Michel Waldschmidt Interpolation, January 2021

Exercices: hints, solutions, comments

Third course

1. Let s_0, s_1, s_2 be three complex numbers. Give a necessary and sufficient condition for the following to hold. There exist three sequences of polynomials $(\Lambda_{n,0}(z))_{n\geq 0}$, $(\Lambda_{n,1}(z))_{n\geq 0}$, $(\Lambda_{n,2}(z))_{n\geq 0}$ such that any polynomial $f \in \mathbb{C}[z]$ can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} \Big(f^{(3n)}(s_0)\Lambda_{n,0}(z) + f^{(3n)}(s_1)\Lambda_{n,1}(z) + f^{(3n)}(s_2)\Lambda_{n,2}(z) \Big).$$

What is the degree of $\Lambda_{n,j}(z)$? The leading term ? Write the six polynomials

$$\Lambda_{0,0}(z), \Lambda_{0,1}(z), \Lambda_{0,2}(z), \Lambda_{1,0}(z), \Lambda_{1,1}(z), \Lambda_{1,2}(z).$$

1. The polynomials Λ_{ni} $(n \geq 0,\ i = 0, 1, 2)$ are defined by

$$\Lambda_{ni}^{(3k)}(s_j) = \delta_{nk}\delta_{ij}, \quad n, k \geq 0,\ i, j = 0, 1, 2.$$

By symmetry, it suffices to deal with $i = 0$.

• We start with $n = 0$: a necessary and sufficient condition for the existence of a polynomial Λ_{00} satisfying $\Lambda_{00}''' = 0$ and

$$\Lambda_{00}(s_0) = 1, \quad \Lambda_{00}(s_1) = \Lambda_{00}(s_2) = 0$$

is $s_0 \neq s_1$ and $s_0 \neq s_2$. Assume from now on that s_0, s_1, s_2 are pairwise distinct. Then there is a unique such polynomial, it has degree 2 and is given by the Lagrange interpolation formula, namely

$$\Lambda_{00}(z) = \frac{(z - s_1)(z - s_2)}{(s_0 - s_1)(s_0 - s_2)}\cdot$$

• For $n \geq 1$ the polynomial Λ_{n0} is the unique polynomial satisfying the differential equation $\Lambda_{n0}''' = \Lambda_{n-1,0}$ with the initial conditions

$$\Lambda_{n0}(s_0) = \Lambda_{n0}(s_1) = \Lambda_{n0}(s_2) = 0.$$

It has degree $3n + 2$ and leading term $\frac{2}{(3n+2)!}z^{3n+2}$.

• We explicit the solution for $n = 1$. The polynomial

$$L(z) = \frac{1}{60}\frac{z^5}{(s_0 - s_1)(s_0 - s_2)} - \frac{1}{24}\frac{(s_1 + s_2)z^4}{(s_0 - s_1)(s_0 - s_2)} + \frac{1}{6}\frac{s_1 s_2 z^3}{(s_0 - s_1)(s_0 - s_2)}$$

satisfies

$$L'''(z) = \frac{z^2}{(s_0 - s_1)(s_0 - s_2)} - \frac{(s_1 + s_2)z}{(s_0 - s_1)(s_0 - s_2)} + \frac{s_1 s_2}{(s_0 - s_1)(s_0 - s_2)} = \Lambda_{00}(z).$$

Hence the unique polynomial Λ_{10} solution of the differential equation $\Lambda_{10}''' = \Lambda_{00}$ with the initial conditions

$$\Lambda_{10}(s_0) = \Lambda_{10}(s_1) = \Lambda_{10}(s_2) = 0$$

is $\Lambda_{10}(z) = L(z) - (c_0z^2 + c_1z + c_2)$ where c_0, c_1, c_2 are the solutions of the system of equations

$$\begin{cases} c_0s_0^2 + c_1s_0 + c_2 & = L(s_0), \\ c_0s_1^2 + c_1s_1 + c_2 & = L(s_1), \\ c_0s_2^2 + c_1s_2 + c_2 & = L(s_2). \end{cases}$$

2. Let s_0, s_1, s_2 be three complex numbers. Give a necessary and sufficient condition for the following to hold. There exist three sequences of polynomials $(M_{n,0}(z))_{n\geq 0}$, $(M_{n,1}(z))_{n\geq 0}$, $(M_{n,2}(z))_{n\geq 0}$ such that any polynomial $f \in \mathbb{C}[z]$ can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} \left(f^{(3n)}(s_0)M_{n,0}(z) + f^{(3n+1)}(s_1)M_{n,1}(z) + f^{(3n+2)}(s_2)M_{n,2}(z) \right).$$

What is the degree of $M_{n,j}(z)$? The leading term ? Write the six polynomials

$$M_{0,0}(z), M_{0,1}(z), M_{0,2}(z), M_{1,0}(z), M_{1,1}(z), M_{1,2}(z).$$

2. The polynomials M_{ni} ($n \geq 0$, $i = 0, 1, 2$) are defined by

$$M_{ni}^{(3k+i)}(s_j) = \delta_{nk}\delta_{ij}, \quad n, k \geq 0, \ i, j = 0, 1, 2.$$

As we will see, there is no condition for the existence and unicity of such polynomials.
• We start with $n = 0$.
The unique polynomial $M_{00}(z)$ satisfying $M_{00}''' = 0$ and $M_{00}(s_0) = 1$, $M_{01}'(s_1) = M_{01}''(s_2) = 0$ is the constant polynomial $M_{00}(z) = 1$.
The unique polynomial $M_{01}(z)$ satisfying $M_{01}''' = 0$ and $M_{01}(s_0) = M_{01}''(s_2) = 0$, $M_{01}'(s_1) = 1$ is $M_{01}(z) = z - s_0$.
The unique polynomial $M_{02}(z)$ satisfying $M_{02}''' = 0$ and $M_{02}(s_0) = M_{02}'(s_1) = 0$, $M_{02}''(s_2) = 1$ is

$$M_{02}(z) = \frac{1}{2}z^2 - s_1z + \frac{1}{2}s_0(2s_1 - s_0).$$

• Let $n \geq 1$. For $i = 0, 1, 2$, the polynomial M_{ni} is the unique solution of the differential equation $M_{ni}''' = M_{n-1,i}$ with the initial condition

$$M_{ni}(s_0) = M_{ni}'(s_1) = M_{ni}''(s_2) = 0.$$

For $n \geq 0$ and $i = 0$ the solution is given by $M_{n0}(z) = \frac{1}{(3n)!}z^{3n}$.

The leading term of M_{n1} is $\frac{1}{(3n+1)!}z^{3n+1}$ and the leading term of M_{n2} is $\frac{2}{(3n+2)!}z^{3n+2}$.
• We explicit the solution for $n = 1$.
The polynomial

$$A(z) = \frac{1}{24}z^4 - \frac{1}{6}s_0z^3$$

satisfies

$$A'''(z) = z - s_0 = M_{01}(z).$$

Hence the unique polynomial M_{11} solution of the differential equation $M_{11}''' = M_{01}$ with the initial conditions

$$M_{11}(s_0) = M_{11}'(s_1) = M_{11}''(s_2) = 0,$$

is $M_{11}(z) = A(z) - (az^2 + bz + c)$ where a, b, c are the solutions of the system of equations

$$\begin{cases} as_0^2 + bs_0 + c &= A(s_0), \\ 2as_1 + b &= A'(s_1), \\ 2a &= A''(s_2). \end{cases}$$

The polynomial

$$B(z) = \frac{1}{120}z^5 - \frac{s_1}{24}z^4 + \frac{s_0(2s_1 - s_0)}{12}z^3$$

satisfies

$$B'''(z) = \frac{1}{2}z^2 - s_1 z + \frac{1}{2}s_0(2s_1 - s_0) = M_{02}(z).$$

Hence the unique polynomial M_{12} solution of the differential equation $M_{12}''' = M_{02}$ with the initial conditions

$$M_{12}(s_0) = M_{12}'(s_1) = M_{12}''(s_2) = 0$$

is $M_{12}(z) = B(z) - (az^2 + bz + c)$ where a, b, c are the solutions of the system of equations

$$\begin{cases} as_0^2 + bs_0 + c &= B(s_0), \\ 2as_1 + b &= B'(s_1), \\ 2a &= B''(s_2). \end{cases}$$

3. Let s_0, s_1, s_2 be three complex numbers. Give a necessary and sufficient condition for the following to hold. There exist three sequences of polynomials $(N_{n,0}(z))_{n\geq 0}$, $(N_{n,1}(z))_{n\geq 0}$, $(N_{n,2}(z))_{n\geq 0}$ such that any polynomial $f \in \mathbb{C}[z]$ can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} \left(f^{(3n)}(s_0)N_{n,0}(z) + f^{(3n)}(s_1)N_{n,1}(z) + f^{(3n+1)}(s_2)N_{n,2}(z) \right).$$

What is the degree of $N_{n,j}(z)$? The leading term ? Write the six polynomials

$$N_{0,0}(z), N_{0,1}(z), N_{0,2}(z), N_{1,0}(z), N_{1,1}(z), N_{1,2}(z).$$

3. The polynomials N_{ni} ($n \geq 0$, $i = 0, 1, 2$) are defined by

$$N_{n0}^{(3k)}(s_0) = N_{n1}^{(3k)}(s_1) = \delta_{nk}, \quad N_{n2}^{(3k+1)}(s_2) = \delta_{nk} \qquad n, k \geq 0.$$

The polynomial N_{n1} is deduced from N_{n0} by permuting s_0 and s_1. So we need to deal only with N_{n0} and N_{n2}.

• Let $n = 0$. The conditions on $N_{00}(z)$ and $N_{02}(z)$ are

$$N_{00}(s_0) = 1, \quad N_{00}(s_1) = N_{00}'(s_2) = 0, \qquad N_{02}(s_0) = N_{02}(s_1) = 0, \quad N_{02}'(s_2) = 1.$$

Write

$$N_{00}(z) = (z - s_1)(az + b), \quad N_{02}(z) = c(z - s_0)(z - s_1),$$

so that $N_{00}'(z) = a(2z - s_1) + b$. The conditions on the numbers a and b arise from the requirement $N_{00}(s_0) = 1$ and $N_{00}'(s_2) = 0$:

$$\begin{cases} (as_0 + b)(s_0 - s_1) &= 1, \\ a(2s_2 - s_1) + b &= 0, \end{cases}$$

while the condition on c come from $N'_{02}(s_2) = 1$:

$$c(2s_2 - s_0 - s_1) = 1.$$

A necessary and sufficient condition for the existence and unicity of a solution to this system is $s_0 \neq s_1$, $s_0 + s_1 \neq 2s_2$. Under this assumption, the solution is

$$\begin{cases} N_{00}(z) &= \dfrac{(z-s_1)(z-s_1+2s_2)}{(s_0-s_1)(s_0+s_1-2s_2)}, \\ \\ N_{02}(z) &= \dfrac{(z-s_0)(z-s_1)}{(s_0-s_1)(2s_2-s_0-s_1)}. \end{cases}$$

• For $n \geq 1$ and $i = 0, 1, 2$, the polynomial N_{ni} is the unique solution of the differential equation $N'''_{ni} = N_{n-1,i}$ with the initial condition

$$N_{ni}(s_0) = N_{ni}(s_1) = N''_{ni}(s_2) = 0.$$

The degree of N_{ni} is $3n + 2$, the leading coefficient of N_{n0} is

$$\frac{2}{(3n+2)!(s_0-s_1)(s_0+s_1-2s_2)}$$

while the leading coefficient of N_{n0} is

$$\frac{2}{(3n+2)!(s_0-s_1)(2s_2-s_0-s_1)}.$$

• We explicit the solution for $n = 1$. The polynomial N_{10} is computed as follows : let $A(z)$ be a primitive of N_{00}. Then $N_{10}(z) = A(z) - (az^2 + bz + c)$ where a, b, c are the solutions of

$$\begin{cases} as_0^2 + bs_0 + c &= A(s_0), \\ as_1^2 + bs_1 + c &= A(s_1), \\ 2as_2 + b &= A'(s_2). \end{cases}$$

In the same way, $N_{12}(z) = B(z) - (\alpha z^2 + \beta z + \gamma)$, where $B(z)$ is a primitive of N_{02} while α, β, γ are the solutions of

$$\begin{cases} \alpha s_0^2 + \beta s_0 + \gamma &= B(s_0), \\ \alpha s_1^2 + \beta s_1 + \gamma &= B(s_1), \\ 2\alpha s_2 + \beta &= B'(s_2). \end{cases}$$

Notice that both systems have the same determinant

$$\begin{vmatrix} s_0^2 & s_0 & 1 \\ s_1^2 & s_1 & 1 \\ 2s_2 & 1 & 0 \end{vmatrix} = (s_0 - s_1)(2s_2 - s_0 - s_1).$$

4. On p. 11, check that if the determinant $D(\mathbf{s})$ does not vanish, then $r_j \leq j$ for all $j = 0, 1, \ldots, m-1$.

4.

Let $z_0, z_1, \ldots, z_{m-1}$ be independent variables. Write $\mathbf{z}$ for $(z_0, z_1, \ldots, z_{m-1})$. Let K be the field $\mathbb{Q}(z_0, z_1, \ldots, z_{m-1})$ and $\mathrm{D}(\mathbf{z})$ be the determinant

$$\det\left(\frac{k!}{(k-r_j)!} z_j^{k-r_j}\right)_{0 \leq j,k \leq m-1} \in \mathbb{Q}[\mathbf{z}] \subset K.$$

Recall $a!/(a-b)! = 0$ for $a < b$.

For $j = 0, 1, \dots, m-1$, the row vector

$$\begin{aligned} v_j &= \left(\frac{k!}{(k-r_j)!} z_j^{k-r_j} \right)_{k=0,1,\dots,m-1} \\ &= \left(0, 0, \dots, 0, r_j!, \frac{(r_j+1)!}{1!} z_j, \frac{(r_j+2)!}{2!} z_j^2, \dots, \frac{(m-1)!}{(m-1-r_j)!} z_j^{m-1-r_j} \right) \end{aligned}$$

belongs to $\{0\}^{r_j} \times K^{m-r_j}$. If $r_j > j$ for some $j \in \{0, 1, \dots, m-1\}$, then the $m-j$ vectors $v_j, v_{j+1}, \dots, v_{m-1}$ all belong to the subspace $\{0\}^{j+1} \times K^{m-j-1}$ of K^m, the dimension of which is $m-j-1$; hence the determinant $\mathrm{D}(\mathbf{z})$ vanishes.

This amounts to say that a triangular matrix with a zero on the diagonal has a zero determinant.

5. Prove the proposition p. 11.

5. *Assume* $\mathrm{D}(\mathbf{s}) \neq 0$. *Then there exists a unique family of polynomials* $(\Lambda_{nj}(z))_{n\geq 0, 0\leq j\leq m-1}$ *satisfying*

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}, \text{ for } n, k \geq 0 \text{ and } 0 \leq j, \ell \leq m-1.$$

For $n \geq 0$ *and* $0 \leq j \leq m-1$ *the polynomial* Λ_{nj} *has degree* $\leq mn+m-1$.

The assumption $\mathrm{D}(\mathbf{s}) \neq 0$ means that the linear map

$$\begin{array}{ccc} \mathbb{C}[z]_{\leq m-1} & \longrightarrow & \mathbb{C}^m \\ L(z) & \longmapsto & \left(L^{(r_j)}(s_j) \right)_{0\leq j\leq m-1} \end{array}$$

is an isomorphism of $\mathbb{C}$–vector spaces, $\mathbb{C}[z]_{\leq m-1}$ being the space of polynomials of degree $\leq m-1$.

`First proof`. Assuming $\mathrm{D}(\mathbf{s}) \neq 0$, we prove by induction on n that the linear map

$$\begin{array}{cccc} \psi_n : & \mathbb{C}[z]_{\leq m(n+1)-1} & \longrightarrow & \mathbb{C}^{m(n+1)} \\ & L(z) & \longmapsto & \left(L^{(mk+r_\ell)}(s_\ell) \right)_{0\leq \ell\leq m-1, 0\leq k\leq n} \end{array}$$

is an isomorphism of $\mathbb{C}$–vector spaces. For $n = 0$ this is the assumption $\mathrm{D}(\mathbf{s}) \neq 0$. Assume ψ_{n-1} is injective for some $n \geq 1$. Let $L \in \ker \psi_n$. Then $L^{(m)} \in \ker \psi_{n-1}$, hence $L^{(m)} = 0$, which means that L has degree $< m$. From the assumption $\mathrm{D}(\mathbf{s}) \neq 0$ we conclude $L = 0$.

The fact that ψ_n is injective for all n implies that if a polynomial $f \in \mathbb{C}[z]$ satisfies $f^{(mk+r_\ell)}(s_\ell) = 0$ for all $k \geq 0$ and all ℓ with $0 \leq \ell \leq m-1$, then $f = 0$. This shows the unicity of the solution Λ_{nj} of the system of equations

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}, \quad \text{for} \quad n, k \geq 0 \quad \text{and} \quad 0 \leq j, \ell \leq m-1.$$

Since ψ_n is injective, it is an isomorphism, and hence surjective : for $0 \leq j \leq n-1$ there exists a unique polynomial $\Lambda_{nj} \in \mathbb{C}[z]_{\leq m(n+1)-1}$ such that $\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}$ for $0 \leq j, \ell \leq m-1$. These conditions show that the set of polynomials Λ_{kj} for $0 \leq k \leq n$, $0 \leq j \leq m-1$, is a basis of $\mathbb{C}[z]_{\leq m(n+1)-1}$: any polynomial $f \in \mathbb{C}[z]$ of degree $\leq m(n+1)-1$ can be written in a unique way

$$f(z) = \sum_{j=0}^{m-1} \sum_{k=0}^{n} a_{kj}\Lambda_{kj}(z),$$

and therefore the coefficients are given by $a_{kj} = f^{(mk+r_j)}(s_j)$.

`Second proof`. The conditions

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}, \quad \text{for} \quad n, k \geq 0 \quad \text{and} \quad 0 \leq j, \ell \leq m-1.$$

mean that any polynomial $f \in \mathbb{C}[z]$ has an expansion

$$f(z) = \sum_{j=0}^{m-1} \sum_{n \geq 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z),$$

where only finitely many terms on the right hand side are nonzero.

Assuming $\mathrm{D}(\mathbf{s}) \neq 0$, we first prove the unicity of such an expansion by induction on the degree of f. The assumption $\mathrm{D}(\mathbf{s}) \neq 0$ shows that there is no nonzero polynomial of degree $< m$ satisfying $f^{(mn+r_j)}(s_j) = 0$ for all (n, j) with $0 \leq n, j \leq m-1$. Now if f is a polynomial satisfying $f^{(mn+r_j)}(s_j) = 0$ for all (n, j) with $n \geq 0$ and $0 \leq j \leq m-1$, then $f^{(m)}$ satisfies the same conditions and has a degree less than the degree of f. By the induction hypothesis we deduce $f^{(m)} = 0$, which means that f has degree $< m$, hence $f = 0$. This proves the unicity.

For the existence, let us show that, under the assumption $\mathrm{D}(\mathbf{s}) \neq 0$, the recurrence relations

$$\Lambda_{nj}^{(m)} = \Lambda_{n-1,j}, \quad \Lambda_{nj}^{(r_\ell)}(s_\ell) = 0 \text{ for } n \geq 1, \quad \Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \text{ for } 0 \leq j, \ell \leq m-1$$

have a unique solution given by polynomials $\Lambda_{nj}(z)$, $(n \geq 0,\ j = 0, \ldots, m-1)$, where Λ_{nj} has degree $\leq mn + m - 1$. Clearly, these polynomials will satisfy

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}, \quad \text{for} \quad n, k \geq 0 \quad \text{and} \quad 0 \leq j, \ell \leq m-1.$$

From the assumption $\mathrm{D}(\mathbf{s}) \neq 0$ we deduce that, for $0 \leq j \leq m-1$, there is a unique polynomial Λ_{0j} of degree $< m$ satisfying

$$\Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \text{ for } 0 \leq \ell \leq m-1.$$

By induction, given $n \geq 1$ and $j \in \{0, 1, \ldots, m-1\}$, once we know $\Lambda_{n-1,j}(z)$, we choose a solution L of the differential equation $L^{(m)} = \Lambda_{n-1,j}$; using again the assumption $\mathrm{D}(\mathbf{s}) \neq 0$, we deduce that there is a unique polynomial $\widetilde{L}$ of degree $< m$ satisfying $\widetilde{L}^{(r_\ell)}(s_\ell) = L^{(r_\ell)}(s_\ell)$ for $0 \leq \ell \leq m-1$; then the solution is given by $\Lambda_{nj} = L - \widetilde{L}$.

6. Poritsky's interpolation p. 31. Prove that the condition $\mathrm{D}(\mathbf{s}) \neq 0$ means that $s_0, s_1, \ldots, s_{m-1}$ are pairwise distinct.
Prove also that the function $\Delta(t)$ has a zero at the origin of multiplicity at least $m(m-1)/2$.
N.B. *The fact that the multiplicity is exactly $m(m-1)/2$ follows from the fact that the coefficient of $t^{m(m-1)/2}$ in the Taylor expansion at the origin of $\Delta(t)$ is given by a product of two Vandermonde determinants*

$$\frac{1}{1!2!\cdots(m-1)!} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^{m-1} \\ 1 & \zeta^2 & \cdots & \zeta^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{m-1} & \cdots & \zeta^{(m-1)^2} \end{pmatrix} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{m-1} \\ s_0^2 & s_1^2 & \cdots & s_{m-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ s_0^{m-1} & s_1^{m-1} & \cdots & s_{m-1}^{m-1} \end{pmatrix}.$$

But this is not so easy to prove [Macintyre 1954, §3].

6. Poritsky interpolation is the case

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

The Vandermonde determinant

$$D(\mathbf{s}) = \det\left(s_j^k\right)_{0\le j,k\le m-1} = \det\begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^{m-1}\\ 1 & s_1 & s_1^2 & \cdots & s_1^{m-1}\\ 1 & s_2 & s_2^2 & \cdots & s_2^{m-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & s_{m-1} & s_{m-1}^2 & \cdots & s_{m-1}^{m-1}\end{pmatrix} = \prod_{0\le j<\ell\le m-1}(s_\ell - s_j)$$

does not vanish if and only if $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct.

The determinant $\Delta(t)$ is the determinant of the following matrix

$$\begin{pmatrix} \mathrm{e}^{ts_0} & \mathrm{e}^{ts_1} & \mathrm{e}^{ts_2} & \cdots & \mathrm{e}^{ts_{m-1}}\\ \mathrm{e}^{\zeta ts_0} & \mathrm{e}^{\zeta ts_1} & \mathrm{e}^{\zeta ts_2} & \cdots & \mathrm{e}^{\zeta ts_{m-1}}\\ \mathrm{e}^{\zeta^2 ts_0} & \mathrm{e}^{\zeta^2 ts_1} & \mathrm{e}^{\zeta^2 ts_2} & \cdots & \mathrm{e}^{\zeta^2 ts_{m-1}}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathrm{e}^{\zeta^{m-1} ts_0} & \mathrm{e}^{\zeta^{m-1} ts_1} & \mathrm{e}^{\zeta^{m-1} ts_2} & \cdots & \mathrm{e}^{\zeta^{m-1} ts_{m-1}}\end{pmatrix}.$$

The value $\Delta(0)$ at $t=0$ is 0. We use the multilinearity of the determinant : the derivative (with respect to t) is the sum of determinants where we derive the rows. The derivative of order k of the row

$$\begin{pmatrix}\mathrm{e}^{\zeta^j ts_0} & \mathrm{e}^{\zeta^j ts_1} & \mathrm{e}^{\zeta^j ts_2} & \cdots & \mathrm{e}^{\zeta^j ts_{m-1}}\end{pmatrix}$$

is the row

$$\begin{pmatrix}(\zeta^j s_0)^k\mathrm{e}^{\zeta^j ts_0} & (\zeta^j s_1)^k\mathrm{e}^{\zeta^j ts_1} & (\zeta^j s_2)^k\mathrm{e}^{\zeta^j ts_2} & \cdots & (\zeta^j s_{m-1})^k\mathrm{e}^{\zeta^j ts_{m-1}}\end{pmatrix}$$

which takes the value

$$\begin{pmatrix}(\zeta^j s_0)^k & (\zeta^j s_1)^k & (\zeta^j s_2)^k & \cdots & (\zeta^j s_{m-1})^k\end{pmatrix}$$

at $t=0$.

If we derive the same number of times two rows, the corresponding determinant vanishes at $t=0$. Hence to get a nonzero derivative at 0 we need to take derivatives of order at least

$$0+1+2+\cdots+(m-1) = \frac{m(m-1)}{2}\cdot$$

7. Let $\mathbf{w} = (w_n)_{n\ge 0}$ be a sequence of complex numbers. Prove that the sequence of polynomials $(\Omega_{w_0,w_1,\dots,w_{n-1}}(z))_{n\ge 0}$ defined by $\Omega_\emptyset = 1$ and

$$\Omega_{w_0,w_1,\dots,w_{n-1}}(z) = \int_{w_0}^{z}\mathrm{d}t_1\int_{w_1}^{t_1}\mathrm{d}t_2\cdots\int_{w_{n-1}}^{t_{n-1}}\mathrm{d}t_n$$

for $n\ge 1$ satisfy $\Omega_{w_0}(z) = z - w_0$ and for $n\ge 0$, $\Omega_{w_0,w_1,w_2,\dots,w_n}(w_0) = 0$,

$$\Omega'_{w_0,w_1,w_2,\dots,w_n}(z) = \Omega_{w_1,w_2,\dots,w_n}(z).$$

What are the degree and the leading term of $\Omega_{w_0,w_1,w_2,\dots,w_n}(z)$? Check

$$\Omega^{(k)}_{w_0,w_1,w_2,\dots,w_n}(w_k) = \delta_{kn}$$

for $n \geq 0$ and $k \geq 0$. Deduce that any polynomial is a finite sum

$$f(z) = \sum_{n\geq 0} f^{(n)}(w_n)\Omega_{w_0,w_1,w_2,\ldots,w_n}(z).$$

Check the formula for the Gontcharoff determinant p. 39.

Give a close formula for these polynomials $\Omega_{w_0,w_1,\ldots,w_{n-1}}(z)$ when

- $w_n = 0$ for all $n \geq 0$.
- $w_n = 1$ for even $n \geq 0$, $w_n = 0$ for odd $n \geq 1$.
- $w_n = n$ for all $n \geq 0$.

7. The definition of these polynomials involving iterated integrals means that the sequence of polynomials $(\Omega_{w_0,w_1,\ldots,w_{n-1}})_{n\geq 0}$ in $\mathbb{C}[z]$ is defined as follows : we set $\Omega_\emptyset = 1$, $\Omega_{w_0}(z) = z - w_0$, and, for $n \geq 1$, the polynomial $\Omega_{w_0,w_1,w_2,\ldots,w_n}(z)$ is the polynomial of degree $n+1$ which is the primitive of $\Omega_{w_1,w_2,\ldots,w_n}$ vanishing at w_0.

For $n \geq 0$, we write $\Omega_{n;\mathbf{w}}$ for $\Omega_{w_0,w_1,\ldots,w_{n-1}}$, a polynomial of degree n which depends only on the first n terms of the sequence $\mathbf{w}$.

By induction we deduce that the leading term of $\Omega_{n;\mathbf{w}}$ is $(1/n!)z^n$.

Starting from $\Omega_{w_0}(w_0)$ and using the differential equation, we deduce by induction

$$\Omega_{n;\mathbf{w}}^{(k)}(w_k) = \delta_{kn}$$

for $n \geq 0$ and $k \geq 0$. It follows that the sequence $(\Omega_{n;\mathbf{w}})_{n\geq 0}$ is the unique sequence of polynomials such that any polynomial P can be written as a finite sum

$$P(z) = \sum_{n\geq 0} P^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z).$$

In particular, for $N \geq 0$ we have

$$\frac{z^N}{N!} = \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n}\Omega_{n;\mathbf{w}}(z).$$

This gives an inductive formula defining $\Omega_{N;\mathbf{w}}$: for $N \geq 0$,

$$\Omega_{N;\mathbf{w}}(z) = \frac{z^N}{N!} - \sum_{n=0}^{N-1} \frac{1}{(N-n)!} w_n^{N-n}\Omega_{n;\mathbf{w}}(z).$$

We also have

$$\Omega_{w_0,w_1,\ldots,w_n}(z) = \Omega_{0,w_1-w_0,w_2-w_0,\ldots,w_n-w_0}(z-w_0).$$

With $w_0 = 0$, the first polynomials are given by

$$\begin{aligned}
2!\Omega_{0,w_1}(z) &= (z-w_1)^2 - w_1^2,\\
3!\Omega_{0,w_1,w_2}(z) &= (z-w_2)^3 - 3(w_1-w_2)^2 z + w_2^3,\\
4!\Omega_{0,w_1,w_2,w_3}(z) &= (z-w_3)^4 - 6(w_2-w_3)^2(z-w_1)^2\\
&\quad - 4(w_1-w_3)^3 z + 6w_1^2(w_2-w_3)^2 - w_3^4.
\end{aligned}$$

Let us check that these polynomials are also given by the following determinant

$$\Omega_{w_0,w_1,\dots,w_{n-1}}(z) = (-1)^n \begin{vmatrix} 1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{n-1}}{(n-1)!} & \frac{z^n}{n!} \\ 1 & \frac{w_0}{1!} & \frac{w_0^2}{2!} & \cdots & \frac{w_0^{n-1}}{(n-1)!} & \frac{w_0^n}{n!} \\ 0 & 1 & \frac{w_1}{1!} & \cdots & \frac{w_1^{n-2}}{(n-2)!} & \frac{w_1^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & \cdots & \frac{w_2^{n-3}}{(n-3)!} & \frac{w_2^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{w_{n-1}}{1!} \end{vmatrix}.$$

Indeed, the right hand side is a polynomial of degree n, vanishing at w_0. Its derivative is obtained by replacing the first row with its derivative, namely

$$\begin{pmatrix} 0 & 1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{n-1}}{(n-1)!} \end{pmatrix}.$$

The determinant that we get reduces to a similar determinant as above but with $w_0, w_1, \dots, w_{n-1}$ replaced with $w_1, \dots, w_{n-1}$. Hence the sequence of determinants satisfies the differential equation characteristic of the sequence $(\Omega_{n;\mathbf{w}})_{n\ge 0}$.

• With the sequence $w_n = 0$ for all $n \ge 0$, we get Taylor polynomials

$$\Omega_{n;\mathbf{w}}(z) = \frac{z^n}{n!}\cdot$$

• With the sequence $\mathbf{w} = (1, 0, 1, 0, \dots, 0, 1, \dots)$, that is $w_n = 1$ for even $n \ge 0$, $w_n = 0$ for odd $n \ge 1$, we recover the Whittaker polynomials

$$\Omega_{2n;\mathbf{w}}(z) = M_n(z), \quad \Omega_{2n+1,\mathbf{w}}(z) = M'_{n+1}(z-1).$$

• With the arithmetic progression

$$(a, a+t, a+2t, \dots, a+nt, \dots),$$

$\mathbf{w} = (a + nt)_{n\ge 0}$ with a in $\mathbb{C}$ and t in $\mathbb{C} \setminus \{0\}$, we get the sequence of Abel polynomials

$$\Omega_{n;\mathbf{w}}(z) = \frac{1}{n!}(z-a)(z-a-nt)^{n-1}$$

for $n \ge 1$. In particular for $a = 0$, $t = 1$, the sequence is $\mathbf{w} = (0, 1, 2, 3, \dots, n, \dots)$, namely $w_n = n$ for all $n \ge 0$, this is

$$\Omega_{n;\mathbf{w}}(z) = \frac{1}{n!} z(z-n)^{n-1}$$

for $n \ge 1$.

Références

[Gontcharoff 1930] W. Gontcharoff, "Recherches sur les dérivées successives des fonctions analytiques. Généralisation de la série d'Abel", *Ann. Sci. École Norm. Sup.* (3) **47** (1930), 1–78. MR Zbl

[Macintyre 1954] A. J. Macintyre, "Interpolation series for integral functions of exponential type", *Trans. Amer. Math. Soc.* **76** (1954), 1–13. MR Zbl