

hence

$$\Omega(S) = \lim_{n \to \infty} \frac{\omega_t(S)}{t} = \frac{3}{2}.$$

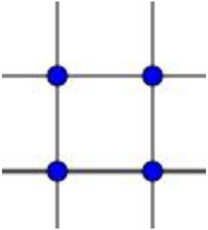
Since $\omega_1(S) = 2$ and $n = 2$, this is an example with

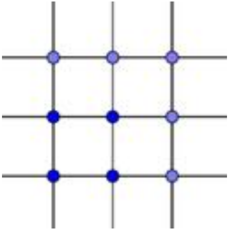
$$\frac{\omega_1(S)}{n} < \Omega(S) < \omega_1(S).$$

# Generic $S \subset \mathbb{K}^2$ with $|S| = 4$

For a generic $S$ in $\mathbb{K}^2$ with $|S| = 4$, we have $\omega_t(S) = 2t$, hence $\Omega(S) = 2$.

Easy for a Cartesian product $S_1 \times S_2$ with $|S_1| = |S_2| = 2$, also true for a generic $S$ with $|S| = 4$.





More generally, for the same reason, when $S$ is a Cartesian product $S_1 \times S_2$ with $|S_1| = |S_2| = m$, we have $\omega_t(S) = mt$ and $\Omega(S) = m = \sqrt{|S|}$. The inequality $\Omega(S) \geq \sqrt{|S|}$ for a generic $S$ with $|S|$ a square follows (Chudnovsky).

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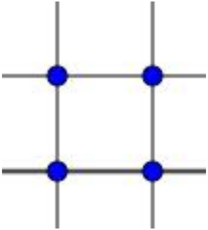

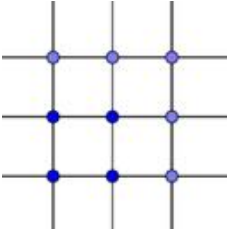

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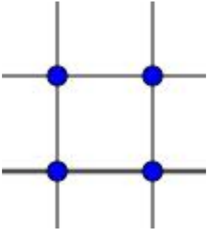

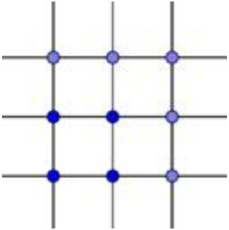

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# Generic $S \subset \mathbb{K}^2$ with $|S| = 5$

Since $5$ points in $\mathbb{K}^2$ lie on a conic, for a generic $S$ with $|S| = 5$ we have $\omega_t(S) = 2t$ and $\Omega(S) = 2$.

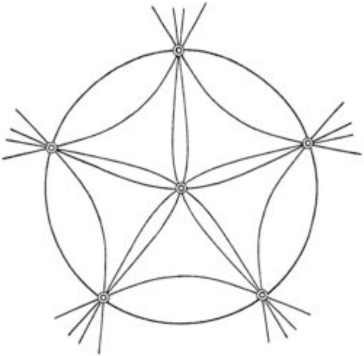
`Remark.` A polynomial in $2$ variables of degree $d$ has

$$\frac{(d+1)(d+2)}{2}$$

coefficients. Hence for $2|S| < (d+1)(d+2)$ we have $\omega_1(S) \leq d$.

For $|S| = 1, 2$ we have $\omega_1(S) = 1$,
for $|S| = 3, 4, 5$ we have $\omega_1(S) \leq 2$,
for $|S| = 6, 7, 8, 9$ we have $\omega_1(S) \leq 3$.

# Generic $S \subset \mathbb{K}^2$ with $|S| = 6$ (Nagata)



Given $6$ generic points $s_1, \dots, s_6$ in $\mathbb{K}^2$, consider $6$ conics $C_1, \dots, C_6$ where $S_i$ passes through the $5$ points $s_j$ for $j \neq i$. This produces a polynomial of degree $12$ with multiplicity $\geq 5$ at each $s_i$. Hence $\omega_5(S) \leq 12$.

In fact $\omega_5(S) = 12$, $\Omega(S) = 12/5$.

# Generic $S \subset \mathbb{K}^2$ with $|S| = 7$ (Nagata)

Given $7$ points in $\mathbb{K}^2$, there is a cubic passing through these $7$ points with a double point at one of them.

Number of coefficients of a cubic polynomial : $10$.

Number of conditions : $6$ for the simple zeros, $3$ for the double zero.

This gives a polynomial of degree $7 \times 3 = 21$ with the $7$ assigned zeroes of multiplicities $8$.

In fact $\omega_8(S) = 21$, $\Omega(S) = 21/8$.

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# Generic $S \subset \mathbb{K}^2$ with $|S| = 8$ (Nagata)

Given $8$ points in $\mathbb{K}^2$, there is a sextic with a double point at $7$ of them and a triple point at $1$ of them.

Number of coefficients of a sextic polynomial : $(6+1)(6+2)/2 = 28$.

Number of conditions : $3 \times 7 = 21$ for the double zeros, $6$ for the triple zero.

This gives a polynomial of degree $8 \times 6 = 48$ with the $8$ assigned zeroes of multiplicities $2 \times 8 + 1 = 17$.

In fact $\omega_{17}(S) = 48$, $\Omega(S) = 47/17$.

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# G.V. Chudnovsky

Gregory Chudnovsky

**Conjecture** :

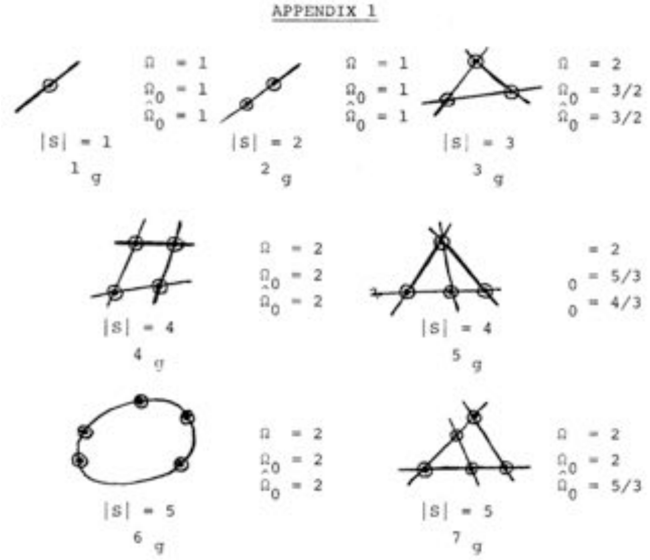
$$\frac{\omega_1 + n - 1}{n} \leq \frac{\omega_t}{t}.$$

True for $n = 2$ (J-P. Demailly).

G.V. Chudnovsky. *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*. M.-J. Bertin (Ed.), Séminaire de Théorie des Nombres Delange-Pisot-Poitou, Paris, 1979–80, Prog. Math., vol. **12**, Birkhäuser.

https://fr.wikipedia.org/wiki/David_et_Gregory_Chudnovsky

# Chudnovsky : $n = 2,\ |S| = 2, 3, 4, 5$



# Chudnovsky : $n = 2,\ |S| = 5, 6$

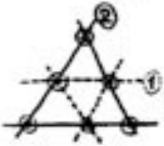
$|S| = 5$

$8_g$

$\Omega = 2$
$\Omega_0 = 7/4$
$\hat{\Omega}_0 = 5/4$

For generic S,
$|S| = 6$
(see $5_g$)

$\Omega(S) = 3$
$\Omega_0(S) = 12/5$
$\hat{\Omega}_0(S) = 12/5$



$|S| = 6$

$10_g$

$\Omega = 3$
$\Omega_0 = 9/4$
$\hat{\Omega}_0 = 2$

$|S| = 6$

$11_g$

$\Omega = 3$
$\Omega_0 = 12/5$
$\hat{\Omega}_0 = 2$

# Chudnovsky : $n = 2$, $|S| = 6, 7$

64

$|S| = 6$

$12_g$

$\Omega = 3$
$\Omega_0 = 2$
$\hat{\Omega}_0 = 2$

$\Omega = 3$
$\Omega_0 = 7/3$
$\hat{\Omega}_0 = 2$

$|S| = 6$

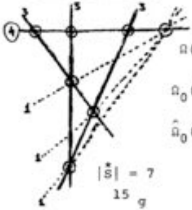
$13_g$

$|S| = 7$

$14_g$

For generic S, $|S| = 7$,
7 cubics with double point
for generic case,
$\Omega(S) = 3$, $\Omega_0(S) = 21/8$,
$\hat{\Omega}_0(S) = 21/8$.

<u>Particular Cases:</u>



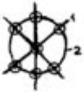
$\Omega(\dot{S}) = 3$

$\Omega_0(\dot{S}) = 16/7$

$\hat{\Omega}_0(\dot{S}) = 7/4$

$|\dot{S}| = 7$

$15_g$



$\Omega = 3$

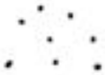
$\Omega_0 = 7/3$

$\hat{\Omega}_0 = 7/3$

$|S| = 7$

$16_g$

# Chudnovsky : $n = 2$, $|S| = 8$



$|S| = 8$

(See 18 - 19 )

For generic S, $|S| = 8$, 8 sextics with 7 double points, 1 triple point. For generic case, $\Omega(S) = 3$, $\Omega_0(S) = 48/17$, $\hat{\Omega}_0(S) = 48/17$.

# Chudnovsky : $n = 2$, $|S| = 9$



$|S| = 9$

$20_g$

For generic S, $|S| = 9$, 1 cubic and $\Omega(S) = 3$, $\Omega_0(S) = 3$, $\hat{\Omega}_0(S) = 3$.

---

$|S| = 9$

$21_g$

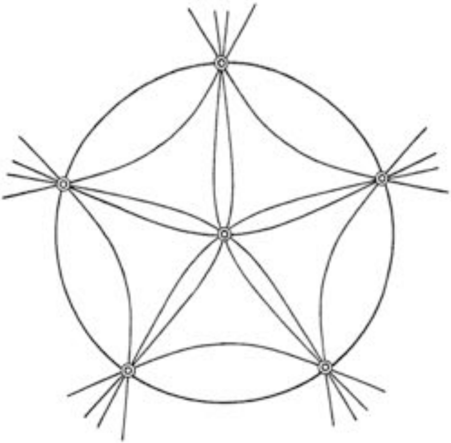
$\Omega = 3$

$\Omega_0 = 5/2$

$\hat{\Omega}_0 = 9/4$

# Chudnovsky : $n = 2$, $|S| = 6$

APPENDIX 2



$|S| = 6$

9 g

Generic S, $|S| = 6$

6 conics

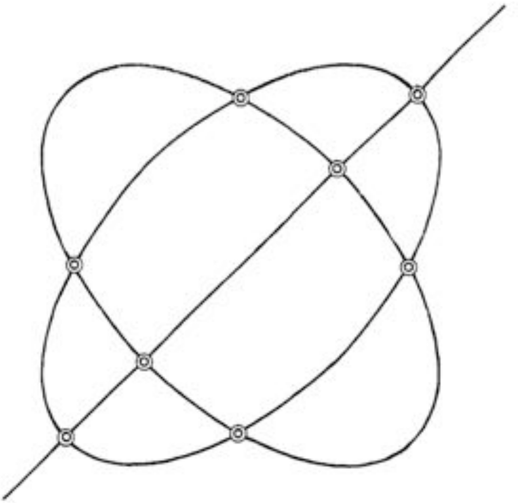
$\Omega(S) = 3$

$\Omega_0(S) = 12/5$

$\Omega_{\Omega}(S) = 12/5$

65

# Chudnovsky : $n = 2,\ |S| = 8$



$|S| = 8$

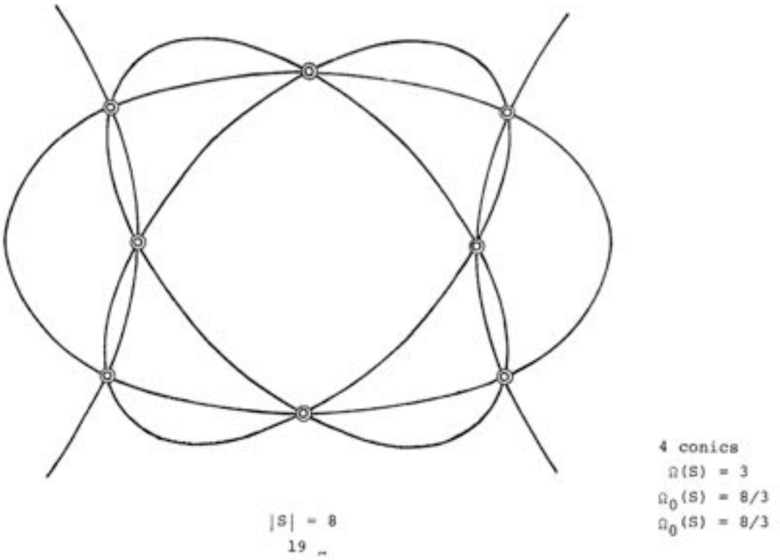
$18_9$

$\Omega(S) = 3$

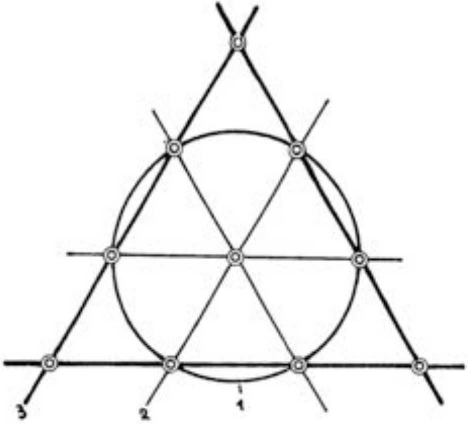
$\hat{\Omega}_0(S) = 5/2$

$\Omega_0(S) = 3/2$

# Chudnovsky : $n = 2,\ |S| = 8$



# Chudnovsky : $n = 2$, $|S| = 10$




$|S_c| = 10$

$22_g$

$\Omega(S_c) = 4$

$\Omega_0(S_c) = 17/6$

$\hat{\Omega}_0(S_c) = 5/2$

# Hélène Esnault and Eckart Viehweg

H. Esnault and E. Viehweg *Sur une minoration du degré d'hypersurfaces s'annulant en certains points.* Math. Ann. **263** (1983), 75 – 86

Methods of projective geometry : for $n \geq 2$,

$$\Omega(S) \geq \frac{\omega_t + 1}{t + n - 1}.$$

# Jean-Pierre Demailly

Jean-Pierre Demailly

Using an appropriate generalization of the Poisson–Jensen formula, proves a new variant of the Schwarz lemma in $\mathbb{C}^n$.

**Consequence** :

$$\Omega(S) \geq \frac{\omega_1(S)\big(\omega_1(S)+1\big)\cdots\big(\omega_1(S)+n-1\big)}{n!\omega_1(S)^{n-1}}$$

**Corollary** : For $n = 1$ or $2$,

$$\Omega(S) \geq \frac{\omega_1(S)+n-1}{n}.$$

# Demailly's Conjecture

Recall the Conjecture of Chudnovsky and the Theorem of Esnault and Viehweg :

$$\Omega(S) \geq \frac{\omega_1 + n - 1}{n}, \qquad \Omega(S) \geq \frac{\omega_t + 1}{t + n - 1}.$$

**Conjecture of Demailly :**

$$\Omega(S) \geq \frac{\omega_t(S) + n - 1}{t + n - 1}.$$

J–P. Demailly. *Formules de Jensen en plusieurs variables et applications arithmétiques.* Bull. Soc. Math. France **110** (1982), 75–102.

https://de.wikipedia.org/wiki/Jean-Pierre_Demailly

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https://de.wikipedia.org/wiki/Jean-Pierre_Demailly

# Abdelhak Azhari

A. Azhari. *Démonstration analytique d'un lemme de multiplicités.* C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 7, 269–272.

A. Azhari. *Sur la conjecture de Chudnovsky – Demailly et les singularités des hypersurfaces algébriques.* Ann. Inst. Fourier **40** (1990), no. 1, 103–116.
http://www.numdam.org/item?id=AIF_1990__40_1_103_0

# Conjecture of André Hirschowitz

Denote by $\omega_t(n,m)$ the maximum of $\omega_t(S)$ over all finite sets $S$ in $\mathbb{K}^n$ with $m$ elements.

**Conjecture :** $\omega_t(n,m)$ is as large as possible.

*For every $n \geq 1$ there is an integer $c(n)$ such that, for every $m \geq c(n)$ and, for all $t$, $\omega_t(n,m)$ is the smallest integer $d$ such that*

$$\binom{d+n}{n} > m\binom{t+n-1}{n}.$$

True for $t=2$ and $n=2$ and $3$, and for $t=3$ and $n=2$.

A. Hirschowitz. *La méthode d'Horace pour l'interpolation à plusieurs variables.* Manuscripta Math. **50** (1985), 337–388.

# Alternate proof of $\Omega(S) \geq \frac{\omega_1(S)}{n}$ (2001, 2002)

**Theorem (Ein-Lazarfeld-Smith, Hochster-Huneke).** *Let $J$ be a homogeneous ideal in $\mathbb{K}[X_0, \ldots, X_n]$ and $t \geq 1$. Then*

$$J^{(tn)} \subset J^t.$$

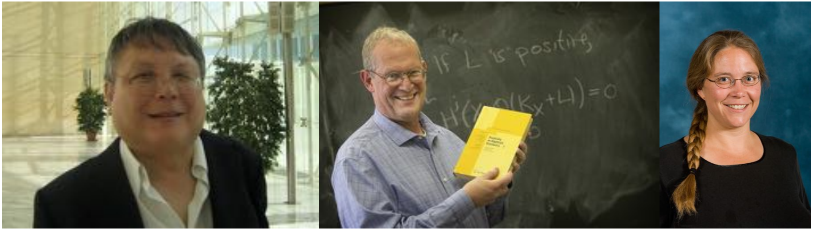
Consequence : From $I(S)^t \supset I(S)^{(tn)}$ we deduce

$$t\omega_1(S) \leq \omega_{tn}(S)$$

and

$$\frac{\omega_1(S)}{n} \leq \frac{\omega_{tn}(S)}{tn} \to \Omega(S) \quad \text{as} \quad t \to \infty.$$

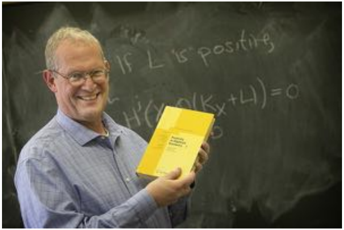
# $J^{(tn)} \subset J^t$ by L. Ein, R. Lazarfeld and K.E. Smith



The proof by Lawrence Ein, Robert Lazarfeld and Karen E. Smith uses multiplier ideals.

L. Ein, R. Lazarfeld and K.E. Smith. *Uniform behavior of symbolic powers of ideals.* Invent. Math. **144** (2001), 241–252.

## $J^{(tn)} \subset J^t$




R. Lazarfeld. *Positivity in algebraic geometry* I – II. Ergeb. Math. **48–49**, Springer, Berlin (2004).

# $J^{(tn)} \subset J^t$ by M. Hochster and C. Huneke

The proof by Melvin Hochster and Craig Huneke uses Frobenius powers and tight closure.

Melvin Hochster

Craig Huneke

M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals.* Invent. Math. **147** (2002), 349–369.

# Briançon-Skoda Theorem

Melvin Hochster

Craig Huneke

For an $m$–generated ideal $\mathfrak{a}$ in the ring of germs of analytic functions at $0 \in \mathbb{C}^n$, the $\nu$–th power of its integral closure is contained in $\mathfrak{a}$, where $\nu = \min\{m, n\}$.

M. Hochster and C. Huneke. *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.

# Symbolic powers

For a homogeneous ideal $J$ in the ring $R = \mathbb{K}[X_0, \ldots, X_n]$ and $m \geq 1$, define the symbolic power $J^{(m)}$ as follows. Write primary decompositions of $J$ and $J^m$ as

$$J = \bigcap_i \mathfrak{Q}_i, \qquad J^m = \bigcap_j \mathfrak{Q}'_j,$$

where $\mathfrak{Q}_i$ is homogeneous and $\mathfrak{P}_i$ primary, $\mathfrak{Q}'_j$ is homogeneous and $\mathfrak{P}'_j$ primary. We set

$$J^{(m)} = \bigcap_j \mathfrak{Q}'_j$$

where the intersection is over the $j$ with $\mathfrak{P}'_j$ contained in some $\mathfrak{P}_i$.

# Symbolic powers

Notice that $J^m \subset J^{(m)}$.

Example of a *fat points* ideal. For $J = \cap_j I(p_j)^{m_j}$,

$$J^{(m)} = \bigcap_j I(p_j)^{mm_j}.$$

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# The containment problem

*Find all* $m$, $t$ *with*

$$J^{(m)} \subset J^t.$$

Brian Harbourne. *Asymptotic invariants of ideals of points.* (2009).
Special Session on Geometry, Syzygies and Computations
Organized by Professors S. Kwak and J. Weyman KMS-AMS joint meeting, December 16–20, 2009.
Slides.
www.math.unl.edu/~bharbourne1/KSSNoPauseRev.pdf

# $\Omega(J)$ for a homogeneous ideal $J$

Cristiano Bocci

Brian Harbourne

For a homogeneous ideal $J$ of $\mathbb{K}[X_0, \ldots, X_n]$, and $t \geq 1$, define $\omega_t(J) = \omega(J^{(t)})$. Then

$$\Omega(J) = \lim_{t\to\infty} \frac{\omega_t(J)}{t}$$

exists and satisfies

$$\Omega(J) \leq \frac{\omega_t(J)}{t} \quad \text{for all} \quad t \geq 1.$$

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# The resurgence of Bocci and Harbourne

Define

$$\varrho(J) = \sup\left\{\frac{m}{r} \;\middle|\; J^{(m)} \not\subset J^r\right\}.$$

Hence, if $\frac{m}{r} > \varrho(J)$, then $J^{(m)} \subset J^r$.

By L. Ein, R. Lazarfeld and K.E. Smith, $\varrho(J) \leq n$.


C. Bocci and B. Harbourne. *Comparing powers and symbolic powers of ideals.* J. Algebraic Geom. **19** (2010), no. 3, 399–417.

C. Bocci and B. Harbourne. *The resurgence of ideals of points and the containment problem.* Proc. Amer. Math. Soc. **138** (2010), no. 4, 1175–1190

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# The resurgence of Bocci and Harbourne

Denote by $\mathrm{reg}(J)$ the Castelnuovo– Mumford regularity of $J$.

**Theorem (Bocci, Harbourne)**. We have

$$\frac{\omega(J)}{\Omega(J)} \leq \varrho(J) \leq \frac{\mathrm{reg}(J)}{\Omega(J)}.$$

Further, if $\omega(J) = \mathrm{reg}(J)$, then

$$J^{(m)} \subset J^t \iff t\omega(J) \leq \omega(J^{(m)}).$$

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# Optimality

Following Bocci and Harbourne, we have

$$\sup_{|S|<\infty} \frac{\omega_1(S)}{\Omega(S)} = n.$$

# Conjecture of Cristiano Bocci and Brian Harbourne

Let $S$ be a finite subset of $\mathbb{P}^2$. Define $\varrho(S) = \varrho(J)$ for $J = I(S)$.

**Conjecture.**

$$\varrho(S) \le 2\frac{\omega_1(S)}{\omega_1(S)+1}.$$

This conjecture implies Chudnovsky's conjecture : from

$$\frac{\omega_1(S)}{\Omega(S)} \le \varrho(S) \le 2\frac{\omega_1(S)}{\omega_1(S)+1}$$

one deduces

$$\frac{\omega_1(S)+1}{2} \le \Omega(S).$$

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# The containment problem (continued)

Let $\mathfrak{M}$ be the homogeneous ideal $(X_0, \dots, X_n)$ in $R$.

Fact. In characteristic zero, the ideal $J = I(S)$ satisfies $J^{(2)} \subset \mathfrak{M}J$.

Proof. Let $P \in J^{(2)}$. Hence $\frac{\partial}{\partial X_i} P \in J$. Use Euler's formula

$$(\deg P)P = \sum_{i=0}^{n} X_i \frac{\partial}{\partial X_i} P.$$

**Question.** *For which* $m$, $t$, $j$ *do we have* $J^{(m)} \subset \mathfrak{M}^j J^t$ *?*

Remark. Since $\mathfrak{M}^j J^t \subset J^t$, the condition $J^{(m)} \subset \mathfrak{M}^j J^t$ implies $J^{(m)} \subset J^t$.

B. Harbourne and C. Huneke. *Are symbolic powers highly evolved?* J. Ramanujan Math. Soc. **28**A (2013), 247–266. arxiv:1103.5809.

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# Conjecture of Brian Harbourne and Craig Huneke

Chudnovsky's result

$$\frac{\omega_1 + n - 1}{n} \leq \frac{\omega_t}{t}.$$

for $n = 2$ follows from

$$J^{(2t)} \subset \mathfrak{M}^t J^t$$

for any homogeneous ideal of points $J = I(S)$ in $\mathbb{K}[X_0, X_1, X_2]$.

Generalization for $n \geq 2$.

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Let $J = \cap_j I(p_j)^{m_j}$ be a fat points ideal in $R$.

**Conjecture (Harbourne and Huneke).** For all $t > 0$,

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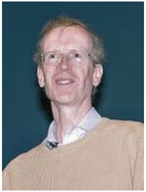
# Evolutions

Andrew Wiles



Richard Taylor

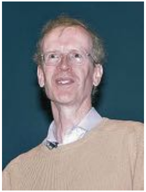
Matthias Flach

*Evolutions* are certain kinds of ring homomorphisms that arose in proving Fermat's last Theorem (A. Wiles, R. Taylor, M. Flach).

An important step in the proof was to show that in certain cases only trivial evolutions occurred.

# Evolutions

Andrew Wiles



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# Evolutions

D. Eisenbud and B. Mazur showed the question of triviality could be translated into a statement involving symbolic powers. They then made the following conjecture in characteristic $0$ :

**Conjecture (Eisenbud–Mazur)** *Let* $\mathfrak{P} \subset \mathbb{C}[[x_1, \ldots, x_d]]$ *be a prime ideal. Let* $\mathfrak{M} = (x_1, \ldots, x_d)$. *Then* $\mathfrak{P}^{(2)} \subset \mathfrak{M}\mathfrak{P}$.

# Evolutions

Heuristically, the main conjecture of Harbourne and Huneke can be thought of as a generalization of the conjecture of Eisenbud and Mazur.

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# Brian Harbourne

Brian Harbourne, Sandra Di Rocco, Tomasz Szemberg, Thomas Bauer

Oberwolfach
Linear Series on Algebraic Varieties : 2010-10-03 – 2010-10-09

M. Dumnicki, B. Harbourne, T. Szemberg and H. Tutaj-Gasińska. *Linear subspaces, symbolic powers and Nagata type conjectures.* Adv. Math. **252** (2014), 471–491.

https://owpdb.mfo.de/detail?photo_id=13201

# Marcin Dumnicki

Chudnovsky's conjecture

$$\Omega(S) \geq \frac{\omega_1(S) + n - 1}{n}$$ holds

for generic finite subsets in $\mathbb{P}^3$.

M. Dumnicki. *Symbolic powers of ideal of generic points in* $\mathbb{P}^3$. J. Pure Applied Algebra **216** (2012), 1410–1417.

# Thomas Bauer and Thomasz Szemberg.

Th. Bauer and T. Szemberg. *The effect of points fattening in dimension three*. Recent advances in Algebraic Geometry. A volume in honor of Rob Lazasfeld's 60th Birthday LMS, Cambridge University Press 2015.

# Further references


M. Baczyńska, M. Dumnicki, A. Habura, G. Malara, P. Pokora, T. Szemberg, J. Szpond and H. Tutaj-Gasińska. *Points fattening on* $\mathbb{P}^1 \times \mathbb{P}^1$ *and symbolic powers of bi–homogeneous ideals.* J. Pure Applied Algebra **218** (2014), 1555–1562.

C. Bocci, S.M. Cooper and B. Harbourne. *Containment results for ideals of various configurations of points in* $\mathbb{P}^N$. J. Pure Applied Algebra **218** (2014), 65–75.

# Jugal Verma, Sylvia Wiegand, Roger Wiegand

e-mail January 24, 2014.

http://www.math.iitb.ac.in/~jkv/

http://www.math.unl.edu/~swiegand1/

https://www.math.unl.edu/~rwiegand1/


http://webusers.imj-prg.fr/~michel.waldschmidt/