Update: *May 5, 2016*

ALGEBRAIC INDEPENDENCE OF PERIODS OF ELLIPTIC FUNCTIONS

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Abstract. This text is based on notes (taken by R. Thangadurai) of three courses given at Goa University on December 15 and 16, 2014, on *Algebraic independence of Periods of Elliptic functions*, during the Winter School on modular functions in one and several variables organized by R. Balasubramanian, S. Gun, A.J. Jayanthan and W. Kohnen.

The main goal of these lectures was to discuss some of the results obtained by G.V. Chudnovsky in the 1970's on algebraic independence of numbers related with elliptic functions.

1. Preliminaries - Complex Analysis

One of many references for this section is Lang's book [11].

A complex function of one complex variable f is *analytic* in an open set $\mathcal{V} \subset \mathbb{C}$ if, for all $z_0 \in \mathcal{V}$, there exists a power series converging in a neighborhood $\mathcal{U}$ of z_0 such that

$$f(z) = \sum_{n\geq 0} a_n(z-z_0)^n \quad \text{for all } z \in \mathcal{U}\cap\mathcal{V}.$$

An analytic function f has a *zero of multiplicity* k at z_0 if $a_n = 0$ for all $n = 0, 1, \dots, k-1$ and $a_k \neq 0$. Since

$$a_n = \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}z^n}(z_0),$$

this number k is the least non negative integer such that $(\mathrm{d}^k f/\mathrm{d}z^k)(z_0) \neq 0$. Equivalently, f has a zero of multiplicity k at z_0 if there exists a function g, analytic in a neighborhood $\mathcal{V}$ of z_0, such that $g(z_0) \neq 0$ and $f(z) = (z-z_0)^k g(z)$ for all $z \in \mathcal{V}$. The set of zeroes of an analytic function $f \neq 0$ in a connected open subset D of $\mathbb{C}$ is a discrete subset of D.

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *entire* if it is analytic in the entire complex plane $\mathbb{C}$. For example, $f(z) = e^z$ is an entire function; this function has the special property that it never vanishes. Here is Theorem 2.1 of [11, Chap. XIII]:

Proposition 1. *An entire function f has no zero in $\mathbb{C}$ if and only if there exists an entire function g such that*

$$f(z) = e^{g(z)}.$$

Let Ω be a discrete subset of $\mathbb{C}$. There exists an entire function f, the set of zeroes of which is Ω, and these zeroes are simple. If we have one solution f, we deduce all solutions by considering fe^g with g an entire function, as shown by Proposition 1. When Ω is finite, a solution is

$$\prod_{\omega\in\Omega}(z-\omega).$$

If Ω is infinite, a good candidate is

$$\prod_{\omega\in\Omega}\left(1-\frac{z}{\omega}\right) \tag{1}$$

when $0\not\in\Omega$, and

$$z\prod_{\omega\in\Omega\setminus\{0\}}\left(1-\frac{z}{\omega}\right)$$

when $0\in\Omega$. This is indeed a solution when the infinite product (1) converges. Recall that an infinite product

$$\prod_{n\geq 0}u_n$$

is *convergent* if the set $\{n\geq 0 \mid u_n=0\}$ is finite and if, for all sufficiently large n_0, the sequence

$$u_{n_0}u_{n_0+1}\dots u_N=\prod_{n=n_0}^{N}u_n \qquad (N\geq n_0),$$

has a finite nonzero limit as $N\to\infty$. A necessary condition for convergence is that the sequence $(u_n)_{n\geq 0}$ tends to 1 as N tends to infinity. For $\Omega=\mathbb{Z}$, the product

$$z\prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(1-\frac{z}{n}\right)$$

is not absolutely convergent, but it becomes convergent by grouping the factors indexed by n and $-n$, which yields the well known formula [11, Chap. XIII, Example 2.4]

$$\sin(\pi z)=\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).$$

In general, when Ω is a discrete set in $\mathbb{C}$, in order to take care of the convergence of the infinite product (1), one modifies it, without changing the set of zeroes, by multiplying by exponential factors. Consider

$$\prod_{\substack{\omega\in\Omega\\ \omega\neq 0}}\left(1-\frac{z}{\omega}\right)e^{g_\omega(z)}$$

where the g_ω's are entire functions. In order to have a convergent product, the idea is to take for g_ω an approximation of

$$-\log\left(1-\frac{z}{\omega}\right),$$

so that the factors in the infinite product

$$\left(1-\frac{z}{\omega}\right)e^{g_\omega(z)}$$

are sufficiently close to 1. Since

$$-\log(1-u) = u + \frac{u^2}{2} + \cdots \quad \text{for} \quad |u| < 1,$$

a suitable choice is of the form

$$\prod_{\substack{\omega\in\Omega\\ \omega\neq 0}} \left(1-\frac{z}{\omega}\right)\exp\left(\frac{z}{\omega}+\frac{z^2}{2\omega^2}+\cdots+\frac{z^{k_\omega}}{k_\omega\omega^{k_\omega}}\right),$$

where k_ω is a positive integer depending only on ω. This expression is called *Weierstrass canonical product*. Sometimes, it is possible to take for k_ω a constant k independent of ω. This happens in particular for entire functions of *finite order*, namely the entire functions f for which

$$\lim_{R\to\infty} \frac{\log\log|f|_R}{\log R} < \infty.$$

When f is a complex function which is continuous in the disk $|z| \le R$, we define

$$|f|_R = \sup_{|z|\le R} |f(z)|.$$

By the maximum modulus principle, if f is analytic in $|z| < R$ and continuous in the disk $|z| \le R$, we have

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

Definition. *Let f be an entire function and $\rho > 0$ a real number. We say that f is of* strict order $\le \rho$, *if there exist constants C and R_0 such that*

$$|f|_R \le \exp(CR^\rho) \text{ for all } R \ge R_0.$$

We say that f is of *strict order* ρ if f is of *strict order* $\le \rho$ and

$$\lim_{R\to\infty} \frac{\log\log|f|_R}{\log R} = \rho.$$

We want to extend the definition to meromorphic functions in $\mathbb{C}$. One of the equivalent definitions of a meromorphic function is the following. A complex function is called *meromorphic* in an open set $\mathcal{V} \subset \mathbb{C}$ if it is a quotient of two functions which are analytic in $\mathcal{V}$.

We need the following lemma.

Lemma 1. *Let g and h be two nonzero entire functions of strict order $\leq \rho$. Assume that the quotient $f = g/h$ is an entire function. Then f has strict order $\leq \rho$.*

This lemma allows us to introduce the definition:

Definition. *Let f be a meromorphic function in $\mathbb{C}$. We say that f is of* strict order $\leq \rho$, *if there exist two entire functions g and h of strict order $\leq \rho$ such that $f = g/h$.*
A meromorphic function in $\mathbb{C}$ is of finite order *if it is a quotient of two entire functions of finite order.*

Hadamard Product. Let Ω be a discrete subset of $\mathbb{C}$. Let $\rho > 0$ be a positive real number and k the least integer $\geq \rho$. Let f be an entire function of strict order $\leq \rho$. Assume that f has zeroes precisely at points in Ω and that these zeroes are simple. Then there exists an entire function g such that

$$f(z) = e^{g(z)} z \prod_{\substack{\omega \in \Omega \\ \omega \neq 0}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2} + \cdots + \frac{z^k}{k\omega^k}\right).$$

If g satisfies this property, then for any $\ell \in \mathbb{Z}$ the function $g + 2i\pi\ell$ also satisfies the same property.

Here are two examples (another one will be discussed in §2). The Hadamard product related with the set of rational integers $\mathbb{Z}$ with $\rho = 1$ (hence $k = 1$) is

$$\sin(\pi z) = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

The second example is the Hadamard product related with the set of negative integers $\Omega = \{\dots, -n, \dots, -2, -1\}$ with $\rho = 1$, hence $k = 1$. Denote the Euler's Gamma function by Γ and the Euler's constant by γ:

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t}, \quad \gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

Then [11, Chap. XV, §2],

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

Definition. *A* lattice *in $\mathbb{C}$ is a discrete subgroup Ω of $\mathbb{C}$ containing a basis of $\mathbb{C}$ over $\mathbb{R}$: this means that there exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, such that*

$$\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2.$$

Denote by $\#(\Omega \cap D(0, r))$ the number of elements of Ω inside a disk of radius r with center 0. Then the limit

$$\lim_{r \to \infty} \frac{1}{r^2} \#(\Omega \cap D(0, r))$$

exists and is positive: it is the area of the parallelogram

$$\left\{a\omega_1 + b\omega_2 \;\mid\; 0 \le a, b < 1\right\};$$

this area does not depend on the choice of the basis $\{\omega_1, \omega_2\}$ of Ω as a $\mathbb{Z}$–module, and this parallelogram is a fundamental domain for the quotient $\mathbb{C}/\Omega$.

Lemma 2. (Schwarz Lemma with Blaschke factors). *Let $0 < r < R$ be real numbers and f a complex function which is continuous in the closed disc $|z| \le R$ and analytic in the open disc $|z| < R$. Assume f has at least N zeroes, counting multiplicities, in the disc $|z| < r$. Then*

$$|f|_r \le \left(\frac{2rR}{R^2 + r^2}\right)^N |f|_R.$$

When $r > 0$ is a positive real number and f is a nonzero analytic function in an open connected set containing the disc $|z| \le r$, we denote by N_r the number of zeroes of f, counting multiplicities, in the disc $|z| < r$.

Corollary 1. *Let f be a nonzero entire function of strict order $\le \rho$. Then there exists a constant C such that*

$$N_r \le C r^\rho \text{ for all } r \ge 1.$$

Combining Lemma 2 with Cauchy's inequalities, we deduce the following estimate.

Corollary 2. *Let $0 < r < R$, f an entire function and $z_0 \in \mathbb{C}$. Assume f has at least N zeroes in the disc $|z - z_0| < r$. Let $t \ge 0$ be an integer. Then*

$$\frac{|z_0|^t}{t!}\left|\frac{\mathrm{d}^t f}{\mathrm{d}z^t}(z_0)\right| \le \left(\frac{2rR}{R^2 + r^2}\right)^N |f|_{R+|z_0|}.$$

2. Elliptic functions

References for this section are [11, Chap. XIV] and [37].

Definition. *Let Ω be a lattice in $\mathbb{C}$. The Hadamard product attached to Ω with $\rho = 2$ is*

$$\sigma(z) = z \prod_{\substack{\omega \in \Omega \\ \omega \ne 0}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right),$$

which is the Weierstrass sigma function attached to Ω.

It is an entire function of strict order 2, with a simple zero at each point of Ω.

Definition. *The* Weierstrass zeta function attached to Ω *is the logarithmic derivative of the sigma function, namely*

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\substack{\omega \in \Omega \\ \omega \ne 0}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

The series of the meromorphic functions in the right hand side is absolutely and uniformly convergent on any compact subset of $\mathbb{C}$ [11, Chap. XIII, Lemma 1.2]. Weierstrass zeta function is a meromorphic function of strict order ≤ 2, quotient of two entire functions of strict order 2, with simple poles at $\omega \in \Omega$ with residue 1.

Definition. *The Weierstrass elliptic function attached to Ω is the derivative of the zeta function, namely*

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{2}$$

The series of the meromorphic functions in the right hand side is absolutely and uniformly convergent on any compact set. For $a \notin \Omega$, we have

$$\wp(z) - \wp(a) = -\frac{\sigma(z+a)\sigma(z-a)}{\sigma(z)^2\sigma(a)^2},$$

hence $\wp$ is a meromorphic function of strict order ≤ 2, quotient of two entire functions of strict order 2. It follows from the formula (2) that

$$\wp(z+\omega) = \wp(z) \text{ for all } \omega \in \Omega, \tag{3}$$

that is, $\wp$ is a periodic function with periods $\omega \in \Omega$. Besides, $\wp$ has double poles at $\omega \in \Omega$ with residue 0.

As a consequence of the periodicity (3) of $\wp$, for $\omega \in \Omega$, the difference $\zeta(z+\omega) - \zeta(z)$ is a constant, which is denoted $\eta(\omega)$. For $\omega \in \Omega$ and $\eta = \eta(\omega)$, we have

$$\frac{\sigma(z+\omega)}{\sigma(z)} = \pm \exp\left(\eta \left(z + \frac{\omega}{2} \right) \right).$$

More precisely, let $\{\omega_1, \omega_2\}$ be a basis of Ω over $\mathbb{Z}$, so that $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Let $\omega \in \Omega$. Write $\omega = a\omega_1 + b\omega_2$ for some integers a and b. Then $\eta(\omega) = a\eta_1 + b\eta_2$ where $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$. For $i = 1, 2$, we have

$$\frac{\sigma(z+\omega_i)}{\sigma(z)} = -\exp\left(\eta_i \left(z + \frac{\omega_i}{2} \right) \right).$$

Near 0, the zeta function has a Laurent series expansion

$$\zeta(z) = \frac{1}{z} - \sum_{k=1}^{\infty} G_{2k+2}(\Omega) z^{2k+1},$$

where, for $\ell \geq 4$,

$$G_\ell(\Omega) = \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \omega^{-\ell}.$$

These numbers $G_\ell(\Omega)$ are the Eisenstein series for Ω. Note that for any $\lambda \in \mathbb{C}^\times$, we have

$$\wp_{\lambda\Omega}(\lambda z) = \lambda^{-2} \wp_\Omega(z).$$

Define

$$g_2 = 60G_4(\Omega) \text{ and } g_3 = 140G_6(\Omega).$$

Then $\wp(z)$ is a solution of the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3.$$

By selecting a suitable path from $\wp(z)$ to ∞, we can write, for $z \notin \Omega$,

$$z = \int_{\wp(z)}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}\cdot$$

Define $\Delta = g_2^3 - 27g_3^2$. Write

$$4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3).$$

Since the discriminant of this polynomial is Δ and since $\Delta \neq 0$, the three roots e_1, e_2, e_3 are distinct and one can order them so that

$$e_i = \wp(\omega_i/2) \text{ for } i = 1, 2 \text{ and } e_3 = -e_1 - e_2 = \wp((\omega_1 + \omega_2)/2).$$

One deduces

$$\omega_i = 2\int_{e_i}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}\cdot$$

The next examples involve Euler's Beta function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} = \int_0^1 x^{a-1}(1 - x)^{b-1}dx.$$

Example 1. Consider the elliptic Weierstrass function $\wp$ satisfying $\wp'^2 = 4\wp^3 - 4\wp$. Here we have $g_2 = 4$ and $g_3 = 0$. Then

$$\omega_1 = \int_1^{\infty} \frac{dt}{\sqrt{t^3 - t}} = B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots \tag{4}$$

and $\omega_2 = i\omega_1$, together with

$$\eta_1 = \frac{\pi}{\omega_1} \text{ and } \eta_2 = -i\eta_1.$$

Example 2. Consider the elliptic Weierstrass function $\wp$ satisfying $\wp'^2 = 4\wp^3 - 4$. Here we have $g_2 = 0$ and $g_3 = 4$. Then

$$\omega_1 = \int_1^{\infty} \frac{dt}{\sqrt{t^3 - t}} = \frac{1}{3}B(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots \tag{5}$$

and $\omega_2 = \rho\omega_1$, where $1 + \rho + \rho^2 = 0$, together with

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} \text{ and } \eta_2 = \rho^2\eta_1.$$

Note. An *elliptic integral of the first kind* is an indefinite integral

$$\int \frac{dx}{y},$$

where x and y are related by $y^2 = 4x^3 - g_2x - g_3$. We can write it as

$$\int \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}.$$

The periods of this integral are the values of the integral on closed loops, and these values are the elements of $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

The *elliptic integral of the second kind* is

$$\int \frac{xdx}{y}$$

which can be written

$$\int \frac{tdt}{\sqrt{4t^3 - g_2t - g_3}}.$$

Setting $t = \wp(z)$, we get

$$\int_{\wp(z)}^{\infty} \frac{tdt}{\sqrt{4t^3 - g_2t - g_3}} = -\zeta(z).$$

If γ is a closed loop and $\omega = \int_\gamma \frac{dx}{y}$, then

$$\int_\gamma \frac{xdx}{y} = \eta(\omega).$$

The *elliptic integrals of the third kind* are the integrals of the form

$$\int \frac{dx}{(x-c)y}$$

with $c \in \mathbb{C}$. For $x_0 = \wp(t_0)$ and $y_0 = \wp'(t_0)$, one has

$$\int \frac{y + y_0}{x - x_0} \cdot \frac{dx}{y} = \log \frac{\sigma(z - t_0)}{\sigma(z)} + z\zeta(t_0).$$

The numbers

$$\zeta(u) - \frac{\eta}{\omega}\, u + \frac{2n\pi i}{\omega} \qquad (n \in \mathbb{Z})$$

are periods of elliptic integrals of the third kind.

3. Elliptic curves over $\mathbb{C}$

Denote by $\mathbb{P}^2(\mathbb{C})$ the projective plane. Let g_2, g_3 be complex numbers such that the discriminant $\Delta = g_2^3 - 27g_3^2$ does not vanish. Define

$$E(\mathbb{C}) = \left\{(t:x:y) \in \mathbb{P}^2(\mathbb{C}) \;\mid\; y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\}.$$

This is a non singular cubic curve. The point $(0:0:1) \in E(\mathbb{C})$ is *the point at infinity* on the curve and will be denoted by O_E.

For a subfield K of $\mathbb{C}$, the elliptic curve E is *defined over* K if g_2 and g_3 are in K. In this case,

$$E(K) = \left\{(t:x:y) \in \mathbb{P}^2(K) \;\mid\; y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\}.$$

Let $\wp$ be the Weierstrass elliptic function with invariants g_2, g_3. The map $\mathbb{C} \to E(\mathbb{C})$ defined by

$$\begin{aligned} z &\mapsto (1 : \wp(z) : \wp'(z)) \quad \text{for } z \notin \Omega \\ \omega &\mapsto O_E \quad \text{for } \omega \in \Omega \end{aligned}$$

is a surjective map and by (3) induces a bijective map $\mathbb{C}/\Omega \to E(\mathbb{C})$.

Since $\mathbb{C}/\Omega$ is an additive group, the bijection $\mathbb{C}/\Omega \to E(\mathbb{C})$ yields a structure of abelian group on $E(\mathbb{C})$, with O_E as neutral element. By construction, the additive groups $\mathbb{C}/\Omega$ and $E(\mathbb{C})$ are isomorphic. These groups are also isomorphic as topological groups.

One can check that for $u, v, w \in \mathbb{C}$, the condition $u+v+w = 0$ is equivalent to

$$\det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0.$$

This means that three points on $E(\mathbb{C})$ add to O_E if and only if they are on a straight line. From the vanishing of this determinant, one also deduces

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$

and

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\cdot$$

Further, we have,

$$\wp(2u) = -2\wp(u) + \frac{1}{4}\left(\frac{\wp''(u)}{\wp'(u)}\right)^2$$

and

$$\zeta(2u) = 2\zeta(u) + \frac{1}{2}\frac{\wp''(u)}{\wp'(u)}\cdot$$

By induction, one deduces that for every integer $m \geq 2$, we have

$$\wp(mu) = \frac{A_m\bigl(\wp(u)\bigr)}{B_m\bigl(\wp(u)\bigr)},$$

where A_m and B_m are polynomials of degrees m^2 and $m^2 - 1$ respectively.

Ring of endomorphisms. We denote by $\mathrm{End}(E)$ the ring of *endomorphisms* of the topological group $E(\mathbb{C})$, namely the maps $E \to E$ which are continuous homomorphisms of the abelian group $E(\mathbb{C})$ to itself. We use the isomorphism between $E(\mathbb{C})$ and $\mathbb{C}/\Omega$ and we lift an endomorphism of $\mathbb{C}/\Omega$ to a linear map $\lambda : z \mapsto \lambda z$ of $\mathbb{C}$ where $\lambda\Omega \subset \Omega$:

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Omega & \longrightarrow & \mathbb{C}/\Omega \end{array}$$

One deduces
$$\mathrm{End}(E) = \{\lambda \in \mathbb{C} \mid \lambda\Omega \subset \Omega\}\,.$$
Since Ω is a lattice, for $m \in \mathbb{Z}$ we have $m\Omega \subset \Omega$, hence $\mathbb{Z} \subseteq \mathrm{End}(E)$ for all E. It turns out $\mathrm{End}(E) \neq \mathbb{Z}$ if and only if $\tau = \omega_2/\omega_1$ is a quadratic irrationality. In this case, we say that the elliptic curve E has *complex multiplication* ; then the ring $\mathrm{End}(E)$ is an *order* of the quadratic field $\mathbb{Q}(\tau)$, that is a subring of the ring of integers of maximal rank 2. The field $\mathbb{Q}(\tau)$ is called the *field of endomorphisms of* E.

A survey on the history of transcendence related with elliptic functions is given in [37]. The very first result goes back to C. L. Siegel (1932). He proved the following. Let $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. Define $\tau = \omega_2/\omega_1$. Let $E : y^2 = 4x^3 - g_2x - g_3$ be the Weierstrass elliptic curve attached to Ω. Then at least one of the numbers g_2, g_3, ω_1, ω_2 is transcendental. As a consequence, if E has complex multiplication and if g_2 and g_3 are algebraic, any non–zero period is a transcendental number.

Much more is known now, as we are going to see.

4. Schneider–Lang Criterion and some of its applications

Criterion of Schneider–Lang. *Let K be a number field. Let $f_1, f_2, \ldots, f_m$ be meromorphic functions. Assume that the functions f_1, f_2 are of finite order and are algebraically independent. Assume that the differential operator* $\mathrm{d}/\mathrm{d}z$ *takes the ring $K[f_1, f_2, \ldots, f_m]$ to itself. Then the set*
$$S = \{u \in \mathbb{C} \mid u \textit{ is not pole of } f_j \textit{ and } f_j(u) \in K \quad \textit{for all} \quad j = 1, 2, \ldots, m\}$$
is finite.

In this statement, it does not make a difference whether we assume the functions f_1, f_2 are algebraically independent over $\mathbb{C}$ or over $\mathbb{Q}$.

Here are some of the many consequences of the criterion of Schneider–Lang.

Theorem 1 (Hermite-Lindemann)**.** *Let u be a nonzero complex number. Then at least one of the two numbers u, e^u is a transcendental number.*

Proof. Take $m = 2$, $f_1(z) = z$ and $f_2(z) = e^z$. These two entire functions are algebraically independent and of finite order. Consider the field $K = \mathbb{Q}(u, e^u)$. The functions f_1 and f_2 satisfy the differential equations $f_1' = 1$ and $f_2' = f_2$, hence $f_j' \in K[f_1, f_2]$ for $j = 1, 2$. Consider the set $S = \{u \in \mathbb{C} \mid f_j(u) \in K \text{ for } j = 1, 2\}$. For any integer $m \in \mathbb{Z}$, $f_1(mu) = mu \in K$ and $f_2(mu) = e^{mu} = (e^u)^m \in K$, hence $mu \in S$ for all integer $m \in \mathbb{Z}$. Since $u \neq 0$, it follows that S is infinite. Therefore K is not a number field, which proves the result. $\square$

Theorem 2 (Gel'fond-Schneider (1934))**.** *Let $u \neq 0$ and $\beta \notin \mathbb{Q}$ be two complex numbers. Then at least one of the three numbers u, β, $e^{\beta u}$ is transcendental.*

Proof. The two functions $f_1(z) = e^z$ and $f_2(z) = e^{\beta z}$ are algebraically independent (for $\beta \notin \mathbb{Q}$) and of finite order. Define $K = \mathbb{Q}(u, \beta, e^{\beta u})$. Then for any integer $m \in \mathbb{Z}$

$$f_1(mu) = e^{mu} = (e^u)^m \in K$$

and

$$f_2(mu) = e^{\beta mu} = (e^{\beta u})^m \in K.$$

From the assumption $u \neq 0$, it follows that the set of mu, $m \in \mathbb{Z}$, is infinite. Therefore, by the Schneider–Lang Criterion, K is not a number field. This proves the result. □

Theorem 3 (Schneider (1937))**.** *Let $\wp$ be the Weierstrass elliptic function with invariants g_2, g_3. Let u be a complex number with $u \notin \Omega$. Then at least one of the numbers $g_2, g_3, u, \wp(u)$ is transcendental.*

Proof. Set $m = 3$ and consider the three functions $f_1(z) = z, f_2(z) = \wp(z), f_3(z) = \wp'(z)$. The two functions f_1 and f_2 are algebraically independent and of finite order. Define $K = \mathbb{Q}(g_2, u, \wp(u), \wp'(u))$. Notice that $g_3 \in K$. From

$$f_1' = 1, f_2' = \wp' = f_3, f_3' = \wp'' = 6\wp^2 - g_2/2 = 6f_2^2 - g_2/2$$

we deduce that $f_j' \in K[f_1, f_2, f_3]$ for $j = 1, 2, 3$. The set of $m \in \mathbb{Z}$ such that mu is not a pole of $\wp$ is infinite, and for each of these m we have $f_j(mu) \in K$ for $j = 1, 2, 3$. Therefore, by the Schneider–Lang criterion, K is not a number field. □

Corollary 3. *Assume g_2 and g_3 are algebraic. Let $\omega \in \Omega$ with $\omega \neq 0$. Then ω is transcendental.*

Proof. Let ℓ be the least integer such that $u = \omega/2^\ell \notin \Omega$. Then $2u \in \Omega$, hence $\wp'(u) = 0$. Therefore $\wp(u)$ is algebraic and by Theorem 3, it follows that u is transcendental. □

Theorem 4 (Schneider (1937))**.** *Let Ω and Ω^* be two lattices, $\wp$ and $\wp^*$ be the corresponding Weierstrass elliptic functions. Denote by g_2, g_3 and g_2^*, g_3^* their invariants. Assume that $\wp$ and $\wp^*$ are algebraically independent. Let $u \in \mathbb{C}$ satisfy $u \notin \Omega$ and $u \notin \Omega^*$. Then, at least one of the numbers g_2, g_3, g_2^*, g_3^*, $\wp(u)$, $\wp^*(u)$ is transcendental.*

Proof. Take $f_1(z) = \wp(z)$, $f_2(z) = \wp^*(z)$, $f_3(z) = \wp'(z)$, $f_4(z) = (\wp^*)'(z)$. By the assumption, f_1 and f_2 are algebraically independent and of finite order. Define

$$K = \mathbb{Q}(g_2, g_3, g_2^*, g_3^*, \wp(u), \wp^*(u), \wp'(u), (\wp^*)'(u)).$$

Clearly $f_j' \in K[f_1, f_2, f_3, f_4]$ for $j = 1, 2, 3, 4$. The set of $m \in \mathbb{Z}$ such that $mu \notin (\Omega \cup \Omega^*)$ is infinite, and for these values of m we have $f_j(mu) \in K$ for $j = 1, 2, 3, 4$. By the Schneider–Lang criterion, it follows that K is not a number field. □

Further results on the transcendence of numbers related with periods of elliptic integrals of the third kind have been derived from the Schneider–Lang Criterion in [31, 32]. See also [12, 13, 14].

Sketch of the proof of Schneider–Lang Criterion. By the assumption, f_1 and f_2 are algebraically independent over $\mathbb{Q}$. The idea is to get a contradiction by assuming the set S is sufficiently large; so we plan to show that, if S has too many elements, there exists a non-zero polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ such that the function $F(z) = P(f_1(z), f_2(z))$ is the zero function. To start with, we show the existence of such a P for which F has many zeroes. Then by induction we show that F has even more zeroes than what was given by the construction, until we reach the conclusion that $F = 0$.

Assume S has at least s elements $u_1, \dots, u_s$, with s sufficiently large (how large it should be can be made explicit, but here we do not address this issue). Choose a positive integer T which is very large. Let L be the least integer with $L^2 \geq 2Ts[K : \mathbb{Q}]$.

Step 1. We show that there exists a non-zero polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$, of partial degrees less than L, such that $F = P(f_1, f_2)$ has a zero of multiplicity $\geq T$ at each point $u_1, \dots, u_s$.

We write the expected polynomial P as

$$P = \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} a_{ij} X^i Y^j,$$

where a_{ij} are unknowns taking their values in $\mathbb{Z}$. Linear algebra shows that, as soon as $(L+1)^2 > Ts[K : \mathbb{Q}]$, there is a non trivial solution to the system of sT equations in L^2 unknowns with coefficients in K:

$$\frac{\mathrm{d}^t F}{\mathrm{d} z^t}(u_h) = 0 \text{ for all } t, s \text{ with } 0 \leq t < T \text{ and } 1 \leq h \leq s.$$

The stronger requirement $L^2 \geq 2Ts[K : \mathbb{Q}]$ enables one to achieve an upper bound for the absolute values of the coefficients a_{ij} of such a P, thanks to the so–called Thue–Siegel lemma.

Step 2. Since F is not the zero function, the set of integers T' such that

$$\frac{\mathrm{d}^t F}{\mathrm{d} z^t}(u_h) = 0 \text{ for all } t, s \text{ with } 0 \leq t < T' \text{ and } 1 \leq h \leq s$$

is finite; by step 1, this set contains T. Denote by T' the largest element in this set. Hence there exists an h_0, $1 \leq h_0 \leq s$, for which the number

$$\gamma = \frac{\mathrm{d}^{T'} F}{\mathrm{d} z^{T'}}(u_{h_0})$$

is not 0.

Step 3. Using Schwarz's Lemma and Cauchy's inequalities (Corollary 2), we deduce an upper bound for $|\gamma|$. The choice of the parameters implies that $|\gamma|$ is very small.

Step 4. Arithmetic arguments (Liouville's inequality) produce a lower bound for $|\gamma|$: it means that $|\gamma|$ cannot be too small.

The conclusion (upper bound for s) follows by comparing the estimates from steps 3 and 4.

□

Further transcendence results related with elliptic functions are known. See for instance [29] and the surveys [33, 36, 37].

5. Linear Independence of Periods

Let Ω be a lattice for which the Weierstrass $\wp$ function has algebraic invariants g_2 and g_3. Let (ω_1, ω_2) be a pair of fundamental periods and let η_1 and η_2 be the associated quasi–periods of the Weierstrass zeta function.

Let $\overline{\mathbb{Q}}$ be denote the field of all algebraic numbers. Thanks to the work of Baker, Coates and Masser [1, 15, 37], all linear relations with algebraic coefficients among the 6 numbers

$$1, \omega_1, \omega_2, \eta_1, \eta_2, 2i\pi$$

are known. The dimension of the $\overline{\mathbb{Q}}$–vector space spanned by these numbers is

$$\begin{cases} 6 & \text{in the non CM case,} \\ 4 & \text{in the CM case.} \end{cases}$$

6. Algebraic Independence of Periods

The main tool in this section is the method which was introduced by A. O. Gel'fond in 1949 when he proved algebraic independence results for values of the exponential function. A typical result he achieved was the algebraic independence of $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$. His method was developed by several authors including Brownawell and Kubota [3] and then Chudnovsky, who proved the following results (see [5, 6, 7, 8, 9, 10, 30]).

Theorem 5 (Chudnovsky (1976))**.** *Let $\wp$ be a Weierstrass elliptic function with invariants g_2 and g_3 and let ω_1, ω_2 be a pair of fundamental periods. Denote by η_1 and η_2 the associated quasi–periods of the Weierstrass zeta function. Then at least two of the numbers $g_2, g_3, \omega_1, \omega_2, \eta_1$ and η_2 are algebraically independent.*

Sketch of the Proof. We first select parameters which will be used as upper bounds for the partial degrees of the auxiliary polynomials and as lower bounds for the multiplicities and the number of points that we will consider.
Step 1. We prove that there exists a non-zero polynomial $P(X, Y, Z) \in \mathbb{Z}[g_2, g_3][X, Y, Z]$ such that the function $F(z) = P\big(z, \wp(z), \zeta(z)\big)$ has zeroes of high multiplicity at points of the form $(\omega_1/2) + m\omega_1 + n\omega_2$ for some set of integers m and n.

The Lemma of Thue–Siegel produces an upper bound for the coefficients of such a P.

Step 2. Select a nonzero value $\gamma = (\mathrm{d}^t F/\mathrm{d}z^t)(z_0)$ of a derivative of F at a point z_0 of the form $(\omega_1/2) + m\omega_1 + n\omega_2$.

This is a so–called *zero estimate* which one proves by means of elimination theory.

Step 3. Using Schwarz's Lemma and Cauchy's inequalities (Corollary 2), we deduce an upper bound for $|\gamma|$.

Step 4. The conclusion follows from Gel'fond's criterion (Proposition 2 below). □

We denote by $H(P)$ the maximum modulus of the coefficients of a polynomial $P \in \mathbb{C}[X]$.

Proposition 2 (Gel'fond's Criterion)**.** *Let $\theta \in \mathbb{C}$. Assume that there exists a sequence of polynomials $Q_N \in \mathbb{Z}[X]$ with $Q_N \neq 0$, $\deg Q_N \leq N$, $\log H(Q_N) \leq N$ such that*

$$|Q_N(\theta)| \leq \exp(-6N^2).$$

Then θ is algebraic and $Q_N(\theta) = 0$ for all sufficiently large N.

The next result assumes that g_2 and g_3 are algebraic.

Theorem 6 (Chudnovsky (1976))**.** *Let $\wp$ be a Weierstrass elliptic function with period lattice Ω and with algebraic invariants g_2 and g_3. Let $\omega \in \Omega$ and η the associated quasi–period. Let $u \in \mathbb{C}$ with $u \notin \Omega$ be such that $\wp(u)$ is algebraic. Assume that u and ω are $\mathbb{Q}$–linearly independent. Then the two numbers*

$$\frac{\eta}{\omega},\ \zeta(u) - \frac{\eta}{\omega}u$$

are algebraically independent.

Notice that for a Weierstrass elliptic function $\wp$, the existence of u not pole of $\wp$ such that $\wp(u)$ and $\wp'(u)$ are algebraic implies that the invariants g_2 and g_3 are algebraic.

Sketch of the Proof. We select auxiliary parameters; next, we show that there exists a non-zero polynomial $P(X,Y) \in \mathbb{Z}[\eta/\omega][X,Y]$ such that $F(z) = P(\wp(z), \zeta(z) - \eta z/\omega)$ has zeroes at several points of the form mu $(m \in \mathbb{Z})$ with high multiplicity. Since ω is a period of F, it follows that F has zeroes at $mu + n\omega$ with the same multiplicity. The rest of the proof is similar to that of Theorem 5. The final step uses Gel'fond's criterion (Proposition 2). □

Corollary 4. *Let $\wp$ be an elliptic function with algebraic invariants. Let ω be a nonzero period of $\wp$ and let η be the associated quasi–period of the Weierstrass zeta function. Then the two numbers*

$$\frac{\eta}{\omega},\ \frac{\pi}{\omega}$$

are algebraically independent.

Proof. We use Theorem 6 with $\omega = \omega_1$, $u = \omega_2/2$ and $\eta = \eta_1$ to deduce the algebraic independence of the two numbers

$$\frac{\eta_1}{\omega_1},\ \zeta(\omega_2/2) - \frac{\eta_1}{\omega_1}\omega_2/2$$

Since

$$2\zeta(\omega_2/2) - \frac{\eta_1}{\omega_1}\omega_2 = \eta_2 - \frac{\eta_1}{\omega_1}\omega_2,$$

Corollary 4 follows from Legendre's relation:

$$\omega_2\eta_1 - \omega_1\eta_2 = 2i\pi.$$

when the imaginary part of ω_2/ω_1 is positive. □

Using either Theorem 5 or Theorem 6 (via Corollary 4) one deduces:

Corollary 5. *Assume g_2 and g_3 are algebraic. Then the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ is ≥ 2.*

The following statement would contain both Theorems 5 and 6. It is a special case of the Conjecture of André–Bertolin on 1 motives [2]:

Conjecture 1. *Let $u \in \mathbb{C} \setminus \Omega$ and $\omega \in \Omega$ be such that u and ω are $\mathbb{Q}$-linearly independent. Then the transcendence degree over $\mathbb{Q}$ of the field*

$$\mathbb{Q}\left(g_2, g_3, \wp(u), \frac{\eta}{\omega}, \zeta(u) - \frac{\eta}{\omega}u\right)$$

is ≥ 2.

We study now the transcendence degree of the field $\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2)$. In the CM case, the transcendence degree may be 2 or 3. When g_2 and g_3 are algebraic, according to Corollary 5, it is 2. If we start with an elliptic curve with algebraic invariants g_2, g_3 and if we select a number c which is transcendental over $\mathbb{Q}(\omega, \pi)$, then for the elliptic function $\wp^*$ associated to the lattice $\Omega^* = c\Omega$, a transcendence basis of the field $\mathbb{Q}(g_2^*, g_3^*, \omega_1^*, \omega_2^*, \eta_1^*, \eta_2^*)$ is $\{c, \omega^*, \pi\}$, hence the transcendence degree is 3.

From Corollary 5 we deduce the next result concerning the CM case. Choose a twelfth root $\Delta^{1/12}$ of the discriminant $\Delta = g_2^3 - 27g_3^2$.

Corollary 6. *Assume $\tau = \omega_2/\omega_1$ is algebraic. Then the two numbers*

$$\Delta^{1/12}\omega,\ \pi$$

are algebraically independent.

Proof. Recall that the invariant $j = j(\tau) = 1728g_2^3/\$ofEisalgebraic.Letc{=}\hat{1}/12$ $betheselectedtwelfthrootof\$$. The elliptic function $\wp^*$ associated to the lattice $\Omega^* = c\Omega$ has algebraic invariants $g_2^* = c^{-4}g_2$ and $g_3^* = c^{-6}g_3$. By Corollary 5, a transcendence basis of the field $\mathbb{Q}(\omega_1^*, \omega_2^*, \eta_1^*, \eta_2^*)$ is $\{\omega^*, \pi\}$. □

In the non CM case, we know only that the transcendence degree of the field $\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2)$ is at least 2, but we expect more, as predicted by is the Conjecture of André–Bertolin on 1 motives [2]:

Conjecture 2. *Let $\wp$ be a Weierstrass elliptic function with invariants g_2 and g_3 and without complex multiplication. Let $\{\omega_1, \omega_2\}$ be a basis of the lattice of periods of $\wp$ and let η_1 and η_2 be the corresponding quasi–periods. Then the transcendence degree of the field*

$$\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2)$$

is ≥ 4.

It would be interesting to know whether the transcendence degree is 6 for a set of g_2, g_3 of full Lebesgue measure in $\mathbb{C}^2$.

Using the relation (4) of Example 1, one deduces that the two numbers π and $\Gamma(1/4)$ are algebraically independent. Similarly, using the relation (5) of Example 2, one deduces that the two numbers π and $\Gamma(1/3)$ are algebraically independent.

S. Bruiltet [4] proved that $\Gamma(1/4)$ and $\Gamma(1/3)$ are not Liouville numbers.

G. Philibert [24] proved that under the assumptions of Theorem 6, there exist two constant c and k such that, for any nonzero polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$, if we set $N = \max\{\deg P, \log H(P)\}$, we have

$$\left| P\left(\zeta(u) - \frac{\eta}{\omega}u, \frac{\eta}{\omega}\right)\right| > \exp(-cN^k).$$

In particular this measure of algebraic independence is valid for the two numbers π and $\Gamma(1/4)$, and also for the two numbers π and $\Gamma(1/3)$.

We should add that further results of algebraic independence for values of elliptic functions are known: see [16, 17, 25, 26, 27, 28, 38, 39], and also the work of Nesterenko [18, 19, 20, 21, 22, 23, 34, 35], who uses a different approach (modular method). These results were not discussed in this Goa workshop.

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Some of these papers are available on the internet. See in particular

`http://www.imj-prg.fr/~michel.waldschmidt/texts.html`

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