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Integer valued entire functions

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Abstract

An integer-valued entire function is an entire function which is analytic in the complex plane and takes integer values at the nonnegative integers ; an example is 2^z.

A Hurwitz function is an entire function with derivatives of any order taking integer values at 0 ; an example is e^z.

Lower bounds for the growth of such functions and similar ones when they are not a polynomial have been investigated.

We start with its connection with transcendental number theory. Next we survey this topic. Finally, we present some new results related with Lidstone, Whittaker, Poritsky and Gontcharoff interpolation.

Introduction : Hilbert's 7th problem (1900)

David Hilbert

(1862 – 1943)

Prove that the numbers

$$e^{\pi} = 23.140\,692\,632\ \ldots$$

and

$$2^{\sqrt{2}} = 2.665\,144\,142\ldots$$

are transcendental.

A *transcendental number* is a number which is not algebraic. The *algebraic numbers* are the roots of the polynomials with rational coefficients.

http://www-history.mcs.st-and.ac.uk/Biographies/Hilbert.html

Values of the exponential function $e^z = \exp(z)$

$$e^\pi = 1 + \frac{\pi}{1} + \frac{\pi^2}{2} + \frac{\pi^3}{6} + \cdots + \frac{\pi^n}{n!} + \cdots$$

The number

$$e = e^1 = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!} + \cdots$$

is transcendental (Hermite, 1873), while

$$e^{\log 2} = 1 + \frac{\log 2}{1} + \frac{(\log 2)^2}{2} + \cdots + \frac{(\log 2)^n}{n!} + \cdots = 2$$

$$e^{i\pi} = 1 + \frac{i\pi}{1} + \frac{(i\pi)^2}{2} + \cdots + \frac{(i\pi)^n}{n!} + \cdots = -1$$

are rational numbers.

Charles Hermite

Charles Hermite

(1822 – 1901)

1873
The number e is transcendental.

Ch. Hermite – *Sur la fonction exponentielle*, C. R. Acad. Sci. Paris, **77** (1873), 18–24 ; 74–79 ; 226–233 ; 285–293 ; *Oeuvres*, Gauthier Villars (1905), III, 150–181.

https://www-history.mcs.st-andrews.ac.uk/Biographies/Hermite.html

Constance Reid : Hilbert

The second problem became known as Hilbert's α^β conjecture. As Hilbert notes, corollaries of this conjecture include the transcendence of $2^{\sqrt{2}}$ and of $e^\pi = (e^{\pi i})^{-i} = (-1)^{-i}$.

An amusing incident concerning this conjecture is related in C. Reid's biography of Hilbert [Rei, C]. Carl Ludwig Siegel came to Gottingen as a student in 1919. He always remembered a lecture by Hilbert who, wanting to give his audience examples of problems in the theory of numbers which seem simple at first glance but which are, in fact, incredibly difficult, mentioned the Riemann Hypothesis, Fermat's Last Theorem and the transcendence of $2^{\sqrt{2}}$. Hilbert said that given recent progress he hoped to see the proof of the Riemann Hypothesis in his lifetime. Fermat's problem required totally new methods and possibly the youngest members of the audience would live to see it solved. As for $2^{\sqrt{2}}$, Hilbert said that no one at the lecture would live to see its proof. Hilbert was wrong! Siegel proved the transcendence of $2^{\sqrt{2}}$ about 10 years later (unpublished) and the solution of the α^β conjecture came shortly afterwards. He was right about Fermat's theorem and the Riemann Hypothesis is still unproved.

- Constance Reid. Hilbert. Springer Verlag 1970.
- Jay Goldman. The Queen of Mathematics : A Historically Motivated Guide to Number Theory. Taylor & Francis, 1998.

George Pólya Aleksandr Osipovich Gel'fond

Growth of integer-valued entire functions.

Pólya : $\mathbb{N}$ Gel'fond : $\mathbb{Z}[i]$

G. Pólya

(1887 – 1985)

A.O. Gel'fond

(1906 – 1968)

http://www-history.mcs.st-and.ac.uk/Biographies/Polya.html

http://www-history.mcs.st-and.ac.uk/Biographies/Gelfond.html

Integer-valued entire functions on $\mathbb{N}$

G. Pólya (1915) :
An entire function f *which is not a polynomial and satisfies* $f(a) \in \mathbb{Z}$ *for all nonnegative integers* a *grows at least like* 2^z*. It satisfies*

$$\limsup_{R\to\infty} \frac{1}{R} \log |f|_R \geq \log 2.$$

G. Pólya

(1887 – 1985)

Notation :

$$|f|_R := \sup_{|z|\leq R} |f(z)|.$$

http://www-history.mcs.st-and.ac.uk/Biographies/Polya.html

Integer-valued entire function on $\mathbb{Z}[i]$

S. Fukasawa (1928), A.O. Gel'fond (1929) :
An entire function f *which is not a polynomial and satisfies* $f(a+ib) \in \mathbb{Z}[i]$ *for all* $a+ib \in \mathbb{Z}[i]$ *grows at least like* e^{cz^2}. *It satisfies*

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

Proof : Expand $f(z)$ into a Newton interpolation series at the Gaussian integers.

A.O. Gel'fond : $\gamma \geq 10^{-45}$.

Entire functions vanishing on $\mathbb{Z}[i]$

The canonical product associated with the lattice $\mathbb{Z}[i]$ is the Weierstrass sigma function

$$\sigma(z) = z \prod_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right),$$

which is an entire function vanishing on $\mathbb{Z}[i]$.

$\sigma(z)$ grows like $e^{\pi z^2/2}$:

$$\limsup_{R \to \infty} \frac{1}{R^2} \log |\sigma|_R = \frac{\pi}{2} \cdot$$

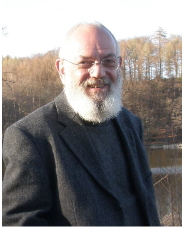
Hence

$$10^{-45} \leq \gamma \leq \frac{\pi}{2} \cdot$$

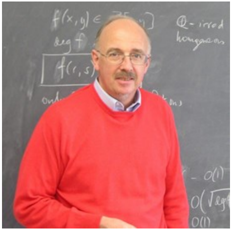
Exact value of the constant γ of Gel'fond

F. Gramain (1981) : $\gamma = \dfrac{\pi}{2e}\cdot$

This is best possible : D.W. Masser (1980).



F. Gramain



D.W. Masser

Irrationality of e^{π}

The function $e^{\pi z}$ takes the value

$$(e^{\pi})^a(-1)^b$$

at the point $a+ib \in \mathbb{Z}[i]$.

If the number

$$e^{\pi} = 23.140\,692\,632\,779\,269\,005\,729\,086\,367\ \ldots$$

were rational, these values would all be rational numbers.

Gel'fond's proof yields the irrationality of e^{π} and more generally the fact that e^{π} is not root of a polynomial $X^N - a$ with $N \geq 1$ and $a \in \mathbb{Q}$.

Transcendence of e^π

A.O. Gel'fond (1929) : e^π is transcendental.

More generally, for α nonzero algebraic number with $\log \alpha \neq 0$ and for β imaginary quadratic number,

$$\alpha^\beta = \exp(\beta \log \alpha)$$

is transcendental.
Example : $\alpha = -1$, $\log \alpha = i\pi$, $\beta = -i$, $\alpha^\beta = (-1)^{-i} = e^\pi$.

R.O. Kuzmin (1930) : $2^{\sqrt{2}}$ is transcendental.
More generally, for α nonzero algebraic number with $\log \alpha \neq 0$ and for β real quadratic number,

$$\alpha^\beta = \exp(\beta \log \alpha)$$

is transcendental.
Example : $\alpha = 2$, $\log \alpha = \log 2$, $\beta = \sqrt{2}$, $\alpha^\beta = 2^{\sqrt{2}}$.

Solution of Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).

Transcendence of α^β
and of $(\log \alpha_1)/(\log \alpha_2)$
for algebraic α, β, α_2 *and* α_2.

Further connection with transcendental number theory

In 1950, E. G. Straus introduced a connection between integer–valued functions and transcendence results, including the Hermite–Lindemann Theorem on the transcendence of e^{α} for $\alpha \neq 0$ algebraic.

However, as he pointed out in a footnote, at the same time, Th. Schneider obtained more far reaching results, which ultimately gave rise to the Schneider–Lang Criterion (1962).

Integer-valued entire functions

An *integer-valued entire function* is an entire function f (analytic in $\mathbb{C}$) which satisfies $f(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \ldots$.

Example : the polynomials

$$\binom{z}{n} = \frac{z(z-1)\cdots(z-n+1)}{n!} \quad (n \geq 0).$$

Any polynomial with complex coefficients which is an integer-valued entire function is a linear combination with coefficients in $\mathbb{Z}$ of these polynomials :

$$u_0 + u_1 z + u_2 \frac{z(z-1)}{2} + \cdots + u_n \frac{z(z-1)\cdots(z-n+1)}{n!} + \cdots$$

(finite sum) with u_i in $\mathbb{Z}$.

G. Pólya (1915)

The function 2^z is a *transcendental* ($=$ not a polynomial) integer-valued entire function.

$$2^{p/q} = \sqrt[q]{2^p} \qquad 2^{\lim p_n/q_n} = \lim 2^{p_n/q_n},$$

$$2^z = \exp(z \log 2) = 1 + \frac{z \log 2}{1} + \frac{(z \log 2)^2}{2} + \frac{(z \log 2)^3}{6} + \cdots$$

G. Pólya (1915) : 2^z is the *smallest* transcendental integer-valued entire function. It has *exponential type*

$$\log 2 = 0.693\,147\,180\ldots$$

Order and type of entire functions

Maximum modulus principle :

$$|f|_r := \sup_{|z|=r} |f(z)| = \sup_{|z|\leq r} |f(z)|.$$

The *order* of an entire function f is

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log |f|_r}{\log r},$$

while the *exponential type* of an entire function is

$$\tau(f) = \limsup_{r\to\infty} \frac{\log |f|_r}{r}.$$

Order and type of entire functions

For $\varrho \in \mathbb{Z}$, $\varrho \geq 0$, the function e^{z^ϱ} is an entire function of order ϱ.

For $\tau \in \mathbb{C}$, $\tau \neq 0$, the function $e^{\tau z}$ is an entire function of order 1 and exponential type $|\tau|$.

For $\tau \in \mathbb{C}$, $\tau \neq 0$, the function

$$\sin(\tau z) = \frac{e^{i\tau\pi z} - e^{-i\tau\pi z}}{2i}$$

has order 1 and exponential type $|\tau|\pi$.

Entire functions of finite exponential type

The exponential type of an entire function is also given by

$$\tau(f) = \limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} \quad (z_0 \in \mathbb{C}).$$

Notation :

$$f^{(n)}(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n f(z).$$

The proof rests on Cauchy's estimate for the coefficients of the Taylor series and on Stirling's formula for $n!$.

Example :

$$(\mathrm{e}^{\tau z})^{(n)} = \tau^n \mathrm{e}^{\tau z}, \qquad \lim_{n\to\infty} |\tau^n \mathrm{e}^{\tau z}|^{1/n} = |\tau|.$$

Order and type of entire functions

If the exponential type is finite, then f has order ≤ 1.
If f has order < 1, then the exponential type is 0.

Examples :
A polynomial has order 0, hence exponential type 0.
The function e^{z^2} has order 2, hence infinite exponential type.
The function $\mathrm{e}^{\mathrm{e}^z}$ has infinite order, hence infinite exponential type.

Integer-valued entire functions on $\mathbb{N}$

Pólya's proof starts by expanding the function f into a *Newton interpolation series* at the points $0, 1, 2, \ldots$:

$$f(z) = \sum_{n \geq 0} u_n \binom{z}{n}, \qquad u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k).$$

Since $f(n)$ is an integer for all $n \geq 0$, the coefficients u_n are integers. If f does not grow fast, for sufficiently large n we have $|u_n| < 1$, hence $u_n = 0$.

I. Newton

(1643– 1727)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Newton.html

Proof of Pólya's Theorem using Laplace transform

For $N \geq 0$ and $t \in \mathbb{C}$ we have

$$\sum_{n=0}^{N} \binom{N}{n} (e^t - 1)^n = e^{Nt}.$$

For $|t| < \log 2$, we have

$$\left|\mathrm{e}^t - 1\right| = \left|\sum_{k=1}^{\infty} \frac{t^k}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|t|^k}{k!} = \mathrm{e}^{|t|} - 1 < 1.$$

Hence for $z \in \mathbb{C}$ and $|t| < \log 2$,

$$\sum_{n=0}^{\infty} \binom{z}{n} (e^t - 1)^n = e^{tz}.$$

Laplace transform

Let f be an entire function of exponential type $< \log 2$. Let r satisfy $\tau(f) < r < \log 2$.
Let $F(t)$ be the Laplace transform of f :

$$f(z) = \frac{1}{2\pi i} \int_{|t|=r} \mathrm{e}^{tz} F(t) \mathrm{d}t = \sum_{n=0}^{\infty} u_n \binom{z}{n}$$

with

$$u_n = \frac{1}{2\pi i} \int_{|t|=r} (e^t - 1)^n F(t) \mathrm{d}t.$$

Proof of Pólya's Theorem

Let f be an entire function of exponential type $< \log 2$. We have

$$f(z) = \sum_{n=0}^{\infty} u_n \binom{z}{n}.$$

Let r satisfy $\tau(f) < r < \log 2$. Then

$$u_n = \frac{1}{2\pi i} \int_{|t|=r} (e^t - 1)^n F(t) \mathrm{d}t.$$

Hence, for sufficiently large n,

$$|u_n| \leq r|F|_r (e^r - 1)^n < 1.$$

Gérard Rauzy. *Les zéros entiers des fonctions entières de type exponentiel.* Séminaire de Théorie des Nombres de Bordeaux, (1976-1977), pp. 1-10 https://www.jstor.org/stable/44165280

Growth of integer-valued entire functions

G. Pólya (1915) : an integral valued entire of exponential type $< \log 2$ is a polynomial.

More precisely, if f is a transcendental integer-valued entire function, then

$$\lim_{r\to\infty} \sqrt{r}2^{-r}|f|_r > 0.$$

Equivalent formulation :

If f is an integer-valued entire function such that

$$\lim_{r\to\infty} \sqrt{r}2^{-r}|f|_r = 0,$$

then f is a polynomial.

Carlson vs Pólya

F. Carlson (1914) : an entire function f of exponential type $< \pi$ satisfying $f(\mathbb{N}) = \{0\}$ is 0.
The function $\sin(\pi z)$ is a transcendental entire function of exponential type π vanishing on $\mathbb{Z}$.

Fritz Carlson

(1888 – 1952)

G. Pólya (1915) : an integer-valued entire function of exponential type $< \log 2$ is a polynomial.
The function 2^z is an integer-valued entire function of exponential type $\log 2$.

G.H. Hardy (1917)

A refinement of Pólya's result was achieved by G.H. Hardy who proved that if f is an integer-valued entire function such that

$$\lim_{r\to\infty} 2^{-r}|f|_r = 0,$$

then f is a polynomial.

G.H. Hardy

(1877 – 1947)

Compare with Pólya's assumption :

$$\lim_{r\to\infty} \sqrt{r}2^{-r}|f|_r = 0.$$

https://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html

A. Selberg (1941)

A. Selberg proved that if an integer–valued entire function f satisfies

$$\tau(f) \leq \log 2 + \frac{1}{1500},$$

then f is of the form $P_0(z) + P_1(z)2^z$, where P_0 and P_1 are polynomials.

A. Selberg

(1917 – 2007)

There are only $\boxed{\text{countably many}}$ such functions.

https://www-history.mcs.st-andrews.ac.uk/Biographies/Selberg.html

Ch. Pisot (1942)

Ch. Pisot proved that if an integer–valued entire function f has exponential type ≤ 0.8, then f is of the form

$$P_0(z) + 2^z P_1(z) + \gamma^z P_2(z) + \overline{\gamma}^z P_3(z),$$

where P_0, P_1, P_2, P_3 are polynomials and $\gamma, \overline{\gamma}$ are the non real roots of the polynomial $z^3 - 3z + 3$.

This contains the result of Selberg, since

$$|\log \gamma| = 0.758\,98 \cdots > \log 2 + \frac{1}{1500} = 0.693$$

Ch. Pisot

(1910 – 1984)

Pisot obtained more general result for functions of exponential type $< 0.9934\ldots$

Completely integer–valued entire function

A *completely integer–valued entire function* is an entire function which takes values in $\mathbb{Z}$ at all points in $\mathbb{Z}$.

Let $u > 1$ be a quadratic unit, root of a polynomial $X^2 + aX + 1$ for some $a \in \mathbb{Z}$. Then the functions

$$u^z + u^{-z} \quad \text{and} \quad \frac{u^z - u^{-z}}{u - u^{-1}}$$

are completely integer–valued entire function of exponential type $\log u$.

Examples of such quadratic units are the roots u and u^{-1} of the polynomial $X^2 - 3X + 1$:

$$u = \frac{3 + \sqrt{5}}{2}, \quad u^{-1} = \frac{3 - \sqrt{5}}{2}.$$

Quizz

Let ϕ be the Golden ratio and let $\tilde{\phi} = -\phi^{-1}$, so that

$$X^2 - X - 1 = (X - \phi)(X - \tilde{\phi}).$$

For any $n \in \mathbb{Z}$ we have

$$\phi^n + \tilde{\phi}^n \in \mathbb{Z}$$

and

$$\log \phi = -\log |\tilde{\phi}| < \log 2.$$

Why is $\phi^z + \tilde{\phi}^z$ not a counterexample to Pólya's result on the growth of transcendental integer-valued entire functions ?

Completely integer–valued entire function

The function

$$\frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{z}-\frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{-z}.$$

is a completely integer–valued transcendental entire function.

In 1921, F. Carlson proved that if the type $\tau(f)$ of a completely integer–valued entire function f satisfies

$$\tau(f)<\log\left(\frac{3+\sqrt{5}}{2}\right)=0.962\ldots,$$

then f is a polynomial.

A. Selberg (1941)

A. Selberg : if the type $\tau(f)$ of a completely integer–valued entire function f satisfies

$$\tau(f) \le \log\left(\frac{3+\sqrt{5}}{2}\right) + 2 \cdot 10^{-6},$$

then f is of the form

$$P_0(z) + P_1(z)\left(\frac{3+\sqrt{5}}{2}\right)^{z} + P_2(z)\left(\frac{3+\sqrt{5}}{2}\right)^{-z}$$

where P_0, P_1, P_2 are polynomials.

Hurwitz functions

A *Hurwitz function* is an entire function f such that $f^{(n)}(0) \in \mathbb{Z}$ for all $n \geq 0$.

A. Hurwitz

(1859 – 1919)

The polynomials which are Hurwitz functions are the polynomials of the form

$$a_0 + a_1 z + a_2 \frac{z^2}{2} + a_3 \frac{z^3}{6} + \cdots + a_n \frac{z^n}{n!}$$

with $a_i \in \mathbb{Z}$.

https://www-history.mcs.st-andrews.ac.uk/Biographies/Hurwitz.html

Hurwitz functions

The exponential function

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$$

is a transcendental Hurwitz function of exponential type 1. For $a \in \mathbb{Z}$, the function e^{az} is also a Hurwitz function of exponential type $|a|$.

Kakeya (1916)

S. Kakeya (1916) : a Hurwitz function of exponential type < 1 is a polynomial.
More precisely, a Hurwitz function satisfying

$$\limsup_{r\to\infty} \sqrt{r}\mathrm{e}^{-r}|f|_r = 0$$

is a polynomial.

Question : is $\sqrt{r}$ superfluous ? Is e^z the *smallest* Hurwitz function ?

Recall Pólya vs Hardy : an integer-valued entire functions of low growth is a polynomial.

Pólya's assumption : $\lim_{r\to\infty} \sqrt{r}2^{-r}|f|_r = 0.$

Hardy's assumption : $\lim_{r\to\infty} 2^{-r}|f|_r = 0.$

Pólya (1921)

G. Pólya refined Kakeya's result in 1921 : a Hurwitz function satisfying

$$\limsup_{r\to\infty} \sqrt{r}\mathrm{e}^{-r}|f|_r < \frac{1}{\sqrt{2\pi}}$$

is a polynomial.
(Kakeya's assumption : $\limsup = 0$).
This is best possible for $\boxed{\text{uncountably many}}$ functions, as shown by the functions

$$f(z) = \sum_{n\geq 0} \frac{e_n}{2^n!} z^{2^n}$$

with $e_n \in \{1, -1\}$ which satisfy

$$\limsup_{r\to\infty} \sqrt{r}\mathrm{e}^{-r}|f|_r = \frac{1}{\sqrt{2\pi}}.$$

A variant of Pólya's result

Let f be an entire function and let $A \geq 0$. Assume

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}.$$

Then there exists $n_0 > 0$ such that, for $n \geq n_0$ and for all $z \in \mathbb{C}$ in the disc $|z| \leq A$, we have

$$|f^{(n)}(z)| < 1.$$

Hint for the proof.

Use Cauchy's inequalities and Stirling's formula. $\square$

Sato and Straus (1964)

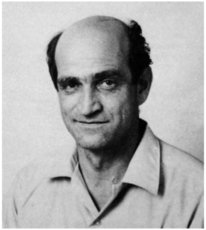
D. Sato and E.G. Straus proved that for every $\epsilon > 0$, there exists a transcendental Hurwitz function with

$$\limsup_{r\to\infty} \sqrt{2\pi r}\, \mathrm{e}^{-r} \left(1 + \frac{1+\epsilon}{24r}\right)^{-1} |f|_r < 1,$$

while every Hurwitz function for which

$$\limsup_{r\to\infty} \sqrt{2\pi r}\, \mathrm{e}^{-r} \left(1 + \frac{1-\epsilon}{24r}\right)^{-1} |f|_r \leq 1$$

is a polynomial.



E.G. Straus

(1922 – 1983)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Straus.html

Integer–valued functions vs Hurwitz functions :

Let us display horizontally the rational integers and vertically the derivatives.

integer–valued functions : horizontal

$$\begin{array}{ccccccc} f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & n & \cdots \end{array}$$

Hurwitz functions : vertical

$$\begin{array}{cc} & \vdots \\ f^{(n)} & \bullet \\ & \vdots \\ f' & \bullet \\ f & \bullet \\ & 0 \end{array}$$

Several points and / or several derivatives

There are several natural ways to mix integer–valued functions and Hurwitz functions :

- *horizontally*, one may include finitely may derivatives in the study of integer–valued functions.

  A k*–times integer–valued function* is an entire function f such that $f^{(j)}(n) \in \mathbb{Z}$ for all $n \geq 0$ and $j = 0, 1, \ldots, k-1$.

- *Vertically*, one may consider entire functions with all derivatives at finitely many points taking integer values.

  A k*–point Hurwitz function* is an entire function having all its derivatives at $0, 1, \ldots, k-1$ taking integer values.

k–times integer–valued functions (horizontal)

$k = 2$: $f(n) \in \mathbb{Z}$, $f'(n) \in \mathbb{Z}$ ($n \geq 0$).

$$\begin{array}{ccccccc} f' & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & n & \cdots \end{array}$$

According to Gel'fond (1929), a k–times integer–valued function of exponential type $< k \log\left(1 + \mathrm{e}^{-\frac{k-1}{k}}\right)$ is a polynomial.

The function $(\sin(\pi z))^k$ has exponential type $k\pi$ and vanishes with multiplicity k on $\mathbb{Z}$.

Two–point Hurwitz functions (vertical)

$k = 2$: $f^{(n)}(0) \in \mathbb{Z}$, $f^{(n)}(1) \in \mathbb{Z}$ ($n \geq 0$).

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
f^{(n)} & \bullet & \bullet \\
\vdots & \vdots & \vdots \\
f' & \bullet & \bullet \\
f & \bullet & \bullet \\
 & 0 & 1
\end{array}
$$

D. Sato (1971) : every two point Hurwitz entire functions for which there exists a positive constant C such that

$$|f|_r \leq C \exp\left(r^2 - r - \log r\right)$$

is a polynomial.

Also, there exist transcendental two point Hurwitz entire functions with

$$|f|_r \leq \exp\left(r^2 + r - \log r + O(1)\right).$$

k–point Hurwitz functions

For $k \geq 3$ our knowledge is more limited.

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
f^{(n)} & \bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f' & \bullet & \bullet & \cdots & \bullet \\
f & \bullet & \bullet & \cdots & \bullet \\
 & 0 & 1 & \cdots & k
\end{array}$$

D. Sato (1971) proved that the order of k–point Hurwitz functions is $\geq k$.
This is best possible, as shown by the function $\mathrm{e}^{z(z-1)\cdots(z-k+1)}$.

k–point Hurwitz functions

For an entire function f of order $\leq \varrho$, define

$$\tau_\varrho(f) = \limsup_{r\to\infty} \frac{\log |f|_r}{r^\varrho}.$$

f grows like $e^{\tau_\varrho(f) z^\varrho}$.

Example : for $k \geq 1$, the function $f(z) = \mathrm{e}^{z(z-1)\cdots(z-k+1)}$ has order k and $\tau_k(f) = 1$: it grows like e^{z^k}.

k–point Hurwitz functions

L. Bieberbach (1953) stated that if a transcendental entire function f of order ϱ is a k–point Hurwitz entire function, then either $\varrho > k$, or $\varrho = k$ and the type $\tau_k(f)$ of f satisfies $\tau_k(f) \geq 1$.

L. Bieberbach

(1886 – 1982)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Bieberbach.html

k–point Hurwitz functions

However, as noted by D. Sato, since the polynomial

$$a(z) = \frac{1}{2}z(z-1)(z-2)(z-3)$$

can be written

$$a(z) = \frac{1}{2}z^4 - 3z^3 - \frac{11}{2}z^2 - 3z,$$

it satisfies $a'(z) \in \mathbb{Z}[z]$.

It follows that the function $e^{a(z)}$ is a 4-point Hurwitz transcendental entire function of order $\varrho = 4$ and $\tau_4(f) = 1/2$.

Utterly integer–valued entire functions

Another way of mixing the horizontal and the vertical generalizations is to introduce *utterly integer–valued entire function*, namely entire functions f which satisfy $f^{(n)}(m) \in \mathbb{Z}$ for all $n \geq 0$ and $m \in \mathbb{Z}$.

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & & \vdots & \\
f^{(n)} & \bullet & \bullet & \cdots & \bullet & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
f' & \bullet & \bullet & \cdots & \bullet & \cdots \\
f & \bullet & \bullet & \cdots & \bullet & \cdots \\
 & 0 & 1 & \cdots & m & \cdots
\end{array}
$$

G.A. Fridman (1968), M. Welter (2005)

E.G. Straus (1951) suggested that transcendental utterly integer–valued entire function may not exist.

G.A. Fridman (1968) showed that there exists transcendental utterly integer–valued function f with

$$\limsup_{r\to\infty} \frac{\log\log|f|_r}{r} \le \pi$$

and proved that a transcendental utterly integer–valued function f satisfies

$$\limsup_{r\to\infty} \frac{\log\log|f|_r}{r} \ge \log(1+1/\mathrm{e}).$$

The bound $\log(1+1/\mathrm{e})$ was improved by M. Welter (2005) to $\log 2$: hence f grows like e^{2^z} (double exponential).

Sato's examples

An utterly integer–valued transcendental entire functions has infinite order : it grows like a double exponential $\mathrm{e}^{\mathrm{e}^{\alpha z}}$.

D. Sato (1985) constructed a nondenumerable set of utterly integer–valued transcendental entire functions.

He selected inductively the coefficients a_n with

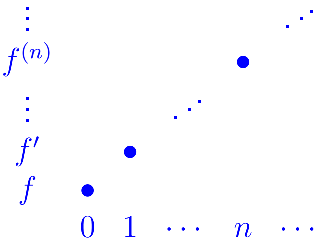
$$\frac{1}{n!(2\pi)^n} \leq |a_n| \leq \frac{3}{n!(2\pi)^n}$$

and defined

$$f(z) = \sum_{n\geq 0} a_n \sin^n(2\pi z).$$

Abel series

There is also a *diagonal* way of mixing the questions of integer–valued functions and Hurwitz functions by considering entire functions f such that $f^{(n)}(n) \in \mathbb{Z}$. The source of this question goes back to N. Abel.



Niels Abel

(1802 – 1829)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Abel.html

Abel polynomials

Recall

$$P_n(z) = \frac{1}{n!} z(z-n)^{n-1} \qquad (n \geq 1).$$

Any polynomial f has a finite expansion

$$f(z) = \sum_{n \geq 0} f^{(n)}(n) P_n(z).$$

G. Halphén (1882) : Such an expansion (with a series in the right hand side which is absolutely and uniformly convergent on any compact of $\mathbb{C}$) holds also for any entire function f of finite exponential type $< \omega$, where $\omega = 0.278\,464\,542\ldots$ is the positive real number defined by $\omega \mathrm{e}^{\omega+1} = 1$.

If an entire function f of exponential type $< \omega$ satisfies $f^{(n)}(n) = 0$ for all sufficiently large n, then f is a polynomial.

F. Bertrandias (1958)

Let $\tau_0 = 0.567\,143\,290\ldots$ be the positive real number defined by $\tau_0 e^{\tau_0} = 1$.
The function $f(z) = e^{\tau_0 z}$ satisfies $f'(z) = f(z-1)$ and $f(0) = 1$, hence $f^{(n)}(n) = 1$ for all $n \geq 0$.

F. Bertrandias (1958) : an entire function f of exponential type $< \tau_0$ such that $f^{(n)}(n) \in \mathbb{Z}$ for all sufficiently large integers $n \geq 0$ is a polynomial.

Let τ_1 be the complex number defined by $\tau_1 e^{\tau_1} = (1 + i\sqrt{3})/2$.
Then an entire function f of exponential type $< |\tau_1| = 0.616\ \ldots$ such that $f^{(n)}(n) \in \mathbb{Z}$ for all sufficiently large integers $n \geq 0$ is of the form $P(z) + Q(z)e^{\tau_0 z}$, where P and Q are polynomials.

Variations on this theme

- q analogues and multiplicative versions (geometric progressions) :
  Gel'fond (1933, 1952), J.A. Kazmin (1973), J.P. Bézivin (1984, 1992) F. Gramain (1990), M. Welter (2000, 2005), J-P. Bézivin (2014).
- analogs in finite characteristic :
  D. Adam (2011), D. Adam and M. Welter (2015).
- congruences :
  A. Perelli and U. Zannier (1981), J. Pila (2003, 2005).
- several variables :
  S. Lang (1965), F. Gross (1965), A. Baker (1967), V. Avanissian and R. Gay (1975), F. Gramain (1977, 1986), P. Bundschuh (1980) ...

The Masser–Gramain–Weber constant

D.W. Masser (1980) and F. Gramain–M. Weber (1985) studied an analog of Euler's constant for $\mathbb{Z}[i]$, which arises in a 2–dimensional analogue of Stirling's formula :

$$\delta = \lim_{n\to\infty}\left(\sum_{k=2}^{n}(\pi {r_k}^2)^{-1} - \log n\right),$$

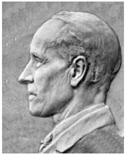
where r_k is the radius of the smallest disc in $\mathbb{R}^2$ that contains at least k integer lattice points inside it or on its boundary.

In 2013, G. Melquiond, W. G. Nowak and P. Zimmermann computed the first four digits :

$$1.819776 < \delta < 1.819833,$$

disproving a conjecture of F. Gramain.

Lidstone and Whittaker interpolation



George James Lidstone

(1870 – 1952)

$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	○	○
$f^{(2n)}$	●	●
$\vdots$	$\vdots$	$\vdots$
f''	●	●
f'	○	○
f	●	●
	s_0	s_1

John Macnaghten Whittaker

(1905 – 1984)

$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	●	○
$f^{(2n)}$	○	●
$\vdots$	$\vdots$	$\vdots$
f''	○	●
f'	●	○
f	○	●
	s_0	s_1

Arithmetic result for Lidstone interpolation

Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n.

$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	$\circ$	$\circ$
$f^{(2n)}$	$\bullet$	$\bullet$
$\vdots$	$\vdots$	$\vdots$
f''	$\bullet$	$\bullet$
f'	$\circ$	$\circ$
f	$\bullet$	$\bullet$
	s_0	s_1

If

$$\tau(f) < \min\left\{1, \frac{\pi}{|s_0 - s_1|}\right\},$$

then f is a polynomial.

This is best possible.

$\bullet$ values in $\mathbb{Z}$ $\qquad$ $\circ$ no condition

Arithmetic result for Lidstone interpolation

If $\tau(f) < \min\left\{1, \dfrac{\pi}{|s_0 - s_1|}\right\}$, $f^{(2n)}(s_0) \in \mathbb{Z}$ *and* $f^{(2n)}(s_1) \in \mathbb{Z}$

for all sufficiently large n*, then* f *is a polynomial.*

The function

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}$$

has exponential type 1 and satisfies $f(s_0) = 1$, $f(s_1) = 0$ and $f'' = f$, hence $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function

$$f(z) = \sin\left(\pi \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\frac{\pi}{|s_1 - s_0|}$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

Sketch of proof

Recall the following variant of Pólya's result :

Let f be an entire function. Let $A \geq 0$. Assume

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}.$$

Then the set

$$\left\{(n, z_0) \in \mathbb{N} \times \mathbb{C} \;\middle|\; |z_0| \leq A,\ f^{(n)}(z_0) \in \mathbb{Z} \setminus \{0\}\right\}$$

is finite.

Consequence. Let s_0 and s_1 be two complex numbers and f an entire function of exponential type < 1 satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Then only finitely many numbers among $f^{(2n)}(s_0)$, $f^{(2n)}(s_1)$ are not 0.

Lidstone interpolation

Lidstone polynomials. Given two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of complex numbers, only finitely many of them are not 0, there is a unique polynomial P such that $P^{(2n)}(0) = a_n$ and $P^{(2n)}(1) = b_n$ for all $n \geq 0$.

Hence, there exists a unique polynomial Q such that the function $\widetilde{f} = f - Q$ satisfies $\widetilde{f}^{(2n)}(s_0) = \widetilde{f}^{(2n)}(s_1) = 0$.

Unicity [H. Poritsky, 1932]. *Let g be an entire function of exponential type $< \pi$ satisfying $g^{(2n)}(0) = g^{(2n)}(1) = 0$ for all sufficiently large n. Then g is a polynomial.*

Hence, if f has exponential type $< \frac{\pi}{|s_0 - s_1|}$, then $\widetilde{f} = 0$.

Arithmetic result for Whittaker interpolation

Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for each sufficiently large n.

$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	●	○
$f^{(2n)}$	○	●
$\vdots$	$\vdots$	$\vdots$
f''	○	●
f'	●	○
f	○	●
	s_0	s_1

Assume

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\}.$$

Then f is a polynomial.

This is best possible.

Arithmetic result for Whittaker interpolation

If $\tau(f) < \min\left\{1, \dfrac{\pi}{2|s_0 - s_1|}\right\}$, $f^{(2n+1)}(s_0) \in \mathbb{Z}$ *and* $f^{(2n)}(s_1) \in \mathbb{Z}$

for each sufficiently large n*, then* f *is a polynomial.*

The function
$$f(z) = \frac{\sinh(z - s_1)}{\cosh(s_0 - s_1)}$$
has exponential type 1 and satisfies $f'(s_0) = 1$, $f(s_1) = 0$ and $f'' = f$, hence $f^{(2n+1)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function
$$f(z) = \cos\left(\frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0}\right)$$
has exponential type $\frac{\pi}{2|s_1 - s_0|}$ and satisfies $f^{(2n+1)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

Poritsky and Gontcharoff–Abel interpolation

Poritsky

$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	○
$f^{(3n+1)}$	○	○	○
$f^{(3n)}$	●	●	●
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\mathrm{iv})}$	○	○	○
f'''	●	●	●
f''	○	○	○
f'	○	○	○
f	●	●	●
	s_0	s_1	s_2

Gontcharoff–Abel

$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+3)}$	●	○	○
$f^{(3n+2)}$	○	○	●
$f^{(3n+1)}$	○	●	○
$f^{(3n)}$	○	○	●
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\mathrm{iv})}$	○	○	●
f'''	●	○	○
f''	○	●	○
f'	○	●	○
f	●	○	○
	s_0	s_1	s_2

Arithmetic result for Poritsky interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem.
If

$$f^{(mn)}(s_j) \in \mathbb{Z}$$

for all sufficiently large n *and for* $0 \leq j \leq m-1$, *then* f *is a polynomial.*

For $m = 2$ with $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ (Lidstone), the assumption on the exponential type $\tau(f)$ of f is

$$\tau(f) < \min\{1, \pi/|s_0 - s_1|\},$$

and this is best possible.

Gontcharoff–Abel interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem.
Assume that for each sufficiently large n, one at least of the numbers

$$f^{(n)}(s_j) \quad j = 0, 1, \ldots, m-1$$

is in $\mathbb{Z}$. Then f is a polynomial.

In the case $m = 2$ with $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ (Whittaker), the assumption is

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\},$$

and this is best possible.

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Integer valued entire functions

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