On some families of binary forms and the integers they represent

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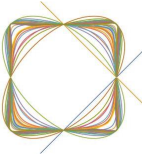


http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

An asymptotic estimate for the number of integers which are represented by a given binary form is due to Landau, Ramanujan and Bernays for positive definite quadratic forms and more recently by Stewart and Xiao for binary forms of higher degree. The purpose of this lecture is to consider the same problem for families of binary forms. In a joint work with E. Fouvry and C. Levesque, we gave an asymptotic estimate for the number of integers which are represented by a cyclotomic form. With E. Fouvry we pursued this study for other families of binary forms.

Étienne Fouvry

Claude Levesque

EF+CL+MW, *Representation of integers by cyclotomic binary forms.* Acta Arithmetica, **184**.1 (2018), 67 – 86. DOI: 10.4064/aa171012-24-12
arXiv: 712.09019 [math.NT]

EF+MW, *Sur la représentation des entiers par des formes cyclotomiques de grand degré.* Bull. Soc. Math. France, **148** (2020), 253–282. DOI: 0.24033/bsmf.2805
arXiv: 1909.01892 [math.NT]

Sums of two squares

A prime number is a sum of two squares if and only if it is either 2 or else congruent to 1 modulo 4.

$$2, 5, 13, 17, 29, 37, 41, 53, 61, 73 \ldots$$

https://oeis.org/A002313

Pierre de Fermat
1607 (?) – 1665

The product of a sum of two squares is a sum of two squares.

Identity of Brahmagupta :

$$(a^2 + b^2)(c^2 + d^2) = e^2 + f^2$$

with either

$$e = ac - bd, \; f = ad + bc$$

or

$$e = ac + bd, \; f = ad - bc.$$

Brahmagupta
598 – 668

Sums of two squares

A positive integer is a sum of two squares if and only if each prime divisor congruent to 3 modulo 4 occurs with an even exponent.

Sums of two squares https://oeis.org/A001481

$$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37 \dots$$

Not sums of two squares https://oeis.org/A022544

$$3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 38 \dots$$

For $N \geq 1$, the number of (x, y) with $x^2 + y^2 \leq N$ is $> N$ (take $\max\{|x|, |y|\} \leq \sqrt{N}$). If an integer m is a sum of two squares, there are many solutions (x, y) to the equation $x^2 + y^2 = m$: if m has s prime divisors which are congruent to 1 modulo 4, there are at least 2^{s-1} solutions.

The Landau–Ramanujan constant

Edmund Landau

1877 – 1938

Srinivasa Ramanujan

1887 – 1920

The number of positive integers $\leq N$ which are sums of two squares is asymptotically $\mathsf{C}_{\Phi_4} N(\log N)^{-\frac{1}{2}}$, where [OEIS A064533]

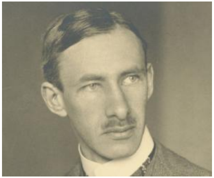
$$\mathsf{C}_{\Phi_4} = \frac{1}{2^{\frac{1}{2}}} \cdot \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} = 0.764\,223\,653\,589\,220\ldots$$

Asymptotic expansion for the number of sums of two squares :
There exist real numbers $\alpha_1, \alpha_2, \ldots$ such that, for any $M \geq 0$, the number of positive integers $\leq N$ which are sums of two squares is asymptotically

$$\frac{N}{\sqrt{\log N}} \left\{ \mathsf{C}_{\Phi_4} + \frac{\alpha_1}{\log N} + \cdots + \frac{\alpha_M}{(\log N)^M} + O\left(\frac{1}{(\log N)^{M+1}}\right)\right\}.$$

Positive definite quadratic forms

Let $F(X,Y) = aX^2 + bXY + cY^2 \in \mathbb{Z}[X,Y]$ be a quadratic form with nonsquare positive discriminant $b^2 - 4ac$. There exists a positive constant C_F such that, for $N \to \infty$, the number of positive integers $m \in \mathbb{Z}$, $m \leq N$ which are represented by F is asymptotically $\mathsf{C}_F N(\log N)^{-\frac{1}{2}}$.



Paul Bernays
1888 – 1977

P. Bernays, *Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht quadratischen Diskriminante*, Ph.D. dissertation,
Advisor : Edmund Landau,
Georg-August-Universität, Göttingen, Germany, 1912.

http://www.ethlife.ethz.ch/archive_articles/120907_bernays_fm/

Earlier results on binary quadratic forms :

Fermat, Lagrange, Legendre, Gauss.

Paul Bernays (1888 – 1977)

https://www.thefamouspeople.com/profiles/paul-bernays-7244.php

• 1912, Ph.D. in mathematics, University of Göttingen, *On the analytic number theory of binary quadratic forms* (Advisor : Edmund Landau).

• 1913, Habilitation, University of Zürich, *On complex analysis and Picard's theorem*, advisor Ernst Zermelo.

• 1912 – 1917, Zürich ; work with Georg Pólya, Albert Einstein, Hermann Weyl.

• 1917 – 1933, Göttingen, with David Hilbert. Studied with Emmy Noether, Bartel Leendert van der Waerden, Gustav Herglotz.

• 1935 – 1936, Institute for Advanced Study, Princeton. Lectures on mathematical logic and axiomatic set theory.

• 1936 —, ETH Zürich.

• With David Hilbert, "Grundlagen der Mathematik" (1934 – 39) 2 vol. — Hilbert–Bernays paradox.

• Axiomatic Set Theory (1958). —
Von Neumann–Bernays–Gödel set theory.

Higher degree

If a positive integer m is represented by a given quadratic form, there are many such representations.
A quadratic form has infinitely many automorphisms, an irreducible binary form of higher degree has a finite group of automorphisms :

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}), \quad F(X_1, X_2) = F(u_1X_1+u_2X_2, u_3X_1+u_4X_2).$$

Stanley Yao Xiao

S. Yao Xiao, *On the representation of integers by binary quadratic forms*. `arXiv:1704.00221`

$$x^k + y^k = m$$

Given an integer $k \geq 3$, that a positive integer is a sum of two k–th powers in more than one way (not counting symmetries) is

- rare for $k = 3$,
- extremely rare for $k = 4$,
- maybe impossible for $k \geq 5$.

1729 : the taxicab number

The smallest positive integer which is sum of two cubes in two essentially different ways :

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Godfrey Harold Hardy

1877–1947

Srinivasa Ramanujan

1887 – 1920

1657 : Frénicle de Bessy (1605 ? – 1675).

Hardy (1917) : 1729 is a rather dull number.
Littlewood : every positive integer was one of Ramanujan's personal friends.

http://www.mathpages.com/home/kmath028/kmath028.htm

Beginning at the 1729th decimal digit of the transcendental number e, the next ten successive digits are 0719425863. It is the first occurrence of all ten digits consecutively in the decimal representation of e.

The sequence of Taxicab numbers

[OEIS A001235] `Taxi-cab numbers: sums of` 2 `cubes in more than` 1 `way.`

$$1729 = 10^3 + 9^3 = 12^3 + 1^3, \quad 4104 = 2^3 + 16^3 = 9^3 + 15^3, \dots$$

$$1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, \dots$$

If n is in this sequence, then nk^3 also, hence this sequence is infinite.

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] `Hardy-Ramanujan numbers:` $\mathrm{Ta}(n)$ `is the smallest number that is the sum of` 2 `positive integral cubes in` n `ways.`

`http://mathworld.wolfram.com/TaxicabNumber.html`

$\mathrm{Ta}(1) = 2$,
$\mathrm{Ta}(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3$,
$\mathrm{Ta}(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$,
$\mathrm{Ta}(4) = 6\,963\,472\,309\,248$,
$\mathrm{Ta}(5) = 48\,988\,659\,276\,962\,496$.

2003 : C. S. Calude, E. Calude and M. J. Dinneen. With high probability,

$$\mathrm{Ta}(6) = 24\,153\,319\,581\,254\,312\,065\,344.$$

Fermat proved that numbers expressible as a sum of two positive integral cubes in n different ways exist for any n.

Hardy and Wright, *An Introduction to the Theory of Numbers*, 1938.

Pierre de Fermat
1607 (?) – 1665

Cubefree taxicab numbers

$$15\,170\,835\,645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3.$$

The smallest cubefree taxicab number with three representations was discovered by Paul Vojta in 1981 while he was a graduate student.

Paul Vojta

Stuart Gascoigne and Duncan Moore (2003) :
$1\,801\,049\,058\,342\,701\,083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3 = 600259^3 + 1165884^3.$

[OEIS A080642] `Cubefree taxicab numbers: the smallest cubefree number that is the sum of` 2 `cubes in` n `ways.`

https://en.wikipedia.org/wiki/Taxicab_number

Taxicabs and Sums of Two Cubes

If the sequence (a_n) of cubefree taxicab numbers with n representations is infinite, then the Mordell-Weil rank of the elliptic curve $x^3 + y^3 = a_n$ tends to infinity with n.

Joseph Silverman

J. H. Silverman, Taxicabs and Sums of Two Cubes, Amer. Math. Monthly, **100** (1993), 331-340.

$$635\,318\,657 = 158^4 + 59^4 = 134^4 + 133^4.$$

Leonhard Euler

1707 – 1783

The smallest integer represented by $x^4 + y^4$ in two essentially different ways was found by Euler, it is $635\,318\,657 = 41 \times 113 \times 241 \times 569$.

[OEIS A216284] Number of solutions to the equation $x^4 + y^4 = n$ with $x \geq y > 0$.

An infinite family with one parameter is known for non trivial solutions to $x_1^4 + x_2^4 = x_3^4 + x_4^4$ (N. Elkies).

http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

Sums of two higher powers

A necessary and sufficient condition for a prime number to be a sum of two squares is given by a congruence.
For $k \geq 3$, there are not enough primes of the form $x^k + y^k$.
[OEIS A334520] `Primes that are the sum of two cubes.`

$$2, 7, 19, 37, 61, 127, 271, 331, 397, 547, 631, 919, 1657, \dots$$

($7 = 2^3 + (-1)^3$).

We believe this list to be infinite, but this is not known.

For an odd integer which is a sum of two 4th powers, each prime number not congruent to 1 modulo 8 has an even exponent. This necessary condition is not sufficient.

[OEIS A004831] `Numbers that are the sum of at most` 2 `nonzero` 4`th powers.`

$$0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, \dots$$

Quartan primes

[OEIS A002645] Quartan primes: primes of the form $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers which are sums of two 4th powers starts with $2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177,$
$4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561,$
$28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161,$
$66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \ldots$

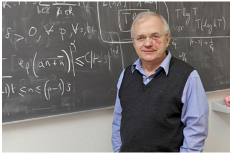
It is not known whether this list is finite or not.

The largest known quartan prime is currently the largest known generalized Fermat prime: The $1\,353\,265$-digit $(145\,310^{65\,536})^4 + 1^4$.

[OEIS A002313] primes of the form $x^2 + y^2$,
[OEIS A002645] primes of the form $x^4 + y^4$,
[OEIS A006686] primes of the form $x^8 + y^8$,
[OEIS A100266] primes of the form $x^{16} + y^{16}$,
[OEIS A100267] primes of the form $x^{32} + y^{32}$.

Primes of the form $X^2 + Y^4$ or $X^3 + 2Y^3$

John Friedlander



Henryk Iwaniec

Roger Heath-Brown

However, it is known that there are infinitely many prime numbers of the form $X^2 + Y^4$ and also infinitely many prime numbers of the form $X^3 + 2Y^3$ – **with the expected asymptotic order !**

Friedlander, J. & Iwaniec, H. *The polynomial* $X^2 + Y^4$ *captures its primes*, Ann. of Math. (2) **148** (1998), no. 3, 945–1040. [A028916]

Heath-Brown, D. R. *Primes represented by* $x^3 + 2y^3$, Acta Mathematica **186** (2001), 1–84. [A173587]

Representation of integers by a binary form of degree ≥ 3

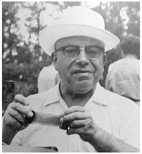
Let F be a binary form of degree $d \geq 3$ with nonzero discriminant. For $N \geq 1$ denote by $R_F(N)$ the number of integers of absolute value at most N which are represented by $F(X, Y)$.
Expected : $R_F(N) \sim \mathsf{C}_F N^{2/d}$.
For $Z > 0$, the number $\mathsf{N}_F(Z)$ of $(x, y) \in \mathbb{Z}^2$ such that $0 < |F(x, y)| \leq Z$ satifies

$$\mathsf{N}_F(Z) = \mathsf{A}_F Z^{\frac{2}{d}} + O(Z^\theta)$$

as $Z \to \infty$, where A_F is the area (Lebesgue measure) of the domain

$$\{(x, y) \in \mathbb{R}^2 \mid F(x, y) \leq 1\}.$$

$\theta = \frac{1}{d}$ if F does not have a linear factor in $\mathbb{R}[X, Y]$, $\theta = \frac{1}{d-1}$ otherwise.



Kurt Mahler
1903 – 1988

Über die mittlere Anzahl der Darstellungen grosser Zahlen durch binäre Formen,
Acta Math. **62** (1933), 91–166.
https://carma.newcastle.edu.au/mahler/biography.html

Representation of integers by a binary form of degree 3 or 4

Cubic forms : $R_F(N) \sim \mathsf{C}_F N^{2/3}$

- On binary cubic forms,
  J. reine angew. Math. 226
  (1967), 30–87.
  irreducible binary cubic forms,
  discriminant not a square :
  automorphism group C_1
- On binary cubic forms : II,
  J. reine angew. Math. 521
  (2000), 185–240.
  irreducible binary cubic forms,
  discriminant a square :
  automorphism group
  conjugate to C_3

Quartic forms : $R_F(N) \sim \mathsf{C}_F N^{1/2}$

Christopher Hooley
1928 – 2018

- On binary quartic forms,
  J. reine angew. Math. 366
  (1986), 32–52.
  irreducible binary quartic
  forms $ax^4 + bx^2y^2 + cy^4$:
  automorphism group
  conjugate to either D_2 or D_4.

Other binary form of degree ≥ 3

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant. Recall

$$R_F(N) = \#\{m \mid 1 \leq m \leq N, \text{ there exists } (x,y) \in \mathbb{Z}^2,\ F(x,y) = m\}.$$

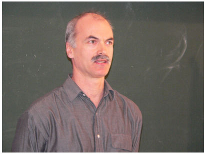
Hooley (1967), Greaves (1994), Skinner and Wooley (1995), Wooley (1995), Heath-Brown (1997) and Browning (2002) have obtained asymptotic estimates for $R_F(N)$ when $F(X,Y)$ is of the form $X^d + Y^d$ with $d \geq 3$.

Bennett, Dummigan and Wooley (1998) have obtained an asymptotic estimate for $R_F(N)$ when $F(X,Y) = aX^d + bY^d$ with $d \geq 3$ and a and b non-zero integers.

Representation of integers by a binary form of degree ≥ 3

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant. There exists $\mathsf{C}_F > 0$ and $\beta_d < \frac{2}{d}$ such that for $N \to \infty$, the number $R_F(N)$ of integers of absolute value at most N which are represented by $F(X,Y)$ satisfies

$$R_F(N) = \mathsf{C}_F N^{\frac{2}{d}} + O(N^{\beta_d}), \qquad \mathsf{C}_F = \mathsf{A}_F W_F.$$



Cam Stewart

Stanley Yao Xiao

W_F depends on the group of automorphisms of F and A_F is the area of the fundamental domain $\{(x,y) \in \mathbb{R}^2 \mid F(x,y) \leq 1\}$.

C.L. Stewart and S. Yao Xiao, *On the representation of integers by binary forms*, Math. Ann. **375** (2019), 133–163.

`DOI: 10.4064/aa171012-24-12` `arXiv:1605.03427v2` https://www.ntwebseminar.org/previous-talks

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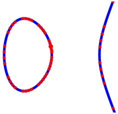

Number Theory Web Seminar

This number theory seminar is purely online. Our talks come from various corners of the field and of the world. They are streamed live on Zoom.

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- Mike Bennett (University of British Columbia)
- Philipp Habegger (University of Basel)
- Alina Ostafe (UNSW Sydney)

There are no fees, but **registration** is necessary. To register please follow this link.

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Please note the usual dates of the seminar in the FAQ Section. Please also note the varying times in the Tuesday slot, to accommodate more time zones.

Talks are usually 50 minutes and then time for some questions.

Contact: ntweb.seminar@gmail.com

Cameron L. Stewart, *On integers represented by binary forms* (University of Waterloo)

Thursday, October 15, 2020 *(8am PDT, 11am EDT, 4pm BST, 5pm CEST, 6pm IDT, 8:30pm IST, 11pm China Standard Time)*
Friday, October 16, 2020 *(2am AEDT, 4am NZDT)*

Abstract: We shall discuss the following results which are joint work with Stanley Xiao.

Let F(x,y) be a binary form with integer coefficients, degree d(>2) and non-zero discriminant. There is a positive number C(F) such that the number of integers of absolute value at most Z which are represented by F is asymptotic to C(F)Z^(2/d).

Let k be an integer with k>1 and suppose that there is no prime p such that p^k divides F(a,b) for all pairs of integers (a,b). Then, provided that k exceeds 7d/18 or (k,d) is (2,6) or (3,8), there is a positive number C(F,k) such that the number of k-free integers of absolute value at most Z which are represented by F is asymptotic to C(F,k)Z^(2/d).

Link to recording (118MB)

Link to lecture notes (PDF)

Link to lecture notes (PDF)

ON THE REPRESENTATION OF INTEGERS BY BINARY FORMS

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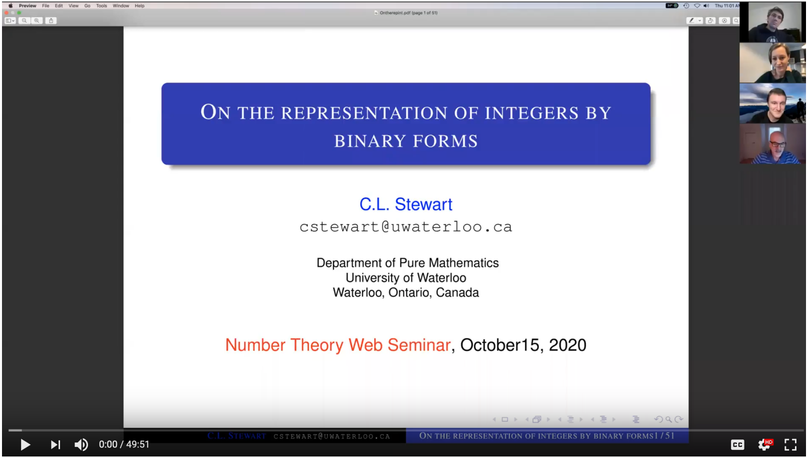
Number Theory Web Seminar, October15, 2020

Link to recording (118MB)



Cyclotomic polynomials

Recall the cyclotomic polynomials, defined by induction :

$$\phi_1(t) = t - 1, \qquad t^n - 1 = \prod_{d|n} \phi_d(t), \qquad \phi_n(t) = \frac{t^n - 1}{\prod_{\substack{d \neq n \\ d|n}} \phi_d(t)}.$$

$$\phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1,$$

$$\phi_2(t) = t + 1, \quad \phi_3(t) = t^2 + t + 1, \quad \phi_5(t) = t^4 + t^3 + t^2 + t + 1,$$

$$\phi_4(t) = t^2 + 1, \quad \phi_6(t) = t^2 - t + 1, \quad \phi_8(t) = t^4 + 1, \quad \phi_{12}(t) = t^4 - t^2 + 1.$$

Also, for m odd,

$$\phi_{2m}(t) = \phi_m(-t).$$

The degree of $\phi_n(t)$ is $\varphi(n)$, where φ is the Euler totient function.

Cyclotomic forms

For $n \geq 1$, define

$$\Phi_n(X,Y) = Y^{\varphi(n)}\phi_n(X/Y).$$

This is a binary form in $\mathbb{Z}[X,Y]$ of degree $\varphi(n)$.

$$\Phi_1(X,Y) = X - Y, \quad \Phi_2(X,Y) = X + Y,$$

$$\Phi_3(X,Y) = X^2 + XY + Y^2, \quad \Phi_4(X,Y) = X^2 + Y^2,$$

$$\Phi_6(X,Y) = \Phi_3(X,-Y) = X^2 - XY + Y^2,$$

$$\Phi_5(X,Y) = X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4,$$

$$\Phi_8(X,Y) = X^4 + Y^4, \quad \Phi_{12}(X,Y) = X^4 - X^2Y^2 + Y^4,$$

$$\Phi_{10}(X,Y) = \Phi_5(X,-Y) = X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4.$$

Integers represented by a given cyclotomic form Φ_n

The result of Stewart and Xiao gives, for the number $R_{\Phi_n}(N)$ of integers $m \le N$ represented by Φ_n for a given n with $\varphi(n) = d \ge 4$,

$$R_{\Phi_n}(N) = \mathsf{C}_{\Phi_n} N^{\frac{2}{d}} + O_\epsilon(N^{\beta_d+\epsilon}) \quad \text{with} \quad \mathsf{C}_{\Phi_n} = w_n \mathsf{A}_{\Phi_n}.$$

Here

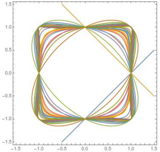
$$\beta_d = \begin{cases} \frac{3}{d\sqrt{d}} & \text{for } d = 4, 6, 8, \\ \frac{1}{d} & \text{for } d \ge 10 \end{cases} \quad \text{and} \quad \mathsf{A}_{\Phi_n} = \iint_{\Phi_n(x,y)\le 1} \mathrm{d}x\mathrm{d}y.$$

The group of automorphisms of Φ_n is isomorphic either to the dihedral group $\mathbb{D}_2$ with 4 elements or to the dihedral group $\mathbb{D}_4$ with 8 elements :

$$\mathrm{Aut}\,\Phi_n = \begin{cases} \mathbb{D}_4 & \text{if } 4 \text{ divides } n, \\ \mathbb{D}_2 & \text{otherwise}, \end{cases} \qquad w_n = \begin{cases} \frac{1}{8} & \text{if } 4 \text{ divides } n, \\ \frac{1}{4} & \text{otherwise}. \end{cases}$$

The cyclotomic fundamental domain

$\Phi_n(x,y) = 1,\ 1 \le n \le 40$

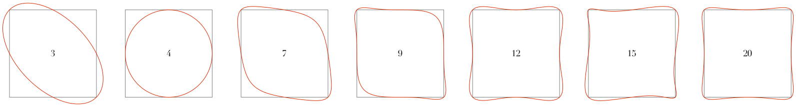


The cyclotomic fundamental domain of the binary form Φ_n is

$\mathcal{O}_n = \{(x,y) \in \mathbb{R}^2 \mid \Phi_n(x,y) \le 1\}.$

Let $\varepsilon > 0$. There exists $n_0 = n_0(\varepsilon)$ such that, for $n \ge n_0$, $\mathcal{O}_n$ contains the square centered at the origin with side $2 - n^{-1+\varepsilon}$ and is contained in the square centered at the origin with side $2 + n^{-1+\varepsilon}$.
Hence

$$\lim_{n\to\infty} \mathsf{A}_{\Phi_n} = 4.$$



The cyclotomic fundamental domain $\mathcal{O}_n$ is convex if and only if n is either a prime, or twice a prime, or a power of 2.

Numbers represented by cyclotomic forms of degree ≥ 2

Theorem 1. *The number of integers $m \leq N$ which are represented by at least one of the binary cyclotomic forms $\Phi_n(X,Y)$ with $n \geq 3$ is asymptotically*

$$\alpha \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{\log N}\right)$$

as $N \to \infty$.

The main term

$$\alpha \frac{N}{\sqrt{\log N}} \quad \text{with} \quad \alpha = \mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3} = 1.403\,133\,059\,034\ \ldots$$

occurs from the contributions of the quadratic forms Φ_4 and Φ_3.

The next term

$$-\beta \frac{N}{(\log N)^{\frac{3}{4}}} \quad \text{with} \quad \beta = 0.302\,316\,142\,357\ldots$$

occurs from the contribution of the numbers which are represented by the form Φ_4 and also by the form Φ_3.

The error term is sharp ; it takes into account all binary cyclotomic forms of degree ≥ 4.

The quadratic form $\Phi_3(X,Y) = X^2 + XY + Y^2$

A prime number is represented by the quadratic form $X^2 + XY + Y^2$ if and only if it is either 3 or else congruent to 1 modulo 3. The quadratic form $X^2 + 3Y^2$ represents the same numbers.

Primes of the form $3m + 1$: https://oeis.org/A002476

$$7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109 \dots$$

Product of two numbers represented by the quadratic form $X^2 + XY + Y^2$:

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = e^2 + ef + f^2$$

with

$$e = ac - bd, \ f = ad + bd + bc.$$

The quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1 + \zeta_3 + {\zeta_3}^2 = 0$:

$$a^2 + ab + b^2 = \mathrm{Norm}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3 b).$$

Loeschian numbers

An integer $m \geq 1$ can be written as

$$m = \Phi_3(x, y) = \Phi_6(x, -y) = x^2 + xy + y^2$$

if and only if the prime divisors of m congruent to 2 modulo 3 occur with an even exponent.

Numbers represented by the quadratic form $X^2 + XY + Y^2$:
https://oeis.org/A003136

$$0, 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31 \dots$$

Numbers not represented by the quadratic form $X^2 + XY + Y^2$:
https://oeis.org/A034020

$$2, 5, 6, 8, 10, 11, 14, 15, 17, 18, 20, 22, 23, 24, 26, 29, 30 \dots$$

Asymptotic expansion for Loeschian numbers

The number of positive integers $\leq N$ which are represented by the quadratic form $X^2 + XY + Y^2$ is asymptotically $\mathsf{C}_{\Phi_3} N(\log N)^{-\frac{1}{2}}$, where

$$\mathsf{C}_{\Phi_3} = \frac{1}{2^{\frac{1}{2}} 3^{\frac{1}{4}}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

[OEIS A301429] `Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers.`

The first decimal digits of C_{Φ_3} are

$$\mathsf{C}_{\Phi_3} = 0.638\,909\,405\,445\,343\,882\,254\,942\,674\ldots$$

There exist real numbers $\alpha'_1, \alpha'_2, \ldots$ such that, for any $M \geq 0$, the number of positive integers $\leq N$ which are represented by the form $X^2 + XY + Y^2$ is asymptotically

$$\frac{N}{(\log N)^{\frac{1}{2}}} \left\{ \mathsf{C}_{\Phi_3} + \frac{\alpha'_1}{\log N} + \cdots + \frac{\alpha'_M}{(\log N)^M} + O\left(\frac{1}{(\log N)^{M+1}}\right) \right\}.$$

S. Ettahri, O. Ramare, L.Surel. Fast multi-precision computation of some Euler products. https://arxiv.org/abs/1908.06808v1

Loeschian numbers which are sums of two squares

An integer $m \geq 1$ is simultaneously of the forms

$$m = \Phi_4(x, y) = x^2 + y^2 \text{ and } m = \Phi_3(u, v) = u^2 + uv + v^2$$

if and only if its prime divisors not congruent to 1 modulo 12 occur with an even exponent.

Sequence : https://oeis.org/A155563

$$1, 4, 9, 13, 16, 25, 36, 37, 49, 52, 61, 64, 73, 81, 97, 100 \dots$$

The number of Loeschian integers $\leq N$ which are sums of two squares is asymptotically

$$\frac{N}{(\log N)^{\frac{3}{4}}} \left\{ \beta + \frac{\alpha_1''}{\log N} + \cdots + \frac{\alpha_M''}{(\log N)^M} + O\left(\frac{1}{(\log N)^{M+1}}\right) \right\}.$$

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2 + \sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5, 7, 11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

$$\beta = 0.302\,316\,142\,357\,065\,637\,947\,769\,900 \dots$$ [OEIS A301430]

S. Ettahri, O. Ramare, L.Surel. Fast multi-precision computation of some Euler products. https://arxiv.org/abs/1908.06808v1

The error term in Theorem 1

Recall Theorem 1 which gives an asymptotic estimate for the number of integers $m \leq N$ which are represented by one at least of the binary cyclotomic forms $\Phi_n(X,Y)$ of degree ≥ 2.

Theorem 1. *The number of integers $m \leq N$ which are represented by at least one of the binary cyclotomic forms $\Phi_n(X,Y)$ with $n \geq 3$ is asymptotically*

$$\alpha \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{\log N}\right)$$

as $N \to \infty$.

$$\alpha = \mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3} = 1.403\,133\,059\,034\ \ldots$$

$$\beta = 0.302\,316\,142\,357\ldots$$

The error term

Any prime number p is represented by a cyclotomic binary form :

$$\Phi_{p^r}(1,1) = \phi_{p^r}(1) = \phi_{2p^r}(-1) = p \text{ for } r \geq 1 \text{ and } p \text{ an odd prime.}$$

For any $d \geq 4$ the number of integers $\leq N$ represented by one at least of the cyclotomic binary forms of degree $\geq d$ is asymptotic to the number $\pi(N)$ of primes $\leq N$.

We now count the representations $\Phi_n(x,y)$ with $\max\{|x|,|y|\} \geq 2$.

Theorem 1'. *The number of integers $m \leq N$ for which there exists $n \geq 3$ and $(x,y) \in \mathbb{Z}^2$ with $\max(|x|,|y|) \geq 2$ and $m = \Phi_n(x,y)$, is asymptotically*

$$\alpha \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right)$$

as $N \to \infty$.

$\mathcal{A}_d(N)$ and $\mathcal{A}_{\geq d}(N)$

Define, for $d \geq 4$,

$$\mathcal{A}_d(N) = \#\{m \mid 1 \leq m \leq N, \text{ there exists } n \geq 3 \text{ and } (x,y) \in \mathbb{Z}^2 \text{ with } \varphi(n) = d \text{ and } \Phi_n(x,y) = m\}$$

and

$$\mathcal{A}_{\geq d}(N) = \#\{m \mid 1 \leq m \leq N, \text{ there exists } n \geq 3 \text{ and } (x,y) \in \mathbb{Z}^2 \text{ with } \max\{|x|,|y|\} \geq 2,\ \varphi(n) \geq d \text{ and } \Phi_n(x,y) = m\}.$$

Theorem 1' states : *Asymptotically, as* $N \to \infty$,

$$\mathcal{A}_{\geq 2}(N) = \alpha \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right).$$

Contribution of the forms of degree ≥ 4

It remains to be shown that

$$\mathcal{A}_{\geq 4}(N) = O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right).$$

Each individual form $\Phi_n(X, Y)$ with $\varphi(n) = d \geq 4$ contributes only to the error term in Theorem 1' with $O(N^{\frac{2}{d}})$.

But there are infinitely many such forms. We need a uniform estimate ; the next one will be good enough.

Proposition 2. *Let $d \geq 4$. For $d \geq 2$ and $N \to \infty$, the number $\mathcal{A}_{\geq d}(N)$ of $m \leq N$ for which there exists n and $(x, y) \in \mathbb{Z}^2$ with $\varphi(n) \geq d$, $\max(|x|, |y|) \geq 2$ and $m = \Phi_n(x, y)$ is bounded by*

$$29N^{\frac{2}{d}}(\log N)^{1.161}.$$

Lower bound for norm forms of CM fields

For $n \geq 3$, the polynomial $\phi_n(t)$ has integer coefficients, hence real coefficients, and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).

For $n \geq 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \geq 2^{-\varphi(n)} \max\{1, |t|\}^{\varphi(n)}.$$

K. Győry

L. Lovász

K. Győry & L. Lovász, *Representation of integers by norm forms II*, Publ. Math. Debrecen **17**, 173–181, (1970).

K. Győry, *Représentation des nombres entiers par des formes binaires*, Publ. Math. Debrecen **24**, 363–375, (1977).

For $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$,

$$\Phi_n(x, y) \geq 2^{-\varphi(n)} \max\{|x|, |y|\}^{\varphi(n)}.$$

Lower bound for $\phi_n(t)$

The lower bound $\Phi_n(x,y) \geq 2^{-\varphi(n)} \max\{|x|,|y|\}^{\varphi(n)}$ is useful only if $\max\{|x|,|y|\} \geq 3$. We need a refinement of the result of K. Győry & L. Lovász for the special case of cyclotomic forms.

Proposition 3. For $n \geq 3$,

$$\inf_{t\in\mathbb{R}} \phi_n(t) \geq \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)}.$$

Hence

$$\Phi_n(x,y) \geq \left(\frac{\sqrt{3}}{2} \max\{|x|,|y|\}\right)^{\varphi(n)}.$$

Corollary. *Let m be a positive integer and let n, x, y be rational integers satisfying $n \geq 3$, $\max(|x|,|y|) \geq 2$ and $\Phi_n(x,y) = m$. Then*

$$\max\{|x|,|y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \textit{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

As a consequence, n is bounded

$$n < 5.383(\log\, m)^{1.161}.$$

Numbers represented by two nonisomorphic binary forms of the same degree

Two binary forms F_1 and F_2 are *isomorphic* if there exists $\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$ such that $F_1(X_1, X_2) = F_2(u_1X_1 + u_2X_2, u_3X_1 + u_4X_2)$.
For $B \geq 2$, let $\mathcal{N}_{F_1,F_2}(B)$ be the number of elements in the set

$$\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid \max_{i=1,2,3,4} |x_i| \leq B,\ F_1(x_1, x_2) = F_2(x_3, x_4)\right\}.$$

Theorem 2. *Let F_1 and F_2 be two nonisomorphic binary forms of the same degree $d \geq 3$. Assume that their discriminants are nonzero. Then for any $\varepsilon > 0$ we have*

$$\mathcal{N}_{F_1,F_2}(B) = O(B^{\gamma_d+\varepsilon}),$$

with

$$\gamma_d = \begin{cases} \frac{2}{3} + \frac{73}{36\sqrt{3}} & \textit{if } d = 3, \\ \frac{1}{2} + \frac{9}{4\sqrt{d}} & \textit{if } 4 \leq d \leq 20, \\ 1 & \textit{for } d \geq 21. \end{cases}$$

Sketch of proof of Theorem 2.

The proof is based on results and ideas of Heath-Brown, Hooley, Salberger, Stewart and Xiao.

Salberger, P. – *Rational points of bounded height on projective surfaces.* Math. Z. **258**, (2008) 805 – 826.

P. Salberger

The goal is to give an upper bound for the number of integral points $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ with $\max_{i=1,2,3,4} |x_i| \le B$ on the hypersurface

$$\mathbb{X}: \qquad F_1(X_1, X_2) = F_2(X_3, X_4).$$

One estimates the number of such points for which the projective point $(x_1 : x_2 : x_3 : x_4)$ does not lie on a complex projective line contained in $\mathbb{X}$ by using a result due to P. Salberger. This produces the main term in the estimate.

Next one estimates the number of points for which the projective point $(x_1 : x_2 : x_3 : x_4)$ lies on a projective line contained in $\mathbb{X}$, and one uses an upper bound for the number of these lines. This produces an error term $O(B)$.

Numbers represented by two nonisomorphic cyclotomic binary forms of the same degree

Corollary. *Let* n_1 *and* n_2 *be two positive integers such that* $\varphi(n_1) = \varphi(n_2) = d \geq 4$. *Assume that the two cyclotomic binary forms* Φ_{n_1} *and* Φ_{n_2} *are not isomorphic. Then for any* $\varepsilon > 0$ *the number of* $m \leq N$ *such that there exists* (a, b) *and* (c, d) *with*

$$m = \Phi_{n_1}(a, b) = \Phi_{n_2}(c, d)$$

is bounded by

$$O_{d,\epsilon}(N^{\eta_d + \epsilon})$$

with

$$\eta_d = \frac{\gamma_d}{d} = \begin{cases} \frac{1}{2d} + \frac{9}{4d\sqrt{d}} & \text{if } 4 \leq d \leq 20, \\ \frac{1}{d} & \text{for } d \geq 22. \end{cases}$$

Isomorphic cyclotomic binary forms

Corollary. *For n_1 and n_2 positive integers with $n_1 < n_2$, the following conditions are equivalent :*
(1) $\varphi(n_1) = \varphi(n_2)$ and the two binary forms Φ_{n_1} and Φ_{n_2} are isomorphic.
(2) The two binary forms Φ_{n_1} and Φ_{n_2} represent the same integers.
(3) n_1 is odd and $n_2 = 2n_1$.

Proof.
We may assume $n_1 \geq 3$.
$(1) \Rightarrow (3)$ If Φ_{n_1} and Φ_{n_2} are isomorphic, the primitive roots of unity ζ_{n_1} and ζ_{n_2} are related by

$$\zeta_{n_1} = \frac{u_1\zeta_{n_2} + u_2}{u_3\zeta_{n_2} + u_4} \quad \text{with} \quad \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}),$$

hence $\mathbb{Q}(\zeta_{n_1}) = \mathbb{Q}(\zeta_{n_2})$. The torsion subgroup of $\mathbb{Q}(\zeta_n)^\times$ is cyclic of order n (resp. $2n$) if n is even (resp. odd).
$(3) \Rightarrow (2)$ For m odd, $\phi_{2m}(t) = \phi_m(-t)$.
$(2) \Rightarrow (1)$ Follows from the corollary above on $\Phi_{n_1}(a, b) = \Phi_{n_2}(c, d)$. $\square$

Even integers not represented by Euler totient function

Let us call *totient* a positive integer which is a value of Euler totient function φ. Let d be a totient and $d^\dagger$ the next totient $> d$.

Always $d + 2 \leq d^\dagger < 2d$.

The list of even integers which are not values of Euler φ function (i.e., for which $C_d = 0$) starts with

$14, 26, 34, 38, 50, 62, 68, \mathbf{74}, \mathbf{76}, 86, 90, 94, 98, 114, 118,$
$\mathbf{122}, \mathbf{124}, 134, 142, 146, \mathbf{152}, \mathbf{154}, 158, 170, 174, 182,$
$\mathbf{186}, \mathbf{188}, 194, 202, 206, 214, 218, 230, \mathbf{234}, \mathbf{236},$
$\mathbf{242}, \mathbf{244}, \mathbf{246}, \mathbf{248}, 254, 258, 266, 274, 278, \mathbf{284}, \mathbf{286},$
$290, 298, \mathbf{302}, \mathbf{304}, 308, 314, 318, \ldots$

[OEIS A005277] `Nontotients: even` n `such that` $\varphi(m) = n$ `has no solution.`

Ford, K, *The distribution of totients. Paul Erdös (1913–1996),* Ramanujan J. (**2**) (1998), no. 1–2, 67–151.
Ford, K, *The number of solutions of* $\varphi(x) = m$, Ann. of Math. (2) (**150**) (1999), no. 1, 283–311.

Numbers represented by cyclotomic forms of degree $\geq d$

Theorem 3. *Let $d \geq 4$. As $N \to \infty$, the number $\mathcal{A}_{\geq d}(N)$ of integers $m \leq N$ for which there exist n and (x, y) with $\Phi_n(x, y) = m$, $\varphi(n) \geq d$ and $\max\{|x|, |y|\} \geq 2$, is asymptotically*

$$\mathcal{A}_{\geq d}(N) = C_d N^{\frac{2}{d}} + \begin{cases} O_\epsilon(N^{\frac{13}{32}+\epsilon}) & \text{for } d = 4, \\ O(N^{\frac{2}{d^\dagger}}) & \text{for } d \geq 6, \end{cases}$$

with

$$C_d = \sum_n \mathsf{C}_{\Phi_n},$$

where the sum is over the set of integers n such that $\varphi(n) = d$ and n is not congruent to 2 modulo 4.

If $d \geq 6$ and $d^\dagger = d + 2$, then the error term is sharp.

Optimality of the error term when $d^\dagger = d+2$

Assume $d \geq 4$ and $d+2$ are totients. Then, among the $\Phi_m(u,v)$ with $\varphi(m) = d+2$, a positive proportion of them is not of the form $\Phi_n(a,b)$ with $\varphi(n) = d$: there exists $v_d > 0$ such that, for sufficiently large N,

$$\mathcal{A}_{\geq d}(N) \geq \mathcal{A}_d(N) + v_d N^{\frac{2}{d+2}}.$$

Lemma (*Confinement*). *Let $n \geq 2$ and let p be a prime number dividing n. Then for all a, b in $\mathbb{Z}$, we have*

$$\Phi_n(a,b) \equiv 0, 1 \mod p.$$

Further similar results are needed modulo 4 and 9.

EF+MW, *Sur la représentation des entiers par des formes cyclotomiques de grand degré.* Bull. Soc. Math. France, **148** 2 (2020), 253–282.

On some families of binary forms and the integers they represent

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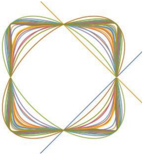

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