Two applications of diophantine approximation to dynamical systems

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Abstract

The behavior of a holomorphic dynamical system near a fixed point depends on a Diophantine condition arising in the works of Liouville, Thue, Siegel and Roth on the rational approximation to algebraic numbers.
The Subspace Theorem of Wolfgang Schmidt is a far reaching generalization of the Thue–Siegel–Roth Theorem ; one of its many consequences is a result on the iterates of an endomorphism of a vector space.
We conclude with the Skolem – Mahler – Lech Theorem.

Celestial mechanics

Classical mechanics :
Sir Isaac Newton
(1643 – 1727)

- The solar system
- The three body problem
- Two body problem

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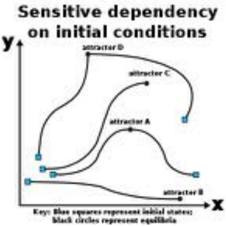
Edward Norton Lorenz (1917 – 2008)



In chaos theory, the *butterfly effect* is the sensitive dependency on initial conditions in which a small change at one place in a deterministic nonlinear system can result in large differences in a later state. The name of the effect, coined by Edward Lorenz, is derived from the theoretical example of the formation of a hurricane being contingent on whether or not a distant butterfly had flapped its wings several weeks earlier.

Lorenz's *butterfly effect*

Two states differing by imperceptible amounts may eventually evolve into two considerably different states. If, then, there is any error whatever in observing the present state — and in any real system such errors seem inevitable — an acceptable prediction of an instantaneous state in the distant future may well be impossible. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting seems to be nonexistent.

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However, recent research shows that *complex systems may not behave like systems with fewer parameters.*

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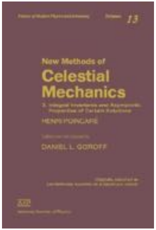
Henri Poincaré (1854 – 1912)



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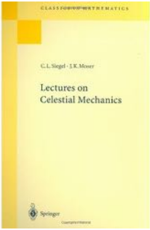
Carl Ludwig Siegel (1896 – 1981)



Dynamical System : iteration of a map

Consider a set X and map $f : X \to X$. We denote by f^2 the composed map $f \circ f : X \to X$.

More generally, we define inductively $f^n : X \to X$ by $f^n = f^{n-1} \circ f$ for $n \geq 1$, with f^0 being the identity.

The orbit of a point $x \in X$ is the sequence

$$(x, f(x), f^2(x), \dots)$$

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Fixed points, periodic points

A fixed point is an element $x \in X$ such that $f(x) = x$. A fixed point is a point, the orbit of which has one element x.

A periodic point is an element $x \in X$ for which there exists $n \geq 1$ with $f^n(x) = x$. The smallest such n is the length of the period of x, and all such n are multiples of the period length. The orbit

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has n elements.

For instance, a fixed point is a periodic point of period length 1.

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Example 1 : endomorphism of a vector space

Take for X a finite dimensional vector space V over a field K and for $f : V \to V$ a linear map.

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If V has dimension d and if we choose a basis of V, then to f is associated a $d \times d$ matrix A with coefficients in K.

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Associated matrix

When f is the linear map associated with the $d \times d$ matrix A, then, for $n \geq 1$, f^n is the linear map associated with the matrix A^n.

To compute A^n, we write the matrix A as a conjugate to either a diagonal or a Jordan matrix

$$A = P^{-1}DP,$$

where P is a regular $d \times d$ matrix. Then, for $n \geq 0$,

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Diagonal form

If D is diagonal with diagonal $(\lambda_1, \ldots, \lambda_d)$, then D^n is diagonal with diagonal $(\lambda_1^n, \ldots, \lambda_d^n)$ and

$$A^n = P^{-1} \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_d^n \end{pmatrix} P.$$

Exercise : compute A^n for $n \geq 0$ and for each of the two matrices

$$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

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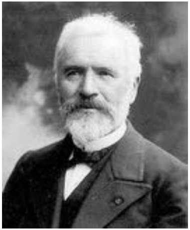
Camille Jordan (1838 – 1922)

If A cannot be diagonalized, it can be put in Jordan form with diagonal blocs like



$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

For instance, for $d = 2$,

$$A = P^{-1}DP \quad \text{with} \quad D = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and

$$A^n = P^{-1} \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} P.$$

Example 2 : holomorphic dynamic

Our second and main example of a dynamical system is with an open set $\mathcal{V}$ in $\mathbf{C}$ and an analytic (=holomorphic) map $f : \mathcal{V} \to \mathcal{V}$. The main goal will be to investigate the behavior of f near a fixed point $z_0 \in \mathcal{V}$. So we assume $f(z_0) = z_0$.

The local behavior of the dynamics defined by f depends on the derivative $f'(z_0)$ of f at the fixed point.

If $|f'(z_0)| < 1$, then z_0 is an attracting point.

If $|f'(z_0)| > 1$, then z_0 is a repelling point.

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Conjugate holomorphic maps

We wish to mimic the situation of an endomorphism of a vector space : in place of a regular matrix P, we introduce a local change of coordinates. Let $\mathcal{D}$ be the open unit disc in $\mathbf{C}$ and $g : \mathcal{D} \to \mathcal{D}$ an analytic map with $g(0) = 0$. We say that f and g are conjugate if there exists an analytic map $h : \mathcal{V} \to \mathcal{D}$, with $h'(z_0) \neq 0$, such that $h(z_0) = 0$ and $h \circ f = g \circ h$.

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Local behavior

Assume $f : \mathcal{V} \to \mathcal{V}$ and $g : \mathcal{D} \to \mathcal{D}$ are conjugate : there exists $h : \mathcal{D} \to \mathcal{D}$, with $h'(z_0) \neq 0$ and $h \circ f = g \circ h$.

From $h'(z_0) \neq 0$, one deduces that h is unique up to a multiplicative nonzero factor.

Further,

$$h \circ f^2 = h \circ f \circ f = g \circ h \circ f = g \circ g \circ h = g^2 \circ h$$

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Linearization of germs of analytic diffeomorphisms of one complex variable

Lemma. *If* f *is conjugate to the homothety* $g(z) = \lambda z$*, then* $\lambda = f'(z_0)$*.*
Hence, in this case, f is conjugate to its linear part. One says that f is *linearizable*.

Proof. Take the derivative of $h \circ f = g \circ h$ at z_0 :

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Johann Samuel König (1712 – 1757)

Define $\lambda = f'(z_0)$.

Theorem (Königs and Poincaré). *For* $|\lambda| \notin \{0, 1\}$*,* f *is linearizable.*

For $\lambda = 0$ and $z_0 = 0$, f has a zero of multiplicity $n \geq 2$ at 0 and is conjugate to $z \mapsto z^n$ (A. Böttcher).

We are interested in the case $|\lambda| = 1$. It was conjectured in 1912 by E. Kasner that f is always linearizable. In 1917, G.A. Pfeiffer produced a counterexample. In 1927, H. Cremer proved that in the *generic* case, f is not linearizable.

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The case $|\lambda| = 1$

Assume $|\lambda| = 1$. Write $\lambda = e^{2i\pi\theta}$. The real number θ is the rotation number of f at z_0.

In 1942, C.L. Siegel proved that if θ satisfies a Diophantine condition, then f is conjugate to the rotation $z \mapsto e^{2i\pi\theta} z$.

In 1965, A.D. Brjuno relaxed Siegel's assumption.

In 1988, J.C. Yoccoz showed that if θ does not satisfies Brjuno's condition, then the dynamic associated with

$$f(z) = \lambda z + z^2$$

has infinitely many periodic points in any neighborhood of 0, hence, is not linearizable.

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Carl Ludwig Siegel
(1896 – 1981)

Jean-Christophe Yoccoz
(1957 — 2016)

Alexander Dmitrijewitsch Brjuno
(1940 –)

1942

1965

1988

KAM Theory

Andrey Nikolaevich Kolmogorov

Vladimir Igorevich Arnold

Jürgen Kurt Moser

(1903 – 1987)

(1937 – 2010)

(1928 – 1999)

Siegel's Diophantine condition : Liouville numbers

Siegel's Diophantine condition on the rotation number θ is that a rational number p/q with a *small denominator* q cannot be too good of a rational approximation of θ.

The same condition was introduced by Liouville, who proved in 1844 that Siegel's Diophantine condition is satisfied if θ is an algebraic number.

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Liouville's inequality (1844)

Liouville's inequality. *Let α be an algebraic number of degree $d \geq 2$. There exists $c(\alpha) > 0$ such that, for any $p/q \in \mathbf{Q}$ with $q > 0$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^d}$$

Joseph Liouville
(1809 - 1882)

The Diophantine condition of Liouville and Siegel

A real number θ satisfies a *Diophantine condition* if there exists a constant $\kappa > 0$ such that

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{q^\kappa}$$

for all $p/q \in \mathbf{Q}$ with $q \geq 2$.

A real number is a *Liouville number* if it does not satisfy a Diophantine condition.

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Generic vs full measure, Baire vs Lebesgue

René Baire
(1874 – 1932)

Henri Léon Lebesgue
(1875 – 1941)

In dynamical systems, a property is satisfied for a *generic rotation number* θ if it is true for all numbers in a countable intersection of dense open sets – these sets are called G_δ sets by Baire who calls *meager* the complement of a G_δ set. The set of numbers which do not satisfy a Diophantine condition is a generic set. For Lebesgue measure, the set of Liouville numbers (i.e. the set of numbers which do not satisfy a Diophantine condition) has measure zero.

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René Baire
(1874 – 1932)

Henri Léon Lebesgue
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In dynamical systems, a property is satisfied for a *generic rotation number* θ if it is true for all numbers in a countable intersection of dense open sets – these sets are called G_δ sets by Baire who calls *meager* the complement of a G_δ set. The set of numbers which do not satisfy a Diophantine condition is a generic set. For Lebesgue measure, the set of Liouville numbers (i.e. the set of numbers which do not satisfy a Diophantine condition) has measure zero.

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Mathematical genealogy

René Baire (1899)

|

Arnaud Denjoy (1909)

|

Charles Pisot (1938)

|

Yvette Amice (1965)

|

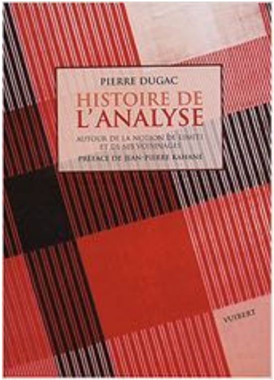
Jean Fresnel (1967)

|

Michel Waldschmidt (1972)

http://genealogy.math.ndsu.nodak.edu

Pierre Dugac (1926 – 2000)



Notes et documents sur la vie et l'œuvre de René Baire. Arch. History Exact Sci. **15** (1975/76), no. 4, 297–383.

https://fr.wikipedia.org/wiki/Pierre_Dugac

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Joseph Louis Lagrange (BA 1754)	Arnaud Denjoy (1909)
Simeon Denis Poisson (1800)	Charles Pisot (1938)
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Wetzlarer Bier Waldschmidt Euler

Brjuno's condition

In terms of continued fraction, the Diophantine condition (of Liouville and Siegel) can be written

$$\sup_{n\geq 1} \frac{\log q_{n+1}}{\log q_n} < \infty.$$

The condition of Brjuno is

$$\sum_{n\geq 1} \frac{\log q_{n+1}}{q_n} < \infty.$$

If a number θ satisfies the Diophantine condition, then it satisfies Brjuno's condition. But there are (transcendental) numbers which do not satisfy the Diophantine condition, but satisfy Brjuno's condition.

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Improvements of Liouville's inequality

In the lower bound

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^d}$$

for α real algebraic number of degree $d \geq 3$, the exponent d of q in the denominator of the right hand side was replaced by κ with

- any $\kappa > (d/2) + 1$ by A. Thue (1909),
- $2\sqrt{d}$ by C.L. Siegel in 1921,
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Thue– Siegel– Roth Theorem

Axel Thue
(1863 – 1922)

Carl Ludwig Siegel
(1896 – 1981)

Klaus Friedrich Roth
(1925 – 2015)

For any real algebraic number α, *for any* $\epsilon > 0$, *the set of* $p/q \in \mathbf{Q}$ *with* $|\alpha - p/q| < q^{-2-\epsilon}$ *is finite.*

Thue– Siegel– Roth Theorem

An equivalent statement is that, for any real algebraic irrational number α and for any $\epsilon > 0$, *the set of* $p/q \in \mathbf{Q}$ *such that*

$$q|q\alpha - p| < q^{-\epsilon}$$

is finite.

The conclusion can be phrased :
the set of $(p, q) \in \mathbf{Z}^2$ *such that*

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Schmidt's Subspace Theorem (1970)

For $m \geq 2$ *let* $L_0, \ldots, L_{m-1}$ *be* m *independent linear forms in* m *variables with algebraic coefficients. Let* $\epsilon > 0$*. Then the set*

$$\{\mathbf{x} = (x_0, \ldots, x_{m-1}) \in \mathbf{Z}^m \ ;$$
$$|L_0(\mathbf{x}) \cdots L_{m-1}(\mathbf{x})| \leq |\mathbf{x}|^{-\epsilon}\}$$

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Wolfgang M. Schmidt
(1933 –)

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Example : $m = 2$, $L_0(x_0, x_1) = x_0$, $L_1(x_0, x_1) = \alpha x_0 - x_1$.
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Specialization arguments

The proof of Schmidt's Subspace Theorem has an arithmetic nature, the fact that the linear forms have algebraic coefficients is crucial.
The subspace Theorem does not hold without this assumption.

However, there are *specializations arguments* which enable one to deduce consequences without any arithmetic assumption, these corollaries ave valid for fields of zero characteristic in general.
An example is the so–called *Theorem of the generalized S–unit equation.*

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An example is the so–called *Theorem of the generalized* S*–unit equation*.

The generalized S–unit equation (1982)

Let K be a field of characteristic zero, let G be a finitely generated multiplicative subgroup of the multiplicative group $K^\times = K \setminus \{0\}$ and let $n \geq 2$.

Theorem (Evertse, van der Poorten, Schlickewei). *The equation*

$$u_1 + u_2 + \cdots + u_n = 1,$$

where the unknowns $u_1, u_2, \cdots, u_n$ take their values in G, for which no nontrivial subsum

$$\sum_{i \in I} u_i \qquad \emptyset \neq I \subset \{1, \ldots, n\}$$

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The generalized S–unit equation (1982)

Jan Hendrick Evertse

Hans Peter Schlickewei

Alf van der Poorten

Linear recurrence sequences

Given a field K (of zero characteristic), a sequence $(u_n)_{n\geq 0}$ is a linear recurrence sequence if there exist an integer $d \geq 1$ and elements $a_0, a_1, \ldots, a_{d-1}$ of K such that, for $n \geq 0$,

$$u_{n+d} = a_{d-1}u_{n+d-1} + \cdots + a_1 u_{n+1} + a_0 u_n.$$

Such a sequence $(u_n)_{n\geq 0}$ is determined by the coefficients $a_0, a_1, \ldots, a_{d-1}$ and by the initial values $u_0, u_1, \ldots, u_{d-1}$.

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Exponential polynomials

If $\alpha_1, \ldots, \alpha_k$ are the distinct roots of the polynomial

$$X^d - a_{d-1}X^{d-1} - \cdots - a_1 X - a_0$$

and $s_1, \ldots, s_k$ their multiplicities, then one can write

$$u_n = \sum_{i=1}^{k} A_i(n)\alpha_i^n,$$

where $A_1, \ldots, A_k$ are polynomials with A_i of degree $< s_i$.

Hence, a linear recurrence sequence is given by an exponential polynomial. Conversely, a sequence given by an exponential polynomial is a linear recurrence sequence.

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Skolem – Mahler – Lech Theorem

The generalized S–unit Theorem yields the following :

Theorem (Skolem 1934 – Mahler 1935 – Lech 1953). *Given a linear recurrence sequence, the set of indices* $n \geq 0$ *such that* $u_n = 0$ *is a finite union of arithmetic progressions.*

Thoralf Albert Skolem (1887 – 1963)

Kurt Mahler (1903 – 1988)

Christer Lech

An *arithmetic progression* is a set of positive integers of the form $\{n_0, n_0 + k, n_0 + 2k, \ldots\}$. Here, we allow $k = 0$.

Another dynamical system

Let V be a finite dimensional vector space over a field of zero characteristic, W a subspace of V, $f : V \to V$ an endomorphism (linear map) and x an element in V.

Corollary of the Skolem – Mahler – Lech Theorem. *The set of* $n \geq 0$ *such that* $f^n(x) \in W$ *is a finite union of arithmetic progressions.*

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By induction, it suffices to consider the case where $W = H$ is an hyperplane of V.

Idea of the proof

Choose a basis of V. The endomorphism f is given by a square $d \times d$ matrix A, where d is the dimension of V. Consider the characteristic polynomial of A, say

$$X^d - a_{d-1}X^{d-1} - \cdots - a_1 X - a_0.$$

By the Theorem of Cayley – Hamilton,

$$A^d = a_{d-1}A^{d-1} + \cdots + a_1 A + a_0 I_d$$

where I_d is the identity $d \times d$ matrix.

Theorem of Cayley – Hamilton

Arthur Cayley
(1821 – 1895)

Sir William Rowan Hamilton
(1805 – 1865)

Hence, for $n \geq 0$,

$$A^{n+d} = a_{d-1}A^{n+d-1} + \cdots + a_1 A^{n+1} + a_0 A^n.$$

It follows that each entry $a_{ij}^{(n)}$, $1 \leq i, j \leq d$, satisfies a linear recurrence sequence, the same for all i, j.

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Hyperplane membership

Let $b_1x_1 + \cdots + b_dx_d = 0$ be an equation of the hyperplane H in the selected basis of V. Let ${}^t\underline{b}$ denote the $1 \times d$ matrix $(b_1, \ldots, b_d)$ (transpose of a column matrix $\underline{b}$). Using the notation $\underline{v}$ for the $d \times 1$ (column) matrix given by the coordinates of an element v in V, the condition $v \in H$ can be written ${}^t\underline{b}\,\underline{v} = 0$.

Let x be an element in V and $\underline{x}$ the $d \times 1$ (column) matrix given by its coordinates. The condition $f^n(x) \in H$ can now be written

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Remark on the theorem of Skolem–Mahler–Lech

T.A. Skolem treated the case $K = \mathbf{Q}$ of in 1934

K. Mahler the case $\mathbf{K} = \overline{\mathbf{Q}}$, the algebraic closure of $\mathbf{Q}$, in 1935

The general case was settled by C. Lech in 1953.

Finite characteristic

C. Lech pointed out in 1953 that such a result may not hold if the characteristic of $\mathbf{K}$ is positive : he gave as an example the sequence $u_n = (1+x)^n - x^n - 1$, a third-order linear recurrence over the field of rational functions in one variable over the field $\mathbf{F}_p$ with p elements, where $u_n = 0$ for $n \in \{1, p, p^2, p^3, \ldots\}$. A substitute is provided by a result of Harm Derksen (2007), who proved that the zero set in characteristic p is a p–automatic sequence. Further results by Boris Adamczewski and Jason Bell.

Harm Derksen

Boris Adamczewski

Jason Bell

Polynomial-linear recurrence relation

A generalization of the Theorem of Skolem–Mahler–Lech has been achieved by Jason P. Bell, Stanley Burris and Karen Yeats who prove that the same conclusion holds if the sequence $(u_n)_{n\geq 0}$ satisfies a polynomial-linear recurrence relation

$$u_n = \sum_{i=1}^{d} P_i(n) u_{n-i}$$

where d is a positive integer and $P_1, \ldots, P_d$ are polynomials with coefficient in the field $\mathbf{K}$ of zero characteristic, provided that $P_d(x)$ is a nonzero constant.

Algebraic maps, algebraic groups

There are also analogues of the Theorem of Skolem–Mahler–Lech for algebraic maps on varieties (Jason Bell).

A version of the Skolem–Mahler–Lech Theorem for any algebraic group is due to Umberto Zannier.

Jason Bell

Umberto Zannier

Open problem

One main open problem related with Theorem of Skolem–Mahler–Lech is that it is not effective : explicit upper bounds for the number of arithmetic progressions, depending only on the order d of the linear recurrence sequence, are known (W.M. Schmidt, U. Zannier), but no upper bound for the arithmetic progressions themselves is known. A related open problem raised by T.A. Skolem and C. Pisot is :

> *Given an integer linear recurrence sequence, is the truth of the statement "$x_n \neq 0$ for all n" decidable in finite time ?*

T. Tao, *Effective Skolem Mahler Lech theorem*. In "Structure and Randomness : pages from year one of a mathematical blog", American Mathematical Society (2008), 298 pages.

http://terrytao.wordpress.com/2007/05/25/open-question-effective-skolem-mahler-lech-theorem/

Zeros of linear recurrence sequences

Jean Berstel et Maurice Mignotte. – Deux propriétés décidables des suites récurrentes linéaires Bulletin de la S.M.F., tome 104 (1976), p. 175-184.
`http://www.numdam.org/item?id=BSMF_1976__104__175_0`
Given a linear recurrence sequence with integer coefficients ; are there finitely or infinitely many zeroes ?

Philippe Robba. – Zéros de suites récurrentes linéaires. Groupe de travail d'analyse ultramétrique (1977-1978) Volume : 5, page 1-5.

L. Cerlienco, M. Mignotte, F. Piras. Suites récurrentes linéaires. Propriétés algébriques et arithmétiques. L'Enseignement Mathématique **33** (1987).

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http://pmb.univ-fcomte.fr/1989/Mignotte.pdf

E. Bavencoffe and J-P. Bézivin *Une famille remarquable de suites recurrentes lineaires*. – Monatshefte für Mathematik, (1995) **120** 3, 189–203

Karim Samake. – Suites récurrentes linéaires, problème d'effectivité. Inst. de Recherche Math. Avancée, 1996 - 62 pages

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Reference

Everest, Graham ; van der Poorten, Alf ; Shparlinski, Igor ; Ward, Tom – *Recurrence sequences*, Mathematical Surveys and Monographs (AMS, 2003), volume 104. `1290 references.`

Graham Everest

Alf van der Poorten

Igor Shparlinski

Tom Ward

Berstel's sequence

http://oeis.org/A007420

$$0,\ 0,\ 1,\ 2,\ 0,\ -4,\ 0,\ 16,\ 16,\ -32,\ -64,\ 64,\ 256,\ 0,\ -768,\ \ldots$$

Jean Berstel

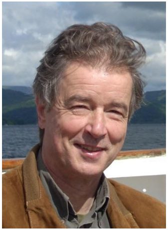
$b_0 = b_1 = 0$, $b_2 = 1$,
$b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n$
for $n \geq 0$.

Linear recurrence sequence of order 3 with exactly 6 zeros : $n = 0,\ 1,\ 4,\ 6,\ 13,\ 52.$

http://www-igm.univ-mlv.fr/~berstel/

Ternary linear recurrences

Berstel's sequence is a linear recurrence sequence of order 3 with 6 zeroes.



Frits Beukers

Frits Beukers (1991) : up to trivial transformation, any other linear recurrence of order 3 with finitely many zeroes has at most 5 zeros.

Edgard Bavencoffe and Jean-Paul Bézivin

Let $n \geq 2$. The sequence with initial values

$$u_0 = 1,\ u_1 = \cdots = u_{n-1} = 0$$

satisfying the recurrence relation of order n with characteristic polynomial

$$\frac{X^{n+1} - (-2)^{n-1}X + (-2)^n}{X + 2}$$

has at least

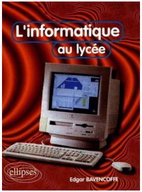
$$\frac{n(n+1)}{2} - 1$$

zeroes.

Edgard Bavencoffe and Jean-Paul Bézivin

For $n = 3$ one obtains Berstel's sequence which happens to have an extra zero.

$$\frac{X^4 + 4X - 8}{X + 2} = X^3 - 2X^2 + 4X - 4.$$



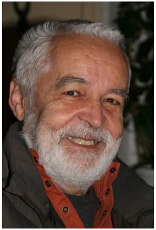
Edgard Bavencoffe

Jean-Paul Bézivin

Berstel's sequence

$$0,\ 0,\ 1,\ 2,\ 0,\ -4,\ 0,\ 16,\ 16,\ -32,\ -64,\ 64,\ 256,\ 0,\ -768,\ \dots$$

$b_0 = b_1 = 0$, $b_2 = 1$, $b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n$ for $n \geq 0$.



Maurice Mignotte

The equation $b_m = \pm b_n$ has exactly 21 solutions (m, n) with $m \neq n$.

The equation $b_n = \pm 2^r 3^s$ has exactly 44 solutions (n, r, s).

Two applications of diophantine approximation to dynamical systems

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