Appendix: Periods on the Kummer surface

Hironori Shiga

1. Periods of a Kummer surface

1.1. Recalling the Kummer surfaces. Let $\omega_1, \ldots, \omega_4$ be points on (z, w)-space $\mathbf{C}^2$ those are independent over $\mathbf{R}$. Let T be a complex torus defined by

$$\mathbf{C}^2/(\mathbf{Z}\omega_1 + \ldots + \mathbf{Z}\omega_4).$$

We consider an involution of $\mathbf{C}^2$

$$\iota : (z, w) \mapsto (-z, -w).$$

We obtain a complex 2-dimensional variety $V = T/\iota$. It has 16 singularities corresponding to 16 fixed points of the involution (namely the half period points and zero). By the resolution of these singularities we get a Kummer surface $S = S(\omega_1, \ldots, \omega_4)$. All these surfaces are diffeomorphic each other. We may construct S by the following alternative manner: Make first the blow up processes at 16 half period points on T, let us denote $\tilde{T}$ the resulting complex surface. The involution ι is still acting on $\tilde{T}$, so we have $S = \tilde{T}/\iota$. In this situation we denote by π the canonical projection $\tilde{T} \to S$.

We have four 1-cycles $\gamma_1, \ldots, \gamma_4$ on T those correspond to $\omega_1, \ldots, \omega_4$, respectively. But as easily checked $\pi(\gamma_i)$ is homotopic to zero, consequently S is simply connected.

As for the 2nd homology group $H_2(S, \mathbf{Z})$, we have the following cycles:

1) the 2-cycles represented by 16 exceptional divisors obtained by the resolution procedure related above , we denote by D the sub $\mathbf{Z}$ -module generated by these divisors,

2) six 2-cycles $\sigma_{ij} = \pi(\gamma_i \times \gamma_j)$.

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These 22 cycles are independent in the $\mathbf{Z}$-module $H_2(S,\mathbf{Z})$, and they give a basis over $\mathbf{Q}$, but not necessarily a basis over $\mathbf{Z}$. We can say $\{\sigma_{ij}\}$ is a $\mathbf{Z}$ basis of the quotient module $H_2(S,\mathbf{Z})/D$. Any way we have rank $H_2(S,\mathbf{Z}) = 22$. It indicates S is a K3 surface.

1.2. Their Periods. The holomorphic 2-form $dz \wedge dw$ on $\mathbf{C}^2$ induces that of S. We denote it by φ, it is the unique holomorphic 2-form on S up to a constant factor. Because π is a $2:1$ map, we have the relation

$$\int_{\sigma_{ij}} \varphi = \frac{1}{2}\int_{\gamma_i \times \gamma_j} dz \wedge dw.$$

So the period on S is essentially the same as the period of the complex torus T.

Let us consider the case when T is a product of two elliptic curves $E = \mathbf{C}/\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ and $E^* = \mathbf{C}/\mathbf{Z}\omega_1^* + \mathbf{Z}\omega_2^*$. Let $\gamma_1, \gamma_2, \gamma_1^*, \gamma_2^*$ be the 1-cycles on $T = E \times E^*$ corresponding to $\omega_1, \omega_2, \omega_1^*, \omega_2^*$, respectively. By putting $C_1 = \gamma_1 \times \gamma_2^*$ and $C_2 = \gamma_2 \times \gamma_1^*$ on T, we have a 2-cycle $\sigma = \pi(C_1 - C_2)$ on S. Then we obtain

$$\int_\sigma \varphi = \frac{1}{2}(\omega_1 \cdot \omega_2^* - \omega_2 \cdot \omega_1^*).$$

Generally it was difficult to show the transcendency of the period of the Kummer surface. As we observed above, even in the case of product type the argument is reduced to the quadratic relation among the periods of elliptic curves. We can refer only one example of explicit transcendental periods of the Kummer surface obtained by classical arguments (see [**1**]).

Let us consider an one parameter family of algebraic surfaces (it is a family of some Kummer sufaces)

$$\Sigma(\mu) : xyz(x + y + z + 1) + \mu^4 = 0, \quad \mu \in \mathbf{C}.$$

We note that it is essentially the same as the family

$$X^4 + Y^4 + Z^4 + 1 + kXYZ = 0, \quad k \in \mathbf{C}.$$

The holomorphic differential is given by

$$\omega = \frac{dx \wedge dy}{f_z(x, y, z)},$$

where f stands for the left hand side of the defining equation of $\Sigma(\mu)$. One 2-cycle on $\Sigma(\mu)$, saying K_0, is given by the lifting (near to the origin) of a torus $\{|y| = 1/4\} \times \{|z| = 1/4\}$ via the natural projection from $\Sigma(\mu)$ to the (x, y)- space. Suppose μ is an algebraic number, then the period

$$\int_{K_0} \omega$$

is always transcendental. To prove it we can reduce the argument to the transcendency of the period of a hypergeometric curve.

References

[1] Narumiya, N and H. Shiga, *The Mirror Map for a Family of $K3$ Surfaces Induced From the Simplest 3-Dimensional Reflexive Polytope*, CRM Proc. Lecture Notes, **30** (2001), 139–161. MR **2002m:**14030

Department of Mathematics, Faculty of Science, Chiba university, 1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan

E-mail address: shiga@math.s.chiba-u.ac.jp