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24

ON THE TRANSCENDENCE METHODS OF GELFOND AND SCHNEIDER IN SEVERAL VARIABLES

M. Waldschmidt

1. Introduction

The methods we consider here were introduced by Gelfond and Schneider in their solutions of Hilbert's seventh problem on the transcendence of α^β (for algebraic α and β). Gelfond's proof [5] involved the two functions e^z and $e^{\beta z}$, with their derivatives, at the multiples of $\log\alpha$, while Schneider's proof [12] involved the two functions z and α^z, evaluated at the points $\mathbb{Z}+\mathbb{Z}.\beta$ (without derivatives).

Both methods have been extensively developed later. In his Bourbaki lecture [2], D. Bertrand pointed out a similarity between two of the most recent results which have been obtained, one by the method of Gelfond - Baker [16], and the other by Schneider's method [15].

The purpose of this paper is to prove a theorem which contains the two above-mentioned results, by combining the methods of Gelfond and Schneider.

Here is a corollary of our main result. Let G be a commutative algebraic group of dimension $d \geq 1$ which is defined over the field of algebraic numbers. We denote by $T_G(\mathbb{C})$ the tangent space of G at the origin, and by $\exp_G : T_G(\mathbb{C}) \longrightarrow G(\mathbb{C})$ the exponential map of the Lie group $G(\mathbb{C})$. Let d_0 (resp. d_1) be the dimension of the maximal unipotent (resp. multiplicative) factor of G, so that $G = \mathbb{G}_a^{d_0} \times \mathbb{G}_m^{d_1} \times G_2$, where G_2 is of dimension $d_2 = d - d_0 - d_1$.

Theorem 1.1 *Let V be a hyperplane of $T_G(\mathbb{C})$, W a subspace of V of dimension $t \geq 0$ over $\mathbb{C}$, and $Y = \mathbb{Z}y_1 + \ldots + \mathbb{Z}y_m$ a finitely generated subgroup of V of rank m over $\mathbb{Z}$. Assume that W is defined over $\overline{\mathbb{Q}}$ in $T_G(\mathbb{C})$, and that $\Gamma = \exp_G Y$ is contained in $G(\overline{\mathbb{Q}})$. Assume further*

$$m > (d_1 + 2d_2).(d - 1 - t). \qquad (1.2)$$

Then V contains a non-zero algebraic Lie sub-algebra of $T_G(\mathbf{C})$ which is defined over $\overline{\mathbf{Q}}$.

The arrangement of this paper is as follows. In §2 we give a refinement of the six exponentials theorem. In §3 we derive further corollaries from Theorem 1.1. In §4 we state our main theorem, and in §5 we show that it contains Theorem 1.1. The proof of the main theorem is given in §7, using an auxiliary function which is constructed in §6.

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§2. A refinement of the six exponentials theorem.

a) *A strong version of the six exponentials theorem*

A well-known open problem is to prove that if t is a real number such that 2^t and 3^t are both rational integers, then t is rational. More generally, the *four exponentials conjecture* [1], [6], [11], [13] states that if x_1, x_2 are $\mathbf{Q}$-linearly independent complex numbers, and y_1, y_2 are $\mathbf{Q}$-linearly independent complex numbers, then one at least of the four numbers

$$e^{x_i y_j}, \qquad i=1,2;\ j=1,2$$

is transcendental.

The best known result in this direction is the so-called *six exponentials theorem* [1], [6], [11]: if x_1, x_2 (resp. y_1, y_2, y_3) are $\mathbf{Q}$-linearly independent complex numbers, then one at least of the six numbers

$$e^{x_i y_j}, \qquad i=1,2;\ j=1,2,3$$

is transcendental.

We refine this result in the following way.

Corollary 2.1 *Let x_1, x_2 be two complex numbers which are $\mathbf{Q}$-linearly independent, and let y_1, y_2, y_3 be three complex numbers which are $\mathbf{Q}$-linearly independent. Further let α_{ij}, $i=1,2$; $j=1,2,3$, be six algebraic numbers. Assume that the six numbers*

$$\exp(x_i y_j - \alpha_{ij}), \qquad i=1,2;\ j=1,2,3,$$

are algebraic. Then

$$x_i y_j = \alpha_{ij} \qquad \text{for } i=1,2 \text{ and } j=1,2,3.$$

If one takes for granted that $\mathbf{Q}$-linearly independent logarithms of algebraic numbers are algebraically independent (a weak form of Schanuel's conjecture), then it is sufficient to consider in corollary 2.1 two numbers y_1, y_2 instead of three (a strong form of the four exponentials conjecture).

We first deduce Corollary 2.1 from Theorem 1.1, then we give some consequences.

b) *Proof of Corollary* 2.1

We choose $G = \mathbf{G}_a^2 \times \mathbf{G}_m^2$, which means $d=4$, $d_0=2$, $d_1=2$, $d_2=0$. We identify $T_G(\mathbf{C})$ with $\mathbf{C}^4$ by

$$\exp_G(u_1,u_2,u_3,u_4) = (u_1,u_2,e^{u_3},e^{u_4}) \in \mathbf{C}^2 \times (\mathbf{C}^\times)^2,$$

and we consider the hyperplane V of $\mathbf{C}^4$ of equation

$$x_2(u_3+u_1) = x_1(u_4+u_2).$$

This hyperplane is the image of the linear map of $\mathbf{C}^3$ into $\mathbf{C}^4$:

$$(z_1,z_2,z_3) \longrightarrow (z_1,z_2,x_1z_3-z_1,x_2z_3-z_2).$$

It contains the points

$$\eta_j = (\alpha_{1j},\alpha_{2j},x_1y_j-\alpha_{1j},x_2y_j-\alpha_{2j}), \qquad j=1,2,3.$$

We take

$$Y = \mathbf{Z}\eta_1 + \mathbf{Z}\eta_2 + \mathbf{Z}\eta_3.$$

We have $m = rk_{\mathbf{Z}}\, Y = 3$, because a relation

$$h_1\eta_1 + h_2\eta_2 + h_3\eta_3 = 0$$

with rational integers h_1, h_2, h_3 implies

$$h_1\alpha_{i1} + h_2\alpha_{i2} + h_3\alpha_{i3} = 0, \qquad i=1,2,$$

and

$$h_1y_1 + h_2y_2 + h_3y_3 = 0,$$

which gives $h_1 = h_2 = h_3 = 0$.

If the six numbers

$$\delta_{ij} = \exp(x_iy_j - \alpha_{ij}), \qquad i=1,2;\ j=1,2,3,$$

are all algebraic, then

$$\exp_G \eta_j \in G(\overline{\mathbf{Q}}), \qquad j = 1,2,3.$$

Finally, we put

$$W = \mathbf{C}(1,0,-1,0) + \mathbf{C}(0,1,0,-1).$$

This is a vector space of dimension $t = 2$, which is defined over $\mathbf{Q}$ in $\mathbf{C}^4$, and which is contained in V.

We use Theorem 1.1: the inequality

$$m > (d_1 + 2d_2)\,(d - 1 - t)$$

is satisfied; therefore V contains a non-zero $\overline{\mathbf{Q}}$-Lie sub-algebra $T_H(\mathbf{C})$ of $T_G(\mathbf{C})$. Since G is linear, V contains such a $T_H(\mathbf{C})$ of dimension 1.

The assumption that x_1, x_2 are $\mathbf{Q}$-linearly independent means that V does not contain a non-zero element of the form $(0,0,a_1,a_2)$ with rational a_1, a_2. Hence V contains a non-zero element $(\gamma_1,\gamma_2,0,0)$ with algebraic γ_1, γ_2. Therefore $\gamma_1 x_2 = \gamma_2 x_1$, and the number $\gamma = x_2/x_1$ is algebraic and irrational.

Define

$$\log \delta_{ij} = x_i y_j - \alpha_{ij}, \qquad i = 1,2;\ j = 1,2,3.$$

Then

$$\gamma \log \delta_{1j} - \log \delta_{2j} = \alpha_{2j} - \gamma\alpha_{1j},\ j = 1,2,3.$$

Since γ is irrational, we deduce from Baker's theorem (see Corollary 3.3 below):

$$\log \delta_{1j} = 0,\ \log \delta_{2j} = 0,\ \text{and}\ \alpha_{2j} = \gamma\alpha_{1j}$$

for $j = 1,2,3$, which is the desired conclusion

c) *Some consequences of the strong six exponentials theorem.*

The next result can be referred to as the *five exponentials theorem.*

Corollary 2.2 *Let x_1, x_2 be two $\mathbf{Q}$-linearly independent complex numbers, and y_1, y_2 be also two $\mathbf{Q}$-linearly independent complex numbers. Further let η be a non-zero algebraic number. Then one at least of the five numbers*

$$e^{x_1y_1}, e^{x_1y_2}, e^{x_2y_1}, e^{x_2y_2}, e^{\eta x_2/x_1}$$

is transcendental.

Remark. Here is the *strong five exponentials conjecture*: under the hypotheses of Corollary 2.2, if $\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22},\beta$ are algebraic numbers, and if the five numbers $\exp(x_iy_j - \alpha_{ij})$, $i = 1,2$; $j = 1,2$, and $\exp(\eta\frac{x_2}{x_1} - \beta)$ are all algebraic then

$$x_iy_j = \alpha_{ij} \qquad i = 1,2;\ j = 1,2, \quad \text{and} \quad \eta x_2 = \beta x_1.$$

This is clearly a weaker statement than the strong four exponentials conjecture, but still it contains non trivial open problems; for instance, if $\log a$, $\log b$, $\log c$ are non-zero logarithms of algebraic numbers, is it true that

$$(\log a).(\log b) \neq \log c?$$

(Choose $x_1 = 1$, $x_2 = \log a$, $y_1 = 1 + \log b$, $y_2 = \log b$, $\alpha_{11} = \eta = 1$, $\alpha_{12} = \alpha_{21} = \alpha_{22} = \beta = 0$).

Proof of Corollary 2.2 Apply Corollary 2.1 with

$$y_3 = \eta/x_1,\ \alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha_{23} = 0,\ \alpha_{13} = \eta.$$

If the two numbers

$$\gamma_1 = e^{x_1y_1} \text{ and } \gamma_2 = e^{x_1y_2}$$

are algebraic, then the theorem of Hermite-Lindemann (which is the case $n = 1$ of Corollary 3.3) implies that η, $\log\gamma_1$ and $\log\gamma_2$ are $\mathbf{Q}$-linearly independent.

Let us give a few special cases of Corollary 2.2.

(2.2.3) Let α_1, α_2, β be non-zero algebraic numbers, with $\log\alpha_1$, $\log\alpha_2$ $\mathbf{Q}$-linearly independent. Let $t \in \mathbf{C}$, $t \neq 0$. Then one at least of the numbers

$$\alpha_1^t, \alpha_2^t, e^{\beta t}$$

is transcendental, and also one at least of the numbers

$$\alpha_1^t, \alpha_2^t, e^{\beta/t}.$$

is transcendental.

(2.2.4) Let α and β be non-zero algebraic numbers with $\log\alpha \neq 0$, and let $t \in \mathbf{C}$ be irrational. Then one at least of

$$\alpha^t, \alpha^{t^2}, e^{\beta t},$$

and one at least of

$$\alpha^t, \alpha^{t^2}, e^{\beta/t}$$

is transcendental. If, further, $\beta t/\log\alpha$ is irrational, then one at least of

$$\alpha^t, \alpha^{t^2}, e^{\beta t^2}$$

is transcendental.

(2.2.5). Let $\alpha_1, \alpha_2, \gamma, \eta$ be non-zero algebraic numbers with $\log\alpha_1$, $\log\alpha_2$ **Q**-linearly independent and $\log\gamma \neq 0$. Then one at least of

$$\alpha_1^{\eta\log\gamma}, \alpha_2^{\eta\log\gamma},$$

and at least one of

$$\alpha_1^{\eta/\log\gamma}, \alpha_2^{\eta/\log\gamma}$$

is transcendental.

For instance, if α and β are non-zero algebraic numbers with $\log\alpha \neq 0$ and $\log\beta \neq 0$, then the numbers

$$\alpha^{\log\beta} \text{ and } \alpha^{(\log\beta)^2}$$

are not both algebraic. In this result, only the case $\alpha = \beta$ was known, as a consequence of some results on algebraic independence [4].

§3. Further corollaries to Theorem 1.1

We first consider the case $t = d - 1$ (Gelfond's method), next the case $t = 0$ (Schneider's method), and finally we give an example with $t = 1$.

a) *Gelfond's method*

If, in Theorem 1.1, the hyperplane V itself is defined over $\overline{\mathbf{Q}}$, then one can choose $W = V$, $t = d - 1$, and the assumption on m reduces to $m > 0$, which means $Y \neq 0$. One deduces the following corollary, which is Wüstholz's result announced in [16] (see [2] Th. 4).

Corollary 3.1 *Let G be a commutative algebraic group defined over $\overline{\mathbf{Q}}$, and let $u \in T_G(\mathbf{C})$ be such that $\exp_G u \in G(\overline{\mathbf{Q}})$. Then the smallest subspace of $T_G(\mathbf{C})$ defined over $\overline{\mathbf{Q}}$ which contains u is an algebraic Lie sub-algebra of $T_G(\mathbf{C})$, defined over $\overline{\mathbf{Q}}$.*

Proof. (See [2] p. 36–37). Let W_0 be the smallest subspace of $T_G(\mathbf{C})$ defined over $\overline{\mathbf{Q}}$ which contains u. We want an algebraic subgroup H_0 of G, defined over $\overline{\mathbf{Q}}$, such that $W_0 = T_{H_0}(\mathbf{C})$. If $W_0 = T_G(\mathbf{C})$ (resp. $W_0 = 0$), take $H_0 = G$ (resp. $H_0 = 0$). Otherwise choose any hyperplane W of $T_G(\mathbf{C})$ defined over $\overline{\mathbf{Q}}$, which contains W_0. By Theorem 1.1, there exists an algebraic subgroup H of G, of positive dimension, such that $T_H(\mathbf{C}) \subset W$. Let H_W be the largest connected algebraic subgroup of G, defined over $\overline{\mathbf{Q}}$, for which

$$T_{H_W} \subset W.$$

By Theorem 1.1 on G/H_W, we deduce that u belongs to $T_{H_W}(\mathbf{C})$. Finally, we define H_0 as the intersection of H_W, when W runs over the hyperplanes of $T_G(\mathbf{C})$, defined over $\overline{\mathbf{Q}}$, which contain W_0. We get $T_{H_0}(\mathbf{C}) \subset W_0$, hence $T_{H_0} = W_0$. This proves Corollary 3.1.

Let us remark that our proof of Corollary 3.1 does not use Baker's method: we do not perform an extrapolation involving the Schwarz lemma on the one-dimensional complex line $\mathbf{C}.u$ in $T_G(\mathbf{C})$; also we do not need to introduce in our proof suitable division points of u, even if $\exp_G u$ is of finite order in $G(\overline{\mathbf{Q}})$ (compare with [2] p. 37). However, if one looks for effective estimates, one gets sharper results in the situation of Corollary 3.1 than in the general case of Theorem 1.1 if one combines the present approach with Baker's extrapolation procedure (see [10]).

b) *Schneider's method*

If we have no arithmetic assumption on V, we can always choose $W = 0$, which means $t = 0$, and the hypothesis on $m = rk_{\mathbf{Z}} Y$ is

$$m > (d_1 + 2d_2)(d - 1).$$

The corresponding statement for the multiplicative case ($d = d_1$) is given in [2]. Here is an example involving a power of an elliptic curve ($d = d_2$).

Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3.$$

A complex number u is an algebraic point of $\wp$ if either u is a pole of $\wp$ or else $\wp(u)$ is an algebraic number. Let k be the field of endomorphisms of the corresponding elliptic curve.

Corollary 3.2. *Let u_{ij}, $1 \leq i \leq n$, $1 \leq j \leq \ell$, be algebraic points of $\wp$, with $n \geq 1$ and $\ell > \frac{2}{[k:\mathbf{Q}]} \cdot n(n+1)$. Further, let $t_1, \ldots, t_n$ be complex* *numbers. Assume that for $1 \leq j \leq \ell$, the point*

$$\sum_{i=1}^{n} t_i u_{ij}$$

is an algebraic point of $\wp$.

a) *If the ℓ points*

$$(u_{1j}, \dots, u_{nj}), \qquad 1 \leq j \leq \ell,$$

in $\mathbf{C}^n$ are k-linearly independent, then the numbers $1, t_1, \dots, t_n$ are k-linearly independent.

b) *If the ℓn numbers*

$$u_{ij}, \qquad 1 \leq i \leq n,\ 1 \leq j \leq \ell,$$

are k-linearly independent, then $t_1, \dots, t_n$ are all in k.

Proof.

(a) Let E be the elliptic curve in $\mathbf{P}_2$ whose exponential map is given by

$$\exp_E(z) = \big(1, \wp(z), \wp'(z)\big).$$

Consider the algebraic group $G = E^{n+1}$ of dimension $d = d_2 = n + 1$. We identify $T_G(\mathbf{C})$ with $\mathbf{C}^{n+1}$ by

$$\exp_G(z_1 \dots z_{n+1}) = (\exp_E z_1, \dots, \exp_E z_{n+1}).$$

Let V be the hyperplane $z_{n+1} = t_1 z_1 + \dots + t_n z_n$. Let

$$u_{n+1,j} = \sum_{i=1}^{n} t_i u_{ij}, \qquad 1 \leq j \leq \ell,$$

and

$$y_j = (u_{1j}, \dots, u_{n+1,j}) \in \mathbf{C}^{n+1}, \qquad 1 \leq j \leq \ell.$$

We denote by σ the ring of endomorphisms of E, and by Y the σ-module generated by $y_1, \dots, y_\ell$. Plainly we have

$$Y \subset V \quad \text{and} \quad rk_{\mathbf{Z}} Y = \ell \,.\, [k : \mathbf{Q}].$$

From Theorem 1.1 we deduce that V contains a non-zero $\overline{\mathbf{Q}}$ algebraic Lie sub-algebra of $T_G(\mathbf{C})$. Since $G = E^{n+1}$, V contains such a $\overline{\mathbf{Q}}$-Lie sub-algebra of dimension 1, hence there exists $(b_1, \dots, b_{n+1}) \in k^{n+1}$ such that $0 \neq (b_1, \dots, b_{n+1}) \in V$. This proves (a).

(b) There is no loss of generality in assuming that the k-vector space $k + kt_1 + \dots + kt_n$ is generated by $1, t_1, \dots, t_r$. Assume $r \geq 1$. Write

$$t_i = b_{i0} + \sum_{\rho=1}^{r} b_{i\rho} t_\rho, \qquad r < i \leq n$$

where $b_{i\rho}$, $0 \leq \rho \leq r$, are in k. Define

$$u'_{\rho j} = u_{\rho j} + \sum_{i=r+1}^{n} b_{i\rho} u_{ij}, \qquad 1 \leq \rho \leq r,\ \ 1 \leq j \leq \ell,$$

and apply (a) with n replaced by r to get a contradiction. Hence $r = 0$ and $t_1 \in k$ for $1 \leq i \leq n$.

c) *Baker's theorem.*

We will deduce from Theorem 1.1 the following result of Baker [1] Chap. 2.

Corollary 3.3. *Let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers such that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\mathbf{Q}$. Then the numbers $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\overline{\mathbf{Q}}$.*

Of course Corollary 3.3 is a special case of Corollary 3.1 (see [16], [2]): we take $G = \mathbf{G}_a \times \mathbf{G}_m^n$, and

$$u = (1, \log \alpha_1, \dots, \log \alpha_n) \in \mathbf{C} \times \mathbf{C}^n;$$

if there is a non-trivial relation

$$\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n = 0,$$

then Corollary 3.1 shows that the hyperplane

$$\beta_0 z_0 + \beta_1 z_1 + \dots + \beta_n z_n = 0$$

contains the smallest $\overline{\mathbf{Q}}$ algebraic Lie sub-algebra of $T_G(\mathbf{C})$ which contains u, hence the point $(\log \alpha_1, \dots, \log \alpha_n)$ in $\mathbf{C}^n$ belongs to a hyperplane which is defined over $\mathbf{Q}$.

We give another proof of Corollary 3.3, which is more close to Schneider's method (see [14] §8.3.b, [7], [17]).

a) We first use Schneider's method to prove that $\log\alpha_1,\dots,\log\alpha_n$ are $\overline{\mathbf{Q}}$-linearly independent. Assume

$$\log\alpha_n = \beta_1\log\alpha_1 + \dots + \beta_{n-1}\log\alpha_{n-1},$$

where $\beta_1,\dots,\beta_{n-1}$ are algebraic (and not all rational). Consider the algebraic group $G = \mathbf{G}_a^{n-1}\times\mathbf{G}_m$, of dimension $d = d_1 = n$, and the hyperplane V of equation

$$z_n = z_1\log\alpha_1 + \dots + z_{n-1}\log\alpha_{n-1}$$

in $\mathbf{C}^{n-1}\times\mathbf{C}$. Further, let

$$Y = \{(h_1 + h_n\beta_1,\dots,h_{n-1}+h_n\beta_{n-1}, h_1\log\alpha_1+\dots+h_n\log\alpha_n);\ (h_1,\dots,h_n)\in\mathbf{Z}^n\}.$$

Therefore Y is of rank n and is contained in V. From Theorem 1.1 with $t=0$ we deduce that V contains a non-zero element $(\gamma_1,\dots,\gamma_{n-1},0)$ where $\gamma_1,\dots,\gamma_{n-1}$ are algebraic. This contradicts the assumption that $\log\alpha_1,\dots,\log\alpha_{n-1}$ are linearly independent over $\overline{\mathbf{Q}}$.

b) We now start from a relation

$$\log\alpha_n = \beta_0 + \beta_1\log\alpha_1 + \dots + \beta_{n-1}\log\alpha_{n-1},$$

where $1, \log\alpha_1,\dots,\log\alpha_{n-1}$ are $\overline{\mathbf{Q}}$-linearly independent. We take $G = \mathbf{G}_a^n\times\mathbf{G}_m$, and V is the hyperplane

$$z_n = \beta_0 z_0 + z_1\log\alpha_1 + \dots + z_{n-1}\log\alpha_{n-1},$$

while

$$Y = \{(h_n, h_1+h_n\beta_1\dots,h_{n-1}+h_n\beta_{n-1}, h_1\log\alpha_1+\dots+h_n\log\alpha_n);\ (h_1,\dots,h_n)\in\mathbf{Z}^n\}.$$

We now take

$$W = \mathbf{C}(1,0,\dots,0,1)\subset\mathbf{C}^n\times\mathbf{C},$$

and we use Theorem 1.1 with $d=n+1$, $d_0=n$, $d_1=1$, $t=1$, $m=n$. We find in V a non-zero element $(\gamma_0,\gamma_1,\dots,\gamma_{n-1},0)$, where the γ's are algebraic. Therefore

$$\beta_0\gamma_0 + \gamma_1\log\alpha_1 + \dots + \gamma_{n-1}\log\alpha_{n-1} = 0.$$

From the initial assumption we deduce $\gamma_1 = \dots = \gamma_{n-1} = 0$, hence $\beta_0 = 0$, which is what we wanted.

Remark. This new proof of Baker's theorem can be refined into an effective lower bound for linear forms in logarithms (see [15] §6.d; compare with [1] Chap. 2). The estimate we get by this method is the same as can be achieved by Gelfond's method alone. As mentioned above, it means that it is weaker than the estimates which arise by combining Gelfond's and Baker's method. It would be interesting to deduce Baker's estimates from Schneider's method.

§4. The main result.

a) *The statement.*

In Theorem 1.1, we assumed that V was a hyperplane of $T_G(\mathbf{C})$. In some cases (e.g. [15]), it is interesting to deal with a subspace of $T_G(\mathbf{C})$ of any dimension $n<d$, and instead of the assumption

$$m > (d_1+2d_2)(d-1-t),$$

we require only

$$m > (d_1+2d_2)\,.\,\frac{n-t}{d-n}.$$

Let us consider again an algebraic group $G = \mathbf{G}_a^{d_0}\times\mathbf{G}_m^{d_1}\times G_2$, defined over $\overline{\mathbf{Q}}$, of dimension $d = d_0+d_1+d_2\ge 1$. Let

$$\pi_0 : G\longrightarrow \mathbf{G}_a^{d_0} \text{ and } \pi_1 : G\longrightarrow\mathbf{G}_m^{d_1}$$

be the corresponding projections. Here we do not assume that d_0 and d_1 are maximal.

Theorem 4.1 *Let V be a subspace of $T_G(\mathbf{C})$ of dimension $n<d$, W a subspace of V, and Y a finitely generated subgroup of V. Assume that W is defined over $\overline{\mathbf{Q}}$, and that $\Gamma = \exp_G Y$ is contained in $G(\overline{\mathbf{Q}})$. Finally, define*

$$\kappa = rk_{\mathbf{Z}}(Y\cap\operatorname{Ker}\exp_G).$$

Then there exists a connected algebraic subgroup G' of G, defined over $\overline{\mathbf{Q}}$, with $G'\neq G$, satisfying the following properties. Define

$$\delta = \dim G/G', \qquad \delta_0 = \dim\mathbf{G}_a^{d_0}/\pi_0(G'),$$

$$\delta_1 = \dim \mathbf{G}_m^{d_1}/\pi_1(G'), \qquad \delta_2 = \delta - \delta_0 - \delta_1,$$
$$\lambda = rk_{\mathbf{Z}}\Gamma/\Gamma \cap G', \qquad \tau = \dim W/W \cap T_{G'}(\mathbf{C}).$$

Then $\delta > \tau$ and

$$(\lambda + \delta_1 + 2\delta_2)(d - n) \leq (\delta - \tau)(d_1 + 2d_2 - \kappa).$$

The conclusion holds trivially with $G' = 0$ if the inequality

$$m > (d_1 + 2d_2 - \kappa) \cdot \frac{n - t}{d - n} \tag{4.2}$$

is not satisfied. On the other hand, if this inequality (4.2) holds, then $\dim G' > 0$.

The special case $t = d - 1$ of Theorem 1.1 (see Corollary 3.1) readily follows from Theorem 4.1: when W is a hyperplane of $T_G(\mathbf{C})$, the condition $\delta > \tau$ means $W \supset T_{G'}(\mathbf{C})$, and the assumption $m > 0$ gives $\dim G' > 0$.

The proof of Theorem 4.1 is given in §7 below. We will deduce Theorem 1.1 from Theorem 4.1 in §5. Now we give some further corollaries to Theorem 4.1.

b) *Schneider's method*

Here, we use only the case $t = 0$ of Theorem 4.1. Let us recall (cf. [14]) that

$$\mu(Y, V) = \min_{E \subset V} \frac{rk_{\mathbf{Z}} Y/Y \cap E}{\dim_{\mathbf{C}} V/E},$$

where E runs over the set of vector subspaces of V with $E \neq V$.

Corollary 4.3 *With the assumptions of Theorem 4.1, if $\exp_G V$ is Zariski dense in $G(\mathbf{C})$, then*

$$\mu(Y, V) \leq (d_1 + 2d_2 - \kappa)/(d - n).$$

In the case $\dim V = 1$, the conclusion is simply

$$m \leq (d_1 + 2d_2 - \kappa)/(d - n),$$

which is equivalent to the results of [14] Chap. 4.

We will deduce Corollary 4.3 from Theorem 4.1 in section e below. We first deduce some consequence from Corollary 4.3.

c) *Algebraic points on the graph of an analytic homomorphism.*

Let us denote by G' a commutative algebraic group defined over $\overline{\mathbf{Q}}$, by $\Psi : \mathbf{C}^n \longrightarrow G'(\mathbf{C})$ an analytic homomorphism, and by Y a subgroup of $\overline{\mathbf{Q}}^n$ such that $\Psi(Y) \subset G'(\overline{\mathbf{Q}})$. We write

$$\rho = \rho(G') = \begin{cases} 1 & \text{if } G' \text{ is linear,} \\ 2 & \text{otherwise.} \end{cases}$$

Corollary 4.4 *Assume that $\dim G' \geq 1$, and that G' does not contain a non-zero unipotent linear subgroup. If Ψ is not constant, then*

$$\mu(Y, \overline{\mathbf{Q}}^n) \leq \rho.$$

We deduce Corollary 4.4 by applying Corollary 4.3 to $G = \mathbf{G}_a^n \times G''$, where G'' is the Zariski closure of $\Psi(\mathbf{C}^n)$ in $G'(\mathbf{C})$, with $d \geq n+1$, $d = d_\rho$, $d_0 = n$.

A special case of Corollary 4.4 was already given in [14] Prop. 8.1.2. Here is a consequence of Corollary 4.4.

Corollary 4.5 *Let L be the maximal (connected) unipotent linear algebraic subgroup of G. Then the image of $\Psi(\overline{\mathbf{Q}}^n) \cap G'(\overline{\mathbf{Q}})$ in G'/L has a finite rank $\leq \rho n$.*

Proof. We proceed by induction on n. Assume $y_1, \ldots, y_{\rho n+1}$ are in $\overline{\mathbf{Q}}^n$, with $\Psi(y_j) \in G'(\overline{\mathbf{Q}})$, $1 \leq j \leq \rho n + 1$, and that their images in G'/L are $\mathbf{Q}$-linearly independent. Let $Y = \mathbf{Z}y_1 + \ldots + \mathbf{Z}y_{\rho n+1}$. From Corollary 4.4 we deduce $\mu(Y, \overline{\mathbf{Q}}^n) \leq \rho$. Let W be a subspace of $\mathbf{C}^n$, defined over $\overline{\mathbf{Q}}$, such that

$$rk_{\mathbf{Z}} Y \cap W \geq 1 + \rho \dim_{\mathbf{C}} W,$$

and $W \neq \mathbf{C}^n$. The restriction of Ψ to W gives a contradiction with the induction hypothesis.

Thanks to Corollary 4.4, we can prove the following result, which was announced in [15] (6.7).

Corollary 4.6 *If $rk_{\mathbf{Z}} Y \geq \rho n + 1$, then there exists $y \in Y$, $y \neq 0$, such that the homomorphism $t \longrightarrow \Psi(yt)$ of $\mathbf{C}$ into $G'(\mathbf{C})$ is rational.*

This result was proved already in [14] Th. 8.1.1 under the assumption $Y \subset \mathbf{R}^n$, and in [14] Th. 6.3.2 under the assumption that G' is an extension by a linear group of an abelian variety which is isogeneous to a product of simple abelian varieties of C.M. type. We could also deduce Corollary 4.6 from Corollary 3.1 following [14] Chap. 6.

Proof of Corollary 4.6. Assume first $\mu(Y, \mathbb{C}^n) > \rho$. Then Corollary 4.4 shows that $\Psi(\mathbb{C}^n)$ is contained in the maximal unipotent linear subgroup of G', hence Ψ is rational. The general case follows from the arguments of [14] p. 150.

d) *The coefficient* $\mu^\sharp(\Gamma, G)$.

Let us introduce the following Dirichlet exponent: let K be a subfield of $\mathbb{C}$, G be a commutative algebraic group of dimension $d \geq 1$, Γ a finitely generated subgroup of $G(K)$, and

$$\pi_0 : G \longrightarrow \mathbb{G}_a^{d_0}, \ \pi_1 : G \longrightarrow \mathbb{G}_m^{d_1}$$

two surjective morphisms, with $d_0 \geq 0$, $d_1 \geq 0$. Further, we set $d_2 = d - d_0 - d_1$. Therefore $G = \mathbb{G}_a^{d_0} \times \mathbb{G}_m^{d_1} \times G_2$, where $\dim G_2 = d_2$. We define

$$\mu^\sharp(\Gamma, G) = \min_{G' \subset G} (\lambda + \delta_1 + 2\delta_2)/\delta,$$

where G' runs over the set of algebraic subgroups of G, defined over K, with $G' \neq G$, and

$$\delta = \dim G/G', \ \delta_0 = \dim \mathbb{G}_a^{d_0}/\pi_0(G'),$$

$$\delta_1 = \dim \mathbb{G}_m^{d_1}/\pi_1(G'), \ \delta_2 = \delta - \delta_0 - \delta_1,$$

$$\lambda = rk_{\mathbb{Z}}\Gamma/\Gamma \cap G'.$$

It should be noted that $\mu^\sharp(\Gamma, G)$ depends not only on Γ and G, but also on K, π_0 and π_1. If G' is any algebraic subgroup of G, $G' \neq G$, *we define* $\mu^\sharp(\Gamma/\Gamma \cap G', G/G')$ by choosing

$$\pi_0' : G' \longrightarrow \mathbb{G}_a^{d_0'}, \ \pi_1' : G' \longrightarrow \mathbb{G}_m^{d_1'},$$

with $\delta_0 = \dim \mathbb{G}_a^{d_0}/\pi_0(G')$ and $\delta_1 = \dim \mathbb{G}_m^{d_1}/\pi_1(G')$ so that we get commutative diagrams:

$$\begin{array}{ccc} G & \xrightarrow{\pi_0} & \mathbb{G}_a^{d_0} \\ \downarrow & & \downarrow \\ G/G' & \xrightarrow{\pi_0'} & \mathbb{G}_a^{d_0}/\pi_0(G') \end{array}$$

and

$$\begin{array}{ccc} G & \xrightarrow{\pi_1} & \mathbb{G}_m^{d_1} \\ \downarrow & & \downarrow \\ G/G' & \xrightarrow{\pi_1'} & \mathbb{G}_m^{d_1}/\pi_1(G') \end{array}$$

Also we define $\mu^\sharp(\Gamma \cap G', G')$ by choosing the restrictions $G' \longrightarrow \pi_0(G')$ and $G' \longrightarrow \pi_1(G')$ with $\pi_0(G') \cong \mathbb{G}_a^{d_0 - \delta_0}$ and $\pi_1(G') \cong \mathbb{G}_m^{d_1 - \delta_1}$, where $\cong$ means *isogeneous to.*

By taking $G' = 0$, we see that

$$\mu^\sharp(\Gamma, G) \leq \frac{\ell + d_1 + 2d_2}{d},$$

where $\ell = \mathrm{rank}_{\mathbb{Z}}\Gamma$.

e) *Proof of Corollary 4.3*

The conclusion of Theorem 4.1 is

$$\mu^\sharp(\Gamma, G) \leq \frac{d_1 + 2d_2 - \kappa}{d - n}. \tag{4.7}$$

If $\mu^\sharp(\Gamma, G) = (\ell + d_1 + 2d_2)/d$, then (4.7) gives

$$\frac{\ell + \kappa}{n} \leq \frac{d_1 + 2d_2 - \kappa}{d - n}.$$

and Corollary 4.3 follows. Otherwise, we write

$$\mu^\sharp(\Gamma, G) = (\lambda + \delta_1 + 2\delta_2)/\delta$$

for some algebraic subgroup G' of G, $G' \neq G$, of dimension $d - \delta > 0$. Clearly we have

$$\mu^\sharp(\Gamma/\Gamma \cap G', G/G') = (\lambda + \delta_1 + 2\delta_2)/\delta.$$

We define

$$E = V \cap T_{G'}, \qquad V' = V/E, \qquad Y' = Y/Y \cap E,$$
$$n' = \dim V', \qquad m' = rk_{\mathbb{Z}}Y', \qquad \kappa' = rk_{\mathbb{Z}}(Y' \cap \ker \exp_{G/G'});$$

therefore

$$\mu(Y, V) \leq m'/n'.$$

Further, let

$$\Gamma' = \exp_{G/G'} Y' = \Gamma/\Gamma \cap G'.$$

We notice that $m' = \lambda + \kappa'$. We apply (4.7) to Γ':

$$(\delta - n')\mu^\sharp(\Gamma', G/G') \le \delta_1 + 2\delta_2 - \kappa'.$$

Hence

$$m'\delta \le (\lambda + \kappa')\delta \le n'(\lambda + \delta_1 + 2\delta_2).$$

We conclude that

$$\mu(\Gamma, V) \le \frac{m'}{n'} \le \frac{\lambda + \delta_1 + 2\delta_2}{\delta} \le \frac{d_1 + 2d_2 - \kappa}{d - n}.$$

§5. Proof of Theorem 1.1

In this section we deduce Theorem 1.1 (with a slight refinement) from Theorem 4.1. We first introduce a generalization of the coefficient $\mu^\sharp$ of §4. Next we prove an auxiliary lemma concerning some problem which arises with the periods of the exponential map, and finally we complete the proof of Theorem 1.1

a) *The coefficient* $\mu^\sharp(\Gamma, G, W)$.

Let K be a subfield of $\mathbf{C}$, G be commutative connected algebraic group of dimension d, $\pi_0 : G \longrightarrow \mathbf{G}_a^{d_0}$ and $\pi_1 : G \longrightarrow \mathbf{G}_m^{d_1}$ two surjective morphisms of algebraic groups, $d_2 = d - d_0 - d_1$, Γ a finitely generated subgroup of $G(K)$, and W a subspace of $T_G(\mathbf{C})$, distinct from $T_G(\mathbf{C})$. We define

$$\mu^\sharp(\Gamma, G, W) = \min_{G'} \frac{\lambda + \delta_1 + 2\delta_2}{\delta - \tau},$$

where G' runs over the set of connected algebraic subgroups of G which are defined over K, with $G' \neq G$ and $\delta > \tau$, and where

$$\delta = \dim G/G', \qquad \delta_0 = \dim \mathbf{G}_a^{d_0}/\pi_0(G'),$$
$$\delta_1 = \dim \mathbf{G}_m^{d_1}/\pi_1(G'), \qquad \delta_2 = \delta - \delta_0 - \delta_1,$$
$$\lambda = rk_{\mathbf{Z}}\Gamma/\Gamma \cap G', \qquad \tau = \dim_{\mathbf{C}} W/W \cap T_{G'}.$$

Remarks

(1) Since $\tau = \dim(T_{G'} + W)/T_{G'}$, we have $\delta - \tau = \dim T_G/(T_{G'} + W)$, and therefore the condition $\tau = \delta$ is equivalent to $T_{G'} + W = T_G$. In any case $\mu^\sharp$ satisfies:

$$\mu^\sharp(\Gamma, G, W) \le \frac{\ell + d_1 + 2d_2}{d - n},$$

where $\ell = rk_{\mathbf{Z}}\Gamma$ and $t = \dim_{\mathbf{C}} W$.

(2) The coefficient $\mu^\sharp$ depends not only on G, Γ and W, but also on the choice of π_0 and π_1. For G' connected algebraic subgroup of G, we define

$$\mu^\sharp(\Gamma/\Gamma \cap G', G/G', W/W \cap T_{G'}), \qquad \text{if } G' \neq G,$$

and

$$\mu^\sharp(\Gamma \cap G', G', W \cap T_{G'}), \qquad \text{if } G' \neq 0,$$

with the same conventions as in §4.*c*.

We need the following generalization of Lemma 1.3.1 of [14] and Lemma 3.2 of [15].

Lemma 5.1 *Assume*

$$\mu^\sharp(\Gamma, G, W) < \frac{\ell + d_1 + 2d_2}{d - t}.$$

Then there exists an algebraic subgroup G' of G, which is defined over K, of dimension $d' \ge 1$, such that either $W \supset T_{G'}$, or

$$\mu^\sharp(\Gamma \cap G', G', W \cap T_{G'}) = \frac{\ell' + d_1' + 2d_2'}{d' - t} > \frac{\ell + d_1 + 2d_2}{d - t},$$

where

$$t' = \dim W \cap T_{G'}, \qquad d_1' = \dim \pi_1(G'),$$
$$d_0' = \dim \pi_0(G'), \qquad \ell' = rk_{\mathbf{Z}}\Gamma \cap G',$$

and

$$d_2' = d' - d_0' - d_1'.$$

Proof. Assume that W does not contain a non-zero K-algebraic Lie subalgebra of $T_G(\mathbf{C})$. We will prove the desired conclusion by induction on d. If $d = 1$ then $t = 0$ and

$$\mu^\sharp(\Gamma, G, 0) = \ell + d_1 + 2d_2.$$

Assume Lemma 5.1 holds for all proper algebraic subgroups of G. By the definition of $\mu^\sharp$, there exists an algebraic subgroup G° of G such that

$$\mu^\sharp(\Gamma, G, W) = (\lambda^\circ + \delta_1^\circ + 2\delta_2^\circ)/(\delta^\circ - \tau^\circ),$$

where

$$\begin{aligned} &\delta^\circ = \dim G/G^\circ, && \delta_0^\circ = \dim \mathbf{G}_a^{d_0}/\pi_0(G^\circ),\\ &\delta_1^\circ = \dim \mathbf{G}_m^{d_1}/\pi_1(G^\circ), && \delta_2^\circ = \delta^\circ - \delta_0^\circ - \delta_1^\circ,\\ &\lambda^\circ = rk_{\mathbf{Z}}\Gamma/\Gamma\cap G^\circ, && \tau^\circ = \dim_{\mathbf{C}} W/W\cap T_{G^\circ}. \end{aligned}$$

We define

$$\begin{aligned} &d^\circ = \dim G^\circ, && d_0^\circ = \dim \pi_0(G^\circ),\\ &d_1^\circ = \dim \pi_1(G^\circ), && d_2^\circ = d^\circ - d_0^\circ - d_1^\circ,\\ &t^\circ = \dim W\cap T_{G^\circ}, && \ell^\circ = rk_{\mathbf{Z}}\Gamma\cap G^\circ. \end{aligned}$$

Hence

$$\begin{aligned} &\delta^\circ + d^\circ = d, && \delta_i^\circ + d_i^\circ = d_i, && i = 0,1,2,\\ &\lambda^\circ + \ell^\circ = \ell, && t^\circ + \tau^\circ = t. \end{aligned}$$

The assumption that W does not contain T_{G° gives $d^\circ > t^\circ$, and the hypothesis

$$\mu^\sharp(\Gamma, G, W) < (\ell + d_1 + 2d_2)/(d-t)$$

is equivalent to

$$\frac{\ell^\circ + d_1^\circ + 2d_2^\circ}{d^\circ - t^\circ} > \frac{\ell + d_1 + 2d_2}{d-t}.$$

If

$$\mu^\sharp(\Gamma\cap G^\circ, G^\circ, W\cap T_{G^\circ}) = (\ell^\circ + d_1^\circ + 2d_2^\circ)/(d^\circ - t^\circ),$$

then the lemma is proved with $G' = G^\circ$. Otherwise we can use the induction hypothesis, since $d^\circ < d$. We deduce that there exists an algebraic subgroup G' of G° such that

$$\mu^\sharp(\Gamma\cap G', G', W\cap T_{G'}) = \frac{\ell' + d_1' + 2d_2'}{d' - t'} > \frac{\ell^\circ + d_1^\circ + 2d_2^\circ}{d^\circ - t^\circ},$$

with

$$\begin{aligned} &d_i' = \dim \pi_i(G'), && (i = 0,1)\\ &d' = \dim G', && d_2' = d' - d_0' - d_1'.\\ &t' = \dim W\cap T_{G'}, && \ell' = rk_{\mathbf{Z}}\Gamma\cap G'. \end{aligned}$$

This completes the proof of Lemma 5.1.

b) *Another auxiliary lemma.*

Let G be a commutative algebraic group over $\mathbf{C}$ of dimension $d = d_0 + d_1 + d_2 \geq 1$, as before. Further let $Y = \mathbf{Z}y_1 + \ldots + \mathbf{Z}y_m$ be a finitely generated subgroup of $T_G(\mathbf{C})$, and G' an algebraic subgroup of G. Define

$$\Gamma = \exp_G Y, \qquad Y' = Y\cap T_{G'}, \qquad \Gamma' = \exp_G Y'.$$

Of course we have $\Gamma' \subset \Gamma\cap G'$, but the rank of $\Gamma\cap G'$ may be larger than the rank of Γ', because of the periods of $\exp_G$.

Let us define $\Omega = \operatorname{Ker}\exp_G$, and

$$\kappa = rk_{\mathbf{Z}}Y\cap\Omega, \qquad \kappa' = rk_{\mathbf{Z}}Y'\cap\Omega.$$

Lemma 5.2. *We have*

$$rk_{\mathbf{Z}}\Gamma' \geq rk_{\mathbf{Z}}\Gamma\cap G' - (d_1 + 2d_2 - \kappa) + d_1' + 2d_2' - \kappa',$$

where, as before,

$$\begin{aligned} &d' = \dim G', && d_0' = \dim \pi_0(G'),\\ &d_1' = \dim \pi_1(G'), && d_2' = d' - d_0' - d_1'. \end{aligned}$$

Proof Let Ω' be the kernel of the exponential map of G/G' in $T_{G/G'} \cong T_G/T_{G'}$. We first remark that G/G' is a product of $\mathbf{G}_a^{d_0-d_0'} \times \mathbf{G}_m^{d_1-d_1'}$ by an algebraic group of dimension $d_2 - d_2'$, hence

$$rk_{\mathbf{Z}}\Omega' \leq (d_1 + 2d_2) - (d_1' + 2d_2').$$

By considering the surjective map

$$Y/Y' \longrightarrow \Gamma/\Gamma\cap G'$$

given by $\exp_{G/G'}$, we find

$$rk_{\mathbf{Z}}Y - rk_{\mathbf{Z}}Y' \leq rk_{\mathbf{Z}}\Gamma - rk_{\mathbf{Z}}\Gamma\cap G' + rk_{\mathbf{Z}}\Omega'.$$

From

$$rk_{\mathbf{Z}}Y = rk_{\mathbf{Z}}\Gamma + \kappa$$

and

$$rk_{\mathbf{Z}} Y' = rk_{\mathbf{Z}} \Gamma' + \kappa'$$

we easily deduce Lemma 5.2.

c) *An upper bound for $\mu^\sharp$.*

We now come back to the arithmetic case where G is defined over $\overline{\mathbf{Q}}$. We can state Theorem 4.1 in the following way.

Corollary 5.3 *With the assumptions of Theorem 4.1,*

$$\mu^\sharp(\Gamma, G, W) \leq (d_1 + 2d_2 - \kappa)/(d - n).$$

d) *Proof of Theorem 1.1*

We proceed by induction on d, the case $d = 1$ being trivial. We assume that the hypotheses of Theorem 1.1 are satisfied, apart from (1.2) which we replace by the weaker assumption

$$m > (d_1 + 2d_2 - \kappa)(d - 1 - t), \tag{5.4}$$

with $\kappa = rk_{\mathbf{Z}}(Y \cap \operatorname{Ker} \exp_G) = \ell - m$.

From (5.4) we have

$$\ell + d_1 + 2d_2 \geq (d - t)(d_1 + 2d_2 - \kappa). \tag{5.5}$$

We assume that the conclusion of Theorem 1.1 does not hold, and we will deduce a contradiction.

By Corollary 5.3 (with $n = d - 1$) and assumption (5.5) we have

$$\mu^\sharp(\Gamma, G, W) \leq d_1 + 2d_2 - \kappa < \frac{\ell + d_1 + 2d_2}{d - t}.$$

Using Lemma 5.1 with the assumption that V (hence W) does not contain a non-zero $\overline{\mathbf{Q}}$-Lie sub-algebra of $T_G(\mathbf{C})$, we find an algebraic subgroup G' of G, of dimension $d' \geq 1$, such that

$$\mu^\sharp(\Gamma \cap G', G', W \cap T_{G'}) = \frac{\ell + d_1' + 2d_2'}{d' - t'} > \frac{\ell + d_1 + 2d_2}{d - t}, \tag{5.6}$$

From Lemma 5.2 we deduce that $\Gamma' = \exp_G Y'$, with $Y' = Y \cap T_{G'}$, satisfies

$$\mu^\sharp(\Gamma', G', W \cap T_{G'}) \geq \mu^\sharp(\Gamma \cap G', G', W \cap T_{G'}) - (d_1 + 2d_2 - \kappa) + d_1' + 2d_2' - \kappa'.$$

From (5.5) and (5.6) we get

$$\mu^\sharp(\Gamma', G', W \cap T_{G'}) > d_1' + 2d_2' - \kappa'.$$

Corollary 5.3 shows that $V \cap T_{G'}$ is not a hyperplane of $T_{G'}$, hence $V \supset T_{G'}$, which is the desired contradiction.

§6. The auxiliary function

The proof of Theorem 4.1 involves a refinement of Proposition 2.4 of [15], which we now give. We consider as before an algebraic group $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ over $\overline{\mathbf{Q}}$, of dimension $d = d_0 + d_1 + d_2$, a vector subspace V of $T_G(\mathbf{C})$ of dimension $n < d$, a subspace W of V, of dimension $t \geq 0$, which is defined over $\overline{\mathbf{Q}}$ in $T_G(\mathbf{C})$, and a finitely generated subgroup $Y = \mathbf{Z}y_1 + \ldots + \mathbf{Z}y_m$ of V of rank m such that $\Gamma = \exp_G Y$ is contained in $G(\overline{\mathbf{Q}})$. For each integer $S \geq 1$ we write

$$Y(S) = \{h_1 y_1 + \ldots + h_m y_m : (h_1, \ldots, h_m) \in \mathbf{Z}^m, \quad 0 \leq h_j \leq S,\ 1 \leq j \leq m\},$$

and

$$\Gamma(S) = \exp_G Y(S).$$

Next, let κ satisfy

$$0 \leq \kappa \leq rk_{\mathbf{Z}} V \cap \operatorname{Ker} \exp_G.$$

Finally, we choose a basis $e_1, \ldots, e_t$ of W, defined over $\overline{\mathbf{Q}}$, and we denote by $\Psi : \mathbf{C}^t \longrightarrow G(\mathbf{C})$ the t-parameters subgroup defined by

$$\mathbf{C}^t \cong W \subset T_G(\mathbf{C}) \xrightarrow{\exp_G} G(\mathbf{C}).$$

Given an embedding of G_2 into a projective space $\mathbf{P}_N$, and a polynomial P in $d_0 + d_1 + N + 1$ unknowns, which is homogeneous in the last $N + 1$ unknowns, we say that P vanishes at a point γ in $G(\mathbf{C})$ with multiplicity T along W if the function $z \longrightarrow P(\Psi(z) + \gamma)$ has a zero of order T at the point $z = 0$ in $\mathbf{C}^t$ (see [9] and [10]).

We choose two real numbers $a \geq 1$ and $b \geq 1$.

Proposition 6.1 *There exist an embedding of G_2 in a projective space $\mathbf{P}_N$ over $\overline{\mathbf{Q}}$, and a constant $C > 0$, satisfying the following properties.*

For each integer $S \geq 2$, define T, D_0, D_1, D_2, Δ as functions of S by

$$\Delta^{d-n} = C \, . \, S^{d_1+2d_2-\kappa} \, . \, (\log S)^{bd_0}$$

and

$$T(\log S)^a = D_0(\log S)^b = D_1 S = D_2 S^2 = \Delta.$$

There exists a sequence $(P_S)_{S \geq S_0}$ of polynomials in the ring,

$$\overline{\mathbb{Q}}[X_1^0, \ldots, X_{d_0}^0, \; X_1^1, \ldots, X_{d_1}^1, \; X_0^2, \ldots, X_N^2],$$

where P_S

- *–is of degree $\leq D_0$ in the variables $X_1^0, \ldots, X_{d_0}^0$,*
- *–is of degree $\leq D_1$ in the variables $X_1^1, \ldots, X_{d_1}^1$,*
- *–is homogeneous of degree $\leq D_2$ in the variables $X_0^2, \ldots, X_N^2$,*
- *–vanishes at all the points of $\Gamma(S)$ with multiplicity $\geq T$ along W,*
- *–but does not vanish everywhere on $G(\mathbf{C})$.*

This result is proved in [15] Proposition 2.4 in the case $W = 0$ and $b = 1$. The estimates for the derivatives are provided by Lemma 7 of D. Bertrand in Appendix 1 of [14] (compare with [3] and [10]).

§7. Philippon's zero estimate.

We quote here a special case of the main result of [8] (see also [9]) which will enable us to complete the proof of Theorem 4.1.

Let K be a subfield of $\mathbf{C}$, $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ a commutative connected algebraic group over K, $\Gamma = \mathbf{Z}\gamma_1 + \ldots + \mathbf{Z}\gamma_m$ a finitely generated subgroup of $G(K)$, and W a subspace of $T_G(\mathbf{C})$ defined over K. We fix an embedding of G_2 into a projective space $\mathbf{P}_N$, defined over K.

Proposition 7.1 *There exists a positive constant c with the following property. Let T, D_0, D_1, D_2, S be positive numbers, with $D_2 \leq D_0$ and $D_2 \leq D_1$. Assume that there exists a hypersurface of $\mathbb{A}_{d_0+d_1} \times \mathbf{P}_N$, of degrees $\leq D_0$, D_1, D_2, which does not contain G, but vanishes along W with order $\geq T+1$ at each point of $\Gamma(S)$.*

Then there exists a connected algebraic subgroup G' of G, defined over K, such that if we set

$$\delta = \dim G/G', \qquad \delta_0 = \dim \mathbf{G}_a^{d_0}/\pi_0(G'),$$
$$\delta_1 = \dim \mathbf{G}_m^{d_1}/\pi_1(G'), \qquad \delta_2 = \delta - \delta_0 - \delta_1,$$

$$\lambda = rk_{\mathbf{Z}}\Gamma/\Gamma \cap G', \qquad \tau = \dim_{\mathbf{C}} W/W \cap T_{G'},$$

then $\delta \geq 1$ and

$$T^\tau S^\lambda \leq c D_0^{\delta_0} D_1^{\delta_1} D_2^{\delta_2}. \tag{7.2}$$

Given our choice of parameters in Section 6, the inequality (7.2) yields

$$S^{\lambda+\delta_1+2\delta_2}(\log S)^{b\delta_0 - a\delta} \leq c\Delta^{\delta-\tau}. \tag{7.3}$$

Therefore

$$(\lambda + \delta_1 + 2\delta_2)(d-n) \leq (\delta - \tau)(d_1 + 2d_2 - \kappa).$$

It remains to check that $\delta > \tau$. But if $\delta = \tau$, then $\lambda + \delta_1 + 2\delta_2 = 0$, hence $\lambda = \delta_1 = \delta_2 = 0$ and $\delta_0 = \delta \geq 1$; then (7.3) gives a contradiction if we choose, say, $a = 1$ and $b > t$.

This completes the proof of Theorem 4.1.

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A NEW APPROACH TO BAKER'S THEOREM ON LINEAR FORMS IN LOGARITHMS III

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1. Introduction

1.1 We fix nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n$ and algebraic numbers $\beta_1, \ldots, \beta_n$ not all zero and consider the linear form

$$L(z_1, \ldots, z_n) = \beta_1 z_1 + \ldots + \beta_n z_n.$$

Let the canonical heights of $\alpha_1, \ldots, \alpha_n$ be bounded by $A_1, \ldots, A_n \geq 4$ and the heights of the $\beta_1, \ldots, \beta_n$ by $B \geq 4$; then Baker in a famous series of papers obtained the remarkable result that if $\Lambda = L(\log \alpha_1, \ldots, \log \alpha_n) \neq 0$, $A_1 \leq \ldots \leq A_n$ and

$$\Omega = \log A_1 \ldots \log A_n = \Omega' \log A_n$$

we have

$$\log |\Lambda| > -(16nd)^{200n}(\log(B\Omega))\Omega \log \Omega', \tag{1.1.1}$$

where d denotes the degree of the field generated by $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ over the rationals. Furthermore Baker obtained

$$\log |\Lambda| > -(16nd)^{200n}(\log B)\Omega \log \Omega', \tag{1.1.2}$$

if all the β's are rational integers. This substantial improvement of (1.1.1) has a lot of important consequences. For a detailed account see [1].

1.2 No substantial improvement of (1.1.1) or (1.1.2) has been made up to now. Looking at Baker's proof of (1.1.1) and (1.1.2), one can divide it into two parts: the constructive and the deconstructive part. Baker's method for the deconstructive part is the so-called Kummer theory, a very ingenious and sophisticated tool. If one studies the constructive