# Exercices: hints, solutions, comments

## First course

**1.** Let $f$ be an entire function. Assume $f$ is algebraic : there exists $P \in \mathbb{C}[X,Y]$, $P \neq 0$, such that $P(z,f(z)) = 0$. Prove that $f$ is a polynomial : $f \in \mathbb{C}[z]$.

$\boxed{\textbf{1.}}$ Write
$$P(X,Y) = a_0(X)Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X),$$
where $a_0(X), a_1(X), \dots, a_d(X)$ are polynomials with $a_0(X) \neq 0$. The assumption $P(z,f(z)) = 0$ is
$$a_0(z)f(z)^d + a_1(z)f(z)^{d-1} + \cdots + a_d(z) = 0.$$
Let $N$ be the maximum of the degrees of $a_1, \dots, a_d$. There exists constants $c_1, c_2, r_0$ such that, for $|z| \geq r_0$, we have
$$|a_0(z)| \geq c_1 \quad \text{and} \quad \max_{1 \leq i \leq d} |a_i(z)| \leq c_2|z|^N.$$
For $r \geq r_0$ set $M_r = \max\{|f|_r, 1\}$. From
$$c_1 M_r^d \leq d c_2 r^N M_r^{d-1}$$
one deduces $M_r \leq c_3 r^N$. From Cauchy's inequalities it follows that $f$ is a polynomial of degree $\leq N$.

`Remark.` *A stronger result is that the meromorphic functions in $\mathbb{C}$ which are algebraic are the rational fractions, namely the elements in $\mathbb{C}(z)$.*

• *Here is a proof, using an argument proposed by Ameya Kadhe on January 18, 2021.*

Let $f$ be meromorphic functions in $\mathbb{C}$ and $P \in \mathbb{C}[X,Y]$ a nonzero polynomial such that $P(z,f(z)) = 0$. Select $P$ of minimal degree in $Y$ for this property. Let $\beta \in \mathbb{C}$. Since the polynomial $P$ has minimal degree, it is not divisible by $Y - \beta$ in $\mathbb{C}[X,Y]$, hence $P(X,\beta) \in \mathbb{C}[X]$ is not the zero polynomial : it has only finitely many roots. Therefore there are only finitely many $z \in \mathbb{C}$ such that $f(z) = \beta$. This implies that the function $g(z) = f(1/z)$ does not have an essential singularity at 0 [Picard Great Theorem states that in every neighborhood of an essential singularity, the function takes on every complex value, except possibly one, infinitely many times]. One deduces that $g$ is a rational function (consider the Laurent expansion of $g$ at the origin), hence $f$ also is a rational function.

• *Here is a proof without an appeal to the Picard Great Theorem.*

We already proved that if $f$ is entire, then $f$ is a polynomial. Write
$$P(X,Y) = a_0(X)Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X),$$
where $a_0(X), a_1(X), \dots, a_d(X)$ are polynomials with $a_0(X) \neq 0$. The assumption $P(z,f(z)) = 0$ is
$$a_0(z)f(z)^d + a_1(z)f(z)^{d-1} + \cdots + a_d(z) = 0.$$
Let $z_0$ be a pole of $f$. Write
$$a_0(z)f(z) = -a_1(z) - a_2(z)f(z)^{-1} - \cdots a_d(z)f(z)^{-d+1}.$$

As $z \to z_0$, the right hand side has a finite limit $-a_1(z_0)$. Hence the function $g(z) = a_0(z)f(z)$ has no pole at $z_0$. As a consequence, $g(z)$ is an entire function. Since

$$g(z)^d + a_1(z)g(z)^{d-1} + a_0(z)a_2(z)g(z)^{d-1} + \cdots + a_0(z)^{d-2}a_{d-1}(z)g(z) + a_0(z)^{d-1}a_d(z) = 0,$$

the function $g(z)$ is algebraic, hence a polynomial, and therefore $f(z) = g(z)/a_0(z)$ is a rational function.

The main characters of this course are entire functions. However, since we just mentioned meromorphic functions, here is another exercise involving meromorphic functions in $\mathbb{C}$.

*We wish to introduce the following definition : a meromorphic function $f$ is of order $\le \varrho$ if there exists two entire functions $f_1$ and $f_2$ of order $\le \varrho$ with $f_2 \ne 0$ such that $f = \frac{f_1}{f_2}$.*
*Before being allowed to do so, a lemma should be proved. State this lemma.*
*A reference will be given for the proof of this lemma.*

`Answer`. We already defined the notion of order for an entire function. We wish to extend this definition to a larger class, namely the class of meromorphic functions. We need to check that the extended definition coincides with the restricted one on the class of entire functions. So the lemma which one needs to prove is the following, using only the definition of order for entire functions :

*Let $f$, $f_1$, $f_2$ be three entire functions with $f_2 \ne 0$ and $f_1 = ff_2$. Assume that $f_1$ and $f_2$ are of order $\le \varrho$. Then $f$ is of order $\le \varrho$.*

For the proof, one may use the Minimum Modulus Theorem (for instance [Lang Analysis] Chap. XIII Theorem 3.4 p. 368).

`An analogy`. One defines the height of a polynomial in $\mathbb{C}[z]$ as the maximum modulus of its coefficients. One would like to extend the definition from $\mathbb{C}[z]$ to $\mathbb{C}(z)$ and say that a rational fraction $R \in \mathbb{C}(z)$ has height $\le H$ if $R$ can be written as $P/Q$ where $P$ and $Q$ are two polynomials with height $\le H$. However this is allowed, because the new definition would not match the earlier one on the set of polynomials. For instance the polynomial $X^2 + 2X + 1$ as height 2, still it is a quotient of two polynomials of height 1, namely $X^3 + X^2 - X - 1$ and $X - 1$.

**2.** Given pairwise distinct complex numbers $\alpha_1, \ldots, \alpha_n$, positive integers $t_1, \ldots, t_n$ and complex numbers $\beta_{j,\tau}$ for $1 \le j \le n$, $0 \le \tau < t_j$, show that there exists a unique polynomial $f$ of degree $< t_1 + \cdots + t_n$ satisfying

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} f(\alpha_j) = \beta_{j,\tau}$$

for $1 \le j \le n$ and $0 \le \tau < t_j$.

$\boxed{\textbf{2.}}$ This statement means that the linear map

$$\mathbb{C}[z]_{<t_1+\cdots+t_n} \longrightarrow \mathbb{C}^{t_1+\cdots+t_n}$$

which maps a polynomial $f$ of degree $\le t_1 + \cdots + t_n - 1$ to the tuple of numbers

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} f(\alpha_j)\right)_{\substack{1\le j\le n\\ 0\le\tau<t_j}}$$

is an isomorphism of vector spaces. The fact that it is injective is clear : an element in the kernel has more zeroes (counting multiplicities) than its degree, hence it is 0. Since the two vector spaces have the same dimension, it is surjective. Hence the result.

Let us prove the surjectivity in a more explicit way. We will prove that for $1 \le i \le n$ and $0 \le \nu < t_i$, there exists a polynomial $A_{i,\nu}(z)$ of degree $< t_1 + \cdots + t_n$ which satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} A_{i,\nu}(\alpha_j) = \delta_{\tau,\nu}\delta_{i,j}$$

for $1 \le i, j \le n$ and $0 \le \tau < t_j$, $0 \le \nu < t_i$. Then the polynomial

$$f(z) = \sum_{i=1}^{n} \sum_{\nu=0}^{t_i-1} \beta_{j,\tau} A_{i,\nu}(z)$$

will be the unique solution to the question.

Fix $1 \le i \le n$ and $0 \le \nu < t_i$ and set

$$P_i(z) = \prod_{\substack{1 \le j \le n \\ j \ne i}} \left(\frac{z - \alpha_j}{\alpha_i - \alpha_j}\right)^{t_j - 1}.$$

Given any polynomial $B_{i,\nu}(z)$ such that $B_{i,\nu}(\alpha_i) = 1$, the polynomial

$$A_{i,\nu}(z) = \frac{1}{\nu!}(z - \alpha_i)^{\nu} B_{i,\nu}(z) P_i(z)$$

satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} A_{i,\nu}(\alpha_j) = 0$$

for $0 \le \tau < t_j$ with $1 \le j \le n$, $j \ne i$, and also for $0 \le \tau < \nu$ and $j = i$. Further,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\nu} A_{i,\nu}(\alpha_i) = B_{i,\nu}(\alpha_i) = 1$$

We only need to select $B_{i,\nu}(z)$ so that the conditions are satisfied for $\nu < \tau < t_i$ and $j = i$. If $\nu = t_i - 1$, then the solution is given by $B_{i,\nu} = 1$. Assume $\nu \le t_i - 2$. The solution is given by the polynomial $B_{i,\nu}(z)$ of degree $t_i - \nu - 1$ having Taylor expansion at $\alpha_i$ satisfying the conditions $B_{i,\nu}(\alpha_i) = 1$ and

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{k} A_i(\alpha_i) = 0$$

for $1 \le k \le t_i - \nu - 1$. The system to be solved is triangular, there is a unique solution.
`Reference:` An explicit formula can be found in
A.J. van der Poorten, *Hermite interpolation and p–adic exponential polynomials*, J. Austral. Math. Soc. 22 (Sries A) (1976), 12–26.

**3.** Let $f$ be a nonzero entire function of order $\le \varrho$. For $r \ge 0$, denote by $n(f, r)$ the number of zeroes (counting multiplicities) of $f$ in the disc $|z| \le r$. Show that there exists a constant $c > 0$, depending only on $f$, such that, for $r \ge 1$,

$$n(f, r) \le c r^{\varrho}.$$

$\boxed{\mathbf{3.}}$ From Schwarz Lemma of p. 22 we deduce for $R = 3er$

$$|f|_r \le \mathrm{e}^{-n(f,r)} |f|_R.$$

The left hand side is bounded from below by a positive constant $c_1$ (since $f$ is not the zero function) while for $R \geq 3e$ we have

$$|f|_R \leq \mathrm{e}^{c_2 R^\varrho}$$

with $c_2 > 0$. The result follows.

**4.** Solve the exercise on Blaschke products p. 24.

$\boxed{4.}$ For $|z| < R^2/|a|$, we have

$$\left|\frac{z-a}{R^2-\overline{a}z}\right|^2 = \frac{(z-a)(\overline{z}-\overline{a})}{(R^2-\overline{a}z)(R^2-a\overline{z})} = \frac{|z|^2-(a\overline{z}+\overline{a}z)+|a|^2}{R^4-R^2(a\overline{z}+\overline{a}z)+|a|^2|z|^2}.$$

Hence

$$\left|\frac{z-a}{R^2-\overline{a}z}\right| = \frac{1}{R} \quad \text{for} \quad |z| = R.$$

An alternative argument which yields this result is to write $R^2 = z\overline{z}$ for $|z| = R$, to deduce

$$\frac{z-a}{R^2-\overline{a}z} = \frac{1}{z}\cdot\frac{z-a}{\overline{z}-\overline{a}}$$

and to notice that $(z-a)/(\overline{z}-\overline{a})$ has modulus 1.

Since

$$\varphi_a(-ar/|a|) = -\frac{a}{|a|}\cdot\frac{r+|a|}{R^2+r|a|}$$

we have

$$|\varphi_a(-ar/|a|)| = \frac{r+|a|}{R^2+r|a|}.$$

From

$$(R^2-r^2)(r-|a|) \geq 0$$

we deduce

$$\frac{r+|a|}{R^2+r|a|} \leq \frac{2r}{R^2+r^2}.$$

Let $|z| = r$ with $|a| \leq r < R$ and $z \neq -ar/|a|$. From the inequality

$$(R^2-r^2)(R^2-|a|^2)(2r|a|+a\overline{z}+\overline{a}z) > 0$$

we deduce

$$[r^2-(a\overline{z}+\overline{a}z)+|a|^2](R^2+r|a|)^2 < (r+|a|)^2[R^4-(a\overline{z}+\overline{a}z)R^2+|a|^2r^2].$$

Hence

$$\left|\frac{z-a}{R^2-\overline{a}z}\right| < \frac{r+|a|}{R^2+r|a|}.$$

`Remark`. Following Lang's book [Lang, Complex analysis][Exercise 1.3, Chap. I § 3 p.12], one can first reduce the problem to the special case $R = 1$, $z = |z|$ nonnegative real.
(1) Replacing $z$ by $z/R$ and $a$ by $a/R$, we may assume without loss of generality $R = 1$.
(2) Write $z = r\mathrm{e}^{i\theta}$ and define $a' = a\mathrm{e}^{-i\theta}$. We have $|z| = r$, $|a| = |a'|$, $z - a = \mathrm{e}^{i\theta}(r-a')$ and $1 - \overline{a}z = 1 - \overline{a'}r$, hence

$$\left|\frac{z-a}{1-\overline{a}z}\right| = \left|\frac{r-a'}{1-\overline{a'}r}\right|.$$

This means that we may assume $z$ is real and positive, $z = r$, with $0 \le r \le 1$.
(3) The rest of the proof is the same. Lang's exercise establishes only a weaker statement

$$\left| \frac{z-a}{1-\overline{a}z} \right| < 1$$

for which the proof is much easier. The function

$$f(r) = (1-ra)(1-r\overline{a}) - (r-a)(r-\overline{a})$$

from $\mathbb{R}$ to $\mathbb{R}$ is quadratic, the graph is a parabola with maximum at $r = 0$ with $f'(0) = 0$, $f(0) > 0$, $f(1) = 0$, hence $f$ is $> 0$ on the interval $[0, 1)$.

We deduce the following refinement (p. 23) to Schwarz Lemma of p. 22 :
*Let $f$ be an analytic function in a disc $|z| \le R$ of $\mathbb{C}$, with at least $N$ zeroes in a disc $|z| \le r$ with $r < R$. Then*

$$|f|_r \le \left( \frac{2rR}{R^2+r^2} \right)^N |f|_R.$$

`Proof` The function

$$g(z) = f(z) \prod_{j=1}^{N} \frac{R^2 - \overline{a}_j z}{z - a_j}$$

is analytic in the disc $|z| \le R$. Using the previous lemma on Blaschke products, from

$$f = g \prod_{j=1}^{N} \varphi_{a_j}$$

we deduce

$$|g|_R = |f|_R R^N$$

and

$$|f|_r \le |g|_r \prod_{j=1}^{N} |\varphi_{a_j}|_r \le |g|_r \left( \frac{2r}{R^2+r^2} \right)^N.$$

The result then follows from the maximum modulus principle

$$|g|_r \le |g|_R.$$

**5.** From the definition of the Euler Gamma function by means of the canonical product p. 32 :

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n \ge 1} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

deduce that $1/\Gamma(z)$ is an entire function of order 1 and infinite exponential type.
`Remark.` *Note that typos on p. 12 of the slides and on the text of the exercise needs to be corrected.*

$\boxed{5.}$ We first prove that $1/\Gamma(z)$ has an order $\le 1$. This is a special case of a general result on canonical products [Lang, Complex analysis][XIII, §3]. In our special case, the main auxiliary result is the following :

*Let $\epsilon > 0$. There exists $C > 0$ such that, for all $z \in \mathbb{C}$,*

$$|(1-z)\mathrm{e}^z| \le C^{|z|^{1+\epsilon}}.$$

This estimate is easy for $|z| > 1$ :

$$|(1-z)\mathrm{e}^z| \le 2|z|\mathrm{e}^{|z|} \le 2\mathrm{e}^{2|z|} \le \mathrm{e}^{3|z|} \le C^{|z|^{1+\epsilon}}.$$

It is trivial for $\frac{1}{2} \le |z| \le 1$, since $|1-z|$ is bounded from above and $|z|^{1+\epsilon}$ is bounded from below. Finally for $|z| < \frac{1}{2}$ one expand $(1-z)\mathrm{e}^z$ into a power series in $z$ as follows : write

$$\mathrm{e}^z = \sum_{k\ge 0} \frac{z^k}{k!} = 1 + \sum_{k\ge 1} \frac{z^k}{k!} \quad \text{and} \quad z\mathrm{e}^z = \sum_{k\ge 0} \frac{z^{k+1}}{k!} = \sum_{k\ge 1} \frac{kz^k}{k!},$$

so that

$$(1-z)\,\mathrm{e}^z = \mathrm{e}^z - z\mathrm{e}^z = 1 - \sum_{k\ge 2} \frac{k-1}{k!} z^k = 1 - \frac{1}{2}z^2 - \frac{2}{3!}z^3 - \cdots$$

Hence for $|z| < \frac{1}{2}$,

$$|(1-z)\,\mathrm{e}^z| \le 1 + c|z|^2 \le C^{|z|^{1+\epsilon}}.$$

Using this auxiliary result, we deduce

$$\left| \prod_{n\ge 1} \left(1 - \frac{z}{n}\right) \mathrm{e}^{z/n} \right| \le C^{\sum_{n\ge 1}(|z|/n)^{1+\epsilon}} \le \mathrm{e}^{C'|z|^{1+\epsilon}}.$$

This complete the proof that $1/\Gamma(z)$ has an order $\le 1$.

Using the canonical product, one checks the equality between the meromorphic functions

$$\Gamma(z+1) = z\Gamma(z).$$

For $0 < a < 1$, we have

$$\frac{1}{\Gamma(-a-n)} = (1+a)(2+a)\cdots(n+a)\frac{1}{\Gamma(-a)}\cdot$$

Using

$$(1+a)(2+a)\cdots(n+a) = n!\left(1+\frac{a}{1}\right)\left(1+\frac{a}{2}\right)\cdots\left(1+\frac{a}{n}\right) \ge n!$$

we deduce from Stirling's formula that the order of $1/\Gamma$ is 1 and the exponential type is infinite.

**6.** Check that Abel's polynomials

$$P_n(z) = \frac{1}{n!}z(z-n)^{n-1} \quad (n \ge 1)$$

satisfy, for $n \ge 1$,

$$|P_n|_r \le \left(1 + \frac{r}{n}\right)^n \mathrm{e}^n.$$

$\boxed{\mathbf{6.}}$ The explicit form of Stirling's formula p. 19 implies

$$n! > n^n \mathrm{e}^{-n}$$

for all $n \geq 1$. For $n \geq 1$, $r \geq 0$ and $|z| = r$, we deduce

$$|P_n(z)| \leq \frac{1}{n!} r(r+n)^{n-1} \leq \frac{n^n}{n!}\left(1+\frac{r}{n}\right)^n \leq \mathrm{e}^n \left(1+\frac{r}{n}\right)^n.$$

**7.** Check the formula on divided differences p. 35.

$\boxed{\mathbf{7.}}$

$$f(x) = f(x_0) + (x-x_0) f[x_0, x] = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x].$$

Hence

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_2] - f[x_0, x_1]}{x_2 - x_1}\cdot$$

The claim is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}\cdot$$

Indeed one checks

$$(x_2 - x_0)(f[x_0, x_2] - f[x_0, x_1]) = (x_2 - x_1)(f[x_1, x_2] - f[x_0, x_1])$$

since

$$(x_2-x_0) f[x_0, x_2] = f(x_2) - f(x_0), \quad (x_2-x_1) f[x_1, x_2] = f(x_2) - f(x_1), \quad (x_1-x_0) f[x_0, x_1] = f(x_1) - f(x_0)$$

and

$$f(x_2) - f(x_0) + f(x_1) - f(x_2) = f(x_1) - f(x_0).$$

In the general case, one checks

$$\sum_{j=1}^{n} \frac{f(x_j)}{\prod\limits_{\substack{1\leq k\leq n\\ k\neq j}} (x_j - x_k)} - \sum_{j=0}^{n-1} \frac{f(x_j)}{\prod\limits_{\substack{0\leq k\leq n-1\\ k\neq j}} (x_j - x_k)} = (x_n - x_0) \sum_{j=0}^{n} \frac{f(x_j)}{\prod\limits_{\substack{0\leq k\leq n\\ k\neq j}} (x_j - x_k)}$$

(check the coefficient of $f(x_j)$ in both sides of the formula), hence

$$f[x_0, x_1, \dots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod\limits_{\substack{0\leq k\leq n\\ k\neq j}} (x_j - x_k)}\cdot$$

and

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}\cdot$$

## Références


[Lang, Complex analysis] S. Lang, *Complex analysis*, vol. 103 of Graduate Texts in Mathematics, Springer-Verlag, New York, fourth ed., 1999.