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The 9th Expert Scientific Research Meeting of the Sanmenxia Suda Energy Conservation & New Energy Technology Research Institute

A basic introduction to some tools from complex analysis

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Content

Fundación CIDAUT

CIDAUT Foundation is a Spanish non-profit Research and Development Centre for Transport and Energy. One of CIDAUT's current lines of work is sustainable mobility, involving the electric vehicle and its infrastructure.

https://www.cidaut.es/en/electric-automatic-and-electronic-systems

E. Cañibano Álvarez, M. I. González Hernández, L. de Prada Martín, J. Romo García, J. Gutiérrez Diez, J.C. Merino Senovilla.
Development of Mathematical Models for an Electric Vehicle With 4 In-Wheel Electric Motors
Chapter 2 of : Advanced Microsystems for Automotive Applications 2011, Springer Verlag

https://link.springer.com/chapter/10.1007/978-3-642-21381-6_2

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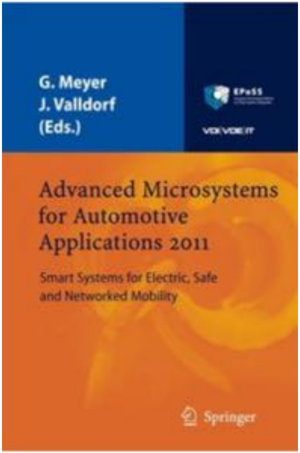

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Maths in Electric Cars - Gillett

GROUP G- Gillett.

GROUP MEMBERS —

- Lukas Milancius
- Muskan Sethi
- Ishwani Patel
- Kritesha.

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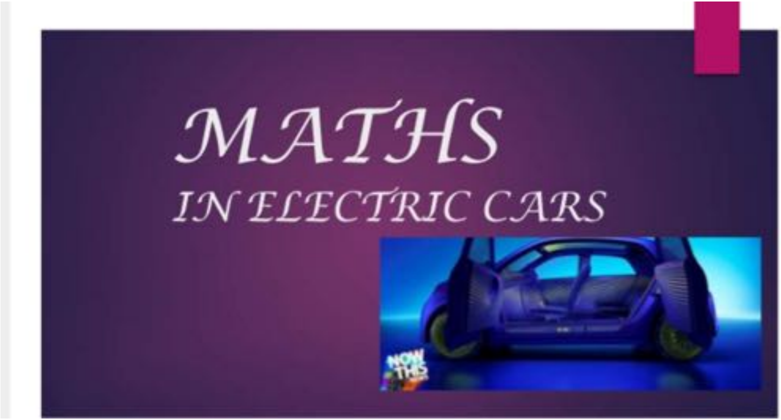

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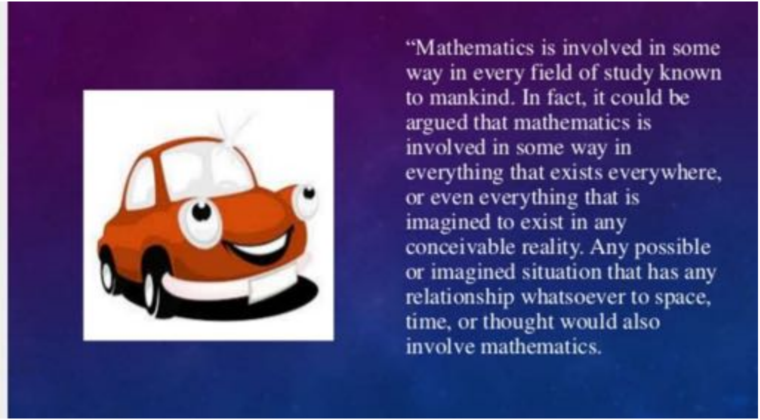

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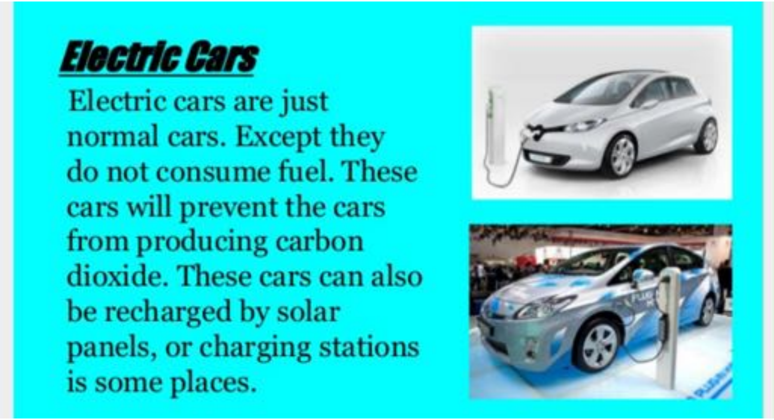

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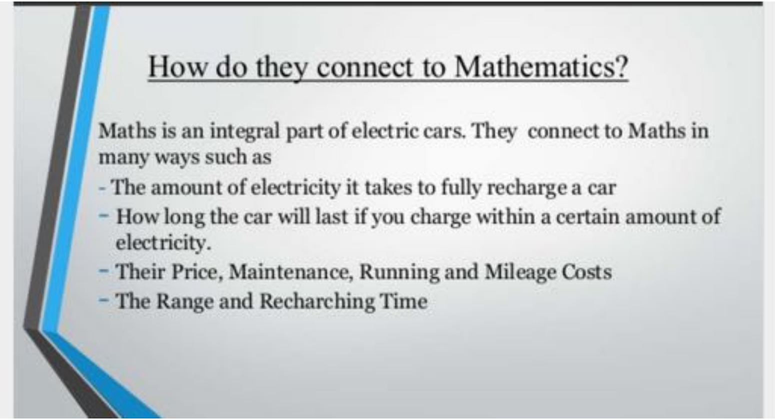

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$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$\mathbb{Q} = \{a/b, a \in \mathbb{Z}, b > 0\}$. $a/b = c/d \Leftrightarrow ad = bc$.

$\mathbb{R}$: Cauchy limits
Cauchy's criterion for convergence of a sequence $(u_n)_{n \geq 0}$:

$$|u_n - u_m| < \epsilon \quad \text{for} \quad n \geq m \geq N(\epsilon).$$

From rational numbers to real numbers

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Convergent sequences, convergent series

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$e = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!} + \cdots$$

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Infinite products, integrals

$$\frac{\pi}{2} = \prod_{n\geq 1}\left(\frac{4n^2}{4n^2-1}\right) = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots},$$

$$\begin{aligned}
\pi &= \int_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y && = 2\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x \\
&= \int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}} && = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^2} \\
&= \frac{22}{7} - \int_0^1 \frac{x^4(1-x^4)\mathrm{d}x}{1+x^2} && = 4\int_0^1\frac{\mathrm{d}x}{1+x^2}.
\end{aligned}$$

Another example : continued fraction

$(\sqrt{2}-1)(\sqrt{2}+1)=1$,

$$\sqrt{2}=1+\frac{1}{1+\sqrt{2}}$$

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Infinitesimal analysis (differential and integral calculus)

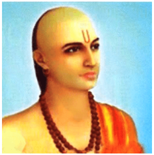
Isaac Newton
(1642 – 1727)

Gottfried Wilhelm Leibniz
(1646 – 1716)

Kerala School of Astronomy and Mathematics

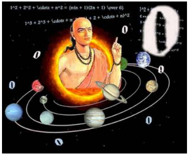
14th – 16th Century : Madhava of Sangamagrama

Madhava
(1340 – 1425)



Parameshvara
(1380 – 1450)

,



Neelakanta Somayaji
(1444 – 1544)

,

Jyeshtadeva, Achyuta Pisharati, Melpathur, Achyuta Panikkar

Narayana Bhattathiri (1559–1632).

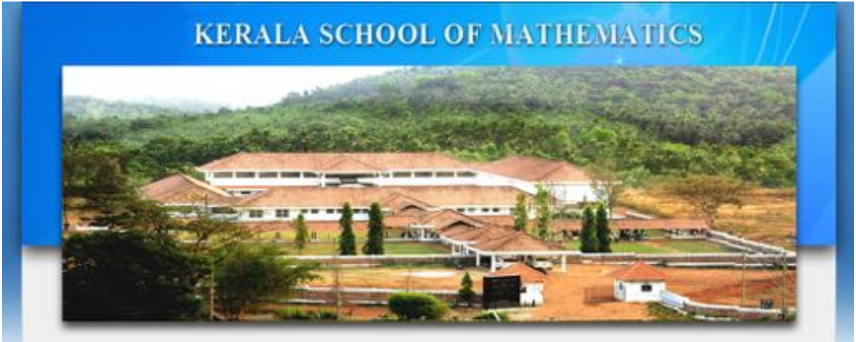
Gregory series for π

James Gregory
(1638 – 1675)

$$\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

https://www.britannica.com/biography/James-Gregory

Kerala School of Mathematics (KSoM)



http://www.ksom.res.in/

Real analysis

Continuous fonctions $\mathbb{R} \to \mathbb{R}$.

C^n functions $\mathbb{R} \to \mathbb{R}$ (n-times continuously derivable)
$1 \leq n \leq \infty$

Analytic functions :
$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$
Taylor series : $a_n = (1/n!) f^{(n)}(x_0)$.

There exist C^∞ functions which are not analytic : for instance $F(x) = e^{-1/x^2}$ with $F(0) = 0$.

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The polynomial $X^2 + 1$ has no real root.

Given a field K and an irreducible polynomial $f \in K[X]$, algebra allows us to construct a field $K(\alpha)$ containing K in which f has a root α, and $K(\alpha)$ is nothing else than the set of $a_0 + a_1\alpha + \cdots + a_n\alpha^n$ with $a_0, a_1, \ldots, a_n$ in K (values at α of a polynomial in $K[X]$).

What is remarkable is that it suffices to add a root i of $X^2 + 1$ to get a field $\mathbb{C} = \mathbb{R}(i)$ in which any non constant polynomial has a root.

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From real numbers to complex numbers

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Complex analysis

Two points of view :

- Cauchy, holomorphic functions of a complex variable
- Weierstrass : analytic functions of a complex variable.

They are the same !

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(1789 – 1857)

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Cauchy – Riemann equations

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Bernhard Riemann
(1826 – 1866)

A function $f : \mathbb{C} \to \mathbb{C}$, $f(x+iy) = u(x,y) + iv(x,y)$, is holomorphic (derivable with respect to the complex variable z) if and only if

$$\frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v \quad \text{and} \quad \frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} v.$$

Differential operators (Fluid dynamics)

Laplacian :

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Given a vector field $\vec{F} = (F_x, F_y, F_z)$,
Gradient :

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z).$$

Divergence :

$$\nabla \cdot \vec{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z.$$

Curl :

$$\nabla \wedge \vec{F} = \begin{pmatrix} \partial F_z/\partial y - \partial F_y/\partial z \\ \partial F_x/\partial z - \partial F_z/\partial x \\ \partial F_y/\partial x - \partial F_x/\partial y \end{pmatrix}.$$

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Maxwell theory of electromagnetism

James Maxwell

(1831 – 1879)

Fourier analysis

An entire function $\mathbb{C} \to \mathbb{C}$ is a sum of a Taylor series $\sum_{n\geq 0} a_n z^n$ which is convergent for all $z \in \mathbb{C}$.

If an entire function $f : \mathbb{C} \to \mathbb{C}$ is periodic of period ω, namely $f(z+\omega) = f(z)$, then there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(e^{2i\pi z/\omega})$.

Joseph Fourier
(1768 – 1830)

Hence f has an expansion as a Fourier series

$$f(z) = \sum_{n\geq 0} a_n e^{2i\pi nz/\omega}.$$

Wavelets

One can take other bases than the sequence $e^{2i\pi nz/\omega}$, $n \geq 0$.
This yields to the theory of wavelets.

Yves Meyer
Abel Prize 2017

Trigonometric series

$$\sum_{n=0}^{\infty}(A_n \cos(nx) + B_n \sin(nx)).$$

$$e^{it} = \cos t + i \sin t.$$

$$e^{int} = \cos(nt) + i \sin(nt) = (\cos t + i \sin t)^n.$$

$$\cos(nt) = T_n(\cos t), \quad \sin(nt) = (\sin t) U_{n-1}(\cos t).$$

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$$\sum_{n=0}^{\infty}(A_n \cos(nx) + B_n \sin(nx)).$$

$$e^{it} = \cos t + i \sin t.$$

$$e^{int} = \cos(nt) + i \sin(nt) = (\cos t + i \sin t)^n.$$

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Chebyshev polynomials of the first kind

$$\cos(nt) = T_n(\cos t).$$

$$T_0(X) = 1. \qquad T_1(X) = X.$$

$$\cos(2t) = 2\cos^2 t - 1, \qquad T_2(X) = 2X^2 - 1.$$

$$\cos(3t) = 4\cos^3 t - 3\cos t, \qquad T_3(X) = 4X^3 - 3X.$$

$$T_4(X) = 8X^4 - 8X^2 + 1, \quad T_5(X) = 16X^5 - 20X^3 + 5X.$$

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$$T_n\left(\frac{z+z^{-1}}{2}\right) = \frac{z^n+z^{-n}}{2}.$$

Proof By analytic continuation, it suffices to check the formula for $|z|=1$.

$z = e^{it} = \cos t + i\sin t$,

$z^{-1} = e^{-it} = \cos t - i\sin t$,

$z + z^{-1} = 2\cos t$,

$z^n + z^{-n} = 2\cos(nt) = 2T_n(\cos t)$. □

The map $z \mapsto (z+z^{-1})/2$ is a 2 to 1 map from the circle $|z|=1$ to the real interval $[-1,1]$.

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Chebyshev polynomials of the first and second kind

First kind : $T_n(x)$

$$T_n(X)^2 - (X^2-1)U_{n-1}(X)^2 = 1.$$

Second kind : $U_n(x)$

$$\sin(nt) = (\sin t)U_{n-1}(\cos t), \qquad T_n'(t) = nU_{n-1}(t).$$

$$T_n(X) + U_{n-1}(X)\sqrt{X^2-1} = (X + \sqrt{X^2-1})^n.$$

Pell – Abel equation :
Given a monic polynomial $D(X)$ over a field k of characteristic $\neq 2$ of non zero discriminant and even degree $2g+2$, consider the equation $P(X)^2 - D(X)Q(X)^2 = 1$, where the unknown P and Q are in $k[X]$

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Pell – Abel equation

Niels Henrik Abel
(1802 – 1829)

1826, integration in 'finite terms' of hyperelliptic differentials.

David Masser

Torsion points on families of simple abelian surfaces and Pell's equation over polynomial rings.

Umberto Zannier

J. Eur. Math. Soc. (JEMS) 17 (2015), no. 9, 2379–2416.

Properties of Chebyshev polynomials

If n is even, then $T_n(X)$ is an even function of X :
$T_n(-X) = T_n(X)$,
T_n is a polynomial in X^2.

If n is odd, then $T_n(X)$ is an odd function of X :
$T_n(-X) = -T_n(X)$,
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$T_n(0) = 0$ is n is odd, $T_n(0) = (-1)^{n/2}$ if n is even.

$T_n(1) = 1, \qquad T_n(-1) = (-1)^n$.

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Explicit formula for Chebyshev polynomials

$$2^{-n+1}T_n(X) = X^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{k} \binom{n-k-1}{k-1} X^{n-2k}.$$

Raphael M. Robinson
1911 – 1995

R.M. Robinson, Intervals containing infinitely many sets of conjugate algebraic integers Studies in Mathematical Analysis and related topics Essays in honor of George Pólya, Stanford 1962, 305 – 215.

Quoted by Serre, Bourbaki Seminar (March 2018).

Properties of Chebyshev polynomials

For $n \geq 1$, the leading coefficient of T_n is 2^{n-1}. Hence $2^{-n+1}T_n(X)$ is a monic polynomial of degree n.

The roots of $T_n(X)$ are

$$\cos\left(\frac{2k-1}{2n}\pi\right), \qquad k = 1, 2, \ldots, n.$$

They all lie in the real interval $[-1, 1]$.

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Extremal values of Chebyshev polynomials on $[-1, 1]$

The roots of $U_n(X)$ are

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The extremal values of T_n on $[-1, 1]$ are all equal to ± 1, they are attained at the points $\cos\left(\frac{k}{n}\pi\right), \qquad k = 0, 1, 2, \ldots, n$.

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Fundamental property of Chebyshev polynomials

Define $c_n = \min_P \|P\|$ where $\|P\| = \sup_{-1\leq x\leq 1} |P(x)|$ and the minimum is over the set of monic polynomials with real coefficients of degree n.
Then

$$c_n = 2^{-n+1}.$$

The Chebyshev polynomial T_n is the polynomial in $\mathbb{Z}[X]$ of degree n, with the largest possible leading coefficient, among the polynomials $P \in \mathbb{Z}[X]$ of degree n such that

$$\|P\| \leq 1.$$

Also, $2^{-n+1}T_n$ is the monic polynomial in $\mathbb{Q}[X]$ of degree n with the smallest $\|P\|$.

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Chebyshev differential equations

First kind :

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

Second kind :

$$(1 - x^2)y'' - 3xy' + n(n+2)y = 0.$$

(Sturm – Liouville differential equations).

Hypergeometric functions :

$$T_n(x) = {}_2F_1(-n, n; \frac{1}{2}; \frac{1}{2}(1-x)).$$

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Chebyshev polynomials

Chebyshev polynomials were first presented in :
Chebyshev, P. L. (1854). "Théorie des mécanismes connus sous le nom de parallélogrammes".
Mémoires des Savants étrangers présentés à l'Académie de Saint-Pétersbourg. 7 : 539–586.
Oeuvres I, 111–143.
https://en.wikipedia.org/wiki/Chebyshev_polynomials

Jean-Pierre Serre Bourbaki Seminar March 2018

Problème (important pour les constructeurs de locomotives) : comment utiliser certains quadrangles articulés (les mécanismes de Chebyshev) pour transformer aussi bien que possible un mouvement circulaire en un mouvement rectiligne, et inversement ? C'est en essayant d'optimiser le "aussi bien que possible" que Chebyshev a été conduit aux polynômes qui portent son nom, ainsi qu'à l'équation $P(x)^2 - D(x)Q(x)^2 = c$. Le lecteur curieux trouvera sur internet des reproductions (avec vidéo) de certains de ces mécanismes.

http://www.bourbaki.ens.fr/TEXTES/1146.pdf

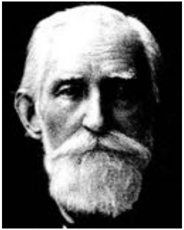
Jean-Pierre Serre

Jean-Pierre Serre

Jean–Pierre SERRE. Distribution asymptotique des valeurs propres des endomorphismes de Frobenius [d'après Abel, Chebyshev, Robinson,...]

Séminaire BOURBAKI, Mars 2018, 70e année, 2017–2018, no 1146.

Pafnouty Lvovich Tchebychev



Pafnouty Chebyshev
(1821 – 1894)

https://en.wikipedia.org/wiki/Pafnuty_Chebyshev
https://www.britannica.com/biography/Pafnuty-Lvovich-Chebyshev
http://www-history.mcs.st-andrews.ac.uk/Biographies/Chebyshev.ht

Chebyshev and prime numbers

Bertrand's Postulate (1845) : between n and $2n$ there is a prime number.

Proved by Chebyshev in 1850.

Joseph Bertrand
(1822 – 1900)

The number $\pi(x)$ of primes $\leq x$ satisfies

$$c_1 x(\log x)^{-1} \leq \pi(x) \leq c_2 x(\log x)^{-1}.$$

If $\pi(x)(\log x)/x$ has a limit, then this limit is 1.

Chebyshev bias

Denote by $\pi(x;4,1)$ the number of prime numbers congruent to 1 modulo 4 and by $\pi(x;4,3)$ the number of prime numbers congruent to 3 modulo 4. Asymptotically, both of them are $\frac{1}{2}x(\log x)^{-1}$. However for $x < 26\,833$ we always have $\pi(x;4,1) \geq \pi(x;4,3)$ with equality only for $x = 5, 17, 41$ and 461.
Assuming Riemann's hypothesis, the inequality $\pi(x;q,a) > \pi(x;q,b)$ occurs more often than the opposite when a is a square modulo q and b is not.

Lettre de M. le Professeur Tchébychev à M. Fuss sur un nouveaux théorème relatif aux nombres premiers contenus dans les formes $4n+1$ et $4n+3$, Bull. Classe Phys. Acad. Imp. Sci. St. Petersburg, **11** (1853), 208.

Mechanisms by Chebyshev http://en.tcheb.ru/

http://en.tcheb.ru/

Etienne Ghys

La tribune des mathématiciens

Les mécanismes de Tchebychev

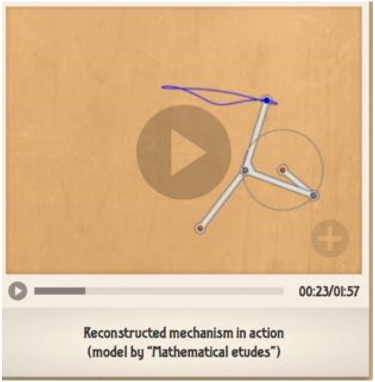
un site remarquable (http ://tcheb.ru)
Le 27 août 2011 - Ecrit par Étienne Ghys

http://images.math.cnrs.fr/+Les-mecanismes-de-Tchebychev+
http://fr.etudes.ru/fr/

Mechanism for transforming rotation into translation motion

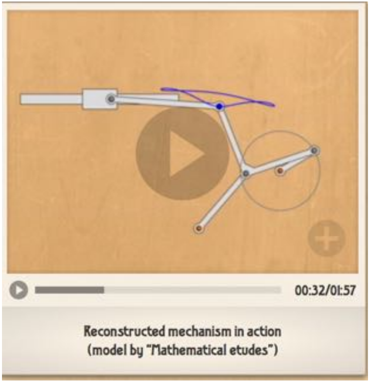
http://en.tcheb.ru/10

Mechanism for transforming rotation into translation motion 1



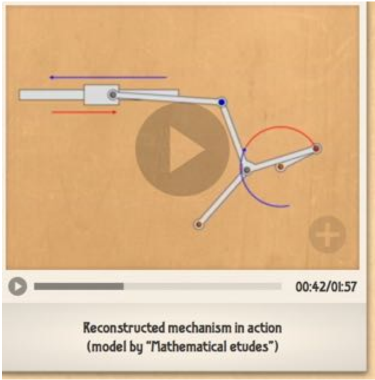
http://en.tcheb.ru/10

Mechanism for transforming rotation into translation motion 2



http://en.tcheb.ru/10

Mechanism for transforming rotation into translation motion 3



http://en.tcheb.ru/10

Connecting rod

Evidence for the connecting rod appears in the late 3rd century Hierapolis sawmill in Roman Asia (modern Turkey). It also appears in two 6th century Byzantine-era saw mills excavated at Ephesus, Asia Minor (modern Turkey) and Gerasa, Roman Syria. The crank and connecting rod mechanism of these Roman-era watermills converted the rotary motion of the waterwheel into the linear movement of the saw blades. Sometime between 1174 and 1206 in the Artuqid State (Turkey), the Arab inventor and engineer Al-Jazari described a machine which incorporated the connecting rod with a crankshaft to pump water as part of a water-raising machine, though the device was complex.

https://en.wikipedia.org/wiki/Connecting_rod

Connecting rod

In Renaissance Italy, the earliest evidence of a (albeit mechanically misunderstood) compound crank and connecting-rod is found in the sketch books of Taccola. A sound understanding of the motion involved is displayed by the painter Pisanello (d. 1455) who showed a piston-pump driven by a water-wheel and operated by two simple cranks and two connecting-rods.
By the 16th century, evidence of cranks and connecting rods in the technological treatises and artwork of Renaissance Europe becomes abundant ; Agostino Ramelli's The Diverse and Artifactitious Machines of 1588 alone depicts eighteen examples, a number which rises in the Theatrum Machinarum Novum by Georg Andreas Böckler to 45 different machines.

https://en.wikipedia.org/wiki/Connecting_rod

Steam engines

Steam engines Beam engine, with twin connecting rods (almost vertical) between the horizontal beam and the flywheel cranks

The first steam engines, Newcomen's atmospheric engine, was single-acting : its piston only did work in one direction and so these used a chain rather than a connecting rod. Their output rocked back and forth, rather than rotating continuously.

Steam engines after this are usually double-acting : their internal pressure works on each side of the piston in turn. This requires a seal around the piston rod and so the hinge between the piston and connecting rod is placed outside the cylinder, in a large sliding bearing block called a crosshead.

https://en.wikipedia.org/wiki/Connecting_rod

Connecting rod (bielle)

https://en.wikipedia.org/wiki/Connecting_rod

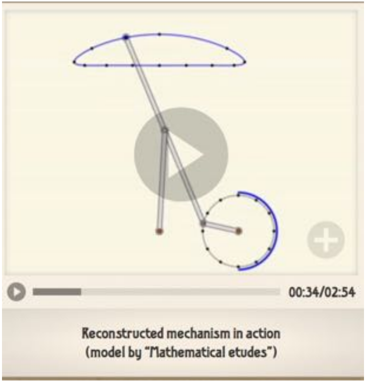
Connecting rod (bielle)

https://commons.wikimedia.org/wiki/File:4-Stroke-Engine.gif

Plantigrade Machine

http://en.tcheb.ru/1

Plantigrade Machine 1



http://en.tcheb.ru/1

Plantigrade Machine 2

http://en.tcheb.ru/1

Plantigrade Machine 3

http://en.tcheb.ru/1

Plantigrade Machine 4

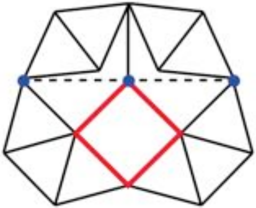
http://en.tcheb.ru/1

Wheelchair

http://en.tcheb.ru/4

A related open problem : Bracing rectangular frameworks

How many non intersecting connected unit rods in the plane are sufficient for making rigid a square ?
Is 23 optimal ?

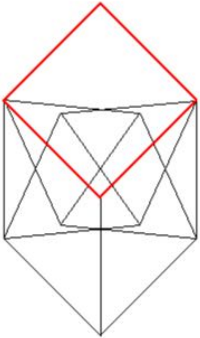


Jean-Paul Delahaye, Pour la Science, N° 490, Août 2018.

Martin Gardner's Sixth Book of Mathematical Diversions from Scientific American, University of Chicago Press, 1971.

Allowing intersections : 19 unit rods are sufficient

Suppose we have a collection of unit rods in the plane that can only be joined at their endpoints. With 3 rods we can make an equilateral triangle. A rigid square can be made using a total of 19 rods.



https://www2.stetson.edu/~efriedma/mathmagic/0100.html

Chebyshev polynomials are orthogonal polynomials

$$\frac{1}{\pi}\int_{-1}^{1} T_n(x)T_m(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \geq 1 \\ \frac{1}{2} & \text{if } n = m = 0. \end{cases}$$

$$\frac{1}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$$

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Orthogonal polynomials

Charles Hermite
(1822 – 1901)

Edmond Laguerre
(1834 – 1886)

Carl Jacobi
(1804 – 1851)

Leopold Gegenbauer
(1849 – 1903)

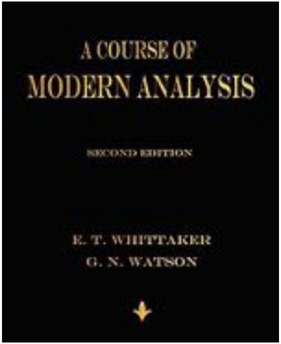
Adrien-Marie Legendre
(1752 – 1833)

E.T. Whittaker and G.N. Watson : A course of Modern Analysis

p.224 : Murphy, Camb. Phil. Soc. Trans. iv (1833) 353 – 408 and v (1835) 113 – 148, 315 – 394.
First systematic study of continuous real orthogonal functions

$$\int_a^b P_m(x) P_n(x) dx = 0$$

for $m \neq n$.

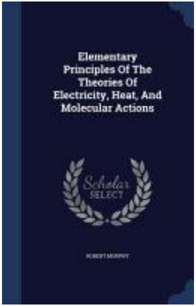


p.311 Murphy's expression of Legendre polynomials as hypergeometric functions : Murphy, Electricity, 1833.

Robert Murphy

In 1830 Murphy was commissioned to write a book on the mathematical theory of electricity, for the use of students at Cambridge. Elementary Principles of Electricity, Heat, and Molecular Actions, part i. On Electricity (Cambridge) was published in 1833 (Deighton, 145 pages).

Robert Murphy
(1806 – 1843)



https://en.wikipedia.org/wiki/Robert_Murphy_(mathematician)

Special polynomials

Dickson polynomials
Fibonacci polynomials, Lucas polynomials, Pell polynomials, Pell – Lucas polynomials, Fermat polynomials polynomials, Fermat – Lucas polynomials, Morgan – Voyce polynomials, Vieta polynomials, Vieta – Lucas polynomials.
Cyclotomic polynomials.

Cyclotomic Dickson polynomials.
Representation of integers by special families of polynomials.
Diophantine equations.

September 3, 2018 Suda Neu-Tech Institute Sanmenxia, Henan, China.
The 9th Expert Scientific Research Meeting of the Sanmenxia Suda Energy Conservation & New Energy Technology Research Institute

A basic introduction to some tools from complex analysis

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http://www.imj-prg.fr/~michel.waldschmidt/