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Linear Forms in Two Logarithms and Schneider's Method*

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We consider an homogeneous linear form in two logarithms of algebraic numbers with algebraic coefficients:

$$\beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 .$$

The first lower bound for such a linear form was obtained by Gel'fond (1935). Baker generalized Gel'fond's method to obtain a result concerning more general linear forms. This result of Baker had such deep consequences that a lot of papers were written on this subject (see [1]); these papers have introduced very important improvements of the original method. But, up to now, the final descent [12], which is an essential characteristic of Baker's method does not enable a very precise dependence on the degree. To obtain such an estimate, we use Schneider's method, which, as far as we know, was never used in this context; this means that no derivative is involved in our proof. However we add also some of the above mentioned ideas which were introduced in connection with Baker's method (cf. in particular [1]). From this point of view, our proof ressembles that of [5].

We apply our lower bound to the simultaneous approximation of numbers (like in [5]); we give an explicit dependence on the degree in the theorem of Franklin and Schneider.

We first state our main result (§ 1) and its corollaries (§§ 1 and 2). After some lemmas (§ 3) we prove the theorem (§ 4). Finally we give the proof of the corollaries (§ 5).

1. The Main Results: A Lower Bound for Linear Forms

For this section, the notations are the following. We denote by $\beta, \alpha_1, \alpha_2$ three non-zero algebraic numbers of exact degrees D_0, D_1, D_2 respectively. Let D be the degree over $\mathbb{Q}$ of the field $K = \mathbb{Q}(\beta, \alpha_1, \alpha_2)$. For $j = 1, 2$, let $\log\alpha_j$ be any determination of the logarithm of α_j, and let A_j be an upper bound for the height of α_j, and for $\exp|\log\alpha_j|$; further define

$$S_j = D_j + \operatorname{Log} A_j .$$

* Dedicated to Professor Th. Schneider

Furthermore let B be an upper bound for the height of β and for e^{D_0}, and let

$$S_0 = D_0 + \operatorname{Log} B.$$

We assume that the number

$$\Lambda = \beta \log\alpha_1 - \log\alpha_2$$

is not zero.

Our main result is the following.

Theorem. *Let $E \geqq e$ be a real number such that*

$$E \leqq \min\left\{e^{2T/5}, e^{DS_1/D_1}, e^{DS_2/D_2}, e\frac{DS_1}{D_1|\log\alpha_1|}, e\frac{DS_2}{D_2|\log\alpha_2|}\right\},$$

where

$$T = 4 + \frac{S_0}{D_0} + \operatorname{Log}\left(D^2\cdot\frac{S_1}{D_1}\cdot\frac{S_2}{D_2}\right).$$

Then

$$|\Lambda| > \exp\left\{-5\cdot 10^8\cdot D^4\cdot\frac{S_1}{D_1}\cdot\frac{S_2}{D_2}\cdot T^2(\operatorname{Log} E)^{-3}\right\}.$$

Remark. A weaker form of this result is the following. Let $E_1 > 1$ be a real number such that

$$E_1 \leqq \min\left\{e^{2T_1/5}, A_1^D, A_2^D, eD\frac{\operatorname{Log} A_1}{|\log\alpha_1|}, eD\frac{\operatorname{Log} A_2}{|\log\alpha_2|}\right\},$$

where

$$T_1 = \operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_1 + \operatorname{Log}\operatorname{Log} A_2 + \operatorname{Log} D.$$

Then

$$|\Lambda| > \exp\{-5\cdot 10^{10}\cdot D^4(\operatorname{Log} A_1)(\operatorname{Log} A_2)T_1^2(\operatorname{Log} E_1)^{-3}\}.$$

We can always choose $E_1 = e$; then we get the inequality

$$|\Lambda| > \exp\{-5\cdot 10^{10}\cdot D^4\cdot(\operatorname{Log} A_1)(\operatorname{Log} A_2)T_1^2\},$$

which is very precise in terms of D; but we get a better result in the particular case when $|\log\alpha_1|$, $|\log\alpha_2|$ are bounded, and still a much more precise result when α_1, α_2 are close to 1. For example the following corollary is an extension of Theorem 4 of [10] (which corresponds to

$$A_1 = B = \exp(\operatorname{Log} A_2)^{1/2}, D = 1;$$

see remark (ii) at the end of [10]).

Corollary 1. *Let $\gamma > 0$; assume, for $j=1$ and $j=2$,*

$$|\log\alpha_j| \leqq B^{-\gamma}, \quad \text{and} \quad B \leqq A_j \leqq e^B.$$

Then

$$|\Lambda| \geqq \exp\left\{-C_1 D^4 \frac{(\operatorname{Log} A_1)(\operatorname{Log} A_2)}{\operatorname{Log} B}\right\},$$

where C_1 is an effectively computable constant depending only on γ.

From an historical point of view, the first appearance of a refined inequality under the assumption that the α's are close to 1 occurs in the paper [11] by Shorey.

In all our other applications the numbers $|\log\alpha_j|$ are bounded. We then obtain the following.

Corollary 2. *Let R be a positive real number; assume $|\log\alpha_j| \leqq R$, $(j=1,2)$, and define*

$$\Sigma = D \max\left\{\frac{S_1}{D_1}, \frac{S_2}{D_2}\right\},$$

$$\sigma = D \min\left\{\frac{S_1}{D_1}, \frac{S_2}{D_2}\right\}.$$

Then

$$|\Lambda| > \exp\left\{-C_2 D^4 \frac{S_1}{D_1} \cdot \frac{S_2}{D_2} \cdot \left(\frac{S_0}{D_0} + \operatorname{Log}\Sigma\right)^2 (\log\sigma)^{-3}\right\},$$

where C_2 is an effectively computable constant depending only on R.

In particular, with the hypotheses of Corollary 2, namely $|\log\alpha_j| \leqq R$, $(j=1,2)$, we get two very simple bounds, namely

$$|\Lambda| \geqq \exp\{-C_3 D^3 S_0^2 S^2\}$$

and

$$|\Lambda| \geqq \exp\{-C_3' D^4 (\operatorname{Log} A)^2 (\operatorname{Log} B)^2\},$$

where C_3 and C_3' depend only on R, and where

$$S = \max\{S_1, S_2\} \quad \text{and} \quad A = \max\{A_1, A_2\}.$$

2. Simultaneous Approximations

We use the same idea as in [5] to apply our Corollary 2 (§ 1) to the simultaneous approximation of certain numbers. More precisely, we have the following result, which gives the dependence on the degree in the theorem of Franklin Schneider.

Proposition 1. *Let a, b be two complex numbers, with $a \neq 0$, and let* $\log a$ *be any non-zero determination of the logarithm of a. There exist effectively computable constants C_4, C_4', depending only on b and* $\log a$, *with the following property.*

Let η_0, η_1, η_2 be algebraic numbers of heights at most H, with $H > e$, and let D be the degree over $\mathbb{Q}$ of the field $\mathbb{Q}(\eta_0, \eta_1, \eta_2)$. Assume that η_0 is irrational. Then

$$|b-\eta_0|+|a-\eta_1|+|a^b-\eta_2| > \exp\{-C_4 D^4 (\mathrm{Log} H)^4 (\mathrm{Log}\,\mathrm{Log} H)^{-2} (\mathrm{Log}\,\mathrm{Log} H + \mathrm{Log} D)^{-1}\}$$

and

$$|b-\eta_0|+|a-\eta_1|+|a^b-\eta_2| > \exp\{-C'_4 D^3 (D + \mathrm{Log} H)^4 (\mathrm{Log}\,\mathrm{Log} H + \mathrm{Log} D)^{-3}\}.$$

For fixed D a better result is known, namely [5]:

$$\exp\{-C_5 (\mathrm{Log} H)^3 (\mathrm{Log}\,\mathrm{Log} H)\},$$

where C_5 depends on b, $\log a$ and D. But the present result gives an information on the "transcendence type" of certain fields. Moreover we shall prove something much more precise than Proposition 1, and also compute C_4, C'_4 explicitely (see § 5 below).

It is possible to improve Proposition 1 when some of the numbers a, b, a^b are algebraic.

Proposition 2. *Let α be a non-zero algebraic number, and b be any complex number. Denote by $\log\alpha$ any non-zero determination of the logarithm of α. There exist two effectively computable constants $C_6 = C_6(\log\alpha, b)$ and $C'_6 = C'_6(\log\alpha, b)$ satisfying the following property.*

Let ξ, η be two algebraic numbers of heights at most H, with $\xi \notin \mathbb{Q}$, $H > e$, and let D be the degree over $\mathbb{Q}$ of $\mathbb{Q}(\xi, \eta)$. Then

$$|b-\xi| + |\alpha^b - \eta| > \exp\{-C_6 D^4 (\mathrm{Log} H)^3 (\mathrm{Log} D)^{-1}\}$$

and

$$|b-\xi| + |\alpha^b - \eta| > \exp\{-C'_6 D^3 (D + \mathrm{Log} H)^3 (\mathrm{Log} D)^{-3}\}.$$

Proposition 3. *Let a be any non-zero complex number, with $\log a \neq 0$, and β be any irrational algebraic number. There exist effectively computable constants C_7, $C'_7(\log a, \beta)$, such that, if ξ, η are two algebraic numbers, of heights at most H, $H > e$, with $D = [\mathbb{Q}(\xi, \eta) : \mathbb{Q}]$, then*

$$|a-\xi| + |a^\beta - \eta| > \exp\{-C_7 D^4 (\mathrm{Log} H)^2 (\mathrm{Log}\,\mathrm{Log} H + \mathrm{Log} D)^{-1}\}$$

and

$$|a-\xi| + |a^\beta - \eta| > \exp\{-C'_7 D^3 (D + \mathrm{Log} H)^2 (\mathrm{Log}\,\mathrm{Log} H + \mathrm{Log} D)^{-1}\}.$$

The following consequence of Proposition 3 was suggested to us by Dale Brownawell, who needs it for his study of pairs of polynomials $R(x, y)$, $S(x, z)$, which are small at numbers $\alpha, \alpha^\beta, \alpha^{\beta^2}$.

We are indebted to Dale Brownawell for his encouragements to finish the present work.

Corollary. *Let* $a \in \mathbb{C}$, $a \neq 0$, $\log a \neq 0$, *and let* β *be an algebraic irrational. Then for any non-zero* $P, Q \in \mathbb{Z}[X]$, *with*

$$\deg P + \deg Q + \operatorname{Log} ht P + \operatorname{Log} ht Q = t \geqq t_0,$$

we have

$$\operatorname{Log} \max\{|P(a)|;\ |Q(a^b)|\} \geqq -t^8.$$

Finally, as a further illustration of our Corollary 2 (§ 1), we give two transcendence measures; they are rather sharp, and proved here in a very simple way.

Proposition 4. *Let* α_1, α_2 *be two non-zero algebraic numbers and* $\log\alpha_1, \log\alpha_2$ *be non-zero determinations of the logarithm of* α_1, α_2 *respectively. For any irrational algebraic number* ξ *of height at most* H *and degree at most* D,

$$\left|\frac{\log\alpha_1}{\log\alpha_2} - \xi\right| > \exp\{-C_8 D^2 (D + \operatorname{Log} H)^2\},$$

where $C_8 = C_8(\log\alpha_1, \log\alpha_2)$ *is effectively computable.*

The best known result (cf. [4]) is $\exp(-C_{15}(\varepsilon) D^2 S^2 (\operatorname{Log} S)^{-1+\varepsilon})$, where $S = D + \operatorname{Log} H$.

Proposition 5. *Let* $\alpha \neq 0$, $\beta \notin \mathbb{Q}$ *be two algebraic numbers, with* $\log\alpha \neq 0$. *For any algebraic number* ξ *of height at most* H *and degree at most* D,

$$|\alpha^\beta - \xi| > \exp\{-C_9 D^3 (D + \operatorname{Log} H)(\operatorname{Log} D + \operatorname{Log}\operatorname{Log} H)^2 (\operatorname{Log} 2D)^{-3}\}$$

where $C_9 = C_9(\log\alpha, \beta)$.

This result was announced in [4] (cf. Theorem 5.15 and the remark before it), but the proof used Gelfond's method. A slightly better result is announced in [6] by Cudnovskii, namely

$$|\alpha^\beta - \xi| > \exp\{-C_9' D^3 \operatorname{Log}(DH)(\operatorname{Log}(D \operatorname{Log} H))^2 (\operatorname{Log} 2D)^{-3}\}.$$

Compare also Propositions 4 and 5 with Gel'fond's book (Theorem III, Chapter III).

3. Some Lemmas

For the convenience of the reader we give a complete list of the lemmas used in the sequel, and some classical definitions.

If $P = a_0 X^d + \ldots + a_d \in \mathbb{C}[X]$ is a polynomial, with roots $z_1, \ldots, z_d$, we define

$$H(P) = \max_{0 \leqq j \leqq d} |a_j|, \qquad \|P\| = \left(\sum_{j=0}^{d} |a_j|^2\right)^{1/2} \quad \text{and}$$

$$M(P) = |a_0| \prod_{j=1}^{d} \max(1, |z_j|);$$

when α is an algebraic number, we define $H(\alpha)=H(P)$, $\|\alpha\|=\|P\|$ and $M(\alpha)=M(P)$ where $P\in\mathbb{Z}[X]$ is the minimal polynomial of α. The inequalities

$$H(P)\leqq\|P\|\leqq(d+1)^{1/2}\cdot H(P)$$

for $P\in\mathbb{C}[X]$ of degree $\leqq d$ imply

$$H(\alpha)\leqq\|\alpha\|\leqq(d+1)^{1/2}H(\alpha)$$

for α algebraic number of degree $\leqq d$.

We begin with three well-known lemmas.

Lemma 1. *If P is a polynomial over $\mathbb{C}$ then*

$$M(P)\leqq\|P\|.$$

Proof. >This is the main result of [8].<

Lemma 2. *Let $P=a_0X^d+\ldots+a_d\in\mathbb{Z}[X]$ be an irreducible polynomial. If $\alpha_1,\ldots,\alpha_k$ are distinct roots of P, then the number*

$$a_0\alpha_1\ldots\alpha_k$$

is an algebraic integer.

Proof. >See for example [4] Lemma 1.8.<

We define the length $L(P)$ of a polynomial

$$P=\sum_{n_1=0}^{N_1}\cdots\sum_{n_q=0}^{N_q}P_{n_1,\ldots,n_q}X_1^{n_1}\ldots X_q^{n_q}\in\mathbb{C}[X_1,\ldots,X_q]$$

by

$$L(P)=\sum_{n_1=0}^{N_1}\cdots\sum_{n_q=0}^{N_q}|P_{n_1,\ldots,n_q}|.$$

Lemma 3. *Let $\alpha_1,\ldots,\alpha_q$ be algebraic numbers of exact degree $d_1,\ldots,d_q$ respectively. Define $D=[\mathbb{Q}(\alpha_1,\ldots,\alpha_q):\mathbb{Q}]$. Let $P\in\mathbb{Z}[X_1,\ldots,X_q]$ have degree at most N_h in X_h, $(1\leqq h\leqq q)$. If $P(\alpha_1,\ldots,\alpha_q)\neq 0$, then*

$$|P(\alpha_1,\ldots,\alpha_q)|\geqq L(P)^{1-D}\cdot\prod_{h=1}^{q}M(\alpha_h)^{-DN_h/d_h}.$$

Proof. >This is a consequence of Lemmas 1 and 2. See for example, [13] p. 30, or look at the proof of the subsequent Lemma 4.<

We now give a refined version of the so-called Siegel's lemma. The following lemma improves earlier results in this direction (especially [4] Lemmas 4.8 and 4.9, and [13] Lemme 1.3.1).

Lemma 4. *Let $\alpha_1,\ldots,\alpha_q$ be algebraic numbers of exact degrees $d_1,\ldots,d_q$ respectively. Define $D=[\mathbb{Q}(\alpha_1,\ldots,\alpha_q):\mathbb{Q}]$. Let*

$$P_{i,j}\in\mathbb{Z}[X_1,\ldots,X_q],\quad(1\leqq i\leqq\nu,1\leqq j\leqq\mu)$$

be polynomials of degree at most N_{jh} *in* X_h *(for* $1 \leqq i \leqq v$*). Define*

$$L_j = \sum_{i=1}^{v} L(P_{i,j})\,; \quad \gamma_{i,j} = P_{i,j}(\alpha_1, \ldots, \alpha_q), \quad (1 \leqq i \leqq v, 1 \leqq j \leqq \mu).$$

If $v > \mu D$, *then there exist rational integers* $x_1, \ldots, x_v$, *not all of which are zero, such that*

$$\sum_{i=1}^{v} \gamma_{i,j} x_i = 0, \quad (1 \leqq j \leqq \mu),$$

and

$$\max_{1 \leqq i \leqq v} |x_i| \leqq 2 + (2^\mu (V_1 \ldots V_\mu)^D)^{1/(v - \mu D)},$$

where

$$V_j = L_j \prod_{h=1}^{q} M(\alpha_h)^{N_{j,h}/d_h}.$$

Proof. >Let σ_1 be an embedding of $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_q)$ into $\mathbb{C}$. Put

$$X = [(\eta^\mu (V_1 \ldots V_\mu)^D)^{1/(v - \mu D)}] + 1$$

where

$$\eta = \begin{cases} 1 & \text{if } \sigma_1 \text{ is real}, \\ 2 & \text{otherwise}. \end{cases}$$

Define the integers $l_1, \ldots, l_\mu$ by

$$l_j < (\eta (V_j (X+2))^D)^{1/\eta} \leqq l_j + 1.$$

Notice that $l_j^\eta > \eta (V_j X)^D$. By the Dirichlet box principle, there exist rational integers $x_1, \ldots, x_v$ such that

$$0 < \max_{1 \leqq i \leqq v} |x_i| \leqq X$$

and

$$\left| \sum_{i=1}^{v} \sigma_1(\gamma_{i,j}) x_i \right| \leqq \frac{\sqrt{\eta}}{l_j} L_j X \prod_{h=1}^{q} \max(1, |\sigma_1(\alpha_h)|)^{N_h},$$

(see for instance Exercise 1.3a of [13]). Now, using Lemma 1, we obtain

$$\left| a_1^{N_1 D/d_1} \ldots a_q^{N_q D/d_q} \prod_{\sigma} \sum_{i=1}^{v} \sigma(\gamma_{i,j}) x_i \right| \leqq \eta V_j^D X^D l_j^{-\eta} < 1$$

where a_h is the leading coefficient of the minimal polynomial of α_h, and σ ranges over the different embeddings of K into $\mathbb{C}$. Lemma 2 shows that the left hand side of the previous inequality is an algebraic integer. The result follows at once.<

Notice that in terms of heights,

$$V_j \leqq v(N_{j,1}+1) \ldots (N_{j,q}+1) \max_i H(P_{i,j}) \prod_{h=1}^{q} ((d_h+1)^{1/2} H(\alpha_h)) N_{j,h}/d_h.$$

Lemma 5. *Let α be an algebraic number of degree d. Then*

$$H(\alpha^2) \leqq (d+1) H(\alpha)^2 .$$

Moreover if α' is an algebraic number of degree d' such that $\alpha'^h = \alpha$, where h is a positiver integer, then

$$\|\alpha'\| \leqq 2^{d'} \|\alpha\| .$$

Proof. >Let P (resp. Q) be the minimal polynomial of α (resp. α^2) over $\mathbb{Z}$, then $Q(X^2) = P(X)$ or $P(X)P(-X)$. In the first case $H(Q) = H(P)$, while $H(Q) \leqq (d+1) H(P)^2$ in the second case. This proves the first assertion.

Consider now α'. Clearly,

$$M(\alpha') \leqq M(\alpha) .$$

The second result follows using the trivial inequality

$$\|\alpha'\| \leqq 2^{d'} M(\alpha')$$

and Lemma 1.<

Lemma 6 (A consequence of Hermite's interpolation formula). *Let F be a complex function, analytic on $|z| \leqq R$. Let S be a set of m points in the disk $|z| \leqq R_1$, $R_1 < R$. Put*

$$\Delta = \min_{z \in S} \prod_{\substack{z' \in S \\ z' \neq z}} |z' - z| , \qquad \delta = \min_{\substack{z, z' \in S \\ z' \neq z}} |z' - z| .$$

Then

$$|F|_{R_1} \leqq |F|_R \left(\frac{2R_1 + 1}{R - R_1} \right)^{m-1} + (\delta \Delta)^{-1} (4R_1 + 2)^{m-1} \sum_{z \in S} |F(z)| ,$$

$$\left(\text{where } |F|_T = \max_{|z| = T} |F(z)| \right).$$

>See Lemma 2′ of [9] and the remark below it.<

Lemma 7 (A lemma in diophantine approximation). *Let θ be a positive real number such that $k\theta$ is not an integer for $1 \leqq k \leqq M$. Let $q_1 < q_2 < \ldots \leqq q_n \leqq M < q_{n+1}$ be the denominators of the principal convergents of θ. Then*

$$\prod_{i=1}^{n} \|q_i \theta\| > 2^{-n} (q_2 \ldots q_{n+1})^{-1} \geqq e^{-3(\mathrm{Log}\, q_n)^2} (2q_{n+1})^{-1} .$$

>It is well-known that $2q_{i+1} \|q_i \theta\| > 1$ (see, for example, [3] inequality (16), page 7). This proves the first inequality.

The definition of the q_i shows that $q_i \geqq F_i$ (the i-th Fibonacci number). This implies

$$q_i \geqq \alpha^{i-1} , \qquad \alpha = (1 + \sqrt{5})/2 .$$

Hence, for $n \geqq 2$,

$$2^{n-1} q_2 \ldots q_n \leqq (2q_n)^{n-1} \leqq (2q_n)^{\mathrm{Log}\, q_n / \mathrm{Log}\, \alpha} \leqq e^{3(\mathrm{Log}\, q_n)^2} ;$$

these inequalities hold also for $n = 1$. This proves the second inequality of the lemma.<

Corollary. *Let θ be a positive real number such that $k\theta$ is not an integer for $1 \leqq k \leqq M$. Then,*

$$\operatorname{Log}(\|\theta\| \dots \|M\theta\|) \geqq -2\operatorname{Log}(2^M \cdot M!) - 3(\operatorname{Log} M)^2 + \min_{1 \leqq k \leqq M} \operatorname{Log}\|k\theta\| .$$

〉Use the fact that $|\theta - p/q| > (2q^2)^{-1}$ if p/q is not a principal convergent of θ.〈

Remark. It is easy to replace the term $\operatorname{Log}(2^M \cdot M!)$ by $\theta(M \log M)$.

For the following lemma, it is convenient to define the size of an algebraic number α by

$$s(\alpha) = \operatorname{Log}\max\{|\overline{\alpha}|, 1\} + \operatorname{Log}\operatorname{den}\alpha ,$$

where $\operatorname{den}\alpha$ is the denominator of α, and $|\overline{\alpha}|$ is the maximum of the absolute values of the conjugates of α.

We remark that

$$s(\alpha) \leqq 1 + \operatorname{Log} H(\alpha)$$

and

$$\operatorname{Log} H(\alpha) \leqq d(s(\alpha) + 1) .$$

Lemma 8. *Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers of size at most $s_1, \dots, s_n$ respectively. If $b_1, \dots, b_n$ are rational integers such that the number*

$$\Lambda = b_1 \log\alpha_1 + \dots + b_n \log\alpha_n$$

is non-zero, then

$$|\Lambda| > \exp\{-D(|b_1|s_1 + \dots + |b_n|s_n + 1)\} ,$$

where $D = [\mathbb{Q}(\alpha_1, \dots, \alpha_n) : \mathbb{Q}]$.

〉We may suppose $|\Lambda| \leqq 1/2$. Then $\Lambda \notin 2\pi i\mathbb{Z}$ and the number $\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1$ is non-zero. Without loss of generality we may suppose $b_j \geqq 0$ for $1 \leqq j \leqq r$ and $b_j \leqq 0$ for $r+1 \leqq j \leqq n$. Take the norm of the number

$$(\alpha_1^{b_1} \dots \alpha_r^{b_r} - \alpha_{r+1}^{-b_{r+1}} \dots \alpha_n^{-b_n}) \prod_{j=1}^{n} (\operatorname{den}\alpha_j)^{|b_j|} ;$$

since each conjugate of this number has absolute value at most

$$2\exp(s_1|b_1| + \dots + s_n|b_n|),$$

we obtain the lower bound

$$|\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1| > 2^{-D+1}\exp\{-D(s_1|b_1| + \dots + s_n|b_n|)\} .$$

The lemma follows using the inequality $|e^z - 1| \leqq 2|z|$ which is true for $|z| \leqq 1/2$.

Lemma 9. *Suppose that R is a polynomial with integer coefficients, degree d and height H. Denote by α a zero of R at minimal distance from γ and let k be the order of α. Then*

$$|\gamma - \alpha|^k \leqq 2^{2d^2/k}(2dH)^{d/k}|R(\gamma)| .$$

〉This is a weak form of the corollary of Theorem 4′ of [7].〈

4. Proof of the Theorem

4.1. The Main Result

We shall first prove the following Proposition 6, and then deduce our theorem

Proposition 6. *Let $\beta, \alpha_1, \alpha_2$ be three non-zero algebraic numbers of degree D_0, D_1, D_2 respectively; denote by D the degree of the field $K = \mathbb{Q}(\beta, \alpha_1, \alpha_2)$. For $j = 1, 2$, let $\log \alpha_j$ be a non-zero determination of the logarithm of α_j, and let $a_j \geqq 1$ be an upper bound for $\frac{1}{D_j} \operatorname{Log} M(\alpha_j)$ and for $\frac{e}{D} |\log \alpha_j|$. Further denote by $b \leqq 1$ an upper bound for $\frac{1}{D_0} \operatorname{Log} M(\beta)$. Furthermore define*

$$G \doteq b + \operatorname{Log} b + 2 \operatorname{Log}(D^2(a_1 + a_2)) + 10,$$

$$Z = \min\left\{\operatorname{Log} \frac{Da_1}{|\log \alpha_1|}\,;\ \operatorname{Log} \frac{Da_2}{|\log \alpha_2|}\,;\ \frac{G}{10}\,;\ Da_1\,;\ Da_2\right\}$$

and

$$U = D^4 a_1 a_2 G^2 Z^{-3}.$$

If α_2/α_1 is not a square in K, and if

$$\Lambda = \beta \log \alpha_1 - \log \alpha_2$$

does not vanish, then

$$|\Lambda| > \exp\{-4 \cdot 10^6 \cdot U\}.$$

Through Sections 4.2 up to 4.10, we assume that

$$0 < |\Lambda| \leqq \exp\{-4 \cdot 10^6 U\},$$

and we shall arrive at a contradiction at the end of Section 4.10. The actual proof of the theorem is given at Section 4.11, as an easy consequence of Proposition 6.

4.2. Notations

We consider two real numbers L_0, L_1, and two positive integers M_1, M_2, namely

$$L_0 = 54000 D^3 a_1 a_2 G Z^{-3},$$

$$L_1 = 47 D G Z^{-1},$$

$$M_1 = 2[950 D^2 G a_2 Z^{-2}],$$

$$M_2 = 2[950 D^2 G a_1 Z^{-2}].$$

The reader should observe that

$$D L_0 G \leqq 54000 U$$

and

$$D L_1 (M_1 a_1 + M_2 a_2) \leqq 178600 U.$$

We note also that the inequality

$$M_1 + M_2 \leqq 3800 D^2 G (a_1 + a_2)$$

together with our definition of G leads to

$$b+\operatorname{Log}(M_1+M_2)\leqq G.$$

On the other hand, since

$$950D^2Ga_2Z^{-2}\geqq 9500,$$

we have

$$\begin{aligned}\tfrac{1}{4}M_1M_2&\geqq(950)^2\cdot(1-\tfrac{1}{9500})^2\cdot U/Z\\&\geqq 902310U/Z.\end{aligned}$$

The set

$$\{\beta^{j_0}\alpha_1^{j_1}\alpha_2^{j_2};(j_0,j_1,j_2)\in\mathbb{N}^3, j_0+j_1+j_2\leqq D-1, j_0\leqq D_0-1, j_1\leqq D_1-1, j_2\leqq D_2-1\}$$

is a set of generators of $K=\mathbb{Q}(\beta,\alpha_1,\alpha_2)$ over $\mathbb{Q}$; we denote by

$$\{\xi_1,\dots,\xi_D\}$$

a subset of it, which is a basis of K over $\mathbb{Q}$; we shall write $\xi_l=\beta_0^{l_0}\alpha_1^{l_1}\alpha_2^{l_2}$ with $0\leqq l_i\leqq D_i-1$, $(i=1,2,3)$. Remark that

$$\max_{1\leqq l\leqq D}\log|\xi_l|\leqq D\max\{D_0b, D_1a_1, D_2a_2\}\leqq U.$$

4.3. *A Reduction*

We claim that the numbers

$$u+v\beta,\qquad (0\leqq u<M_1, 0\leqq v<M_2)$$

are pairwise distinct (the letters u,v denote rational integers). Otherwise β is a rational number b_1/b_2, with $|b_1|<M_1$, $0<|b_2|<M_2$, b_1,b_2 rational integers; hence by Lemma 8,

$$|b_2\Lambda|>\exp\{-2D(|b_1|(1+D_1a_1)+|b_2|(1+D_2a_2)+1)\},$$

and therefore

$$\begin{aligned}|\Lambda|&>\exp\{-4D(M_1D_1a_1+M_2D_2a_2)\}\\&>\exp\{-1520U\}\end{aligned}$$

contrary to our assumption.

This argument shows that in the sequel, we may suppose that either $D_0\geqq 2$ or $b\geqq 16$, hence $DG\geqq 26$ because otherwise the theorem holds trivially.

4.4. *The Auxiliary Function*

For brevity we write α_1^z for $\exp(z\log\alpha_1)$. We shall construct an auxiliary function of the form

$$f(z)=\sum_{0\leqq h<L_0}\sum_{0\leqq k<L_1}p_{h,k}z^h\alpha_1^{kz},$$

where
$$p_{h,k} = \sum_{l=1}^{D} p_{h,k,l}\xi_l,$$

and $p_{h,k,l}$ are rational integers that we have to choose. For $(u,v)\in\mathbb{C}\times\mathbb{C}$, the number $f(u+v\beta)$ is related to the number
$$\varphi_{u,v} = \sum_{0\leq h<L_0}\ \sum_{0\leq k<L_1} p_{h,k}(u+v\beta)^h\alpha_1^{ku}\alpha_2^{kv}$$
by
$$f(u+v\beta)-\varphi_{u,v} = \sum_{0\leq l<L_0}\ \sum_{0\leq k<L_1} p_{h,k}(u+v\beta)^h\cdot\alpha_1^{ku}\alpha_2^{kv}(e^{kv\Lambda}-1).$$

We choose $p_{h,k,l}$ in $\mathbb{Z}$, not all zero, such that
$$\varphi_{u,v}=0 \quad \text{for} \quad 0\leq u<M_1, \quad 0\leq v<M_2,$$
with u and v odd integers.

This is equivalent to solving in $\mathbb{Z}$ a linear system of $\frac{1}{4}M_1M_2$ equations with at least DL_0L_1 unknowns $p_{h,k,l}$, and whose coefficients belong to a number field of degree D. These coefficients are polynomials in $\beta_0,\alpha_1,\alpha_2$, namely
$$\sum_{t=0}^{h}\binom{h}{t}u^{h-t}\cdot v^t\cdot\beta_0^{t+l_0}\cdot\alpha_1^{ku+l_1}\cdot\alpha_2^{kv+l_2}.$$

We use Lemma 4 with
$$V=(L_0+1)(L_1+1)(M_1+M_2)^{L_0}M(\beta_0)M(\alpha_1)^{1+\frac{M_1L_1}{4D_1}}\cdot M(\alpha_2)^{1+\frac{M_2L_1}{4D_1}}$$
$$\leq 98653UD^{-1}.$$

Notice that $\mu=\frac{1}{4}M_1M_2$ and $\nu=DL_0L_1$ satisfy
$$\frac{\mu D}{\nu-\mu D}\leq 0.552.$$

Therefore we obtain a non trivial solution $p_{h,k,l}\in\mathbb{Z}$ with
$$\operatorname{Log}\max_{h,k,l}|p_{h,k,l}|\leq 54457U\cdot D^{-1}.$$

Notice also that
$$\operatorname{Log}\sum_{0\leq h<L_0}\ \sum_{0\leq k<L_1}|p_{h,k}|\leq 54460U.$$

4.5. *Statement of the Inductive Argument*

Let J be an integer with
$$0\leq J\leq\left[\frac{\operatorname{Log}L_1}{\operatorname{Log}2}\right]+1.$$

We shall prove that there exist rational integers
$$p_{h,k,l}^{(J)}, \quad (0\leq h<L_0, 0\leq k<L_1 2^{-J}, 1\leq l\leq D),$$

not all zero, of absolute value bounded by

$$\exp(54457UD^{-1})$$

such that the numbers

$$\varphi_{u,v}^{(J)} = \sum_{0 \leqq h < L_0} \sum_{0 \leqq k < L_1 2^{-J}} \left(\sum_{l=1}^{D} p_{h,k,l}^{(J)} \xi_l \right) 2^{-Jh} \cdot (u+v\beta)^h \alpha_1^{ku} \alpha_2^{kv}$$

are zero for all odd integers u, v satisfying

$$0 < u < M_1, \quad 0 < v < M_2.$$

For $J=0$, this statement is a consequence of the construction of the auxiliary function, with $p_{h,k,l}^{(0)} = p_{h,k,l}$.

For now on, we assume that the assertion is correct for some integer J, with

$$0 \leqq J \leqq \left[\frac{\operatorname{Log} L_1}{\operatorname{Log} 2} \right],$$

and we manage to prove it for $J+1$ (we shall succeed in Section 4.9).

We define

$$p_{h,k}^{(J)} = \sum_{l=1}^{D} p_{h,k,l}^{(J)} \xi_l,$$

and

$$f_J(z) = \sum_{0 \leqq h < L_0} \sum_{0 \leqq k < L_1^{(J)}} p_{h,k}^{(J)} 2^{-Jh} z^h \alpha_1^{kz},$$

with $L_1^{(J)} = L_1 \cdot 2^{-J}$,

4.6. An Upper Bound for $f_J(u+v\beta)$

We prove that

$$|f_J(u+v\beta)| < \exp(-3825000U)$$

for all odd integers u, v in the range $0 < u < M_1$, $0 < v < M_2$.

By the inequality

$$|e^w - 1| < |w| \cdot e^{|w|} \quad \text{for all} \quad w \in \mathbb{C},$$

we have, for real u, v in the ranges $0 < u < M_1$ and $0 < v < M_2$,

$$|f_J(u+v\beta) - \varphi_{u,v}^{(J)}| \leqq \sum_h \sum_k |p_{h,k}^{(J)}| 2^{-Jh} \cdot (u+v|\beta|)^h$$

$$\cdot \exp(ku|\log\alpha_1| + kv|\log\alpha_2|) \cdot kv|\Lambda| e^{kv|\Lambda|}$$

$$\leqq e^{54460U} \cdot (M_1 + M_2|\beta|)^{L_0} \cdot \exp\left(\frac{L_1 D}{e} (M_1 a_1 + M_2 a_2) \right) \cdot 2M_2 L_1 |\Lambda|$$

$$\leqq |\Lambda| \cdot \exp(175000U) \leqq \exp(-3825000U).$$

By the induction hypothesis, $\varphi_{u,v}^{(J)} = 0$ for all odd integers u, v in the considered ranges.

4.7. An Upper Bound for $|f_J|_{R_1}$.

We define $R_1 = M_1 + M_2|\beta|$, *and we prove*

$$|f_J|_{R_1} \leqq \exp\{-405000U\}.$$

For this we shall use Lemma 6, with

$$S = \{u + v\beta\,; 0 \leqq u < M_1, 0 \leqq v < M_2, u \text{ and } v \text{ odd}\},$$

and

$$R = (m-1-L_0)(L_1|\log\alpha_1|)^{-1} + R_1, \qquad m = M_1M_2/4 = \operatorname{Card} S,$$

and we write the conclusion of Lemma 6

$$|f_J|_{R_1} \leqq E_1 + E_2.$$

The first term on the right E_1 is defined by

$$E_1 = |f_J|_R \left(\frac{2R_1+1}{R-R_1}\right)^{m-1}.$$

We have

$$\begin{aligned} \operatorname{Log} E_1 &\leqq \operatorname{Log}|f_J|_R - (m-1)\operatorname{Log}((R-R_1)/(2R_1+1)) \\ &\leqq U + m\operatorname{Log}5 + RL_1|\log\alpha_1| \\ &\quad - (m-1)\operatorname{Log}((R-R_1)/(2R_1+1)). \end{aligned}$$

And

$$\begin{aligned} \frac{R-R_1}{R_1} &= \frac{m-1-L_0}{m}\,\frac{M_1M_2}{4(M_1+M_2|\beta|)L_1|\log\alpha_1|} \\ &\geqq 0.996\,\frac{M_1M_2}{4L_1(M_1|\log\alpha_1| + M_2|\log\alpha_2|)} \\ &\geqq 0.993\,\frac{950}{188}\,\frac{e^Z}{Z} \geqq 0.993(950e/188) \end{aligned}$$

since

$$Z \leqq \operatorname{Min}\operatorname{Log}\frac{Da_j}{|\log\alpha_j|}.$$

We get

$$\begin{aligned} \operatorname{Log} E_1 &\leqq 54460U + L_0\operatorname{Log}\frac{2RR_1}{R-R_1} \\ &\quad + R_1L_1|\log\alpha_1| - (m-1-L_0)\operatorname{Log}\frac{R-R_1}{2e(R_1+1)} \\ &\leqq 54460U + 59000U + DL_1(M_1a_1 + M_2a_2)e^{-Z} \\ &\quad - \left(\frac{m-1-L_0}{m}\right)(949.9)^2\left(1 - \frac{\operatorname{Log}(1.085Z)}{Z}\right)U \\ &\leqq 113460U + 178600Ue^{-Z} - 0.996 \times 902310\left(1 - \frac{\operatorname{Log}(1.085Z)}{Z}\right)U. \end{aligned}$$

A tedious, but elementary, study of this function of Z gives

$$\operatorname{Log} E_1 \leqq 113460U + 178600e^{-2.24}U - 898700\left(1-\frac{1.085}{e}\right)U$$
$$< -406000U.$$

The second term E_2 is defined by

$$E_2 = (\delta\Delta)^{-1}(4R_1+2)^{m-1} \sum_{u+v\beta\in S} |f_J(u+v\beta)|$$

where

$$\Delta = \min_{u,v} \prod_{(u',v')\neq(u,v)} |u-u'-(v-v')\beta|\,; \quad \delta = \min_{(u',v')\neq(u,v)} |u-u'-(v-v')\beta|.$$

We obtain a lower bound for $|\Delta|$ as follows. For a certain (u,v)

$$|\Delta| = \prod_{v',u'} |(u-u')-(v-v')\beta|,$$

where u, v, u', v' are odd integers. This formula leads to the inequalities

$$|\Delta| \geqq \prod_{v'}\left(\|(v-v')\beta\| \left[\frac{M_1-6}{4}\right]!^2\, 2^{(M_1-6)/2}\right)$$
$$\geqq \min_{-M_2\leqq 2n\leqq 0}\left(\prod_{\substack{w=n\\ w\neq 0}}^{M_2/2+n-1} \|2w\beta\|\right)\cdot\left(\left[\frac{M_1-6}{4}\right]!\,2^{(M_1-6)/4}\right)^{M_2}$$
$$\geqq \left(\prod_{0<2n<M_2} \|2n\beta\|\right)^2 \left(\left[\frac{M_1-6}{4}\right]!\,2^{(M_1-6)/4}\right)^{M_2}$$

[for the first inequality we use the fact that, for fixed v', there are at least M_1-1 different numbers $(u-u')-(v-v')\beta$ and that the distance between two of these numbers is at least 2].

Then, the corollary of Lemma 7 and the inequalities $(n/e)^n \leqq n! \leqq (n/2)^n$ for $n\geqq 6$ give (with $m' := m-3M$, $M := \max(M_1, M_2)$)

$$\operatorname{Log}|\delta\Delta| \geqq m' \operatorname{Log}((M_1-9)/2e) - 4\operatorname{Log}(2^{M/2}\cdot(M/2)!) - 6(\operatorname{Log}(M/2))^2$$
$$+3\min_{1\leqq n\leqq M_2} \operatorname{Log}\|(n|\beta|)\|$$
$$\geqq m' \operatorname{Log}((M_1-9)/2e) - 3M\operatorname{Log}(M/2) - 8D^2(b+\operatorname{Log} M)$$
$$\geqq m' \operatorname{Log}((M_1-9)/2e) - 6000U$$

[if β is not real then $|\beta|$ is an algebraic number of degree at most D^2, and size at most $s(\beta)$].

If $M_1 \geqq |\beta| M_2$, we obtain

$$\operatorname{Log}|\delta\Delta| \geqq m' \operatorname{Log}\left(\frac{R_1-18}{4e}\right) - 6000U.$$

In the other case we use the formula

$$\Delta = \beta^{m-1} \prod ((u-u')\beta^{-1} - (v-v'))$$

and proceed in the same manner as before to estimate the product [notice that $s(\beta^{-1}) \leqq 2D(s(\beta)+1)$], this gives

$$\operatorname{Log}|\delta\Delta| \geqq m \operatorname{Log}|\beta| + m' \operatorname{Log}((M_2-9)/2e) - 6000U.$$

So, in both cases we have,

$$\begin{aligned}\operatorname{Log}|\delta\Delta| &\geqq m \operatorname{Log}\left(\frac{R_1}{4e}\right) - 2M \operatorname{Log}\frac{M}{2e} - m \operatorname{Log}\frac{R_1}{R_1-18} - 6000U \\ &\geqq m \operatorname{Log}(R_1/4e) - 8900U.\end{aligned}$$

Hence

$$\operatorname{Log} E_2 \leqq m \operatorname{Log} 16e + 10000U - 3825000U \leqq -410000U.$$

Finally, we obtain

$$\operatorname{Log}|f_J|_{R_1} \leqq -405000U.$$

4.8. A New Set of Equations

We prove that for all integers u', v' satisfying $0 < u' < M_1$, $0 < v' < M_2$,

$$\varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}} = 0.$$

First from the proof of Section 4.6 we deduce

$$\left| f_J\left(\frac{u'+v'\beta}{2}\right) - \varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}} \right| \geqq \exp(-3825000 \cdot U).$$

By Section 4.7 we obtain

$$\left|\varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}}\right| \leqq \exp(-404000 \cdot U).$$

Now $\varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}}$ is a polynomial in $\beta, \alpha_1, \alpha_2, (\alpha_1/\alpha_2)^{1/2}$, of degree $\leqq 1$ in $(\alpha_1/\alpha_2)^{1/2}$:

$$\begin{aligned}2^{L_0(J+1)} \cdot \varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}} = &\sum_h \sum_{k \text{ even}} \sum_l p^{(J)}_{h,k,l} 2^{(L_0-h)(J+1)} (u'+v'\beta)^h \\ &\cdot \beta^{\lambda_0} \cdot \alpha_1^{\frac{ku'}{2}+\lambda_1} \alpha_2^{\frac{kv'}{2}+\lambda_2} + \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2} \sum_h \sum_{k \text{ odd}} \sum_l p^{(J)}_{h,k,l} \\ &\cdot 2^{(L_0-h)(J+1)} \cdot (u'+v'\beta)^h \cdot \beta^{\lambda_0} \cdot \alpha_1^{\frac{ku'-1}{2}+\lambda_1} \cdot \alpha_2^{\frac{kv'+1}{2}+\lambda_2}.\end{aligned}$$

We now use Lemma 3. We bound the length of this polynomial by

$$\begin{aligned}&\exp(54460UD^{-1} + L_0 \operatorname{Log}(M_1+M_2) + L_0(J+1)\operatorname{Log} 2) \\ &\leqq \exp\{54460UD^{-1} + L_0 \operatorname{Log}(2L_1(M_1+M_2))\}.\end{aligned}$$

On the other hand we use the bound

$$\|\beta\|^{1+\frac{L_0}{D_0}}\cdot\|\alpha_1\|^{1+\frac{L_1^{(J)}M_1}{2D_1}}\cdot\|\alpha_2\|^{1+\frac{L_1^{(J)}M_2}{2D_2}}\cdot\|(\alpha_1/\alpha_2)^{1/2}\|^{1/2}$$
$$\leqq\exp\{L_0\operatorname{Log}B+\tfrac{1}{2}L_1^{(J)}(M_1a_1+M_2a_2)+4UD^{-1}\}.$$

From Lemma 3 we conclude that either $\varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}}=0$ or

$$\operatorname{Log}\left|\varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}}\right|\geqq-(108920U+2DL_0G+178600\cdot2^{-J}+D(J+1)L_0\operatorname{Log}2)$$
$$\geqq-400000U,$$

since $J\operatorname{Log}2\leqq\operatorname{Log}L_1$. This lower bound does not hold, hence $\varphi^{(J)}_{\frac{u'}{2},\frac{v'}{2}}=0$.

4.9. End of the Inductive Argument

We now proceed to prove the assertion of Section 4.5 for $J+1$. We use Section 4.8 for odd integers u',v'; we write $u'=2u''+1$, $v'=2v''+1$:

$$\sum_{h=0}^{L_0-1}\sum_{k=0}^{L_1^{(J)}}p_{h,k}^{(J)}2^{-Jh}\left(\frac{u'+v'\beta}{2}\right)^h\cdot\alpha_1^{(u''+1)k}\cdot\alpha_2^{v''k}\cdot(\alpha_2/\alpha_1)^{k/2}=0.$$

As $(\alpha_2/\alpha_1)^{1/2}\notin K$, we obtain, by writing the considered numbers in the basis $(1,\sqrt{\alpha_2/\alpha_1})$ of $K(\sqrt{\alpha_2/\alpha_1})$ over K:

$$\sum_{h=0}^{L_0-1}\sum_{0\leq k\leq\frac{L_1^{(J)}}{2}}p_{h,2k}^{(J)}2^{-Jh}\left(\frac{u'+v'\beta}{2}\right)^h\cdot\alpha_1^{(2u''+1)k}\cdot\alpha_2^{(2v''+1)k}=0$$

and

$$\sum_{h=0}^{L_0-1}\sum_{0\leq k<\frac{L_1^{(J)}}{2}}p_{h,2k+1}^{(J)}2^{-Jh}\left(\frac{u'+v'\beta}{2}\right)^h\alpha_1^{(2u''+1)k}\cdot\alpha_2^{(2v''+1)k}=0.$$

One at least of the two sets

$$\left\{p_{h,2k}^{(J)}\cdot0\leqq h<L_0,0\leqq k\leqq\frac{L_1^{(J)}}{2}\right\};\quad\left\{p_{h,2k+1}^{(J)};0\leqq h<L_0,0\leqq k<\frac{L_1^{(J)}}{2}\right\}$$

has at least one non-zero element. We denote this set by

$$\{p_{h,k}^{(J+1)};0\leqq h<L_0,0\leqq k\leqq L_1^{(J+1)}\},$$

with $L_1^{(J+1)}\leqq L_1^{(J)}/2$; we deduce:

$$\sum_{h=0}^{L_0-1}\sum_{k=0}^{L_1^{(J+1)}}p_{h,k}^{(J+1)}2^{-(J+1)h}\cdot(u'+v'\beta)^h\alpha_1^{u'k}\alpha_2^{v'k}=0$$

for all odd integers u',v' with $0\leqq u'<M_1$, $0\leqq v'<M_2$. This proves the claim of Section 4.5 for $J+1$.

4.10. The Contradiction

For $J_1 = \left[\frac{\operatorname{Log} L_1}{\operatorname{Log} 2}\right] + 1$, we have $2^{J_1} > L_1$, hence $L_1^{(J_1)} = 0$, and the numbers

$$\varphi_{u,v}^{(J_1)} = \sum_{h=0}^{L_0-1} p_{h,0}^{(J_1)} 2^{-J_1 h} (u + v\beta)^h$$

are zero for all odd integers u, v with

$$0 \leqq u < M_1, \quad 0 \leqq v < M_2.$$

By Section 4.3, the polynomial

$$\sum_{h=0}^{L_0-1} p_{h,0}^{(J_1)} X^h \in \mathbb{C}[X]$$

has at least $\frac{1}{4} M_1 M_2$ zeros, and its degree is less than L_0. Since $M_1 M_2 > 4L_0$, all the numbers

$$p_{h,0}^{(J_1)} = \sum_{l=1}^{D} p_{h,0,l}^{(J_1)} \xi_l$$

are zero, and this is a contradiction.

4.11. The General Case

We now consider the linear form

$$\Lambda = \beta \log \alpha_1 - \log \alpha_2$$

given by the hypotheses of our theorem (§ 1). We assume, as we may without loss of generality, that

$$\beta \log \alpha_1 \neq 0, \quad \log \alpha_2 \neq 0, \quad \log \alpha_1 \neq \log \alpha_2,$$

for otherwise we have $\Lambda = \beta' \log \alpha$, with β' equal to β, 1 or $\beta + 1$, and α equal to α_1 or α_2, hence by Lemma 8:

$$\begin{aligned} |\Lambda| &> \frac{1}{H(\beta')+1} \exp\{-D(s(\alpha)+1)\} \\ &> \exp\{-4D(\operatorname{Log} B + \operatorname{Log} A_1 + \operatorname{Log} A_2)\} \\ &> \exp\left\{-12D^4 \cdot \frac{\operatorname{Log} A_1}{D_1} \cdot \frac{\operatorname{Log} A_2}{D_2} \right. \\ &\quad \left. \cdot \left(\frac{\operatorname{Log} B}{D_0} + \operatorname{Log}\left(\frac{\operatorname{Log} A_1}{D_1} + \frac{\operatorname{Log} A_2}{D_2} + \operatorname{Log} D\right)\right)^2 (\operatorname{Log} E)^{-3}\right\}. \end{aligned}$$

Let $h \geqq 0$ be the largest integer such that both numbers

$$\exp\{2^{-h} \log \alpha_1\} \quad \text{and} \quad \exp\{2^{-h} \log \alpha_2\}$$

belong to K. The existence of such an h is straightforward; in fact it can be proved that (see the appendix)

$$h \leqq 80D^4(s(\alpha_1)+2)+D^2|\log\alpha_1|,$$

but an interesting feature of our proof is that we shall not use any upper bound for h.

We now define a new linear form $\Lambda'=\beta'\log\alpha'_1-\log\alpha'_2$ by considering two cases:

Case 1. The number $\exp\{2^{-h}\log\alpha_1\}$ is not a square in K; then we put

$$\log\alpha'_1=2^{-h}\log\alpha_1, \quad \log\alpha'_2=2^{-h+1}\log\alpha_2, \quad \beta'=2\beta.$$

Case 2. The number $\exp\{2^{-h}\log\alpha_1\}$ is a square in K; therefore $\exp\{2^{-h}\log\alpha_2\}$ is not a square in K, and we put

$$\log\alpha'_1=2^{-h}\log\alpha, \quad \log\alpha'_2=2^{-h}\log\alpha_2, \quad \beta'=\beta.$$

We remark that in both cases the numbers $\beta',\alpha'_1,\alpha'_2$ are in K, that α'_2/α'_1 is not a square in K, and that

$$|\Lambda|\geqq\tfrac{1}{2}|\Lambda'|.$$

The degree of $\mathbb{Q}(\beta',\alpha'_1,\alpha'_2)$ is at most D, and the degree of β' is equal to D_0, while the degree D'_1 of α'_1 (resp. D'_2 of α'_2) is at least D_1 (resp. $D_2/2$). Moreover, by Lemma 5,

$$\|\beta'\|\leqq e^{D_0B}, \quad \|\alpha'_1\|\leqq e^{D'_1}H(\alpha_1), \quad \|\alpha'_2\|\leqq e^{D'_2}(H(\alpha_2))^2.$$

We use our Proposition 6 for the linear form Λ', with

$$a_1=1+e\frac{\operatorname{Log}A_1}{D_1}, \quad a_2=1+4e\frac{\operatorname{Log}A_2}{D_2},$$

and $G=2F$.

Notice that

$$\frac{Da_1}{|\log\alpha'_1|}\geqq e\frac{D\operatorname{Log}A_1}{D_1|\log\alpha_1|}, \quad \frac{Da_2}{|\log\alpha'_2|}\geqq 2e\frac{D\operatorname{Log}A_2}{D_2|\log\alpha_2|},$$

hence $\operatorname{Log}E\leqq Z$.

Since $1+4^2e^2\cdot 2^2\cdot 10^6<5\cdot 10^8$, we easily deduce the desired result.

5. Proof of Corollaries and Propositions

5.1. We First Prove the Remark which Follows the Statement of the Theorem

Since our definitions of A_1, A_2, B involve only lower bounds [like $A_j\geqq e$, $A_j\geqq H(\alpha_j)$, $A_j\geqq\exp|\log\alpha_j|$], we get a weaker result if we replace in our theorem the numbers $\log A_1$, $\log A_2$, $\log B$ by $D_1\log A_1$, $D_2\log A_2$, $\frac{D'_0}{2}\log B$ respectively, where $D'_0=\max\{D_0,2\}$. Hence T is replaced by

$$5+\frac{D'_0}{2D_0}\operatorname{Log}B+2\operatorname{Log}D+\operatorname{Log}(1+\operatorname{Log}A_1)+\operatorname{Log}(1+\operatorname{Log}A_2),$$

and this number is at most $5T_1$ (using Liouville inequality which enables us to assume $\operatorname{Log} B + \operatorname{Log} D \geqq 1 + \operatorname{Log} 2$). On the other hand S_1 (resp. S_2) is replaced by $D_1 + D_1 \operatorname{Log} A_1$ (resp. $D_2 + D_2 \operatorname{Log} A_2$), which is at most $2D_1 \operatorname{Log} A_1$ (resp. $2D_2 \operatorname{Log} A_2$). This proves the result for $E_1 \geqq e$, while the case $1 < E_1 \leqq e$ is a weaker claim.

5.2. Proof of Corollary 1

We choose $E_1 = (DB)^\delta$, with $\delta = \min\{\frac{2}{5}, \gamma\}$. Since $T_1 \leqq 3 \operatorname{Log} B + \operatorname{Log} D$, we get

$$|\Lambda| > \exp\left\{-45 \cdot 10^{10} \delta^{-3} D^4 \frac{(\operatorname{Log} A_1)(\operatorname{Log} A_2)}{\operatorname{Log} B + \operatorname{Log} D}\right\},$$

and we deduce the desired result with, say,

$$C_1 = 5 \cdot 10^{11} \cdot \max\{16, \gamma^{-3}\}.$$

5.3. Proof of Corollary 2

Put $R_1 = \max\{e, R\}$, and take $E = \max\left\{e, \left(\frac{e\sigma}{R_1}\right)^{2/5}\right\}$. Since

$$\max\{1, 1 + y - x\} \geqq y/x \quad \text{for} \quad x \geqq 1 \quad \text{and} \quad y \geqq 1,$$

we have

$$\tfrac{5}{2} \operatorname{Log} E \geqq (\operatorname{Log} \sigma)/\operatorname{Log} R_1.$$

On the other hand

$$T \leqq 5\left(\frac{S_0}{D_0} + \operatorname{Log} \Sigma\right),$$

therefore we get the result with

$$C_2 = 2 \cdot 10^{11} \cdot (\max\{1, \operatorname{Log} R\})^2 \leqq 2 \cdot 10^{11} \cdot \max\{1, R\}.$$

When we choose $S_1 = S_2$, we obtain $\sigma \geqq (DS)^{1/2}$, and

$$|\Lambda| \geqq \exp\left\{-8C_2 D^4 \frac{S_1}{D_1} \frac{S_2}{D_2} \left(\frac{S_0}{D_0} + 1\right)^2\right\},$$

therefore we can take $C_3 = 32 C_2$ (since $D_0 D_1 D_2 \geqq D$); also $C'_3 = 16 C_2$.

5.4. An Explicit Lower Bound for $|b - \eta_0| + |a - \eta_1| + |a^b - \eta_2|$

Proposition 7. *Let a, b be two complex numbers, with $a \neq 0$, and let* $\log a$ *be any non zero determination of the logarithm of a. Let η_0, η_1, η_2 be algebraic numbers of degrees D_0, D_1, D_2 respectively, and let*

$$D = [\mathbb{Q}(\eta_0, \eta_1, \eta_2) : \mathbb{Q}].$$

Let S_0, S_1^*, S_2^* *be such that*

$$S_0 \geqq D_0 + \log \max\{H(\eta_0), e\},$$
$$S_1^* \geqq D_1 + \log \max\{H(\eta_1), e\},$$
$$S_2^* \geqq D_2 + \log \max\{H(\eta_2), e\},$$

and define

$$\Sigma^* = D \cdot \max\left\{\frac{S_1^*}{D_1}, \frac{S_2^*}{D_2}\right\},$$
$$\sigma^* = D \min\left\{\frac{S_1^*}{D_1}, \frac{S_2^*}{D_2}\right\}.$$

Then the number

$$\Xi = |b - \eta_0| + |a - \eta_1| + |a^b - \eta_2|$$

satisfies

$$\Xi > \exp\left\{-C_{10} \cdot D^4 \cdot \frac{S_1^*}{D_1} \cdot \frac{S_2^*}{D_2} \cdot \left(\frac{S_0}{D_0} + \operatorname{Log}\Sigma^*\right)^2 \cdot (\log\sigma^*)^{-3}\right\},$$

with

$$C_{10} = 3 \cdot 10^{11} \cdot (1 + \max\{|\log a|, |b \log a|\})^5 + \operatorname{Log}\frac{1}{|\log a|}.$$

5.5. Proof of Proposition 7

We assume, as we may without loss of generality,

$$|a - \eta_1| \leqq |a|/2 \quad \text{and} \quad |a^b - \eta_2| \leqq |a^b|/2.$$

Using the inequality

$$|e^z - e^{z_0}| \geqq \tfrac{1}{2}|e^{z_0}| \cdot |z - z_0| \quad \text{for} \quad |z - z_0| \leqq \tfrac{1}{2},$$

we choose $\log \eta_1$, $\log \eta_2$ in such a way that

$$|\log a - \log \eta_1| \leqq \frac{2}{|a|} \cdot |a - \eta_1|$$

and

$$|b \log a - \log \eta_2| \leqq \frac{2}{|a^b|} \cdot |a^b - \eta_2|.$$

Notice that

$$|\log \eta_1| \leqq 1 + |\log a|$$

and

$$|\log \eta_2| \leqq 1 + |b \log a|.$$

Since η_0 is irrational, the linear form

$$\Lambda = \eta_0 \log \eta_1 - \log \eta_2$$

vanishes only in the case $\log\eta_1 = \log\eta_2 = 0$, and in this case the result holds trivially thanks to the extra term $\operatorname{Log}\frac{1}{|\log a|}$.

We now suppose $\Lambda \neq 0$, and we apply Corollary 2:

$$|\Lambda| > \exp\left\{-C_2 D^4 \frac{S_1}{D_1}\cdot\frac{S_2}{D_2}\cdot\left(\frac{S_0}{D_0}+\operatorname{Log}\Sigma\right)^2\cdot(\operatorname{Log}\sigma)^{-3}\right\},$$

with $C_2 = C_2(R)$ and

$$R = 1 + \max\{|\log a|, |b\log a|\},$$

$$S_1 = R\cdot S_1^*,$$

$$S_2 = R\cdot S_2^*.$$

Therefore

$$\Sigma = R\Sigma^* \leqq (\Sigma^*)^R,$$

$$\sigma = R\sigma^* \geqq \sigma^*,$$

and

$$|\Lambda| > \exp\left\{-C_2 R^4 D^4\cdot\frac{S_2^*}{D_1}\cdot\frac{S_2^*}{D_2}\left(\frac{S_0}{D_0}+\operatorname{Log}\Sigma^*\right)^2(\operatorname{Log}\sigma^*)^{-3}\right\}.$$

Since

$$|\Lambda| \leqq \Xi \max\left\{1+|\log a|;\ \frac{2|b|}{|a|};\ \frac{2}{|a^b|}\right\},$$

$$\leqq \Xi \exp\left\{2R + \operatorname{Log}\frac{1}{|\log a|}\right\}$$

we easily get the desired result.

5.6. *Proof of Proposition 1*

We use Proposition 7 with

$$S_0 = 2D_0 \operatorname{Log} H,$$

$$S_1^* = 2D_1 \operatorname{Log} H,$$

$$S_2^* = 2D_2 \operatorname{Log} H,$$

therefore

$$\Sigma^* = \sigma^* = 2D \operatorname{Log} H.$$

Since

$$2\operatorname{Log} H + \operatorname{Log}(2D) + \operatorname{Log}\operatorname{Log} H \leqq 3(\operatorname{Log} H + \operatorname{Log} D),$$

we get

$$\Xi > \exp\{-36\cdot C_{10}\cdot D^4\cdot(\operatorname{Log} H)^2\cdot(\operatorname{Log} H + \operatorname{Log} D)^2 \cdot(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} D)^{-3}\}.$$

Therefore

$$\Xi > \exp\{-C_4 D^4 (\operatorname{Log} H)^4 (\operatorname{Log}\operatorname{Log} H)^{-2} (\operatorname{Log}\operatorname{Log} H + \operatorname{Log} D)^{-1}\}$$

with

$$C_4 = 11 \cdot 10^{12} (1 + \max\{|\log a|, |b \log a|\})^5 + 36 \cdot \operatorname{Log} \frac{1}{|\log a|}.$$

For the second inequality we use once more Proposition 7 with

$$S_0 = D_0 + \operatorname{Log} H,$$
$$S_1^* = D_1 + \operatorname{Log} H,$$
$$S_2^* = D_2 + \operatorname{Log} H,$$

and

$$\Sigma^* \leqq 2D \operatorname{Log} H, \qquad \sigma^* \geqq \max\{D, \operatorname{Log} H\} \geqq (D \operatorname{Log} H)^{1/2}.$$

Since $D_0 \geqq 2$, we have

$$D_0 + \operatorname{Log} H + D_0 \operatorname{Log}(2D \operatorname{Log} H) \leqq 3\sqrt{D_0 (D + \operatorname{Log} H)},$$

and we obtain

$$\Xi \geqq \exp\left\{-72 C_{10} \frac{D^4}{D_0 D_1 D_2} (D + \operatorname{Log} H)^4 \cdot (\operatorname{Log}\operatorname{Log} H + \operatorname{Log} D)^{-3}\right\}.$$

The desired result follows from the remark that $D_0 D_1 D_2 \geqq D$, and we conclude with

$$C_4' = 72 C_{10}.$$

5.7. Proof of Proposition 2

We choose first

$$S_1^* = D_1 + \operatorname{Log} \max\{H(\alpha), e\},$$
$$S_2^* = 2D_2 \operatorname{Log} H,$$
$$S_0 = 2D_0 \operatorname{Log} H,$$

and we remark that $[\mathbb{Q}(\alpha, \eta, \xi) : \mathbb{Q}] \leqq D \cdot D_1$. Since

$$\Sigma^* \leqq 2S_1^* \cdot D \operatorname{Log} H$$

and

$$\sigma^* \geqq D,$$

from the inequalities

$$2 \operatorname{Log} H + \operatorname{Log} \Sigma^* \leqq (2 + \operatorname{Log} 2S_1^*) \cdot (\operatorname{Log} H + \operatorname{Log} D)$$

and

$$(2 + \operatorname{Log}(2S_1^*))^2 \leqq 4S_1^*$$

we get

$$|b-\xi|+|\alpha^b-\eta|>\exp\{-C_6\cdot D^4\cdot(\operatorname{Log}H)(\operatorname{Log}H+\operatorname{Log}D)^2(\operatorname{Log}D)^{-3}\}$$
$$>\exp\{-C_6\cdot D^4(\operatorname{Log}H)^3\cdot(\operatorname{Log}D)^{-1}\}$$

with

$$C_6=8D_1^3S_1^{*2}C_{10}.$$

We prove the second inequality in the same way, with

$$S_0=D_0+\operatorname{Log}H,\quad S_2^*=D_2+\operatorname{Log}H,$$

and we use the bounds

$$\Sigma^*\leqq 2S_1^*D^2\operatorname{Log}H$$

and

$$D_0+\operatorname{Log}H+D_0\operatorname{Log}\Sigma^*\leqq(1+\operatorname{Log}(2S_1^*))(D+\operatorname{Log}H)\sqrt{D_0}$$

(recall that $D\geqq D_0\geqq 2$). Therefore

$$|b-\xi|+|\alpha^b-\eta|>\exp\left\{-C_6'\cdot\frac{D^4}{D_0D_2}(D+\operatorname{Log}H)^3\cdot(\operatorname{Log}D)^{-3}\right\}$$

with

$$C_6'=3D_1^3\cdot S_1^{*2}\cdot C_{10}.$$

5.8. Proof of Proposition 3

We choose first

$$S_1^*=2D_1\operatorname{Log}H,\quad\text{where}\quad D_1=\deg(\xi),$$
$$S_2^*=2D_2\operatorname{Log}H,\quad D_2=\deg(\eta),$$
$$S_0=D_0+\operatorname{Log}\max\{H(\beta),e\},$$

and remark that $[\mathbb{Q}(\beta,\eta,\xi):\mathbb{Q}]\leqq DD_0$.

Here

$$\Sigma^*=\sigma^*=2D\operatorname{Log}H,$$

hence Proposition 7 gives

$$|a-\xi|+|a^\beta-\eta|>\exp\{-C_7D^4(\operatorname{Log}H)^2(\operatorname{Log}D+\operatorname{Log}\operatorname{Log}H)^{-1}\},$$

with

$$C_7=2^9\cdot C_{10}D_0^2(D_0+\operatorname{Log}\max\{H(\beta),e\})^2.$$

To obtain the second inequality, take

$$S_1^*=S_2^*=D+\operatorname{Log}H,$$

and notice that in this case

$$\Sigma^*\leqq D(D+\operatorname{Log}H)\leqq(D\operatorname{Log}H)^2$$

and

$$\sigma^*\geqq D+\operatorname{Log}H\geqq\sqrt{D\operatorname{Log}H}.$$

5.9. Proof of the Corollary of Proposition 3

Assume that

$$\max(|P(a)|, |Q(a^b)|) \leqq t^{-8}.$$

Let ξ be a root of P at minimal distance to a, h the multiplicity of ξ and η a root of Q at minimal distance to a^b, k the multiplicity of η. We may assume $k \geqq h$. Lemma 9 gives (for $t \geqq 6$)

$$|\xi - a| \leqq e^{-t^8/2k}, \qquad |\eta - a^b| \leqq e^{-t^8/2k},$$

and we have

$$\max(H(\xi), H(\eta)) \leqq e^t, \qquad D \leqq t^2/hk.$$

From the second assetion of Proposition 3 we get

$$\max(|\xi - a|, |\eta - a^b|) \geqq \exp(-C_{14} t^8 (k \operatorname{Log} t)^{-1}).$$

Contradiction for $t \geqq t_0$.

5.10. Proof of Proposition 4

We apply again Proposition 7 with

$$S_i^* = D_i + \operatorname{Log} \max\{H(\eta_i), e\}, \qquad i = 1, 2 (D_i = \deg \alpha_i),$$
$$S_0 = D + \operatorname{Log} H,$$

and we get

$$\Xi > \exp(-C_8 D^2 (D + \operatorname{Log} H)^2)$$

with

$$C_8 = 16 C_{10} D_1^3 D_2^3 (D_1 + \log \max\{H(\eta_1), e\})(D_2 + \log \max\{1 + H(\eta_2), e\}) \cdot \operatorname{Log}^2(\operatorname{Log} \max\{H(\eta_1), H(\eta_2), e^e\}).$$

5.11. Proof of Proposition 5

We take

$$S_0 = 2D_0 \operatorname{Log} \max\{H(\beta), e\},$$
$$S_1^* = 2D_1 \operatorname{Log} \max\{H(\alpha), e\},$$
$$S_2^* = D + \operatorname{Log} H$$

and we get

$$\Xi > \exp(-C_9 D^3 (D + \operatorname{Log} H)(\operatorname{Log} D + \operatorname{Log} \operatorname{Log} H)^2 (\operatorname{Log} 2D)^{-3})$$

with

$$C_9 = 2^{10} C_{10} D_1^3 D_0^2 (\log \max\{H(\beta), e\} + \operatorname{Log} \max\{H(\alpha), e\})^2.$$

6. Appendix

Lemma A. *Let γ be a non-zero algebraic integer of degree at most D and which is not a root of unity. Then there exists a conjugate γ' of γ such that*

$$|\gamma'| > 1 + (30D^2 \operatorname{Log}(6D))^{-1}.$$

⟩This is the main results of [2].⟨

Lemma B. *Let α be a non-zero algebraic number which is not a root of unity. If $\alpha = \beta^m$, $\deg\beta \leqq d$, then*

$$m < 40d^2(\operatorname{Log} 6d) \operatorname{Log}(2H(\alpha)),$$

where $H(\alpha)$ is the height of α.

⟩If α is not a unit and if $a = d(\alpha)$ then the ideal $(a\alpha)$ of $\mathbb{Q}(\alpha)$ is divisible by the m-th power of some prime ideal of $\mathbb{Q}(\alpha)$. Taking the norm we get

$$2^m \leqq H(\alpha)^{2d}.$$

and the inequality of the lemma is satisfied.

If α is a unit, thanks to Lemma 5 we may suppose $\beta > 1$, and we obtain

$$1 + (30d^2 \operatorname{Log}(6d))^{-1} < |\beta| = |\alpha|^{1/m} \leqq |2H(\alpha)|^{1/m}.$$

The result follows at once.⟨

Lemma C. *Let K be a number field of degree D over $\mathbb{Q}$, and let α be a non-zero element of K. We denote by* $\log\alpha$ *any fixed non-zero determination of the logarithm of α.*

Let m be a positive integer such that the number

$$\exp\left\{\frac{1}{m}\log\alpha\right\}$$

belongs to K. Then

$$m \leqq 80D^4(s(\alpha)+2) + D^2|\log\alpha|.$$

Proof. ⟩If α is not a root of unity, we use Lemma 6 and we bound

$$(\operatorname{Log}(6D)(\operatorname{Log}(2H(\alpha)) \quad \text{by} \quad 2D^2(s(\alpha)+2).$$

Now assume that α belongs to the cyclic group of the roots of unity of K; let ζ be a generator of this group. The order n of ζ is an even integer satisfying $n \leqq 2D^2$ $\left(\text{the degree of } \zeta \text{ is } \varphi(n) \geqq \sqrt{\frac{n}{2}}\right)$. There exists a rational integer k such that

$$\log\alpha = 2i\pi k/n.$$

If $m > 0$ is such that $\exp\left\{\frac{1}{m}\log\alpha\right\}$ is in K, then m divides k, hence

$$m \leqq n\frac{|\log\alpha|}{2\pi} \leqq \frac{D^2}{\pi}|\log\alpha|.$$ ⟨

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