

## Diophantine equations and Diophantine approximations

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The pdf file of this talk can be downloaded at URL  
<http://www.imj-prg.fr/~michel.waldschmidt/>

The study of Diophantine equations is among the oldest topics investigated by mathematicians. It is known that some problems will never be solved, yet fundamental progress has been achieved recently. We survey some of the main results and some of the main conjectures.

## Diophantus of Alexandria



## Diophantine equations

A **Diophantine** equation is an equation of the form

$$f(X_1, \dots, X_n) = 0$$

where  $f(X_1, \dots, X_n) \in \mathbf{Z}[X_1, \dots, X_n]$  is a given polynomial and the variables (sometimes called *unknowns*)  $X_1, \dots, X_n$  take their values  $x_1, \dots, x_n$  in  $\mathbf{Z}$  (integer points) or in  $\mathbf{Q}$  (rational points).

We will mainly consider integral points.

## Pierre Fermat (1601 ? –1665)

*Fermat's* Last Theorem.



## Diophantine equations: historical survey

Pierre Fermat (1601 ? – 1665)

Leonhard Euler (1707 – 1783)

Joseph Louis Lagrange (1736 – 1813)

XIXth Century: Adolf Hurwitz, Henri Poincaré



## Hilbert's 8th Problem

August 8, 1900



David Hilbert (1862 – 1943)

Second International Congress  
of Mathematicians in Paris.

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

## Hilbert's 10th problem

<http://logic.pdmi.ras.ru/Hilbert10/stat/stat.html>  
D. Hilbert (1900) —

*Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoeffizienten sei vorgelegt: man soll ein Verfahren angeben, nach welchem sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.*

Determination of the solvability of a Diophantine equation.

Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

# Negative solution to Hilbert's 10th problem

Julia Robinson (1952)

Julia Robinson, Martin Davis, Hilary Putnam (1961)

Yuri Matijasevic (1970)



**Remark:** the analog for *rational points* of Hilbert's 10th problem is not yet solved:  
*Does there exist an algorithm in order to decide whether a Diophantine equation has a rational solution or not?*

# Axel Thue

Thue (1908): there are only finitely many integer solutions of

$$F(x, y) = m,$$

when  $F$  is homogeneous irreducible form over  $\mathbb{Q}$  of degree  $\geq 3$ .

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**Über Annäherungswerte algebraischer Zahlen.**  
 Von Herrn Axel Thue in Kristiania.

*Theorem I. Bedeutet  $\varrho$  eine positive Wurzel einer ganzen Funktion vom Grade  $r$  mit ganzen Koeffizienten, so hat die Relation*

$$(1.) \quad 0 < |q\varrho - p| < \frac{c}{q^{1+\epsilon}},$$

*wo  $c$  und  $\epsilon$  zwei beliebig gegebene positive Größen bezeichnen, nicht unendlich viele Auflösungen in ganzen positiven Zahlen  $p$  und  $q$ .*

*Die Richtigkeit hiervon ergibt sich gleich, wenn  $r = 1$  und wenn  $r = 2$ . Wir brauchen folglich nur zu zeigen, daß der Satz immer richtig ist, wenn die genannte Funktion irreduktibel ist und  $r > 2$ .*

*Um dieses Ziel zu erreichen, wollen wir zuerst zwei Hilfsätze entwickeln.*

*Erster Hilfsatz. Es sei  $\varrho$  eine beliebige Wurzel einer ganzen irreduktiblen Funktion  $F$  mit ganzen Koeffizienten und vom Grade  $r > 2$ .*

*Es seien ferner  $\theta$  eine beliebig gewählte positive Größe  $> \frac{2}{r}$  und  $n$  eine solche beliebige ganze positive Zahl, daß*

$$(2.) \quad \frac{2}{r-2} - \frac{\theta}{n-1} > \omega,$$

*wo  $\omega$  eine beliebig gegebene positive Größe  $< \frac{2}{r-2}$  bedeutet.*

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*wo  $c$  und  $\epsilon$  zwei beliebig gegebene positive Größen bezeichnen, nicht unendlich viele Auflösungen in ganzen positiven Zahlen  $p$  und  $q$ .*

## JFM 40.0256.01 [Lampe, Prof. (Berlin)]

Thue, A.

Om en general i store hele tal ulösbar ligning. (Norwegian)  
Christiania Vidensk. Selsk. Skr., Nr. 7, 15 S. Published: 1908

Die Gleichung  $q^n F(p/q) = c$ , wo  $F(x)$  eine beliebige ganze irreduzible Funktion  $r$ -ten Grades ( $r > 2$ ) in  $x$  mit ganzzahligen Koeffizienten,  $c$  eine beliebige ganze Zahl bezeichnet, hat nur eine beschränkte Anzahl von ganzzahligen Lösungen in  $p$  und  $q$  (Rev. sem. 18<sub>2</sub>, 104).

## JFM 40.0265.01 [Fueter, Prof. (Basel)]

Thue, A.

Über Annäherungswerte algebraischer Zahlen. (German)  
J. für Math. 135, 284-305. Published: (1909)

Bedeutet  $\varrho$  eine positive Wurzel einer ganzen Funktion vom Grade  $r$  mit ganzen Koeffizienten, so hat die Relation

$$0 < |q\varrho - p| < \frac{c}{q^{\frac{r}{2}+k}},$$

wo  $c$  und  $k$  zwei beliebige gegebene positive Größen bezeichnen, nicht unendlich viele Auflösungen in ganzen positiven Zahlen  $p$  und  $q$ . Nach dem Beweise dieses Satzes wendet der Verf. denselben auf Kettenbrüche und auf die Frage nach der Auflösbarkeit einer in bezug auf  $p$  und  $q$  homogenen und irreduziblen Funktion  $U(p, q) = c$  in ganzen positiven Zahlen  $p$  und  $q$  an.

## Liouville's inequality

**Liouville's inequality.** Let  $\alpha$  be an algebraic number of degree  $d \geq 2$ . There exists  $c(\alpha) > 0$  such that, for any  $p/q \in \mathbf{Q}$  with  $q > 0$ ,

$$\left| \alpha - \frac{p}{q} \right| > \frac{c(\alpha)}{q^d}.$$

Joseph Liouville, 1844



## Liouville's estimate for $\sqrt[3]{2}$ :

For any  $p/q \in \mathbf{Q}$ ,

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{6q^3}.$$

*Proof.*

Since  $\sqrt[3]{2}$  is irrational, for  $p$  and  $q$  rational integers with  $q > 0$ , we have  $p^3 - 2q^3 \neq 0$ , hence

$$|p^3 - 2q^3| \geq 1.$$

Write

$$p^3 - 2q^3 = (p - \sqrt[3]{2}q)(p^2 + \sqrt[3]{2}pq + \sqrt[3]{4}q^2).$$

If  $p \leq (3/2)q$ , then

$$p^2 + \sqrt[3]{2}pq + \sqrt[3]{4}q^2 < 6q^2.$$

Hence

$$1 \leq 6q^2 |p - \sqrt[3]{2}q|.$$

## Liouville's estimate for $\sqrt[3]{2}$ :

For any  $p/q \in \mathbf{Q}$ ,

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{6q^3}.$$

*Proof.*

We completed the proof in the case  $p \leq (3/2)q$ .

If  $p > (3/2)q$ , then

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{3}{2} - \sqrt[3]{2} > \frac{1}{6}.$$

## Improving Liouville's inequality

If we can improve the lower bound

$$|p^3 - 2q^3| \geq 1,$$

then we can improve Liouville's estimate

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{6q^3}.$$

What turns out to be much more interesting is the converse:

*If we can improve Liouville's estimate*

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{6q^3},$$

*then we can improve the lower bound*

$$|p^3 - 2q^3| \geq 1.$$

## Mike Bennett <http://www.math.ubc.ca/~bennett/>



For any  $p/q \in \mathbf{Q}$ ,

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{1}{4q^{2.5}}.$$

For any  $(x, y) \in \mathbf{Z}^2$  with  $x > 0$ ,

$$|x^3 - 2y^3| \geq \sqrt{x}.$$

## Consequence of an improvement of Liouville

Assume  $(x, y) \in \mathbf{Z}^2$  with  $x > 0$  satisfy

$$|x^3 - 2y^3| < \sqrt{x}.$$

Since

$$x^3 - 2y^3 = (x - \sqrt[3]{2}y)(x^2 + \sqrt[3]{2}xy + \sqrt[3]{4}y^2),$$

we deduce that  $x$  is close to  $\sqrt[3]{2}y$ . Hence  $x^2 + \sqrt[3]{2}xy + \sqrt[3]{4}y^2$  is close to  $3x^2$ . Being more careful, we deduce

$$x^2 + \sqrt[3]{2}xy + \sqrt[3]{4}y^2 \geq 4x^{0.5}y^{1.5}$$

and therefore

$$\left| \sqrt[3]{2} - \frac{x}{y} \right| \leq \frac{1}{4y^{2.5}},$$

a contradiction with Bennett's improvement of Liouville's inequality.

## Connection between Diophantine approximation and Diophantine equations

Let  $\kappa$  satisfy  $0 < \kappa \leq 3$ .

The following conditions are equivalent:

(i) There exists  $c_1 > 0$  such that

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{c_1}{q^\kappa}$$

for any  $p/q \in \mathbf{Q}$ .

(ii) There exists  $c_2 > 0$  such that

$$|x^3 - 2y^3| > c_2 x^{3-\kappa}$$

for any  $(x, y) \in \mathbf{Z}^2$  having  $x > 0$ .

## Thue's equation and approximation

When  $f \in \mathbf{Z}[X]$  is a polynomial of degree  $d$ , we let

$F(X, Y) = Y^d f(X/Y)$  denote the associated homogeneous binary form of degree  $d$ .

Assume  $f$  is irreducible. Then the following two assertions are equivalent:

(i) For any integer  $k \neq 0$ , the set of  $(x, y) \in \mathbf{Z}^2$  verifying

$$F(x, y) = k$$

is finite.

(ii) For any real number  $c > 0$  and for any root  $\alpha \in \mathbf{C}$  of  $f$ , the set of rational numbers  $p/q$  verifying

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{c}{q^d}$$

is finite.

## Improvements of Liouville's inequality

In the lower bound

$$\left| \alpha - \frac{p}{q} \right| > \frac{c(\alpha)}{q^d}$$

for  $\alpha$  real algebraic number of degree  $d \geq 2$ , the exponent  $d$  of  $q$  in the denominator is best possible for  $d = 2$ , not for  $d \geq 3$ .

In 1909, A. Thue succeeded to prove that it can be replaced by  $\kappa$  with any  $\kappa > (d/2) + 1$ .

## Thue's inequality

Let  $\alpha$  be an algebraic number of degree  $d \geq 3$  and let  $\kappa > (d/2) + 1$ . Then there exists  $c(\alpha, \kappa) > 0$  such that, for any  $p/q \in \mathbf{Q}$  with  $q > 0$ ,

$$\left| \alpha - \frac{p}{q} \right| > \frac{c(\alpha, \kappa)}{q^\kappa}.$$

## Thue equation

Thue's result

For any integer  $k \neq 0$ , the set of  $(x, y) \in \mathbf{Z}^2$  verifying

$$F(x, y) = k$$

is finite.

can also be phrased by stating that for any positive integer  $k$ , the set of  $(x, y) \in \mathbf{Z}^2$  verifying

$$0 < |F(x, y)| \leq k$$

is finite.

## Thue equation

For any number field  $K$ , for any non-zero element  $m$  in  $K$  and for any elements  $\alpha_1, \dots, \alpha_n$  in  $K$  with  $\text{Card}\{\alpha_1, \dots, \alpha_n\} \geq 3$ , the Thue equation

$$(X - \alpha_1 Y) \cdots (X - \alpha_n Y) = m$$

has but a finite number of solutions  $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ .

## Improvements of Liouville's inequality

In the lower bound

$$\left| \alpha - \frac{p}{q} \right| > \frac{c(\alpha)}{q^d}$$

for  $\alpha$  real algebraic number of degree  $d \geq 3$ , the exponent  $d$  of  $q$  in the denominator of the right hand side was replaced by

- any  $\kappa > (d/2) + 1$  by A. Thue (1909),
- $2\sqrt{d}$  by C.L. Siegel in 1921,
- $\sqrt{2d}$  by F.J. Dyson and A.O. Gel'fond in 1947,
- any  $\kappa > 2$  by K.F. Roth in 1955.

## Thue–Siegel–Roth Theorem

Axel Thue  
(1863 – 1922)



Carl Ludwig Siegel  
(1896 – 1981)



Klaus Friedrich  
Roth (1925 – )



For any real algebraic number  $\alpha$ , for any  $\epsilon > 0$ , the set of  $p/q \in \mathbf{Q}$  with  $|\alpha - p/q| < q^{-2-\epsilon}$  is finite.

## Diophantine equations: historical survey

Thue (1908): there are only finitely many integer solutions of

$$F(x, y) = m,$$

when  $F$  is homogeneous irreducible form over  $\mathbf{Q}$  of degree  $\geq 3$ .

Mordell's Conjecture (1922): rational points on algebraic curves

Siegel's Theorem (1929): integral points on algebraic curves



## Diophantine equations: historical survey

Faltings's Theorem (1983): finiteness of rational points on an algebraic curve of genus  $\geq 2$  over a number field.

A. Wiles (1993): proof of Fermat's last Theorem

$$a^n + b^n = c^n \quad (n \geq 3)$$

G. Rémond (2000): explicit upper bound for the number of solutions in Faltings's Theorem.



## Effectivity

The Theorem of Thue–Siegel–Roth is non-effective: upper bounds for the number of solutions can be derived, but no upper bound for the solutions themselves.

Faltings's Theorem is not effective: so far, there is no known effective bound for the solutions  $(x, y) \in \mathbf{Q}^2$  of a Diophantine equation  $f(x, y) = 0$ , where  $f \in \mathbf{Z}[X, Y]$  is a polynomial such that the curve  $f(x, y) = 0$  has genus  $\geq 2$ .

Even for integral points, there is no effective version of Siegel's Theorem on integral points on a curve of genus  $\geq 2$ .

## Gel'fond–Baker method

While Thue's method was based on the non effective Thue–Siegel–Roth Theorem, Baker and Fel'dman followed an effective method introduced by A.O. Gel'fond, involving lower bounds for linear combinations of logarithms of algebraic numbers with algebraic coefficients.





## Lower bound for linear combinations of logarithms

A lower bound for a nonvanishing difference

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$$

is essentially the same as a lower bound for a nonvanishing number of the form

$$b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

since  $e^z - 1 \sim z$  for  $z \rightarrow 0$ .

The first nontrivial lower bounds were obtained by [A.O. Gel'fond](#). His estimates were effective only for  $n = 2$ : for  $n \geq 3$ , he needed to use estimates related to the [Thue–Siegel–Roth](#) Theorem.

## Explicit version of Gel'fond's estimates

[A. Schinzel](#) (1968) computed explicitly the constants introduced by [A.O. Gel'fond](#) in his lower bound for

$$|\alpha_1^{b_1} \alpha_2^{b_2} - 1|.$$



He deduced explicit Diophantine results using the approach introduced by [A.O. Gel'fond](#).

## Alan Baker



In 1968, [A. Baker](#) succeeded to extend to any  $n \geq 2$  the transcendence method used by [A.O. Gel'fond](#) for  $n = 2$ . As a consequence, effective upper bounds for the solutions of [Thue's](#) equations have been derived.

## Thue's equation and Siegel's unit equation

The main idea behind the [Gel'fond–Baker](#) approach for solving [Thue's](#) equation is to exploit [Siegel's](#) unit equation.

Assume  $\alpha_1, \alpha_2, \alpha_3$  are algebraic integers and  $x, y$  rational integers such that

$$(x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y) = 1.$$

Then the three numbers

$$u_1 = x - \alpha_1 y, \quad u_2 = x - \alpha_2 y, \quad u_3 = x - \alpha_3 y,$$

are units. Eliminating  $x$  and  $y$ , one deduces [Siegel's unit equation](#)

$$u_1(\alpha_2 - \alpha_3) + u_2(\alpha_3 - \alpha_1) + u_3(\alpha_1 - \alpha_2) = 0.$$

## Siegel's unit equation

Write Siegel's unit equation

$$u_1(\alpha_2 - \alpha_3) + u_2(\alpha_3 - \alpha_1) + u_3(\alpha_1 - \alpha_2) = 0$$

in the form

$$\frac{u_1(\alpha_2 - \alpha_3)}{u_2(\alpha_1 - \alpha_3)} - 1 = \frac{u_3(\alpha_1 - \alpha_2)}{u_2(\alpha_1 - \alpha_3)}.$$

The quotient

$$\frac{u_1(\alpha_2 - \alpha_3)}{u_2(\alpha_1 - \alpha_3)}$$

is the quantity

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n}$$

in Gel'fond–Baker Diophantine inequality.

## Work on Baker's method:

A. Baker (1968), N.I. Feldman (1971), V.G. Sprindžuck and H.M. Stark (1973), K. Györy and Z.Z. Papp (1983), E. Bombieri (1993), Y. Bugeaud and K. Györy (1996), Y. Bugeaud (1998)...

Solving Thue equations:

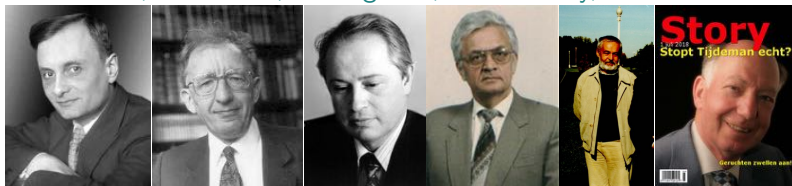
A. Pethő and R. Schulenberg (1987), B. de Weger (1987), N. Tzanakis and B. de Weger (1989), Y. Bilu and G. Hanrot (1996), (1999)...

Solving Thue–Mahler equations:

J.H. Coates (1969), S.V. Kotov and V.G. Sprindžuk (1973), A. Bérczes–Yu Kunrui– K. Györy (2006)...

## Diophantine equations

A.O. Gel'fond, A. Baker, V. Sprindžuk, K. Györy, M. Mignotte, R. Tijdeman, M. Bennett, P. Voutier, Y. Bugeaud, T.N. Shorey, S. Laishram...



## N. Saradha, T.N. Shorey, R. Tijdeman



Survey by T.N. Shorey

*Diophantine approximations, Diophantine equations, transcendence and applications.*

## Thue's Fundamentaltheorem



Paul Voutier (2010)  
 Thue's fundamentaltheorem.  
 I. *The general case.*  
 II: *Some New Irrationality Measures*

## Back to the Thue–Siegel–Roth Theorem

For any real algebraic irrational number  $\alpha$  and for any  $\epsilon > 0$ , there exists  $q_0 > 0$  such that, for  $p/q \in \mathbf{Q}$  with  $q \geq q_0$ , we have

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{2+\epsilon}}.$$

In other terms, the set of  $(q, p) \in \mathbf{Z}^2 \setminus \{(0, 0)\}$  where the two independent linear forms

$$L_0(x_0, x_1) = x_0, \quad L_1(x_0, x_1) = x_0\alpha - x_1$$

satisfy

$$|L_0(x_0, x_1)L_1(x_0, x_1)| \leq \max\{|x_0|, |x_1|\}^{-\epsilon}$$

is contained in a finite union of lines in  $\mathbf{Q}^2$ .

## Schmidt's Subspace Theorem (1970)

For  $m \geq 2$  let  $L_0, \dots, L_{m-1}$  be  $m$  independent linear forms in  $m$  variables with algebraic coefficients. Let  $\epsilon > 0$ . Then the set

$$\{\mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbf{Z}^m ;$$

$$|L_0(\mathbf{x}) \cdots L_{m-1}(\mathbf{x})| \leq |\mathbf{x}|^{-\epsilon}\}$$

is contained in the union of finitely many proper subspaces of  $\mathbf{Q}^m$ .

W.M. Schmidt



## Subspace Theorem

W.M. Schmidt

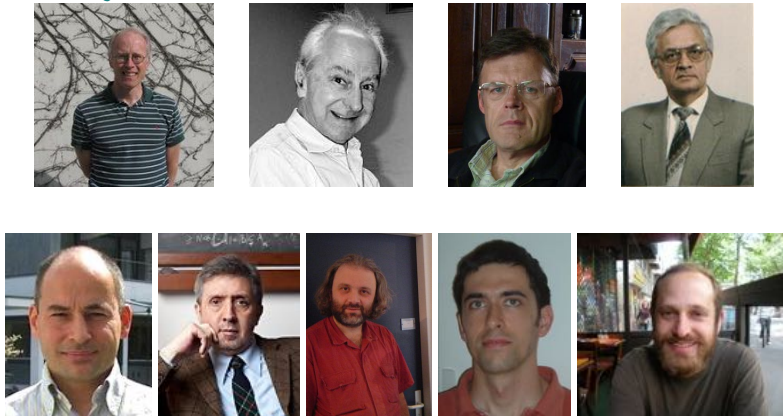


H.P. Schlickewei



## Consequences of the Subspace Theorem

Work of P. Vojta, S. Lang, J-H. Evertse, K. Györy,  
P. Corvaja, U. Zannier, Y. Bilu, P. Autissier, A. Levin ...



## Thue–Mahler equation



Let  $K$  be a number field,  $G$  a finitely generated subgroup of  $K^\times$ ,  $\alpha_1, \dots, \alpha_n$  elements in  $K$  with  $\text{Card}\{\alpha_1, \dots, \alpha_n\} \geq 3$ . Then there are only finitely many  $(x, y) \in \mathbf{Z} \times \mathbf{Z}$  with  $\text{gcd}(x, y) = 1$  satisfying the Thue–Mahler equation

$$(x - \alpha_1 y) \cdots (x - \alpha_n y) \in G.$$

(Kurt Mahler 1933)

## An exponential Diophantine equation

The only solutions of the equation

$$2^a + 3^b = 5^c$$

where the values of the unknowns  $a, b, c$  are nonnegative integers are  $(a, b, c) = (1, 1, 1), (2, 0, 1), (4, 2, 2)$ :

$$2 + 3 = 5, \quad 4 + 1 = 5, \quad 16 + 9 = 25.$$

The more general exponential Diophantine equation

$$2^{a_1} 3^{a_2} + 5^{b_1} 7^{b_2} = 11^{c_1} 13^{c_2}$$

has only finitely many solutions  $(a_1, a_2, b_1, b_2, c_1, c_2)$ .

## $S$ -unit equations – rational case

Let  $S = \{p_1, \dots, p_s\}$  be a finite set of prime numbers. Then the equation

$$u_1 + u_2 = u_3,$$

where the values of the unknowns  $u_1, u_2, u_3$  are relatively prime integers divisible only by the prime numbers in  $S$ , has only finitely many solutions.

Notice that for any prime number  $p$ , the equation

$$u_1 + u_2 + u_3 = u_4$$

has infinitely many solutions in rational integers  $u_1, u_2, u_3$  divisible only by  $p$  and  $\text{gcd}(u_1, u_2, u_3, u_4) = 1$ : for instance

$$p^a + (-p^a) + 1 = 1 \quad \text{for any } a \geq 0.$$

## A consequence of Schmidt's Subspace Theorem

Let  $S = \{p_1, \dots, p_s\}$  be a finite set of prime numbers and let  $n \geq 2$ . Then the equation

$$u_1 + u_2 + \dots + u_n = 1,$$

where the values of the unknowns  $u_1, u_2, \dots, u_n$  are rational numbers with numerators and denominators divisible only by the prime numbers in  $S$  for which no nontrivial subsum

$$\sum_{i \in I} u_i \quad \emptyset \neq I \subset \{1, \dots, n\}$$

vanishes, has only finitely many solutions.

## Finitely generated subgroup of $\mathbf{Q}^\times = \mathbf{Q} \setminus \{0\}$

If  $S = \{p_1, \dots, p_s\}$  is a finite set of prime numbers, the set of rational numbers with numerators and denominators divisible only by the prime numbers in  $S$  is a finitely generated subgroup of  $\mathbf{Q}^\times$ .

Indeed it is generated by  $-1, p_1, \dots, p_s$ .

Conversely, if  $G$  is a finitely generated subgroup of  $\mathbf{Q}^\times$ , then there exists a finite set  $S = \{p_1, \dots, p_s\}$  of prime numbers such that  $G$  is contained in the set of rational numbers with numerators and denominators divisible only by the prime numbers in  $S$ .

Indeed, if  $g_1, \dots, g_t$  is a set of generators of  $G$ , then the set of prime divisors of the numerators and denominators of the  $g_i$  is a solution.

## The generalized $S$ -unit equation

Let  $K$  be a field of characteristic zero, let  $G$  be a finitely multiplicative subgroup of the multiplicative group  $K^\times = K \setminus \{0\}$  and let  $n \geq 2$ . Then the equation

$$u_1 + u_2 + \dots + u_n = 1,$$

where the values of the unknowns  $u_1, u_2, \dots, u_n$  are in  $G$  for which no nontrivial subsum

$$\sum_{i \in I} u_i \quad \emptyset \neq I \subset \{1, \dots, n\}$$

vanishes, has only finitely many solutions.

## Families of Thue equations

The first families of Thue equations having only trivial solutions were introduced by A. Thue himself.

$$(a + 1)X^n - aY^n = 1.$$

He proved that the only solution in positive integers  $x, y$  is  $x = y = 1$  for  $n$  prime and  $a$  sufficiently large in terms of  $n$ . For  $n = 3$  this equation has only this solution for  $a \geq 386$ . M. Bennett (2001) proved that this is true for all  $a$  and  $n$  with  $n \geq 3$  and  $a \geq 1$ . He used a lower bound for linear combinations of logarithms of algebraic numbers due to T.N. Shorey.



## E. Thomas's family of Thue equations

E. Thomas in 1990 studied the families of Thue equations  $x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 = 1$



Set

$$F_n(X, Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3.$$

The cubic fields  $\mathbf{Q}(\lambda)$  generated by a root  $\lambda$  of  $F_n(X, 1)$  are called by D. Shanks the *simplest cubic fields*. The roots of the polynomial  $F_n(X, 1)$  can be described via homographies of degree 3.

## Simplest fields.

When the following polynomials are irreducible for  $s, t \in \mathbf{Z}$ , the fields  $\mathbf{Q}(\omega)$  generated by a root  $\omega$  of respectively

$$\begin{cases} sX^3 - tX^2 - (t + 3s)X - s, \\ sX^4 - tX^3 - 6sX^2 + tX + s, \\ sX^6 - 2tX^5 - (5t + 15s)X^4 - 20sX^3 + 5tX^2 + (2t + 6s)X + s, \end{cases}$$

are cyclic over  $\mathbf{Q}$  of degree 3, 4 and 6 respectively. For  $s = 1$ , they are called *simplest fields* by many authors. For  $s \geq 1$ , I. Wakabayashi call them *simplest fields*.

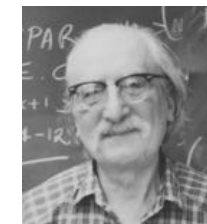
In each of the three cases, the roots of the polynomials can be described via homographies of  $PSL_2(\mathbf{Z})$  of degree 3, 4 and 6 respectively.

## D. Shanks's simplest cubic fields $\mathbf{Q}(\lambda)$ .

Let  $\lambda$  be one of the three roots of

$$F_n(X, 1) = X^3 - (n-1)X^2 - (n+2)X - 1.$$

Then  $\mathbf{Q}(\lambda)$  is a real Galois cubic field.



Write

$$F_n(X, Y) = (X - \lambda_0 Y)(X - \lambda_1 Y)(X - \lambda_2 Y)$$

with

$$\lambda_0 > 0 > \lambda_1 > -1 > \lambda_2.$$

Then

$$\lambda_1 = -\frac{1}{\lambda_0 + 1} \quad \text{and} \quad \lambda_2 = -\frac{\lambda_0 + 1}{\lambda_0}.$$

## E. Thomas's family of Thue equations

In 1990, E. Thomas proved in some effective way that the set of  $(n, x, y) \in \mathbf{Z}^3$  with

$$n \geq 0, \quad \max\{|x|, |y|\} \geq 2 \quad \text{and} \quad F_n(x, y) = \pm 1$$

is finite.

In his paper, he completely solved the equation  $F_n(x, y) = 1$  for  $n \geq 1.365 \cdot 10^7$ : the only solutions are  $(0, -1)$ ,  $(1, 0)$  and  $(-1, +1)$ .

Since  $F_n(-x, -y) = -F_n(x, y)$ , the solutions to  $F_n(x, y) = -1$  are given by  $(-x, -y)$  where  $(x, y)$  are the solutions to  $F_n(x, y) = 1$ .



## Exotic solutions found by E. Thomas in 1990

$$F_0(X, Y) = X^3 + X^2Y - 2XY^2 - Y^3$$

Solutions  $(x, y)$  to  $F_0(x, y) = 1$ :

$$(-9, 5), (-1, 2), (2, -1), (4, -9), (5, 4)$$

$$F_1(X, Y) = X^3 - 3XY^2 - Y^3$$

Solutions  $(x, y)$  to  $F_1(x, y) = 1$ :

$$(-3, 2), (1, -3), (2, 1)$$

$$F_3(X, Y) = X^3 - 2X^2Y - 5XY^2 - Y^3$$

Solutions  $(x, y)$  to  $F_3(x, y) = 1$ :

$$(-7, -2), (-2, 9), (9, -7)$$

## M. Mignotte's work on E. Thomas's family

In 1993, M. Mignotte completed the work of E. Thomas by solving the problem for each  $n$ .

For  $n \geq 4$  and for  $n = 2$ , the only solutions to  $F_n(x, y) = 1$  are  $(0, -1)$ ,  $(1, 0)$  and  $(-1, +1)$ , while for the cases  $n = 0, 1, 3$ , the only nontrivial solutions are the ones found by E. Thomas.



## E. Thomas's family of Thue equations

For the same family

$$F_n(X, Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3,$$

given  $m \neq 0$ , M. Mignotte A. Pethő and F. Lemmermeyer (1996) studied the family of Diophantine equations

$$F_n(X, Y) = m.$$



## M. Mignotte A. Pethő and F. Lemmermeyer (1996)

For  $n \geq 2$ , when  $x, y$  are rational integers verifying

$$0 < |F_n(x, y)| \leq m,$$

then

$$\log |y| \leq c(\log n)(\log n + \log m)$$

with an effectively computable absolute constant  $c$ .

One would like an upper bound for  $\max\{|x|, |y|\}$  depending only on  $m$ , not on  $n$ .

## M. Mignotte A. Pethő and F. Lemmermeyer

Besides, M. Mignotte A. Pethő and F. Lemmermeyer found all solutions of the Thue inequality  $|F_n(X, Y)| \leq 2n + 1$ .

As a consequence, when  $m$  is a given positive integer, there exists an integer  $n_0$  depending upon  $m$  such that the inequality  $|F_n(x, y)| \leq m$  with  $n \geq 0$  and  $|y| > \sqrt[3]{m}$  implies  $n \leq n_0$ .

Note that for  $0 < |t| \leq \sqrt[3]{m}$ ,  $(-t, t)$  and  $(t, -t)$  are solutions. Therefore, the condition  $|y| > \sqrt[3]{m}$  cannot be omitted.

## E. Thomas's family of Thue inequations

In 1996, for the family of Thue inequations

$$0 < |F_n(x, y)| \leq m,$$

Chen Jian Hua has given a bound for  $n$  by using Padé's approximations. This bound was highly improved in 1999 by G. Lettl, A. Pethő and P. Voutier.



## Homogeneous variant of E. Thomas family

I. Wakabayashi, using again the approximants of Padé, extended these results to the families of forms, depending upon two parameters,



$$sX^3 - tX^2Y - (t + 3s)XY^2 - sY^3,$$

which includes the family of Thomas for  $s = 1$  (with  $t = n - 1$ ).

## May 2010, Rio de Janeiro What were we doing on the beach of Rio?





## Suggestion of Claude Levesque

Consider Thomas's family of cubic Thue equations

$F_n(X, Y) = \pm 1$  with

$$F_n(X, Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3.$$

Write

$$F_n(X, Y) = (X - \lambda_{0n}Y)(X - \lambda_{1n}Y)(X - \lambda_{2n}Y)$$

where  $\lambda_{in}$  are units in the totally real cubic field  $\mathbf{Q}(\lambda_{0n})$ . Twist these equations by introducing a new parameter  $a \in \mathbf{Z}$ :

$$F_{n,a}(X, Y) = (X - \lambda_{0n}^a Y)(X - \lambda_{1n}^a Y)(X - \lambda_{2n}^a Y) \in \mathbf{Z}[X, Y].$$

Then we get a family of cubic Thue equations depending on two parameters  $(n, a)$ :

$$F_{n,a}(x, y) = \pm 1.$$

## Thomas's family with two parameters

Joint work with Claude Levesque

**Main result** (2014): *there is an effectively computable absolute constant  $c > 0$  such that, if  $(x, y, n, a)$  are nonzero rational integers with  $\max\{|x|, |y|\} \geq 2$  and*

$$F_{n,a}(x, y) = \pm 1,$$

then

$$\max\{|n|, |a|, |x|, |y|\} \leq c.$$

For all  $n \geq 0$ , trivial solutions with  $a \geq 2$ :

$$(1, 0), (0, 1) \\ (1, 1) \text{ for } a = 2$$

## Exotic solutions to $F_{n,a}(x, y) = 1$ with $a \geq 2$

$(n, a)$	$(x, y)$
(0, 2)	(-14, -9) (-3, -1) (-2, -1) (1, 5) (3, 2) (13, 4)
(0, 3)	(2, 1)
(0, 5)	(-3, -1) (19, -1)
(1, 2)	(-7, -2) (-3, -1) (2, 1) (7, 3)
(2, 2)	(-7, -1) (-2, -1)
(4, 2)	(3, 2)

No further solution in the range

$$0 \leq n \leq 10, \quad 2 \leq a \leq 70, \quad -1000 \leq x, y \leq 1000.$$

**Open question:** are there further solutions?

## Computer search by specialists





## Conjecture on a family $F_{n,s,t}(x, y)$

**Conjecture.** For  $s, t$  and  $n$  in  $\mathbf{Z}$ , define

$$F_{n,s,t}(X, Y) = (X - \lambda_{0n}^s \lambda_{1n}^t Y)(X - \lambda_{1n}^s \lambda_{2n}^t Y)(X - \lambda_{2n}^s \lambda_{0n}^t Y).$$

There exists an effectively computable positive absolute constant  $\kappa$  with the following property: If  $n, s, t, x, y, m$  are integers satisfying

$$\max\{|x|, |y|\} \geq 2, \quad (s, t) \neq (0, 0) \quad \text{and} \quad 0 < |F_{n,s,t}(x, y)| \leq m,$$

then

$$\max\{\log |n|, |s|, |t|, \log |x|, \log |y|\} \leq \kappa(1 + \log m).$$

## Sketch of proof

We want to prove the **Main result**: *there is an effectively computable absolute constant  $c > 0$  such that, if  $(x, y, n, a)$  are nonzero rational integers with  $\max\{|x|, |y|\} \geq 2$  and*

$$F_{n,a}(x, y) = \pm 1,$$

then

$$\max\{|n|, |a|, |x|, |y|\} \leq c.$$

We may assume  $a \geq 2$  and  $y \geq 1$ .

We first consider the case where  $n$  sufficiently large.

## Sketch of proof (continued)

Write  $\lambda_i$  for  $\lambda_{in}$ , ( $i = 0, 1, 2$ ):

$$\begin{aligned} F_n(X, Y) &= X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 \\ &= (X - \lambda_0 Y)(X - \lambda_1 Y)(X - \lambda_2 Y). \end{aligned}$$

We have

$$\left\{ \begin{array}{l} n + \frac{1}{n} \leq \lambda_0 \leq n + \frac{2}{n}, \\ -\frac{1}{n+1} \leq \lambda_1 \leq -\frac{1}{n+2}, \\ -1 - \frac{1}{n} \leq \lambda_2 \leq -1 - \frac{1}{n+1}. \end{array} \right.$$

## Sketch of proof (continued)

Define

$$\gamma_i = x - \lambda_i^a y, \quad (i = 0, 1, 2)$$

so that  $F_{n,a}(x, y) = \pm 1$  becomes  $\gamma_0 \gamma_1 \gamma_2 = \pm 1$ .

One  $\gamma_i$ , say  $\gamma_{i_0}$ , has a small absolute value, namely

$$|\gamma_{i_0}| \leq \frac{m}{y^2 \lambda_0^a},$$

the two others, say  $\gamma_{i_1}, \gamma_{i_2}$ , have large absolute values:

$$\min\{|\gamma_{i_1}|, |\gamma_{i_2}|\} > y |\lambda_2|^a.$$

## Sketch of proof (continued)

Use  $\lambda_0, \lambda_2$  as a basis of the group of units of  $\mathbf{Q}(\lambda_0)$ : there exist  $\delta = \pm 1$  and rational integers  $A$  and  $B$  such that

$$\begin{cases} \gamma_{0,a} &= \delta \lambda_0^A \lambda_2^B, \\ \gamma_{1,a} &= \delta \lambda_1^A \lambda_0^B = \delta \lambda_0^{-A+B} \lambda_2^{-A}, \\ \gamma_{2,a} &= \delta \lambda_2^A \lambda_1^B = \delta \lambda_0^{-B} \lambda_2^{A-B}. \end{cases}$$

We can prove

$$|A| + |B| \leq \kappa \left( \frac{\log y}{\log \lambda_0} + a \right).$$

## End of the proof when $n$ is large

We complete the proof when  $n$  is large by means of a lower bound for a linear form in logarithms of algebraic numbers (Baker's method).

Next we need to consider the case where  $n$  is bounded. We have results which are valid not only for the Thue equations of the family of Thomas. The next result completes the proof of our main theorem.

## Sketch of proof (continued)

The Siegel equation

$$\gamma_{i_0,a}(\lambda_{i_1}^a - \lambda_{i_2}^a) + \gamma_{i_1,a}(\lambda_{i_2}^a - \lambda_{i_0}^a) + \gamma_{i_2,a}(\lambda_{i_0}^a - \lambda_{i_1}^a) = 0$$

leads to the identity

$$\frac{\gamma_{i_1,a}(\lambda_{i_2}^a - \lambda_{i_0}^a)}{\gamma_{i_2,a}(\lambda_{i_1}^a - \lambda_{i_0}^a)} - 1 = -\frac{\gamma_{i_0,a}(\lambda_{i_1}^a - \lambda_{i_2}^a)}{\gamma_{i_2,a}(\lambda_{i_1}^a - \lambda_{i_0}^a)}$$

and the estimate

$$0 < \left| \frac{\gamma_{i_1,a}(\lambda_{i_2}^a - \lambda_{i_0}^a)}{\gamma_{i_2,a}(\lambda_{i_1}^a - \lambda_{i_0}^a)} - 1 \right| \leq \frac{2}{y^3 \lambda_0^a}.$$

## Twists of a given cubic Thue equation

Consider a monic irreducible cubic polynomial  $f(X) \in \mathbf{Z}[X]$  with  $f(0) = \pm 1$  and write

$$F(X, Y) = Y^3 f(X/Y) = (X - \epsilon_1 Y)(X - \epsilon_2 Y)(X - \epsilon_3 Y).$$

For  $a \in \mathbf{Z}$ ,  $a \neq 0$ , define

$$F_a(X, Y) = (X - \epsilon_1^a Y)(X - \epsilon_2^a Y)(X - \epsilon_3^a Y).$$

Then there exists an effectively computable constant  $\kappa > 0$ , depending only on  $f$ , such that, for any  $m \geq 2$ , any  $(x, y, a)$  in the set

$$\{(x, y, a) \in \mathbf{Z}^2 \times \mathbf{Z} \mid xya \neq 0, \max\{|x|, |y|\} \geq 2, |F_a(x, y)| \leq m\}$$

satisfies

$$\max\{|x|, |y|, e^{|a|}\} \leq m^\kappa.$$

## Twists of a given Thue equation

Much more general results can be proved for the twists of a given Thue equation. In particular:

Let  $\alpha$  be an algebraic number of degree  $n \geq 3$  and  $K$  be the field  $\mathbf{Q}(\alpha)$ . When  $\varepsilon$  is a unit of  $K$  such that  $\alpha\varepsilon$  has degree  $n$ , let  $f_\varepsilon(X)$  be the irreducible polynomial of  $\alpha\varepsilon$  and let  $F_\varepsilon(X, Y)$  be its homogeneous version. Then for all but finitely many of these units, Thue equation  $F_\varepsilon(x, y) = \pm 1$  has only the trivial solutions  $x, y$  in  $\mathbf{Z}$  where  $xy = 0$ .

This last result rests on Schmidt's subspace Theorem and is not effective.

## A conjecture

The goal is to obtain effective results.

**Conjecture.** There exists a constant  $\kappa > 0$ , depending only on  $\alpha$ , such that, for any  $m \geq 2$ , all solutions  $(x, y, \varepsilon)$  in  $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_K^\times$  of the inequality

$$|F_\varepsilon(x, y)| \leq m, \text{ with } xy \neq 0 \text{ and } [\mathbf{Q}(\alpha\varepsilon) : \mathbf{Q}] \geq 3,$$

satisfy

$$\max\{|x|, |y|, e^{h(\alpha\varepsilon)}\} \leq m^\kappa.$$

## Twists of a given Thue equation (effective results)

With Claude Levesque we obtained effective partial results in several cases:

- Our first paper (Springer Proceedings in Mathematics & Statistics, 2013) was dealing with non totally real cubic fields.
- Our second one (Ramanujan Math. Soc. Lecture Notes, published in 2016) was dealing with Thue equations attached to a number field having at most one real embedding.
- In the third paper (MJCNT, 2013), for each (irreducible) binary form attached to an algebraic number field, which is not a totally real cubic field, we exhibited an infinite family of equations twisted by units for which Baker's method provides effective bounds for the solutions.
- In a fourth paper (Contemporary Mathematics, 2015), we go one step further by considering twists by a power of a totally real unit.
- In a paper in JTNBx (2015), we solve the problem for the family obtained by twisting Thomas's equations related with the simplest cyclic cubic fields.

## Families of Thue equations (continued)

E. Lee and M. Mignotte with N. Tzanakis studied in 1991 and 1992 the family of cubic Thue equations

$$X^3 - nX^2Y - (n+1)XY^2 - Y^3 = 1.$$

The left hand side is  $X(X+Y)(X-(n+1)Y) - Y^3$ .

For  $n \geq 3.33 \cdot 10^{23}$ , there are only the solutions  $(1, 0)$ ,  $(0, -1)$ ,  $(1, -1)$ ,  $(-n-1, -1)$ ,  $(1, -n)$ .

In 2000, M. Mignotte proved the same result for all  $n \geq 3$ .



## Families of Thue equations (continued)

I. Wakabayashi proved in 2003 that for  $n \geq 1.35 \cdot 10^{14}$ , the equation

$$X^3 - n^2XY^2 + Y^3 = 1$$

has exactly the five solutions  $(0, 1)$ ,  $(1, 0)$ ,  $(1, n^2)$ ,  $(\pm n, 1)$ .

A. Togbé considered the family of equations

$$X^3 - (n^3 - 2n^2 + 3n - 3)X^2Y - n^2XY^2 - Y^3 = \pm 1$$

in 2004. For  $n \geq 1$ , the only solutions are  $(\pm 1, 0)$  and  $(0, \pm 1)$ .



## Families of Thue equations (continued)

I. Wakabayashi in 2002 used Padé approximation for solving the Diophantine inequality

$$|X^3 + aXY^2 + bY^3| \leq a + |b| + 1$$

for arbitrary  $b$  and  $a \geq 360b^4$  as well as for  $b \in \{1, 2\}$  and  $a \geq 1$ .



## Families of Thue equations (continued)

E. Thomas considered some families of Diophantine equations

$$X^3 - bX^2Y + cXY^2 - Y^3 = 1$$

for restricted values of  $b$  and  $c$ .

Family of quartic equations:

$$X^4 - aX^3Y - X^2Y^2 + aXY^3 + Y^4 = \pm 1$$

(A. Pethő 1991, M. Mignotte, A. Pethő and R. Roth, 1996).

The left hand side is  $X(X - Y)(X + Y)(X - aY) + Y^4$ .



## Families of Thue equations (continued)

Split families of E. Thomas (1993):

$$\prod_{i=1}^n (X - p_i(a)Y) - Y^n = \pm 1,$$

where  $p_1, \dots, p_n$  are polynomials in  $\mathbf{Z}[a]$ .

Further results by J.H. Chen, B. Jadrijević, R. Roth, P. Voutier, P. Yuan, V. Ziegler...

## Surveys

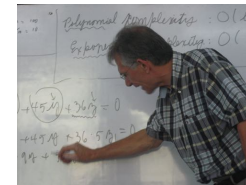
Surveys by **I. Wakabayashi** (2002) and **C. Heuberger** (2005).



## Families of Thue equations (continued)

Further contributors are :

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November 15, 2016

## Diophantine equations and Diophantine approximations

*Michel Waldschmidt*

Université P. et M. Curie (Paris 6)

The pdf file of this talk can be downloaded at URL  
<http://www.imj-prg.fr/~michel.waldschmidt/>