Surat University, SVNIT *December 1, 2017*

Is the Euler constant a rational number,
an algebraic irrational number
or else a transcendental number ?

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http://www.math.jussieu.fr/~miw/

Abstract

To decide the arithmetic nature of a constant from analysis is almost always a difficult problem. Most often, the answer is not known. This is indeed the case for Euler's constant, the value of which is approximately

$$0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\,1\ldots$$

However we know several properties of this number. We survey a few of them.

Reference

Jeffrey C. Lagarias
Euler's constant : Euler's work and modern developments
Bulletin Amer. Math. Soc. **50** (2013), No. 4, 527–628.

arXiv:1303.1856 [math.NT]
Bibliography : 314 references.

Archives Euler and index Eneström

http://eulerarchive.maa.org/

Gustaf Eneström (1852–1923)
Die Schriften Euler's chronologisch nach den Jahren geordnet, in denen sie verfasst worden sind
Jahresbericht der Deutschen Mathematiker–Vereinigung, 1913.

http://www.math.dartmouth.edu/~euler/index/enestrom.html

Harmonic numbers

$$H_1 = 1, \quad H_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.$$

Sequence :

$$1, \quad \frac{3}{2}, \quad \frac{11}{6}, \quad \frac{25}{12}, \quad \frac{137}{60}, \quad \frac{49}{20}, \quad \frac{363}{140}, \quad \frac{761}{280}, \quad \frac{7129}{2520}, \ldots$$

The online encyclopaedia of integer sequences

`https://oeis.org/`

Neil J. A. Sloane

Numerators et denominators

Numerators : `https://oeis.org/A001008`

$$1, 3, 11, 25, 137, 49, 363, 761, 7129, 7381, 83711, 86021, 1145993,$$

$$1171733, 1195757, 2436559, 42142223, 14274301, 275295799,$$

$$55835135, 18858053, 19093197, 444316699, 1347822955, \ldots$$

Denominators : `https://oeis.org/A002805`

$$1, 2, 6, 12, 60, 20, 140, 280, 2520, 2520, 27720, 27720, 360360,$$

$$360360, 360360, 720720, 12252240, 4084080, 77597520,$$

$$15519504, 5173168, 5173168, 118982864, 356948592, \ldots$$

Euler (1731)

De progressionibus harmonicis observationes

The sequence

$$H_n - \log n$$

has a limit $\gamma = 0,577\,21\underline{8}\ldots$
when n tends to infinity.

Leonhard Euler
(1707–1783)

Moreover,

$$\gamma = \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}.$$

Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Euler : $s \in \mathbf{R}$.

Riemann : $s \in \mathbf{C}$.

Numerical value of the Euler constant

The online encyclopaedia of integer sequences
https://oeis.org/A001620
Decimal expansion of Euler's constant
(or Euler–Mascheroni constant) gamma.

Yee (2010) computed 29 844 489 545 decimal digits of gamma.

$$\gamma = 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\ \ldots$$

Nicholas Mercator (1668)

Nicholas Mercator (1620–1687)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k\geq 1} (-1)^{k+1} \frac{x^k}{k}.$$

Gerardus Mercator (1512–1594)

Nicholas is not Gerardus, the Mercator of the eponymus projection :

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Mercator_Gerardus.html

Computation of his constant by Euler in 1731

Euler replaces x by $1/m$ with $m = 1, 2, 3, 4\dots$ in Mercator's formula for $\log(1+x)$:

$$\log 2 = \frac{1}{1} - \frac{1}{2}\left(\frac{1}{1}\right)^2 + \frac{1}{3}\left(\frac{1}{1}\right)^3 - \cdots$$

$$\log\frac{3}{2} = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \cdots$$

$$\log\frac{4}{3} = \frac{1}{3} - \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^3 - \cdots$$

$$\log\frac{5}{4} = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3 - \cdots$$

Adding the first n terms of this sequence of formulae (telescoping series), Euler finds

$$\log(n+1) = H_n - \frac{1}{2}H_{n,2} + \frac{1}{3}H_{n,3} - \cdots$$

Euler's m–harmonic numbers

We have

$$\log(n+1) = H_n - \frac{1}{2}H_{n,2} + \frac{1}{3}H_{n,3} - \cdots$$

with

$$H_{n,m} = \sum_{j=1}^{n} \frac{1}{j^m}$$

for $n \geq 1$ and $m \geq 1$.

Hence, $H_{n,1} = H_n$ and, for $m \geq 2$,

$$\lim_{n\to\infty} H_{n,m} = \zeta(m).$$

Euler's proof (1731)

In the formula

$$H_n - \log(n+1) = \frac{1}{2}H_{n,2} - \frac{1}{3}H_{n,3} + \cdots,$$

when n tends to infinity, the right hand side tends to

$$\sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}$$

which is the sum of an alternating series with a decreasing general term. Hence the left hand side has a limit, which is γ.

Lorenzo Mascheroni (1792)

He produced 32 decimals

$$\gamma = 0,577\,215\,664\,901\,532\,860\,6\underline{1}8\,112\,090\,082\,39$$

the first 19 of them are correct ; the first 15 decimal were already found by Euler in 1755 and then in 1765.

Von Soldner (1809) : 22 decimals

$$\gamma = 0,577\,215\,664\,901\,532\,860\,606\,5$$

C.F. Gauss, F.G.B. Nicolai : 40 decimals

Computation of the decimal of Euler's constant

1872 :	J.W.L. Glaisher	100 decimals
1878 :	J.C. Adams	263 decimals
1952 :	J.W. Wrench Jr	328 decimals
1962 :	D. Knuth	1272 decimals
1963 :	D.W. Sweeney	3566 decimals
1964 :	W.A. Beyer and M.S. Waterman	7114 decimals (4879 correct)
1977 :	R.P. Brent	$20\,700$ decimals
1980 :	R.P. Brent and E.M. McMillan	$30\,000$ decimals
2010 :	Yee	$29\,844\,489\,545$ decimals.

Euler Gamma function (1765)

De curva hypergeometrica hac aequationes expressa $y = 1 \cdot 2 \cdot 3 \cdots x$.

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} \\ &= e^{-\gamma z} \frac{1}{z} \prod_{n=1}^\infty \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.\end{aligned}$$

$$\Gamma'(1) = -\gamma = \int_0^\infty e^{-x} \log x \, dx.$$

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(n+1) = n!$$

Letter from Daniel Bernoulli to Christian Goldbach Octobre 6, 1729

http://fr.wikipedia.org/wiki/Fonction_gamma

Daniel Bernoulli (1700 - 1782)

Christian Goldbach (1690 - 1764)

Euler's formulae (1768)

$$\gamma = \int_0^\infty \left(\frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-t}}{t} \right) dt.$$

$$\gamma = \int_0^1 \left(\frac{1}{1-z} + \frac{1}{\log z} \right) dz.$$

$$\gamma = \sum_{n=2}^\infty \frac{n-1}{n} \left(\zeta(n) - 1 \right).$$

$$\gamma = \frac{3}{4} - \frac{1}{2}\log 2 + \sum_{k=1}^\infty \left(1 - \frac{1}{2k+1}\right) \left(\zeta(2k+1) - 1 \right).$$

Quoting Euler (1768)

"$\mathcal{O} = 0,5772156649015325$ qui numerus eo maiori attentione dignus videtur, quod eum, cum olim in hac investigatione multum studii consumsissem, nullo modo ad cognitum quantitatum genus reducere valui."

> *This number seems also the more noteworthy because even though I have spent much effort in investigating it, I have not been able to reduce it to a known kind of quantity.*

"Manet ergo quaestio magni momenti, cujusdam indolis sit numerus iste $\mathcal{O}$ et ad quodnam genus quantitatum sit referendus."

> *Therefore the question remains of great moment, of what character the number $\mathcal{O}$ is and among what species of quantities it can be classified.*

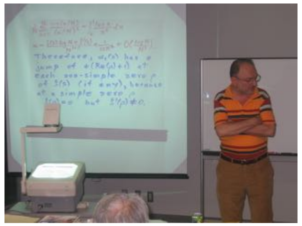
Jonathan Sondow http://home.earthlink.net/~sondow/

$$\gamma = \int_0^\infty \sum_{k=2}^{\infty} \frac{1}{k^2\binom{t+k}{k}} dt$$

$$\gamma = \lim_{s\to 1+} \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{s^n}\right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)}\, {}_2F_3\left(\begin{matrix}1, & 2, & 2\\ 3, & t+2 & \end{matrix}\middle|\, 1\right) dt.$$

Jonathan Sondow and Wadim Zudilin



Jonathan Sondow & Wadim Zudilin, *Euler's constant, q-logarithms, and formulas of Ramanujan and Gosper*, Ramanujan J. **12** (2006), 225–244.

Irrationality of Euler's constant

Conjecture. *Euler constant is irrational.*

If $\gamma = p/q$, then $q > 10^{15\,000}$.
Continued fraction expansion : $30\,000$ first terms have been computed.
http://oeis.org/A002852

$\gamma = [0, 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, 40, 1, 11, 3, \ldots]$

http://mathworld.wolfram.com/Euler-MascheroniConstant.html

> *The famous English mathematician G.H. Hardy is alleged to have offered to give up his Savilian Chair at Oxford to anyone who proved gamma to be irrational, although no written reference for this quote seems to be known. Hilbert mentioned the irrationality of gamma as an unsolved problem that seems "unapproachable" and in front of which mathematicians stand helpless. Conway and Guy (1996) are "prepared to bet that it is transcendental," although they do not expect a proof to be achieved within their lifetimes.*

Hendrik W. Lenstra (1977)

At least one of the two numbers γ, e^γ is transcendental.

Euclidische getallenlichamen
Ph.D. thesis,
Mathematisch Centrum,
Universiteit van Amsterdam,
1977.

http://www.math.leidenuniv.nl/~hwl/PUBLICATIONS/1977c/art.pdf

Stellingen. Behorende bij het proefschrift van H.W. Lenstra Jr.

Theorems of Hermite and Lindemann

Charles Hermite (1873) :
transcendence of e.

Ferdinand Lindemann (1882)
transcendence of π.

Hermite–Lindemann Theorem
For any non–zero complex number z, one at least of the two numbers z, e^z is transcendental.

Corollaries : transcendence of $\log\alpha$ and of e^β for α and β nonzero algebraic numbers with $\log\alpha \neq 0$.

e^γ

http://oeis.org/A073004

$e^\gamma = 1,781\,072\,417\,990\,197\,985\,236\,504\,103\,107\,179\,549\,169\ \dots$

Conjecture. *The number e^γ is irrational.*

If $e^\gamma = p/q$, then $q > 10^{15\,000}$.
Continued fraction expansion of e^γ : $30\,000$ first terms computed.
http://oeis.org/A094644

$e^\gamma = [1, 1, 3, 1, 1, 3, 5, 4, 1, 1, 2, 2, 1, 7, 9, 1, 16, 1, 1, 1, 2, 6, 1, \dots]$

Conjectures on the arithmetic nature of γ

Conjecture 1. *The Euler constant is irrational.*

Conjecture 2. *The Euler constant is transcendental.*

Conjecture 3. *The Euler constant is not a period in the sense of Kontsevich and Zagier.*

Periods : Maxime Kontsevich and Don Zagier

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

A *period* is a complex number with real and imaginary parts given by absolutely convergent integrals of rational fractions with rational coefficients on domains of $\mathbf{R}^n$ defined by (in)equalities involving polynomials with rational coefficients.

Periods

Benjamin Friedrich
Periods and Algebraic de Rham Cohomology
Diplomarbeit im Studiengang Diplom-Mathematik
Universität Leipzig, Fakultät für Mathematik und Informatik
Mathematisches Institut
http://arxiv.org/abs/math/0506113

Joseph Ayoub
Periods and the Conjectures of Grothendieck and Kontsevich–Zagier
European Mathematical Society, Newsletter N°91, March 2014, 12–18.
http://www.ems-ph.org/journals/journal.php?jrn=news

Examples of periods

$$\sqrt{2} = \int_{2x^2 \le 1} dx$$

and all algebraic numbers are periods.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers are periods.

$$\pi = \frac{1}{2i} \int_{|z|=1} \frac{dz}{z} = 2 \int_0^\infty \frac{dt}{1+t^2}.$$

The set of periods is a subalgebra of the field of complex numbers over the field of algebraic numbers ; it is expected that it is not a field.

Euler Gamma and Beta functions

For $p/q \in \mathbf{Q}$,

$$\Gamma\left(\frac{p}{q}\right)^q$$

is a period.

For a and b rational numbers with $\Gamma(a+b) \neq 0$),

$$\begin{aligned} B(a,b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\ &= \int_0^1 x^{a-1}(1-x)^{b-1}dx. \end{aligned}$$

is a period.

$\zeta(s)$ is a period

For s an integer ≥ 2,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

is a period.

`Proof`: by induction.

$$\int_{t_1>t_2\cdots>t_s>0} \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

Maxime Kontsevich and Francis Brown

Multizeta values MZV

Numbers which are not periods ?

Problem (Kontsevich – Zagier) : *Produce an explicit example of a number which is not a period.*

Several levels :

- *analog of Cantor* : the set of periods is countable.
- *analog of Liouville* : find a property which is satisfied by all periods and construct a number which does not satisfy it.
- *analog of Hermite* : prove that given constants arising from analysis are not periods.
  Candidates : $1/\pi,\ e,\ \gamma,\ e^\gamma,\ \Gamma(p/q),\ \Gamma(1/2) = \sqrt{\pi}, \ldots$

Elementary numbers Masahiko Yoshinaga

Analog of Liouville : find a property which is satisfied by all periods and construct a number which does not satisfy it.

Masahiko Yoshinaga (2008)

- defines the class of elementary functions and the class of elementary numbers
- proves that any real period is an elementary number
- produces an example of a number which is not an elementary number (hence is not a period).

http://arxiv.org/abs/0805.0349v1

Masahiko Yoshinaga

The set of elementary functions is countable, the construction of a number which is not a period rests on an enumeration of this set.

Euler constant and arithmetic functions

The function sum of divisors

$$\sigma(n) = \sum_{d|n} d.$$

T.H. Grönwall
(1877- 1932)

T.H. Grönwall (1913)

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}.$$

Guy Robin

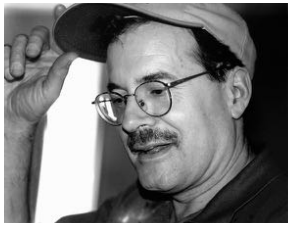
Criterion of Guy Robin (1984) : *Riemann hypothesis is equivalent to*

$$\sigma(n) < e^{\gamma} n \log\log n$$

for all $n \geq 5\,041$.

Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. Pures Appl. **63** (1984), 187–213.

Jeffrey C. Lagarias (2001)



Riemann hypothesis is equivalent to

$$\sigma(n) < H_n + e^{H_n} \log H_n$$

for all $n > 1$.

http://arxiv.org/pdf/math/0008177v2.pdf

The function number of divisors

The function *number of divisors* $d(n)$ is defined for n a positive integer by

$$d(n) = \sum_{d|n} 1 = \mathrm{Card}\{d \mid d|n,\, 1 \le d \le n\}.$$

https://oeis.org/A000005

$$1,\ 2,\ 2,\ 3,\ 2,\ 4,\ 2,\ 4,\ 3,\ 4,\ 2,\ 6,\ 2,\ 4,\ 4,\ 5,\ 2,\ 6,\ 2,\ 6,\ 4,\ 4,$$

$$2,\ 8,\ 3,\ 4,\ 4,\ 6,\ 2,\ 8,\ 2,\ 6,\ 4,\ 4,\ 4,\ 9,\ 2,\ 4,\ 4,\ 8,\ 2,\ 8,\ldots$$

Average value of the function number of divisors

In 1849, Dirichlet gave an estimate for the average value of this function

$$\sum_{k=1}^{n} d(k) = n\log n + (2\gamma - 1)n + \mathcal{O}(\sqrt{n}).$$

J.P.G. Lejeune Dirichlet (1805–1859)

sequence $\sum_{k=1}^{n} d(k)$, $n \ge 0$: http://oeis.org/A006218

$$0,\ 1,\ 3,\ 5,\ 8,\ 10,\ 14,\ 16,\ 20,\ 23,\ 27,\ 29,\ 35,\ 37,\ 41,\ 45,\ 50,\ \ldots$$

Dirichlet's proof (1849)

Denote by $\lfloor x \rfloor$ the integral part of x :

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} \sum_{d|k} 1 = \sum_{\substack{1 \le j, d \le n \\ jd \le n}} 1 = \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor$$

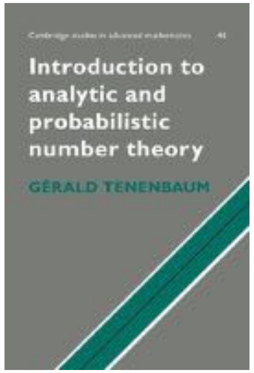
The right hand side is approximately

$$\sum_{j=1}^{n} \frac{n}{j} = nH_n = n\log n + \gamma n + \mathcal{O}(1).$$

Method of the hyperbola (Dirichlet)

The difference between the sum of the integral parts and the harmonic sum is the sum of the fractional parts that Dirichlet estimates using his *hyperbola method* :

$$\sum_{j=1}^{n} \left\{ \frac{n}{j} \right\} = (1-\gamma)n + \mathcal{O}(\sqrt{n}).$$



Dirichlet divisor problem

Let θ be the infimum of the exponents β for which

$$\sum_{k=1}^{n} d(k) = n \log n + (2\gamma - 1)n + \mathcal{O}(n^{\beta}).$$

Dirichlet's Theorem yields $\theta \le \frac{1}{2}\cdot$

This estimate was improved by Voronoi in 1903 : $\theta \le \frac{1}{3}$,

and van der Corput in 1922 : $\theta \le \frac{33}{100}\cdot$

In 1915, Hardy and Landau proved $\theta \ge \frac{1}{4}\cdot$

The exact value of θ is not yet known.

$0,25 \le \theta \le 0,33$

Georgy Voronoy (1868 - 1908)

Johannes van der Corput (1890 - 1975)

Edmund Landau (1877 - 1938)

Godfrey Harold Hardy (1877 - 1947)

$0,25 \le \theta \le 0,3149$

θ is the infimum of the numbers β for which

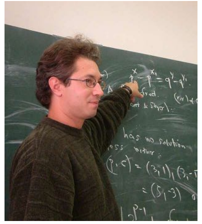
$$\sum_{k=1}^{n} d(k) = n \log n + (2\gamma - 1)n + \mathcal{O}(n^{\beta}).$$

The best known upper bound is due to Martin Huxley in 2003 :

$$\theta \le \frac{131}{416} \sim 0,314\,903\,8\ldots$$

One conjectures $\theta = \frac{1}{4}\cdot$

Florian Luca and Jorge Jimenez Urroz (2012)




F. Luca, J.J. Urroz & M. Waldschmidt
Gaps in binary expansions of some arithmetic functions, and the irrationality of the Euler constant,
Journal of Prime Research in Mathematics, GCU, Lahore, Pakistan, Vol. **8** (2012), 28–35.

The sequence T_k

For $k \geq 0$, set

$$T_k = \sum_{n \leq 2^k} d(n).$$

Consider the binary expansion

$$T_k = \sum_{i=0}^{v_k} a_i 2^i.$$

If $a_{\ell+i} = 0$ for $0 \leq i \leq L-1$, we say that *the binary expansion of T_k has a gap of length at least L starting with ℓ.*

Connection with the irrationality of Euler's constant

Proposition. *Assume that for infinitely many positive k, there exist ℓ and L satisfying*

$$2 + \frac{3\log k}{\log 2} \leq k - \ell \leq L$$

and that the binary expansion of T_k has a gap of length at least L starting at ℓ. Then Euler's constant is irrational.

In other terms, one at least of the following two properties is true :
(i) the binary expansion of T_k does not have extremely long gaps ;
(ii) the Euler constant is irrational.
One expects that both properties are true !

Proof

The relation

$$\sum_{j=1}^{n} d(j) = n\log n + (2\gamma - 1)n + \mathcal{O}(n^\theta)$$

for $n = 2^k$ and $\theta = 1/2$ can be written

$$T_k = 2^k k \log 2 + 2^k(2\gamma - 1) + \mathcal{O}(2^{k/2}).$$

To say that the binary expansion of T_k has a gap of length at least L starting at ℓ means

$$T_k = \sum_{i=\ell+L}^{v_k} a_i 2^i + \sum_{i=0}^{\ell-1} a_i 2^i.$$

Proof (continued)

Setting

$$b = 1 + \sum_{i=\ell+L}^{v_k} a_i 2^{i-k}$$

and dividing by 2^k yields

$$|k \log 2 + 2\gamma + b| < 2^{\ell-k} + c 2^{-k/2}$$

with a constant $c > 0$.
Using the irrationality measure for $\log 2$:

$$\left| \log 2 - \frac{p}{q} \right| \geq \frac{1}{q^{3,58}}$$

which is valid for sufficiently large q, we deduce, under the assumptions of the proposition, that the number γ is irrational.

Irrationality measure for $\log 2$

$$\left| \log 2 - \frac{p}{q} \right| \geq \frac{1}{q^{\kappa}} \quad \text{pour} \quad q \geq q_0.$$

D. Mordukhai-Boltovskoi (1923), K. Mahler (1932),
N.I. Fel'dman (1949 – 1966)
A. Baker (1964) : $\kappa = 12, 5$
E.A. Rukhadze (1987) : $\kappa = 3,891\,399\,78\ldots$
R. Marcovecchio (2009) : $\kappa = 3,574\,553\,91\ldots$

Method of Rhin–Viola (1996)


F. Luca, J.J. Urroz, M. Waldschmidt (2012)

More generally, assume that there exist $\kappa > 0$ and $B_0 > 0$ such that, if b_0, b_1, b_2 are integers with $b_1 \neq 0$, we have

$$|b_0 + b_1 \log 2 + b_2 \gamma| \geq B^{-\kappa}$$

with

$$B = \max\{|b_0|, |b_1|, |b_2|, B_0\}.$$

Then for sufficiently large k, if ℓ and L satisfy

$$2 + \frac{\kappa \log k}{\log 2} \leq k - \ell \leq L,$$

the binary expansion of T_k does not have a gap of length at least L starting at ℓ.

Vincel Hoang Ngoc Minh (2013)

http://hal.archives-ouvertes.fr/hal-00423455

On a conjecture by Pierre Cartier about a group of associators.
Acta Math. Vietnam (2013) **38** :339–398.

. . . we give a complete description of the kernel of polyzêta and draw some consequences about a structure of the algebra of convergent polyzêtas and about the arithmetical nature of the Euler constant.

Irrationality

Lemma. *Let γ be a real number. Assume that for any subfield K of $\mathbf{R}$, the number γ is either in K, or else is transcendental over K. Then γ is a rational number.*

Proof. If the number γ is irrational, from the hypothesis it follows that it is transcendental over $\mathbf{Q}$. In this case γ is algebraic over the field $K = \mathbf{Q}(\gamma^2)$ and does not belong to K.

Divergent series

Euler (1760) : *On divergent series.* Four methods for evaluating

$$1 - 1 + 2 - 6 + 24 - 120 + \cdots$$

$$=$$

$$0! - 1! + 2! - 3! + 4! - 5! + \ldots$$

Wallis hypergeometric series

John Wallis
(1616 - 1703)

Hypergeometric series of Wallis

The divergent power series

$$0! - 1!x + 2!x^2 - 3!x^3 + 4!x^4 - 5!x^5 + \ldots$$

satisfies the linear differential equation

$$y' + \frac{1}{x^2}y = \frac{1}{x};$$

a solution which is convergent at $x = 1$ is given by the integral

$$e^{\frac{1}{x}} \int_0^x \frac{1}{t} e^{-\frac{1}{t}} dt$$

which can be expanded into a continued fraction

$$[1, x, x, 2x, 2x, 3x, 3x, \ldots]$$

for which Euler gives the value at $x = 1$

$$0,596\,347\,362\,\underline{1}23\,7\ldots$$

Benjamin Gompertz (1779–1865)

$$\gamma = -\int_0^\infty e^{-t} \log t \, dt$$

$$\delta = \int_0^\infty e^{-t} \log(t+1) \, dt$$

(A.I. Aptekarev)

The Euler–Gompertz constant

$$0! - 1! + 2! - 3! + 4! - 5! + \cdots$$

$$\delta = \int_0^1 \frac{dt}{1 - \log t} = \int_0^\infty e^{-t} \log(t+1)\, dt =$$

$$0,596\,347\,362\,323\,194\,074\,341\,078\,499\,369\,279\,376\,074\,177\ \ldots$$

https://oeis.org/A073003

Letter of Ramanujan to Hardy (January 16, 1913)

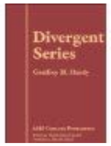
Srinivasa Ramanujan
(1887 – 1920)

Godfrey Harold Hardy
(1877 – 1947)

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$1 - 1! + 2! - 3! + \cdots = 0,596\ldots$$

G.H. Hardy : Divergent Series (1949)



Niels Henrik Abel
(1802 – 1829)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.

Andrei Borisovich Shidlovskii (1959)

One at least of the two numbers γ, δ is irrational. .

K. Mahler (1968)

The number

$$\frac{\pi}{2}\,\frac{Y_0(2)}{J_0(2)} - \gamma$$

is transcendental.

The Bessel functions of first and second kind

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n},$$

$$Y_0(z) = \frac{2}{\pi}\left(\log\left(\frac{z}{2}\right) + \gamma\right) J_0(z) + \frac{2}{\pi}\left(\sum_{n=0}^{\infty} (-1)^n \frac{H_n}{(n!)^2}\left(\frac{z^2}{4}\right)^n\right).$$

Alexander Ivanovich Aptekarev (2007)

A.I. Aptekarev

Quantitative version of the irrationality result due to A.B. Shidlovskii for at least one of the two numbers γ, δ.

Construction of (linear recurrent) sequences $(u_n)_{n\ge 0}$, $(v_n)_{n\ge 0}$ and $(w_n)_{n\ge 0}$ of rational integers with upper bounds for

$$\max\{|u_n|, |v_n|, |w_n|\}$$

and for

$$\max\{|w_n + u_n(e\gamma+\delta)|\ ,\ |v_n + eu_n|\}.$$

Tanguy Rivoal (2009)

Approximation of the function $\gamma + \log x$.
Consequence : rational approximations for γ and $\zeta(2) - \gamma^2$.

T. Rivoal, Kh. Pilehrood, T. Pilehrood (2012)

At least one of the two numbers γ, δ is transcendental.

Tanguy Rivoal

Khodabakhsh Hessami Pilehrood

Tatiana Hessami Pilehrood

Tanguy Rivoal (2012)

Simultaneous rational approximations for the Euler constant and for the Euler–Gompertz constant.

$$\left|\gamma - \frac{p}{q}\right| + \left|\delta - \frac{r}{q}\right| > \frac{C(\epsilon)}{q^{3+\epsilon}}.$$

Method of Mahler :
Two of the numbers e, γ, δ are algebraically independent.

Peter Bundschuh (1979)

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For $p/q \in \mathbf{Q} \setminus \mathbf{Z}$, the number

$$\frac{\Gamma'}{\Gamma}\left(\frac{p}{q}\right) + \gamma$$

is transcendental.

The harmonic series

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots, \qquad \int_0^1 x^j dx = \frac{1}{j+1},$$

hence

$$H_n = \sum_{j=1}^n \frac{1}{j} = \int_0^1 \frac{1-x^n}{1-x} dx.$$

L. Euler (1729) : for $z \geq 0$,

$$H_z = \int_0^1 \frac{1-x^z}{1-x} dx.$$

$$H_{\frac{1}{2}} = 2 - 2\log 2 = 0,613\,705\,638\,880\ldots$$

The harmonic series and the digamma function

The function

$$H_z = \int_0^1 \frac{1-x^z}{1-x} dx$$

which is defined for $z \geq 0$ and satisfies

$$H_n = \sum_{j=1}^n \frac{1}{j} \qquad \text{for } n \in \mathbf{Z},\ n \geq 0$$

is related with the digamma function

$$\psi(z) = \frac{d}{dz} \log \Gamma(z)$$

by

$$\psi(z+1) = -\gamma + H_z.$$

The digamma function

For $z \in \mathbf{C} \setminus \{0, -1, -2, \ldots\}$,

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}\cdot$$

$$\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}.$$

Special values of the digamma function

$$\psi(1) = -\gamma = -0,577\,215\ldots,$$

$$\psi\left(\frac{1}{2}\right) = -2\log(2) - \gamma = -1,963\,510\ldots,$$

$$\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - 3\log(2) - \gamma = -4,227\,453\ldots,$$

$$\psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - 3\log(2) - \gamma = -1,085\,860\ldots.$$

Hence

$$\psi(1) + \psi(1/4) - 3\psi(1/2) + \psi(3/4) = 0.$$

Ram Murty and N. Saradha (2007)

Conjecture (2007) : Let K be a number field over which the q-th cyclotomic polynomial is irreducible. Then the $\varphi(q)$ numbers $\psi(a/q)$ with $1 \le a \le q$ and $(a, q) = 1$ are linearly independent over K.

Baker periods, following (Ram Murty and N. Saradha)

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

A Baker period is an element of the $\overline{\mathbf{Q}}$–vector space spanned by the logarithms of nonzero algebraic numbers.

A Baker period is a period in the sense of Kontsevich and Zagier.
According to Baker's Theorem, such a number is either 0 or transcendental.

Ram Murty and N. Saradha (2007)

Murty and Saradha : at least one of the following statement is true :

- The Euler constant γ is not a Baker period.

- The $\varphi(q)$ numbers $\psi(a/q)$ with $1 \leq a \leq q$ and $(a,q)=1$ are linearly independent over any number field over which the q–th cyclotomic polynomial is irreducible.

Transcendental Numbers

Ram Murty and Purusottam Rath, Springer–Verlag, (2014), 217 p.

Let $q>1$. For any integer a satisfying $\gcd(a,q)=1$, the number

$$-\frac{\Gamma'}{\Gamma}\left(\frac{a}{q}\right)+\gamma$$

is transcendental (it is a Baker period and >0) and at most one of the $\varphi(q)$ numbers

$$\frac{\Gamma'}{\Gamma}\left(\frac{a}{q}\right)$$

($1 \leq a \leq q$ satisfying $\gcd(a,q)=1$) is algebraic.

Euler–Lehmer constants

$\gamma(h,k)=$

$$\lim_{x\to\infty}\left(\sum_{\substack{1\leq n\leq x\\ n\equiv h \mod k}}\frac{1}{n}-\frac{\log x}{k}\right)$$

$$\gamma(2,4)=\frac{1}{4}\gamma$$

Derrick Henry Lehmer (1905 - 1991)

At most one of the numbers

$$\gamma(h,k), \quad 1\leq h<k, \quad k\geq 2$$

is algebraic (Ram Murty and N. Saradha, 2010).

Euler and the digamma function (1765)

$\psi(n)=-\gamma+H_{n-1}$
for $n\geq 1$, with
$H_0=H_{-1}=0$.
For $n\geq 0$,

$$\psi\left(n+\frac{1}{2}\right)=-\gamma-2\log 2+2H_{2n-1}-H_{n-1}.$$

For $|z|<1$,

$$\psi(z+1)=-\gamma+\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+1)z^k.$$

$\zeta(1) = \gamma$?

We have

$$\Gamma(1+t) = \exp\left(-\gamma t + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} t^n\right).$$

We can write

$$\Gamma(1+t) = \exp\left(\sum_{n=1}^{\infty} (-1)^n \frac{\zeta(n)}{n} t^n\right).$$

provided that we set $\zeta(1) = \gamma$.
This normalisation is sometimes used in the study of *multizeta values*; another option is to replace $\zeta(1)$ by an unknown in the formulae involving $\zeta(n)$.

Thomas Johannes Stieltjes (1885)

The Laurent expansion of the Riemann zeta function at the pole $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n$$

with $\gamma_0 = \gamma$ and, for $n \geq 1$,

T. Stieltjes (1856- 1894)

$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1}\right)$$

Exponential periods

Paper by Kontsevich and Zagier :

The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only.

> *There have been some recent indications that one can extend the exponential motivic Galois group still further, adding as a new the Euler constant γ, which is, incidentally, the constant term of $\zeta(s)$ at $s = 1$. Then all classical constants are periods in an appropriate sense.*

Exponential periods

Lagarias quotes Kontsevich : the Euler constant is an exponential period :

$$\gamma = \int_0^1 \int_x^1 \frac{e^{-x}}{y} dy dx - \int_1^{\infty} \int_1^{x} \frac{e^{-x}}{y} dy dx.$$

Rests on

$$-\gamma = \int_0^{\infty} e^{-x} \log x dx.$$

The Euler–Gompertz constant is an exponential period :

$$\delta = \int_0^{\infty} \frac{e^{-t}}{1+t} dt,$$

One conjectures that δ is not a period.

Is the Euler constant a rational number, an algebraic irrational number or else a transcendental number ?

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http://www.math.jussieu.fr/~miw/