Introduction to Diophantine methods *Michel Waldschmidt*
http://www.math.jussieu.fr/∼miw/coursHCMUNS2007.html

Exercises - second sheet

Solve as many as you can, but at least 2, of the following exercises.

Exercise 1. Let $b \ge 2$ be an integer. Show that a real number x is rational if and only if the sequence $(d_n)_{n\ge 1}$ of digits of x in the expansion in basis b

$$x = [x] + d_1 b^{-1} + d_2 b^{-2} + \cdots + d_n b^{-n} + \cdots \qquad (0 \le d_n < b)$$

is ultimately periodic (see § 1.1).
Deduce another proof of Lemma 1.17 in § 1.3.5.

Exercise 2. Let $b \ge 2$ be an integer. Let $(a_n)_{n\ge 0}$ be a bounded sequence of rational integers and $(u_n)_{n\ge 0}$ an increasing sequence of positive numbers. Assume there exists $c > 0$ and $n_0 \ge 0$ such that, for all $n \ge n_0$,

$$u_{n+1} - u_n \ge cn.$$

a) Deduce, for all $k \ge 1$ and $n \ge n_0$,

$$u_{n+k} - u_n \ge cnk + c \cdot \frac{k(k-1)}{2}\cdot$$

b) Show that the number

$$\vartheta = \sum_{n\ge 0} a_n b^{-u_n}$$

is irrational if and only if the set $\{n \ge 0\ ;\ a_n \ne 0\}$ is infinite.
c) Deduce another proof of Lemma 1.17 in § 1.3.5.

Exercise 3. Recall the proof, given in in § 1.1 of the irrationality of the square root of an integer n, assuming n is not the square of an integer: *by contradiction, assume $\sqrt{n}$ is rational and write $\sqrt{n} = a/b$ as an irreducible fraction; notice that b is the least positive integer such that $b\sqrt{n}$ is an integer; denote by m the integral part of $\sqrt{n}$ and consider the number $b' = (\sqrt{n} - m)b$. Since $0 < b' < b$ and $b'\sqrt{n}$ is an integer, we get a contradiction.*

Extend this proof to a proof of the irrationality of $\sqrt[k]{n}$, when n and k are positive integers and n is not the k-th power of an integer.
`Hint.` *Assume that the number $x = \sqrt[k]{n}$ is rational. Then the numbers*

$$x^2, x^3, \ldots, x^{k-1}$$

are also rational. Denote by d the least positive integer such that the numbers $dx, dx^2, \ldots, dx^{k-1}$ are integers. Further, denote by m the integral part of x and consider the number $d' = (x - m)d$.

Exercise 4. Let α be a complex number. Show that the following properties are equivalent.
(i) The number α is algebraic.
(ii) The numbers $1, \alpha, \alpha^2, \dots$ are linearly dependent over $\mathbb{Q}$.
(iii) The $\mathbb{Q}$-vector subspace of $\mathbb{C}$ spanned by the numbers $1, \alpha, \alpha^2, \dots$ has finite dimension.
(iv) There exists an integer $N \geq 1$ such that the $\mathbb{Q}$-vector subspace of $\mathbb{C}$ spanned by the N numbers $1, \alpha, \alpha^2, \dots, \alpha^{N-1}$ has dimension $< N$.
(v) There exists positive integers $n_1 < n_2 < \dots < n_k$ such that $\alpha^{n_1}, \dots, \alpha^{n_k}$ are linearly dependent over $\mathbb{Q}$.

Exercise 5. a) Use the geometrical proof of the irrationality of e in § 1.2.7 to deduce, without computation, that for any integer $n > 1$,

$$\frac{1}{(n+1)!} < \min_{m \in \mathbb{Z}} \left| e - \frac{m}{n!} \right| < \frac{1}{n!} \cdot$$

b) Recall the definition of the Smarandache function : $S(q)$ is the least positive integer such that $S(q)!$ is a multiple of q. Prove that for any $p/q \in \mathbb{Q}$ with $q \geq 2$,

$$\left| e - \frac{p}{q} \right| > \frac{1}{(S(q)+1)!} \cdot$$

Exercise 6. Let $(a_n)_{n \geq 0}$ be a bounded sequence of rational integers.
a) Prove that the following conditions are equivalent:
(i) The number

$$\vartheta_1 = \sum_{n \geq 0} \frac{a_n}{n!}$$

is rational.
(ii) There exists $N_0 > 0$ such that $a_n = 0$ for all $n \geq N_0$.
b) Prove that these properties are also equivalent to
(iii) The number

$$\vartheta_2 = \sum_{n \geq 0} \frac{a_n 2^n}{n!}$$

is rational.

Exercise 7. This exercise extends the irrationality criterion Lemma 1.6 by replacing $\mathbb{Q}$ by $\mathbb{Q}(i)$. The elements in $\mathbb{Q}(i)$ are called the *Gaussian numbers*, the elements in $\mathbb{Z}(i)$ are called the *Gaussian integers*. The elements of $\mathbb{Q}(i)$ will be written p/q with $p \in \mathbb{Z}[i]$ and $q \in \mathbb{Z}$, $q > 0$.

Let ϑ be a complex number. Check that the following conditions are equivalent.
(i) $\vartheta \notin \mathbb{Q}(i)$.
(ii) For any $\epsilon > 0$ there exists $p/q \in \mathbb{Q}(i)$ such that

$$0 < \left| \vartheta - \frac{p}{q} \right| < \frac{\epsilon}{q} \cdot$$

(iii) For any rational integer $N \geq 1$ there exists a rational integer q in the range $1 \leq q \leq N^2$ and a Gaussian integer p such that

$$0 < \left| \vartheta - \frac{p}{q} \right| < \frac{\sqrt{2}}{qN}.$$

(iv) There exist infinitely many Gaussian numbers $p/q \in \mathbb{Q}(i)$ such that

$$\left| \vartheta - \frac{p}{q} \right| < \frac{\sqrt{2}}{q^{3/2}}.$$

Exercise 8. Recall Liouville's inequality in Lemma 2.12 :
For any algebraic number α there exist two positive constants κ and d such that, for any rational number $p/q \neq \alpha$,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{\kappa q^d}.$$

Denote by $P \in \mathbb{Z}[X]$ the minimal polynomial of α.
a) Prove this result with d the degree of P and κ given by

$$\kappa = \max\bigl\{1 \,;\, \max_{|t-\alpha|\leq 1} |P'(t)|\bigr\}.$$

b) Check also that the same estimate is true with again d the degree of P and κ given by

$$\kappa = a_0 \prod_{i=2}^{d} (|\alpha_j - \alpha| + 1),$$

where a_0 is the leading coefficient and $\alpha_1, \ldots, \alpha_d$ the roots of P with $\alpha_1 = \alpha$:

$$P(X) = a_0(X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_d).$$

`Hint:` *For both parts of this exercise one may distinguish two cases, whether $|\alpha - (p/q)|$ is ≥ 1 or < 1.*

Exercise 9. Let $(a_n)_{n\geq 0}$ be a bounded sequence of rational integers and $(u_n)_{n\geq 0}$ be an increasing sequence of integers satisfying

$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty.$$

Assume that the set $\{n \geq 0 \,;\, a_n \neq 0\}$ is infinite.
Define

$$\vartheta = \sum_{n\geq 0} a_n 2^{-u_n}.$$

Show that ϑ is a Liouville number .
`Hint`: *compare with (2.32).*

Exercise 10. a) Let b be a positive integer. Give the continued fraction expansion of the number

$$\frac{-b+\sqrt{b^2+4}}{2}.$$

b) Let a, b and c be positive integers. Write a degree 2 polynomial with integer coefficients having a root at the real number whose continued fraction expansion is

$$[0; \overline{a, b, c}].$$