The Four Exponentials Problem and the Schanuel Conjecture

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Abstract

Let (*) $\boldsymbol{t}$ be a real number such that $\boldsymbol{2^t}$ and $\boldsymbol{3^t}$ are integers; does it follow that $\boldsymbol{t}$ is a nonnegative integer? A positive answer would follow from the solution of the four exponentials Problem, a very special case of the so–called *weak Schanuel's Conjecture*, namely the conjecture of algebraic independence of logarithms of algebraic numbers. These questions are open both in the complex case and in the $\boldsymbol{p}$–adic case. One of the main motivations for investigating the $\boldsymbol{p}$–adic situation is Leopoldt's Conjecture, which will be sort of a *leitmotiv* in this survey.

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1 Introduction

This Festschrift in honor of Springer's Editorial Director Dr. Catriona Byrne is a good opportunity for me to thank Catriona for her support in publishing

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*Note to the Editor: the mathematical symbols in the abstract should not be boldface. Please also remove the semi colon at the end of my e-mail address above.

a good part of my works with Springer Verlag. I had the pleasure of working with her also as an editor of this publisher. I am thankful also to the editors of this special volume of the Lectures Notes in Mathematics for their invitation to share my *favorite open problems and the ones dear to my heart, with some background and context.*

([1]) I have worked on several conjectures. The one on which I spent much more time than the others is the so–called *four exponentials Problem* [Lang (1966), Chap. II § 1 p. 11], which is also the first of the eight problems in Schneider's book on transcendental numbers [Schneider (1957)]. This question is a very special case, arguably one of the easiest unsolved cases so far, of Schanuel's Conjecture [Lang (1966), Chap. III, Historical note p. 30]. Over the years, I tried to prove Schanuel's Conjecture; since very few results are known, in the process of trying to solve it, I added some hypotheses which might help, and, quite often, after some time, I came back trying to solve the four exponentials Problem. Without success so far! It is hard to predict whether the special case of the four exponentials Problem will be solved before the very general case of Schanuel's Conjecture. As an example, the following simple looking statement is open:

Let t be a real number such that 2^t and 3^t are integers. Prove that t is a nonnegative integer.

While the four exponentials Problem is still open, a weaker statement, the *six exponentials Theorem* (Theorem 3), is known to be true; a special case is the following:

Let t be a real number and p_1, p_2, p_3 be three distinct primes. Assume that the three numbers p_1^t, p_2^t and p_3^t are integers. Then t is a nonnegative integer.

For a complete proof of this result using interpolation determinants, see [Waldschmidt (2021)].

The present paper deals only with Schanuel's Conjecture and some of its consequences, including the four exponentials Problem and the problem of algebraic independence of logarithms of algebraic numbers (of which Leopoldt's Conjecture is a special case in the p–adic case). Further conjectures (including Grothendieck's Conjecture on Abelian periods, André's Conjecture on motives, the Conjecture of Kontsevich–Zagier on periods,...) would deserve to be discussed also – see for instance [Waldschmidt (2012)].

2 Leopoldt's Conjecture on the p–adic rank of the group of units of an algebraic number field

When I started to do research in 1969 in Bordeaux, my thesis advisor, Jean Fresnel, suggested me to study Leopoldt's Conjecture [Leopoldt (1962)]. At that time, Fresnel was interested in p–adic L–functions [Amice and Fresnel (1972)] and Leopoldt's Conjecture was comparatively recent. The goal is to prove that the p–adic rank of the group of units of an algebraic number field is the same as the usual rank given by Dirichlet's unit Theorem. It amounts

[1] Note to the Editor: I do not understand why this section is not right justified.

to say that the p–adic regulator, which is defined as the usual regulator by replacing logarithms with p–adic logarithms, does not vanish. According to Fresnel, *since it amounts to prove that a determinant does not vanish, it should not be so difficult!*

For a subfield of an abelian extension of an imaginary quadratic field, the decomposition, due to Frobenius, of the *Gruppendeterminant* of the Galois group – see for instance [Fresnel (1969), Waldschmidt (1971), Kanemitsu and Waldschmidt (2013)] – shows that the regulator splits into a product of linear forms with algebraic coefficients of logarithms of algebraic numbers. As a consequence, in this special case, as shown by J. Ax [Ax (1965), Conjecture p. 587], Leopoldt's Conjecture is a consequence of the p–adic version of a conjecture of A.O. Gel'fond on the linear independence, over the field $\overline{\mathbb{Q}}$ of algebraic numbers, of $\mathbb{Q}$–linearly independent logarithms of algebraic numbers. This linear independence result in the complex case has been achieved by the seminal work of A. Baker [Baker (1966)], by means of a far reaching development of Gel'fond's method.

The p–adic analog of Baker's result was proved the year after by A. Brumer [Brumer (1967)], who therefore solved Leopoldt's Conjecture for these abelian extensions. As pointed out by Brumer, the translation to the p–adic case of transcendence methods had been worked out by J-P. Serre [Serre (1967)]. As a matter of fact, Serre was interested in an extension to several variables (in the p–adic case) of the six exponentials theorem for an application to ℓ–adic abelian representations [Serre (1968), Henniart (1982)].

In the general case, Leopoldt's Conjecture is a special case of the p–adic version of the conjecture on algebraic independence of logarithms of algebraic numbers. This is how Fresnel suggested me to study the theory of transcendental numbers.

3 Conjecture on the algebraic independence of logarithms of algebraic numbers

According to A.O. Gel'fond [Gel'fond (1952), Chap. III § 5 p.177], *one may assume . . . that the most pressing problem in the theory of transcendental numbers is the investigation of the measures of transcendence of finite sets of logarithms of algebraic numbers.* From a qualitative point of view, the statement is the following one [Lang (1966), Chap. III, Historical note p. 31], which, according to Lang, *has been conjectured for a long time (by anybody who has looked at the subject).*

Conjecture 1 (Algebraic independence of logarithms of algebraic numbers) *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbb{Q}$–linearly independent complex numbers, such that the numbers $\alpha_i = \mathrm{e}^{\lambda_i}$ $(i = 1, \ldots, n)$ are algebraic numbers. Then $\lambda_1, \ldots, \lambda_n$ are algebraically independent.*

In [Calegari and Mazur (2009), Conjecture 3.9], this conjecture is called *weak Schanuel*.

Under the assumptions of Conjecture 1, the conclusion of Baker's Theorem [Baker (1966)] is that the numbers $1, \lambda_1, \ldots, \lambda_n$ are $\overline{\mathbb{Q}}$–linearly independent, while the conclusion of Conjecture 1 is that, for any nonzero polynomial P (with rational or algebraic coefficients) in n variables, the number $P(\lambda_1, \ldots, \lambda_n)$ does not vanish.

By abuse of language, we sometimes write $\lambda_i = \log \alpha_i$ $(i = 1, \ldots, n)$; the way Conjecture 1 is stated avoids the need to select a branch of the complex logarithm. For instance with $\lambda_1 = \log 2$ and $\lambda_2 = \log 2 + 2\pi \mathrm{i}$, hence $\alpha_1 = \alpha_2 = 2$, Baker's Theorem yields the linear independence of the numbers $1, \log 2, \pi$ over $\overline{\mathbb{Q}}$, while Conjecture 1 claims that $\log 2$ and π are algebraically independent (which is not yet proved).

Conjecture 1 is true for $n = 1$: a nonzero logarithm of an algebraic number is transcendental, according to the Theorem of Hermite–Lindemann [Schneider (1957), Chap. II § 4], [Lang (1966), Chap. III Corollary 1], [Waldschmidt (1974), Th. 3.1.1], [Waldschmidt (2000), Th. 1.2]. This is essentially the only case where Conjecture 1 has been proved. Under the assumptions of Conjecture 1, the conclusion should be that the transcendence degree of the field $\mathbb{Q}(\lambda_1, \ldots, \lambda_n)$ is n. As a matter of fact, it is not yet known that the field generated by all logarithms of all nonzero algebraic numbers has a transcendence degree over $\mathbb{Q}$ at least 2. However, for a conjecture which is equivalent to Conjecture 1, half of the result is proved (see inequalities (2) in § 5 and (3) in § 6). Hence, depending on the point of view, one may consider that we are half way on proving Conjecture 1.

4 Four exponentials Problem and six exponentials Theorem

We would like to solve at least some special cases of Conjecture 1. For instance we would like to prove that there are no algebraic relations like

$$(\log \alpha_1)^2 = \log \alpha_2$$

involving nonzero logarithms of algebraic numbers $\log \alpha_1$ and $\log \alpha_2$. Very few results are known even for this very specific case. We will mainly work with homogeneous relations; among the simplest nonlinear ones is the following:

$$(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3),$$

which amounts to consider the vanishing of the determinant

$$\begin{vmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \alpha_3 & \log \alpha_4 \end{vmatrix}. \tag{1}$$

Here, $\log \alpha_i$ denote complex numbers such that $\alpha_i = \mathrm{e}^{\log \alpha_i}$ are algebraic numbers. The four exponentials Problem states that such a determinant can vanish if and only if either the two rows are linearly dependent over the rational number field $\mathbb{Q}$, or the two columns are linearly dependent over $\mathbb{Q}$. Since a 2×2 matrix has rank ≤ 1 if and only if it can be written as

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix},$$

an equivalent form of the four exponentials problem is the following:

Conjecture 2 (Four exponentials Problem) *Let x_1, x_2 be two complex numbers which are linearly independent over $\mathbb{Q}$ and let y_1, y_2 be two complex numbers which are linearly independent over $\mathbb{Q}$. Then one at least of the four numbers*

$$\mathrm{e}^{x_1 y_1},\ \mathrm{e}^{x_1 y_2},\ \mathrm{e}^{x_2 y_1},\ \mathrm{e}^{x_2 y_2}$$

is transcendental.

Here is a sketch of proof of Conjecture 2 as a consequence of Conjecture 1 on algebraic independence of logarithms of algebraic numbers [Waldschmidt (2000) Exercise 1.8]. As pointed out by D. Roy [Roy (1995), p. 52], Conjecture 1 is equivalent to the following statement: *Let $\lambda_1, \ldots, \lambda_n$ be logarithms of algebraic numbers and let $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ be a nonzero polynomial with algebraic coefficients such that $P(\lambda_1, \ldots, \lambda_n) = 0$. Then there is a vector subspace $\mathcal{V}$ of $\mathbb{C}^n$, rational over $\mathbb{Q}$, which is contained in the set of zeroes of P and contains the point $(\lambda_1, \ldots, \lambda_n)$.* To completes the proof, one uses the fact that if $\mathcal{V}$ a vector subspace of $\mathbb{C}^4$, which is rational over $\mathbb{Q}$ and is contained in the hypersurface $z_1 z_4 = z_2 z_3$, then there exists $(a\colon b) \in \mathbb{P}_1(\mathbb{Q})$ such that $\mathcal{V}$ is included either in the plane

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4;\ a z_1 = b z_2,\ a z_3 = b z_4\}$$

or in the plane

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4;\ a z_1 = b z_3,\ a z_2 = b z_4\}.$$

This four exponentials problem was proposed explicitly by S. Lang [Lang (1966), Chap. II § 1 p. 11] and K. Ramachandra [Ramachandra (1968) p. 87–88]; it is also the first of the eight problems at the end of Schneider's book [Schneider (1957)].

The following statement is weaker than Conjecture 2 but is proved:

Theorem 3 (Six exponentials Theorem) *Let x_1, x_2 be two complex numbers which are linearly independent over $\mathbb{Q}$, and let y_1, y_2, y_3 be three complex numbers which are linearly independent over $\mathbb{Q}$. Then one at least of the six numbers*

$$\mathrm{e}^{x_1 y_1},\ \mathrm{e}^{x_1 y_2},\ \mathrm{e}^{x_1 y_3},\ \mathrm{e}^{x_2 y_1},\ \mathrm{e}^{x_2 y_2},\ \mathrm{e}^{x_2 y_3}$$

is transcendental.

Equivalently, a 2×3 matrix with entries logarithms of algebraic numbers, having its two rows linearly independent over $\mathbb{Q}$ and its three columns linearly independent over $\mathbb{Q}$, has rank 2.

The six exponentials Theorem 3 was proved by [Lang (1966), Chap. II Th. 1] and [Ramachandra (1968)]. The footnote p. 67 of [Ramachandra (1968)] reads:

> After writing this manuscript I came to know from professor C.L. Siegel that this is a result first due to Schneider and Siegel. The result is unpublished. This result may also be found in a recent paper by S. Lang, *Algebraic values of meromorphic functions*, Topology **5** (4), (1966), pp. 363–370. The results of this paper have something in common with Lang's results.

Indeed, one can infer from [Alaoglu and Erdős (1944) p. 455] that the six exponentials Theorem 3 and the four exponentials Problem (Conjecture 2) were also known to C.L. Siegel. When I met A. Selberg at a conference organized by Kai-Man Tsang in Hong Kong in December 1993, he told me that he knew the proof of the six exponentials Theorem 3, but he did not publish it because it was too easy. He said he tried to solve the four exponentials Problem (Conjecture 2), which was much more interesting, but he did not succeed.

A proof of Theorem 3 is given in [Waldschmidt (1974), Chapter 2 (Schneider's method)]. In his plenary lecture for the Journées Arithmétiques in Luminy in 1989, M. Laurent introduced a new idea for transcendence proofs, by means of interpolation determinants in place of an auxiliary function; the example he worked out was the six exponentials Theorem [Laurent (1991)]. See also [Waldschmidt (2021)] for the simplest case of rational integers.

The four exponentials Problem (Conjecture 2) has been solved under the extra assumption that the field generated by the four numbers x_1, x_2, y_1, y_2 has transcendence degree ≤ 1 [Brownawell (1974), Corollary 7], [Waldschmidt (1973), Corollary 4]. The proof uses a method of algebraic independence devised by A.O. Gel'fond. This result has been extended in [Roy and Waldschmidt (1995)], where the determinant $X_1X_4 - X_2X_3$ is replaced by any homogeneous quadratic form; for the proof, Gel'fond's criterion is replaced by a Diophantine approximation result due to Wirsing.

For an explanation of the fact that the transcendence machinery fails so far to solve the four exponentials Problem, see Corollary 8.3 of [Roy (2002)].

As mentioned above, the p–adic analog of the six exponentials Theorem has been proved by J-P. Serre [Serre (1967)]. As shown in [Roy (1993), Corollary p. 450], a positive solution of the p–adic version of the four exponentials Conjecture implies Leopoldt's Conjecture for Galois extensions of $\mathbb{Q}$ with Galois group a dihedral group of order 6, 8 or 12 (hence, in particular, it implies Leopoldt's Conjecture for number fields which are Galois extensions of $\mathbb{Q}$ of degree ≤ 7).

5 Rank of matrices

We have seen that the four exponentials Problem can be stated as the non-vanishing of the determinant (1). More generally, Conjecture 1 shows that a determinant, the entries of which are logarithms of algebraic numbers, can vanish only in *trivial* cases. A precise statement, with a definition of the meaning of *trivial*, is the following ([Roy (1995), p. 54] and [Waldschmidt (2000), Lemma 12.8]).

Definition 1 Let M be a matrix with entries in $\mathbb{C}$ and K a subfield of $\mathbb{C}$. Let $e_1, \dots, e_t$ be a basis of the K–vector space spanned by the entries of M. Hence M can be written as

$$M = M_1 e_1 + \cdots + M_t e_t,$$

where the matrices $M_1, \dots, M_t$ have entries in K. Let $X_1, \dots, X_t$ be indeterminates. The rank of the matrix $M_1 X_1 + \cdots + M_t X_t$, with coefficients in the ring $K[X_1, \dots, X_n]$ of polynomials in n variables, does not depend on the choice of the basis $e_1, \dots, e_t$ and is denoted as $r_{\mathrm{str},K}(M)$, which is called the *structural rank of* M with respect to K.

For any matrix M with complex coefficients and any field K, the upper bound $\mathrm{rk}(M) \le r_{\mathrm{str},K}(M)$ is plain. Assume now that the entries of M are logarithms of algebraic numbers. Conjecture 1 implies $\mathrm{rk}(M) = r_{\mathrm{str},K}(M)$. From the six exponentials Theorem, one deduces that when $r_{\mathrm{str}}(M) \ge 3$, then $\mathrm{rk}(M) \ge 2$. More generally, the lower bound

$$\mathrm{rk}(M) \ge \frac{1}{2} r_{\mathrm{str},\mathbb{Q}}(M) \tag{2}$$

follows from [Waldschmidt (1981)]. The lower bound (2) also holds in the p–adic case; it proves that the p–adic rank of the group of units of an algebraic number field is at least half of its usual rank [Waldschmidt (1984)].

The proof of [Waldschmidt (1981)] also yields an answer to the above mentioned question on ℓ–adic representations [Serre (1968), Henniart (1982)], while the complex version answers a question of A. Weil [Weil (1956)] on the characters of the idèle class group of an algebraic number field [Waldschmidt (1982)].

6 Strong six exponentials Theorem and strong four exponentials Problem

There is room between the four exponentials Problem and the six exponentials Theorem for a result involving five numbers [Waldschmidt (1988)]:

Theorem 4 (Five exponentials Theorem) *Let γ be a nonzero algebraic number, x_1, x_2 be two complex numbers which are linearly independent over $\mathbb{Q}$, and y_1, y_2 be* *two complex numbers which are linearly independent over* $\mathbb{Q}$. *Then one at least of the five numbers*

$$\mathrm{e}^{x_1 y_1},\ \mathrm{e}^{x_1 y_2},\ \mathrm{e}^{x_2 y_1},\ \mathrm{e}^{x_2 y_2},\ \mathrm{e}^{\gamma x_1/x_2}$$

is transcendental.

This result is weaker than the four exponentials Problem (because of the assumption $\gamma \neq 0$) but does not imply the six exponentials Theorem. A result which includes both the five and the six exponentials theorems (Theorems 4 and 3) is the next one [Waldschmidt (1990), Corollary 2.3], [Waldschmidt (1988), Corollary 2.1], [Waldschmidt (2000), § 11.3.3, example 2, p. 386].

Let x_1, x_2 be two complex numbers which are linearly independent over $\mathbb{Q}$, let y_1, y_2, y_3 be three complex numbers which are linearly independent over $\mathbb{Q}$ and let β_{ij} $(i = 1, 2,\ j = 1, 2, 3)$ be six algebraic numbers. Then one at least of the six numbers

$$\mathrm{e}^{x_1 y_1 - \beta_{11}},\ \mathrm{e}^{x_1 y_2 - \beta_{12}},\ \mathrm{e}^{x_1 y_3 - \beta_{13}},\ \mathrm{e}^{x_2 y_1 - \beta_{21}},\ \mathrm{e}^{x_2 y_2 - \beta_{22}},\ \mathrm{e}^{x_2 y_3 - \beta_{23}}$$

is transcendental.

A first generalization of this result has been achieved by D. Roy [Roy (1990)], [Roy (1992b), Corollary 2 p. 38], [Waldschmidt (2000), Corollary 11.16], who considers matrices with entries which are linear combinations, with algebraic coefficients, of 1 and of logarithms of algebraic numbers. Denote by $\mathcal{L}$ the $\overline{\mathbb{Q}}$–vector space spanned by 1 and all logarithms of all nonzero algebraic numbers. A typical element of $\mathcal{L}$ is of the form

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

with algebraic numbers α_i and β_j.

Theorem 5 (D. Roy, strong six exponentials Theorem) *Let* x_1, x_2 *be two complex numbers which are linearly independent over* $\overline{\mathbb{Q}}$ *and let* y_1, y_2, y_3 *be three complex numbers which are linearly independent over* $\overline{\mathbb{Q}}$. *Then one at least of the six numbers*

$$x_1 y_1,\ x_1 y_2,\ x_1 y_3,\ x_2 y_1,\ x_2 y_2,\ x_2 y_3$$

does not belong to $\mathcal{L}$.

The *strong four exponentials Problem* is the same statement as Theorem 5 with only two numbers y_1, y_2 instead of three.

Several consequences of the strong four exponentials Problem are stated in [Waldschmidt (2005a)][2] and corollaries of the strong six exponentials Theorem are derived in [Waldschmidt (2005b)].

[2] Erratum: The right assumption in corollary 2.12 p. 346 of Waldschmidt (2005a) is that the three numbers 1, Λ_{11} and Λ_{21} are linearly independent over the field of algebraic numbers.

A much more general statement than Theorem 5 is an extension by D. Roy [Roy (1992a)] of the lower bound (2) to matrices having entries in $\mathcal{L}$: for such a matrix,

$$\mathrm{rk}(M) \geq \frac{1}{2} r_{\mathrm{str},\overline{\mathbb{Q}}}(M) \tag{3}$$

see [Waldschmidt (2000), Th. 1.17 and Corollary 12.18].

As pointed out by D. Roy ([Roy (1992a)], [Roy (1995), Conjecture 1.1] and [Waldschmidt (2000), Lemma 12.14]), Conjecture 1 on the algebraic independence of logarithms of algebraic numbers is equivalent to the statement that if the entries of M are in $\mathcal{L}$, then the rank $\mathrm{rk}(M)$ of the matrix M is always equal to its structural rank $r_{\mathrm{str},\overline{\mathbb{Q}}}(M)$ with respect to $\overline{\mathbb{Q}}$. From this point of view, we can consider (3) as proving half of Conjecture 1.

As always, the situation is the same in the p–adic case, both for results and for conjectures. Applications to Leopoldt's Conjecture on the p–adic rank of the units of a number field have been derived in the following references: [Emsalem, Kisilevsky, and Wales (1984), Jaulent (1985), Emsalem (1987), Laurent (1989), Laurent (1990), Roy (1993)]. See also [Calegari and Mazur (2009), § 3 Remark p. 127] and [Maksoud (2022)].

7 Schanuel's Conjecture

Conjecture 1 on the algebraic independence of logarithms of algebraic numbers is a special case of Schanuel's Conjecture, which was proposed by Stephen Schanuel during a course given by Serge Lang at Columbia in the 1960's [Lang (1966), Chap. III, Historical Note, p. 30–31].

Conjecture 6 (Schanuel's Conjecture) *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$–linearly independent complex numbers. Then at least n of the $2n$ numbers*

$$x_1, \ldots, x_n, \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$$

are algebraically independent over $\mathbb{Q}$.

The conclusion is that the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(x_1, \ldots, x_n, \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n})$ is at least n. This result is known when $x_1, \ldots, x_n$ are algebraic numbers: this is the Lindemann–Weierstrass Theorem. Conjecture 1 is the special case of Conjecture 6 where the n numbers $\mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$ are assumed to be algebraic.

8 Roy's conjecture

In his plenary talk at the Journées Arithmétiques in Rome in 1999 [Roy (2001a), Roy (2001b)], D. Roy proposed a new conjecture of his own and proved the remarkable and surprising result that it is equivalent to Schanuel's Conjecture 6.

Denote by $\mathcal{D}$ the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

on the field $\mathbb{C}(X_0, X_1)$.

Conjecture 7 (Conjecture of D. Roy) *Let ℓ be a positive integer, $y_1, \ldots, y_\ell$ $\mathbb{Q}$-linearly independent complex numbers, $\alpha_1, \ldots, \alpha_\ell$ nonzero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} < u \quad \textit{and} \quad \max\{s_0, s_1+t_1\} < u < \frac{1}{2}(1+t_0+t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0, partial degree $\leq N^{t_1}$ in X_1 and height $\mathrm{H}(P_N) \leq e^N$, which satisfies

$$\left| \left(\mathcal{D}^k P_N\right)\Big(\sum_{j=1}^{\ell} m_j y_j, \prod_{j=1}^{\ell} \alpha_j^{m_j}\Big) \right| \leq \exp(-N^u)$$

for any integers k, $m_1, \ldots, m_\ell$ in $\mathbb{N}$ with $k \leq N^{s_0}$ and $\max\{m_1, \ldots, m_\ell\} \leq N^{s_1}$. Then, we have the following lower bound for the transcendence degree:

$$\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}(y_1, \ldots, y_\ell,\ \alpha_1, \ldots, \alpha_\ell) \geq \ell.$$

See also [Waldschmidt (2000), Conjecture 15.36]. Hence Schanuel's Conjecture is equivalent to a purely algebraic statement, which bears some similarity with the available criteria of algebraic independence.

The proof of the equivalence between Schanuel's Conjecture 6 and Roy's Conjecture 7 involves a new interpolation formula for holomorphic functions of two complex variables [Roy (2001a), Roy (2001b)]. Refined interpolation formulae are proved in [Roy (2002), Nguyen and Roy (2016)].

Several significant steps in the direction of Conjecture 7 were performed by D. Roy, first for the multiplicative group [Roy (2008)], next for the additive group [Roy (2010)] and then for the product of the additive group by the multiplicative group [Roy (2013)]. A refinement of Conjecture 7, again equivalent to Schanuel's Conjecture, is devised by Nguyen Ngoc Ai Van in [Nguyen (2009)]. The statement which is proved in [Nguyen and Roy (2016)] is similar to Conjecture 7 and is not restricted to the one parameter subgroup $t \mapsto (t, \exp(t))$.

In [Ghidelli (2015)], Luca Ghidelli refines the results of [Roy (2013)] and [Nguyen and Roy (2016)], replacing the total degree with multidegrees; his tool [Ghidelli (2019)] is an extension of Roy's multiplicity lemma for the resultant, using the theory of multiprojective elimination initiated by P. Philippon and developed by G. Rémond.

This original point of view of D. Roy suggests a promising approach for proving Schanuel's Conjecture: so far it is the only available strategy towards a proof of it.

Transcendence theory is going forward.

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