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Introduction to Transcendental Number Theory **7**

Transcendence of values of special functions

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Abstract

There is no general result on the irrationality or transcendence of values of analytic functions : one needs to restrict to special functions. A number of results are known concerning analytic functions satisfying differential equations. We survey this topic.

We consider the values of the E and G functions introduced by Siegel in 1929, the values of the exponential function, of elliptic or more generally abelian functions, of modular functions, and also the values of functions satisfying functional equations.

Extended abstract

- Irrationality of the values of special functions, transcendence, algebraic independence.
- The exponential function. Hermite–Lindemann Theorem. Trigonometric functions, logarithms.
- Weierstrass dream : transcendental functions take transcendental values. Need to restrict to special functions.
- Siegel E and G–functions. Bessel functions. Hypergeometric functions.
- Differential equations : the Schneider–Lang Theorem. Elliptic functions ; elliptic integrals ; Weierstrass $\wp$ function, zeta function, sigma function ; Jacobi functions sn and cn. Modular invariants j and J, related by $j(\tau) = J(\mathrm{e}^{2i\pi\tau})$. Theta functions. Special values of Euler Gamma and Beta function. Abelian functions and integrals.
- Riemann zeta function
- Functional equations. Mahler's functions.

Numbers : irrationality, transcendence, algebraic independence

We are interested with arithmetic properties of values of special functions.

Irrationality : *is* $f(z_0)$ *rational or irrational ?*

Transcendence : *is* $f(z_0)$ *algebraic (root of a nonzero polynomial with integer coefficients) ?*

Algebraic independence : *are* $f(z_1), f(z_2), \ldots, f(z_m)$ *algebraically dependent ?*
i.e. *does there exist a nonzero polynomial* P *with integer coefficients such that* $P\big(f(z_1), f(z_2), \ldots, f(z_m)\big) = 0$ *?*

Functions : rational, algebraic, transcendental, algebraically independent

Rational functions : $\mathbb{C}(z)$.
Algebraic functions : $P\bigl(z, f(z)\bigr) = 0$
Example :

$$\frac{1}{\sqrt{1-4z}} = \sum_{n\geq 0} \binom{2n}{n} z^n \qquad (|z| < 1).$$

Transcendental functions. Examples :

$$\mathrm{e}^z = \sum_{n\geq 0} \frac{z^n}{n!}.$$

$$\frac{1}{z}\log(1-z) = -\sum_{n\geq 0} \frac{z^n}{n+1} \qquad (|z| < 1).$$

The exponential function

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$$

$$\frac{d}{dz}e^z = e^z, \qquad e^0 = 1.$$

L. Euler

$$e^{i\pi} = -1.$$

Leonhard Euler

1707 – 1783

Charles Hermite and Ferdinand Lindemann

Hermite (1873)

Transcendence of e

$\mathrm{e} = 2.718\,281\,828\,4\ldots$

Lindemann (1882)

Transcendence of π

$\pi = 3.141\,592\,653\,5\ldots$

Hermite – Lindemann Theorem (1882)

Ch. Hermite
1822 – 1901

von Lindemann
1852 – 1939

Theorem. *If w is a nonzero complex number, one at least of the two numbers w, e^w is transcendental.*

Consequences : transcendence of e, π, $\log \alpha$, e^β, for algebraic α and β assuming $\alpha \neq 0$, $\beta \neq 0$, $\log \beta \neq 1$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hermite.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lindemann.html

Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbb{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbb{C}[X, Y]$ is a non-zero polynomial, then the function $P\bigl(z, f(z)\bigr)$ is not 0.

Exercise. *An entire function (analytic in $\mathbb{C}$) is transcendental if and only if it is not a polynomial.*
A meromorphic function in $\mathbb{C}$ is transcendental if and only if it is not rational.

Example. *The transcendental entire function e^z takes an algebraic value at an algebraic argument z only for $z = 0$.*

Transcendence, linear independence, algebraic independence

• An analytic function f which is algebraic over $\overline{\mathbb{Q}}$ takes algebraic values at all algebraic points where f is defined.

• If $f_1, \dots, f_n$ are analytic functions which are linearly dependent over $\overline{\mathbb{Q}}$, then for all $\alpha \in \mathbb{C}$ (where they are defined) the values $f_1(\alpha), \dots, f_n(\alpha)$ are linearly dependent over $\overline{\mathbb{Q}}$.

• If $f_1, \dots, f_n$ are analytic functions which are algebraically dependent over $\overline{\mathbb{Q}}$, then for all $\alpha \in \mathbb{C}$ the values $f_1(\alpha), \dots, f_n(\alpha)$ are algebraically dependent over $\overline{\mathbb{Q}}$.

Weierstrass question

Is it true that a transcendental entire function f takes usually *transcendental values at algebraic arguments?*

Karl Weierstrass

1815 - 1897

Exceptional set of f : bi-algebraic points on the graph

$$E(f) = \left\{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}}\right\}$$

Examples :

- For $f(z) = \mathrm{e}^z$, $E(f) = \{0\}$.
- For $f(z) = \mathrm{e}^{P(z)}$, where $P \in \mathbb{Q}[z]$ is a non–constant polynomial, $E(f)$ is the set of roots of P (Hermite–Lindemann).
- For $f(z) = \mathrm{e}^z + \mathrm{e}^{1+z}$, $E(f) = \emptyset$ (Lindemann–Weierstrass).
- For $f(z) = 2^z$ or $\mathrm{e}^{i\pi z}$, $E(f) = \mathbb{Q}$ (Gel'fond and Schneider).

Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain, Surroca...
If S *is a countable subset of* $\mathbb{C}$ *and* T *is a dense subset of* $\mathbb{C}$*, there exist transcendental entire functions* f *mapping* S *into* T*, as well as all its derivatives.*

Any set of algebraic numbers is the exceptional set of some transcendental entire function.
Also multiplicities can be included.

van der Poorten : *there are transcendental entire functions* f *such that* $D^k f(\alpha) \in \mathbb{Q}(\alpha)$ *for all* $k \geq 0$ *and all algebraic* α*.*

Further results on exceptional sets

For each countable subset A of $\mathbb{C}$ and each family of dense subsets $E_{\alpha,s}$ of $\mathbb{C}$ indexed by $(\alpha, s) \in A \times \mathbb{N}$, there exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f^{(s)}(\alpha) \in E_{\alpha,s}$ for each $(\alpha, s) \in A \times \mathbb{N}$.

Jingjing Huang, Diego Marques & Martin Mereb ;
Algebraic values of transcendental functions at algebraic points.
Bull. Aust. Math. Soc. **82** (2010), 322–327

http://dx.doi.org/10.1017/S0004972710000316

Example : $A = \overline{\mathbb{Q}}$, $E_{\alpha,s} = \overline{\mathbb{Q}}$ or $\mathbb{C} \setminus \overline{\mathbb{Q}}$.
For each $\alpha \in \overline{\mathbb{Q}}$ and $s \geq 0$, choose whether $f^{(s)}(\alpha)$ is algebraic or transcendental.

Recent results on exceptional sets

There exists uncountably many transcendental entire functions f with the property that both f and its inverse function assume algebraic values at algebraic points.

Diego Marques & Carlos Gustavo Moreira ;
A positive answer for a question proposed by K. Mahler.
Math. Ann. **368**, No. 3-4, 1059–1062 (2017).
On a stronger version of a question proposed by K. Mahler.
J. Number Theory **194**, 372–380 (2019).

Diego Marques

Gustavo Moreira

Differential equations, functional equations

Siegel : E and G functions : linear differential equations ; hypergeometric functions, Bessel functions.

Schneider – Lang : nonlinear differential equations ; exponential function, elliptic functions, abelian functions.

Mahler : functional equations

C.L. Siegel
1896 - 1981

Th. Schneider
1903 - 1988

S. Lang
1927 – 2005

L. Mahler
1903 - 1988

Siegel E–functions

Let

$$f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n \in \mathbb{Q}[[z]]$$

be such that

• a_n increases at most exponentially in n (hence f is an entire function)

• f satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$

• The common denominator of $a_0, a_1, \ldots, a_n$ increases at most exponentially in n.

Examples. Algebraic constants, polynomials with algebraic coefficients, the exponential function e^z, the trigonometric functions $\cos z$ and $\sin z$.

Bessel function

The Bessel function

$$J_0(z) = \sum_{n \geq 0} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

$$= 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{36}\left(\frac{z}{2}\right)^6 + \cdots,$$ F.W. Bessel

1784 - 1846

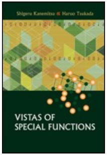
is an E function, solution of the Bessel differential equation

$$y'' + \frac{1}{z}y' + y = 0.$$

References on special functions

Shigeru Kanemitsu, Haruo Tsukada,
Vistas of Special Functions, World Scientific, 2007
Kalyan Chakraborty, Shigeru Kanemitsu & Haruo Tsukada,
Vistas of Special Functions II World Scientific, Singapore, 2009.
Kalyan Chakraborty, Shigeru Kanemitsu & T. Kuzumaki,
A Quick Introduction to Complex Analysis, World Scientific, 2016.


K. Chakraborty




S. Kanemitsu

Srinivasa Ramanujan and Friedrich Wilhelm Bessel

Henri Cohen,
Some formulas of Ramanujan involving Bessel functions
Publications mathématiques de Besançon (2010), 59–68.

http://pmb.univ-fcomte.fr/2010/Cohen.pdf

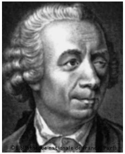
F.W. Bessel
1784 - 1846

S. Ramanujan
1887 - 1920

H. Cohen

Ramanujan also investigated Bessel q series.

Euler Gamma and Beta functions



L. Euler
1707 - 1783

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t}$$
$$= e^{-\gamma z} z^{-1} \prod_{n\geq 1} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

Pochhammer symbol : *rising factorial power*

$$
\begin{aligned}
(x)_n &= x(x+1)\cdots(x+n-1) \\
&= \frac{\Gamma(x+n)}{\Gamma(x)}.
\end{aligned}
$$

Leo August Pochhammer
1841-1920

http://scienceworld.wolfram.com/biography/Pochhammer.html

Siegel hypergeometric E functions

Let $a_1, \dots, a_\ell$, $b_1, \dots, b_m$ be rational numbers with $m > \ell$ and $b_1, \dots, b_m$ not in $\{0, -1, -2, \dots\}$ and $b_m = 1$. Define

$$c_n = \frac{(a_1)_n \cdots (a_\ell)_n}{(b_1)_n \cdots (b_m)_n}$$

Set $t = m - \ell$.

Then

$$f(z) = \sum_{n \geq 1} c_n z^{tn}$$

is an E–function.

Siegel's Theorem (1929)

For $\lambda \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$, consider the E–function

$$K_\lambda(z) = \sum_{n \geq 0} \frac{(-1)^n}{(\lambda+1)_n n!} \left(\frac{z}{2}\right)^{2n},$$

solution of the second order differential equation

$$y'' + \frac{2\lambda+1}{z} y' + y = 0.$$

For $\lambda \in \mathbb{Q}$ not in $\{\pm\frac{1}{2}, -1, \pm\frac{3}{2}, -2, \ldots\}$, for any algebraic number $\alpha \neq 0$, the two numbers $K_\lambda(\alpha)$ and $K'_\lambda(\alpha)$ are algebraically independent.

Bessel functions

Bessel functions of the first kind :

$$J_\lambda(z) = \sum_{n\geq 1} \frac{(-1)^n (z/2)^{2n+\lambda}}{n!\Gamma(n+1+\lambda)}$$

$$= \frac{1}{\Gamma(\lambda+1)} \left(\frac{z}{2}\right)^\lambda K_\lambda(z),$$

solution of the differential equation

$$z^2 y'' + zy' + (z^2 - \lambda^2)y = 0.$$

Also $J_{-\lambda}(z)$ is a solution of the same differential equation.
Modified Bessel functions of the first kind :

$$I_\lambda(z) = \sum_{n\geq 1} \frac{(z/2)^{2n+\lambda}}{n!\Gamma(n+1+\lambda)} = i^{-\lambda} J_\lambda(iz).$$

Bessel functions and continued fractions

From Siegel's 1929 Theorem, it follows that the number

$$\frac{I_1(2)}{I_0(2)} = [0; 1, 2, 3, \dots] = 0.697774658\dots$$

(Sloane's A052119, A001053 and A001040)
is transcendental.

Weinstein, Eric W. *Continued Fraction Constant.* From MathWorld – A Wolfram Web Resource.
http://mathworld.wolfram.com/ContinuedFractionConstant.html

Bessel functions and continued fractions

$$I_{-1/2}(1) = \frac{\sqrt{2}}{\pi} \cdot \frac{e + e^{-1}}{2}, \quad I_{1/2}(1) = \frac{\sqrt{2}}{\pi} \cdot \frac{e - e^{-1}}{2}.$$

$$[1; 3, 5, 7 \ldots] = \frac{e^2 + 1}{e^2 - 1} = \frac{I_{-1/2}(1)}{I_{1/2}(1)},$$

$$[2; 6, 10, 14 \ldots] = \frac{e + 1}{e - 1} = \frac{I_{-1/2}(1/2)}{I_{1/2}(1/2)}$$

B. Sury

B. Sury, Bessel contain continued fractions of progressions ; Resonance **10** 3 (2005) 80–87.

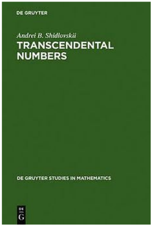
http://www.ias.ac.in/article/fulltext/reso/010/03/0080-0087

Siegel–Shidlovskiĭ theory

Generalization of C.L. Siegel 1929 results by C.L. Siegel himself in 1949, by A.B. Shidlovskiĭ in 1953 – 1955.

Given a set $\{f_1, \dots, f_n\}$ *of* E*–functions satisfying a system of linear differential equations and an algebraic number* α*, the transcendence degree of the field* $\mathbb{Q}\big(f_1(\alpha), f_2(\alpha), \dots, f_n(\alpha)\big)$ *over* $\mathbb{Q}$ *is equal to the transcendence degree of the field* $\mathbb{C}\big(z, f_1, f_2, \dots, f_n\big)$ *over* $\mathbb{C}(z)$*.*

A.B. Shidlovskiĭ.
Transcendental numbers.
Studies in mathematics, **12**,
Walter de Gruyter (1989).



A refined version of the Siegel–Shidlovskiĭ theorem

Algebraic relations among values of E–functions at an algebraic point arise from algebraic relations among the functions.

Linear independence of the values $f_i(\alpha)$ when $f_1, \dots, f_n$ are linearly independent E–functions.

F. Beukers

F. Beukers. *A refined version of the Siegel-Shidlovskiĭ theorem*. Ann. Math. (2) **163**, No. 1, 369-379 (2006).
https ://arxiv.org/abs/math/0405549

Algorithm for determining the exceptional set of an E–function

B. Adamczewski & T. Rivoal *Exceptional values of E-functions at algebraic points*. Bull. Lond. Math. Soc. **50** (2018), no. 4, 697–708.

https://doi.org/10.1112/blms.12168

B. Adamczewski

T. Rivoal

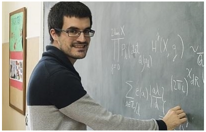
Siegel's Conjecture (1949)

Is-it true that any E–function can be represented as a polynomial with algebraic coefficients in a finite number of hypergeometric E- functions of the form ${}_pF_q(z^{q-p+1})$, $q \geq p \geq 1$, with rational parameters ?

Answer : No.

S. Fischler

T. Rivoal



J. Fresán

P. Jossen

S. FISCHLER & T. RIVOAL. *On Siegel's problem for E-functions*, Rend. Semin. Mat. Univ. Padova, (2021).
https://arxiv.org/pdf/1910.06817.pdf

J. FRESÁN & P. JOSSEN. *A non-hypergeometric E-function*, (2021).
https://arxiv.org/abs/2012.11005

Siegel G–functions

Let

$$g(z) = \sum_{n \geq 0} a_n z^n \in \mathbb{Q}[[z]]$$

be such that

• g has a positive radius of convergence

• g satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$

• The common denominator of $a_0, a_1, \ldots, a_n$ increases at most exponentially in n.

Examples.

• Algebraic functions

• Hypergeometric functions with rational parameters

• Solutions of Picard–Fuchs equations over $\mathbb{Q}(z)$.

Gauss Hypergeometric function

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix} \middle| z\right) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n. $$

It satisfies second order linear differential equation

$$ z(z-1)y'' + \big((a+b+1)z - c\big)y' + aby = 0. $$

Resources on this topic are available on the website (in French) :
`https://danielduverney.fr/fh/`

Examples of hypergeometric functions

$$ {}_2F_1\left(\begin{matrix}1 & 1\\ & 2\end{matrix}\middle| z\right) = -\frac{1}{z}\log(1-z) $$

$$ {}_2F_1\left(\begin{matrix}1/2 & 1\\ & 1\end{matrix}\middle| z\right) = (1-z)^{-1/2} $$

$$ {}_2F_1\left(\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\middle| z\right) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}}. $$

F. Beukers. *Hypergeometric Functions, How Special Are They?*
Notices Am. Math. Soc. **61**, No. 1, 48–56 (2014).
http://dx.doi.org/10.1090/noti1065

Eisenstein hypergeometric function

The G function

$$\sum_{n\geq 0}(-1)^n \frac{\binom{5n}{n}}{4n+1} z^{4n+1}$$

F.G.M. Eisenstein

1823 - 1852

$$= z - z^5 + 10\frac{z^9}{2!} - 15\cdot 14\frac{z^{13}}{3!} + \cdots ,$$

which converges for $|z| < 5^{-5/4}$, is a solution of the quintic equation $x^5 + x = z$.

J. Stillwell, Eisenstein's footnote, Math. Intelligencer **17** (1995), no. 2, 58–62.

F.G.M. Eisenstein, Allgemeine Auflösung der Gleichungen von den ersten vier Graden. J. Reine Angew. Math. 27 (1844), 81–83. Mathematische Werke I, 7–9.

Connections between E and G functions

If

$$g(z) = \sum_{n \geq 0} a_n z^n$$

is a G–function, then

$$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

is an E–function and conversely.

Remarks:

• Siegel's definitions involve algebraic coefficients, not only rational coefficients.

• Apéry's proof of the irrationality of $\zeta(3)$ is related with the theory of G–functions.

S. Fischler & T. Rivoal, Approximants de Padé et séries hypergéométriques équilibrées, J. Math. Pures Appl. **82** (2003), no. 10, 1369–1394.

The ring $\mathbb{G}$

S. Fischler and T. Rivoal introduce the set $\mathbb{G}$ of all values taken by any analytic continuation of any G–function at any algebraic point.

S. Fischler

T. Rivoal

They prove that $\mathbb{G}$ is a countable subring of $\mathbb{C}$ which contains the field $\overline{\mathbb{Q}}$ of algebraic numbers and the logarithms of algebraic numbers.

Conjecturally, $\mathbb{G}$ is not a field.

S. Fischler & T. Rivoal, On the values of G-functions, Comment. Math. Helv. **89** (2014), 313–341.

http://dx.doi.org/10.4171/CMH/321

Euler–Mascheroni constant

Lorenzo Mascheroni
(1750 – 1800)

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx$$
$$= -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

Euler's constant

Sondow's proof of the double integral formula was inspired by the work of F. Beukers related with Apéry's proof of the irrationality of $\zeta(3)$.

F. Beukers

Jonathan Sondow

http://home.earthlink.net/~jsondow/

Jonathan Sondow http://home.earthlink.net/~jsondow/

$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2 \binom{t+k}{k}} dt$$

$$\gamma = \lim_{s \to 1+} \sum_{n=1}^\infty \left(\frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)} F\begin{pmatrix} 1, & 2, & 2 \\ 3, & t+2 \end{pmatrix} dt.$$

Quoting Tanguy Rivoal

It is believed that $\gamma \notin \mathbb{Q}$. Why ? Because if $\gamma = p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$, then $|q| \geq 10\,242\,080$.

It is also believed that $\gamma \notin \mathbb{G}$. Why ? Because Euler and Ramanujan would have found various formulas proving this fact.
It is also plausible that γ does not even belong to the field of fractions of $\mathbb{G}$.

Further, it is expected that $\mathrm{e} = \exp(1)$ does not belong to the field of fractions of $\mathbb{G}$.

Conjecture of Bombieri and Dwork

According to a conjecture of Bombieri and Dwork, $\mathbb{G}$ should coincide with the set of *periods* of algebraic varieties defined over $\overline{\mathbb{Q}}$.

E. Bombieri

B. Dwork

1923–1998

Connection with the *periods* of Kontsevich and Zagier

M. Kontsevich

D. Zagier

The units of the ring $\mathbb{G}$

The group of units of $\mathbb{G}$ contains $\overline{\mathbb{Q}}^\times$ and the values $\mathrm{B}(a,b)$, (a, b in $\mathbb{Q}$) of Euler Beta function.

The numbers $\Gamma(a/b)^b$, $a/b \in \mathbb{Q} \setminus \{0, -1, -2, \ldots\}$, are units in the ring $\mathbb{G}$.

For instance, $\pi = \Gamma(1/2)^2$ is a unit.
Proof:

$$\pi = \sum_{n\geq 1} \frac{4(-1)^n}{2n+1}, \quad \frac{1}{\pi} = \sum_{n\geq 1} \frac{(42n+5)\binom{2n}{n}^3}{2^{12n+4}}.$$

Walt Disney Productions and $1/\pi$

The formula

$$\frac{16}{\pi} = \sum_{n\geq 0} (42n+5) \frac{{(1/2)_n}^3}{n!^3 2^{6n}}$$

appeared in the Walt Disney film *High School Musical*, starring Vanessa Anne Hudgens, who plays an exceptionally bright high school student named Gabriella Montez. Gabriella points out to her teacher that she had incorrectly written the left-hand side as $\frac{8}{\pi}$ instead of $\frac{16}{\pi}$ on the blackboard. After first claiming that Gabriella is wrong, her teacher checks (possibly Ramanujan's Collected Papers ?) and admits that Gabriella is correct.

N.D. Baruah, B. Berndt & H.H. Chan.
Ramanujan's Series for $1/\pi$: A Survey.
American Mathematical Monthly **116** (2009) 567–587.

http://www.jstor.org/stable/40391165

Ramanujan series for $1/\pi$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{k\geq 0} \frac{(4k)!}{k!^4} \cdot \frac{26390k + 1103}{396^{4k}}.$$

N.D. Baruah

B. Berndt

Chan Heng Huat

Nayandeep Deka Baruah, Bruce C. Berndt & Heng Huat Chan.
Ramanujan's Series for $1/\pi$: A Survey.
American Mathematical Monthly **116** (2009) 567–587.

https://en.wikipedia.org/wiki/Ramanujan-Sato_series

Algebraic values of Siegel G functions

Let f be a G–function which is not algebraic. Is–it true that $f(\alpha)$ is algebraic for at most finitely many algebraic α ?

F. Beukers

$$
{}_2F_1\left(\begin{matrix}\frac{1}{12} & \frac{5}{12} \\ & \frac{1}{2}\end{matrix}\,\middle|\,\frac{1323}{1331}\right)=\frac{3}{4}\sqrt[4]{11}.
$$

F. Beukers. *Algebraic values of G-functions*. J. Reine Angew. Math. **434** (1993), 45–65.

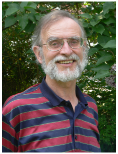
Wolfart's work (1988)

Let $f(z) = {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right)$ with a, b, c in $\mathbb{Q}$. Let Δ be the monodromy group and

$$E = \{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}}\}$$

the exceptional set of f.

(1) If f is algebraic (Δ finite), then $E = \overline{\mathbb{Q}}$.
(2) If f is *arithmetic*, then E is dense in $\overline{\mathbb{Q}}$.
(3) Otherwise, E is finite.

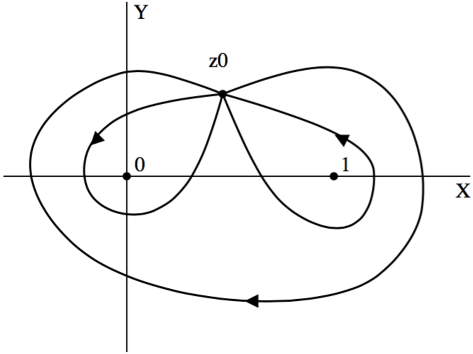


Jürgen Wolfart

A. Yafaev & B. Edixhoven : complete the proof of (3).
Subvarieties of Shimura varieties. Ann. of Math. (2) **157** (2003), no. 2, 621–645.

Monodromy

Singular points 0, 1, ∞



Monodromy matrices : M_0, M_1, M_∞ with $M_0 M_1 M_\infty = \mathrm{Id}$.

http://math.uchicago.edu/~drinfeld/Hypergeometric_motives/Beukers-Heckman.pdf

Arithmetic groups

We assume that $0 < a, b, c \leq 1$ and either $a, b < c$ or $c < a, b$. Then the monodromy modulo scalars embeds in $\mathrm{PSL}(2, \mathbb{R})$.

Let $g_i \in \mathrm{SL}(2, \mathbb{R})$ be a lift of the monodromy around $i = 0, 1, \infty$. Then $\mathrm{Id}_2, {g_0}^2, {g_1}^2, g_\infty^2$ generate a quaternion algebra H defined over

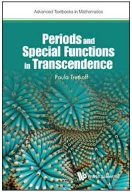
$$k = \mathbb{Q}\big(\cos^2 \pi c, \cos^2 \pi(a-b), \cos^2 \pi(c-a-b),\\ \cos \pi c \cos \pi(a-b) \cos \pi(c-a-b)\big).$$

We say that our monodromy is *arithmetic* if H is split at exactly one infinite places of k.

Reference

Paula Tretkoff,
Periods and Special Functions
in Transcendence ;
Advanced Textbooks in
Mathematics (2017)
World Scientific.

https://doi.org/10.1142/q0085



Paula Tretkoff

Modular group

Let $a = 1/12$, $b = 5/12$, $c = 1/2$. Then the quaternion algebra is $M(2, \mathbb{Q})$ and the monodromy group is $\mathrm{SL}(2, \mathbb{Z})$. It can be shown that

$$ {}_2F_1\left(\begin{matrix} 1/12 \ \ 5/12 \\ 1/2 \end{matrix}\middle| 1 - \frac{1}{J(\tau)}\right)^4 = \frac{E_4(\tau)}{E_4(i)} $$

where

$$ E_4(\tau) = 1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n}, \quad \Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} $$

and $q = \mathrm{e}^{2i\pi\tau}$, $J(\tau) = E_4(\tau)^3/1728\Delta(\tau)$. In particular $J(i) = 1$. From CM-theory it follows that if $\tau_0 \in \mathbb{Q}(i)$, $\mathrm{Im}(\tau_0) > 0$, then both $J(\tau_0)$ and $E_4(\tau_0)/E_4(i)$ are algebraic.

Further examples

$$ {}_2F_1\left(\begin{matrix}1/4 & 1/2\\ \multicolumn{2}{c}{3/4}\end{matrix}\middle|\,\frac{80}{81}\right) = \frac{9}{5}. $$

$$ {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ \multicolumn{2}{c}{5/6}\end{matrix}\middle|\,\frac{27}{32}\right) = \frac{8}{5}. $$

$$ {}_2F_1\left(\begin{matrix}1/12 & 1/4\\ \multicolumn{2}{c}{5/6}\end{matrix}\middle|\,\frac{135}{256}\right) = \frac{2}{5}\sqrt[6]{270}. $$

Akihito Ebisu (2014)

$$ {}_2F_1\left(\begin{matrix}1/24 & 7/24\\ \multicolumn{2}{c}{5/6}\end{matrix}\middle|\,-\frac{2^{10}3^35}{11^4}\right) $$

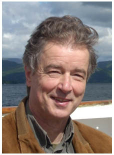
$$ = \sqrt{6}\sqrt[6]{\frac{11}{5^5}}. $$

Yifan Yang (2015)

Non arithmetic examples

$$ {}_2F_1\left(\begin{matrix}1-3a & 3a\\ & a\end{matrix}\,\middle|\,\frac{1}{2}\right)=2^{3-2a}\cos\pi a. $$

$$ {}_2F_1\left(\begin{matrix}2a & 1-4a\\ & 1-a\end{matrix}\,\middle|\,\frac{1}{2}\right)=4^{a}\cos\pi a. $$



F. Beukers

$$ {}_2F_1\left(\begin{matrix}7/48 & 31/48\\ & 29/24\end{matrix}\,\middle|\,-\frac{1}{3}\right)=2^{5/24}3^{-11/12}5\cdot\sqrt{\frac{\sin\pi/24}{\sin 5\pi/24}}. $$

Schneider – Lang Theorem (1949, 1966)

Theodor Schneider

1911 – 1988

Serge Lang

1927 – 2005

Let $f_1, \ldots, f_m$ *be meromorphic functions in* $\mathbb{C}$*. Assume* f_1 *and* f_2 *are algebraically independent and of finite order. Let* $\mathbb{K}$ *be a number field. Assume* f'_j *belongs to* $\mathbb{K}[f_1, \ldots, f_m]$ *for* $j = 1, \ldots, m$*. Then the set*

$$S = \{w \in \mathbb{C} \mid w \text{ not pole of } f_j, \; f_j(w) \in \mathbb{K} \text{ for } j = 1, \ldots, m\}$$

is finite.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Schneider.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lang.html

Hermite – Lindemann Theorem (again)

Charles Hermite

1822 – 1901

von Lindemann

1852 – 1939

Carl Louis Ferdinand

Corollary. *If w is a nonzero complex number, one at least of the two numbers w, e^w is transcendental.*

Proof. Let $\mathbb{K} = \mathbb{Q}(w, \mathrm{e}^w)$. The two functions $f_1(z) = z$, $f_2(z) = \mathrm{e}^z$ are algebraically independent, of finite order, and satisfy the differential equations $f_1' = 1$, $f_2' = f_2$. The set S contains $\{\ell w \mid \ell \in \mathbb{Z}\}$. Since $w \neq 0$, this set is infinite ; it follows that $\mathbb{K}$ is not a number field. $\square$

The exponential function (again)

$$\frac{d}{dz}\mathrm{e}^z = \mathrm{e}^z, \qquad \mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$$

$$\begin{array}{rccc} \exp : & \mathbb{C} & \rightarrow & \mathbb{C}^\times \\ & z & \mapsto & \mathrm{e}^z \end{array}$$

$$\ker \exp = 2i\pi\mathbb{Z}.$$

The function $z \mapsto \mathrm{e}^z$ is the exponential map of the multiplicative group $\mathbb{G}_m$.
The exponential map of the additive group $\mathbb{G}_a$ is

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{C} \\ z & \mapsto & z \end{array}$$

The only period is 0.

Elliptic curves and elliptic functions

Elliptic curves : $\Delta = {g_2}^3 - 27{g_3}^2 \neq 0$.

$$E = \left\{(t:x:y)\ ;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\} \subset \mathbb{P}_2(\mathbb{C}).$$

Elliptic functions

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\wp(z_1+z_2) = R\big(\wp(z_1), \wp(z_2)\big)$$

$$\begin{array}{rccc} \exp_E : & \mathbb{C} & \to & E(\mathbb{C}) \\ & z & \mapsto & \big(1:\wp(z):\wp'(z)\big) \end{array}$$

$$\ker\exp_E = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Weierstraß elliptic function

$$\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

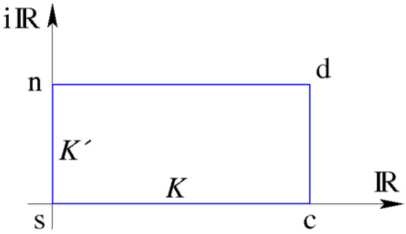
Karl Weierstrass

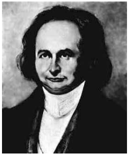
1815–1897

$$\wp'(z) = \sum_{\omega \in \Omega} \frac{-2}{(z-\omega)^3}.$$

Jacobi 12 elliptic functions

Elliptic curve as an intersection of quadrics : the functions sn and cn.





Karl Jacobi

1804–1851

sn sc sd ns nc nd cs cn cd ds dn dc

https://en.wikipedia.org/wiki/Jacobi_elliptic_functions

Periods of a Weierstrass elliptic function

The set of periods of an elliptic function is a *lattice* :

$$\Omega = \{\omega \in \mathbb{C}\,;\ \wp(z+\omega) = \wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

A pair of fundamental periods (ω_1, ω_2) is given by

$$\omega_i = 2\int_{e_i}^{\infty} \frac{\mathrm{d}t}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad (i = 1, 2)$$

where

$$4t^3 - g_2 t - g_3 = 4(t-e_1)(t-e_2)(t-e_3).$$

Examples

Example 1 : $g_2 = 4$, $g_3 = 0$, $j = 1728$

A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2.$$

is given by

$$\omega_1 = \int_1^\infty \frac{\mathrm{d}t}{\sqrt{t^3 - t}} = \frac{1}{2}\mathrm{B}(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} = 2.6220575542\ldots$$

and

$$\omega_2 = i\omega_1.$$

Examples (continued)

Example 2 : $g_2 = 0$, $g_3 = 4$, $j = 0$

A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4t^3.$$

is

$$\omega_1 = \int_1^\infty \frac{\mathrm{d}t}{\sqrt{t^3 - 1}} = \frac{1}{3}\mathrm{B}(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

and

$$\omega_2 = \varrho\omega_1,$$

where $\varrho = \mathrm{e}^{2i\pi/3}$.

Chowla–Selberg Formula

S. Chowla

1907 - 1995

A. Selberg

1917 - 2007

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6\cdot 3\cdot 5\cdot \pi^2}$$

and

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\pi^6}.$$

Formula of Chowla and Selberg (1966) : *the periods of elliptic curves with complex multiplication are products of values of the Gamma function at rational points.*

Chowla–Selberg Formula : an example

F. Adiceam

Faustin Adiceam (2011) :

$$\Gamma\left(\frac{1}{5}\right) = \sqrt{\frac{\pi}{2^{19/5}} \cdot \frac{1}{\sin\left(\frac{3\pi}{5}\right)\sin\left(\frac{9\pi}{20}\right)\sin\left(\frac{7\pi}{20}\right)\sin\left(\frac{\pi}{10}\right)} \cdot \frac{\Gamma\left(\frac{1}{20}\right)\times\Gamma\left(\frac{3}{20}\right)}{\Gamma\left(\frac{9}{20}\right)\times\Gamma\left(\frac{7}{20}\right)}}$$

$$= \sqrt{\frac{\pi}{2^{9/5}} \cdot \frac{(5+\sqrt{5})\left(2\sqrt{5}-\sqrt{2(5+\sqrt{5})}\right)}{5} \cdot \frac{\Gamma\left(\frac{1}{20}\right)\times\Gamma\left(\frac{3}{20}\right)}{\Gamma\left(\frac{9}{20}\right)\times\Gamma\left(\frac{7}{20}\right)}}.$$

Elliptic integrals and ellipses

An ellipse with radii a and b has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the length of its perimeter is

$$2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}}\,\mathrm{d}x.$$

In the same way, the perimeter of a lemniscate

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

is given by an elliptic integral

$$4a\int_0^1 (1 - t^4)^{-1/2}\,\mathrm{d}x.$$

Hypergeometry and elliptic integrals

Recall Gauss Hypergeometric series

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!} $$

C.F. Gauss

1777 - 1855

$$ \begin{aligned} K(z) &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2}\cdot {}_2F_1\left(\begin{matrix} 1/2 & 1/2 \\ & 1 \end{matrix}\middle| z^2\right). \end{aligned} $$

Weierstrass sigma function

Karl Weierstrass

1815–1897

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.
The canonical product of Weierstraß associated with Ω is the sigma function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

This function has a simple zero at each point of Ω.

Hadamard canonical products

J. Hadamard

1865 - 1963

For $\mathbb{N} = \{0, 1, 2, \ldots\}$:

$$-\frac{e^{\gamma z}}{\Gamma(-z)} = z \prod_{n \geq 1} \left(1 - \frac{z}{n}\right) e^{-z/n}.$$

For $\mathbb{Z}$:

$$\frac{\sin \pi z}{\pi} = z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right).$$

Wallis formula for π

John Wallis (Arithmetica Infinitorum 1655)

$$\frac{\pi}{2} = \prod_{n\geq 1} \left(\frac{4n^2}{4n^2-1} \right)$$

$$= \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}.$$

J. Wallis

1616 - 1703

Weierstraß sigma function : an example

For $\Omega = \mathbb{Z} + \mathbb{Z}i$:

$$\sigma_{\mathbb{Z}[i]}(z) = z \prod_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right) \cdot$$

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}\mathrm{e}^{\pi/8}\Gamma(1/4)^{-2} = 0.4749493799\ldots$$

For $\alpha \in \mathbb{Q}(i)$, the number $\sigma_{\mathbb{Z}[i]}(\alpha)$ is algebraic over

$$\mathbb{Q}\bigl(\pi,\ \mathrm{e}^{\pi},\ \Gamma(1/4)\bigr).$$

D.W. Masser, Auxiliary Polynomials in Number Theory, Cambridge Tracts in Mathematics **207** (2016), Cambridge University Press, Chap. 20.

Weierstraß zeta function

The logarithmic derivative of the Weierstraß sigma function is the *Weierstraß* zeta function

$$\frac{\sigma'}{\sigma} = \zeta$$

and the derivative of ζ is $-\wp$. The minus sign is selected so that

$$\wp(z) = \frac{1}{z^2} + \text{a function analytic at } 0.$$

The function ζ is therefore *quasi–periodic* : for any $\omega \in \Omega$ there exists $\eta = \eta(\omega)$ such that

$$\zeta(z + \omega) = \zeta(z) + \eta.$$

Legendre relation (1811, 1825)

The numbers $\eta(\omega)$ are the *quasi–periods* of the elliptic curve.
When (ω_1, ω_2) is a pair of fundamental periods, we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$.
Legendre relation :

$$\omega_2\eta_1 - \omega_1\eta_2 = 2i\pi.$$

this is Louis Legendre and not Adrien Marie Legendre (1752 - 1833)

Legendre and Fourier

Peter Duren, Changing Faces : The Mistaken Portrait of Legendre.
Notices of American Mathematical Society, **56** (2009) 1440–1443.

The Mathematics Consortium Bulletin, October 2020, vol.2, Issue 2, p. 34–35.
https://www.themathconsortium.in/publication/timcbulletin/vol2/issue2

Examples

For the curve $y^2t = 4x^3 - 4xt^2$ the quasi–periods associated to the previous fundamental periods are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1,$$

while for the curve $y^2t = 4x^3 - 4t^3$ they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1.$$

Transcendence and elliptic functions

Siegel (1932) : elliptic analog of Lindemann's Theorem on the transcendence of π.

Schneider (1937) : elliptic analog of Hermite–Lindemann Theorem. General transcendence results on values of elliptic functions, on periods, on elliptic integrals of the first and second kind.

C.L. Siegel
1896 - 1981

Th. Schneider
1911 - 1988

Elliptic analog of Hermite–Lindemann Theorem

Let $w \in \mathbb{C}$, not pole of $\wp$. Then one at least of the numbers g_2, g_3, w, $\wp(w)$ is transcendental.

`Proof` as a consequence of the Schneider–Lang Theorem.
Let $\mathbb{K} = \mathbb{Q}(g_2, w, \wp(w), \wp'(w))$. The two functions $f_1(z) = z$, $f_2(z) = \wp(z)$ are algebraically independent, of finite order. Set $f_3(z) = \wp'(z)$. From ${\wp'}^2 = 4\wp^3 - g_2\wp - g_3$ one deduces

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6{f_2}^2 - (g_2/2).$$

The set S contains

$$\{\ell w \mid \ell \in \mathbb{Z},\ \ell w \text{ not pole of } \wp\}$$

which is infinite. Hence $\mathbb{K}$ is not a number field. $\square$

Elliptic integrals of the third kind

Quasi–periodicity of the Weierstraß sigma function :

$$\sigma(z+\omega_i) = -\sigma(z)\mathrm{e}^{\eta_i(z+\omega_i/2)} \quad (i=1,2).$$

The function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)}\mathrm{e}^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z)\mathrm{e}^{\eta_i u - \omega_i\zeta(u)}.$$

J-P. Serre (1979)

Periods of elliptic integrals of the third kind

Theorem (1979). *Assume* g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β *are algebraic and* $\mathbb{Z}u_1 \cap \Omega = \{0\}$. *Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)}\mathrm{e}^{\bigl(\beta-\zeta(u_1)\bigr)u_2}$$

is transcendental.

Corollary. *Transcendence of periods of elliptic integrals of the third kind :*

$$\mathrm{e}^{\omega\zeta(u)-\eta u+\beta\omega}.$$

Schneider's Theorem on Euler's Beta function

Th. Schneider

1911 - 1988

Let a, b be rational numbers, not integers. Then the number $\mathrm{B}(a, b)$ is transcendental.

Further results by Th. Schneider (1941) and S. Lang (1960's) on abelian functions, abelian varieties and commutative algebraic groups.

Linear independence of transcendental numbers

A. Baker, J. Coates, D.W. Masser, G. Wüstholz,...

Values of Euler Beta and Gamma functions

J. Wolfart

G. Wüstholz

G. Wüstholz : any $\overline{\mathbb{Q}}$–linear relation among periods of an abelian variety arises from its endomorphisms.

(J. Wolfart & G. Wüstholz) : linear independence over the field of algebraic numbers of the values of the Euler Beta function at rational points (a, b).

Transcendence of values at algebraic points of hypergeometric functions with rational parameters.

J. Wolfart & G. Wüstholz. *Der Überlagerungsradius gewisser algebraischer Kurven und die Werte der Betafunktion an rationalen Stellen*. Math. Ann. **273** (1985), no. 1, 1–15.

Transcendence of $\Gamma(1/4)$ and $\Gamma(1/3)$ (1976)

Algebraic independence of two numbers among $\omega_1, \omega_2, \eta_1, \eta_2$ when g_2 and g_3 are algebraic.

$\Gamma(1/4)$ and π are algebraically independent.

$\Gamma(1/3)$ and π are algebraically independent.

G.V. Chudnovsky

Modular functions

$$P(q) = E_2(q) = 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1-q^n},$$

$$Q(q) = E_4(q) = 1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1-q^n},$$

$$R(q) = E_6(q) = 1 - 504 \sum_{n \geq 1} \frac{n^5 q^n}{1-q^n}.$$

S. Ramanujan

1887 - 1920

$$\Delta = \frac{1}{1728}\left(Q^3 - R^2\right) \quad \text{and} \quad J = \frac{Q^3}{\Delta}.$$

Nesterenko's Theorem

1996, Yu. V. Nesterenko :
Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Then three at least of the numbers

$$q, P(q), Q(q), R(q)$$

are algebraically independent.

Y. Nesterenko

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of
$\Gamma(1/4)$, π and e^{π}.
Also : Algebraic independence of
$\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.141\,592\,653\,5\ldots$ *and* $e^{\pi} = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

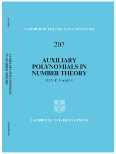
Open problem :
Show that e *and* π *are algebraically independent.*

Transcendence of values of Dirichlet's L–functions :
Sanoli Gun, Ram Murty and Purusottam Rath (2009).

Transcendence of $\Gamma(i)$ following D.W. Masser

$$\Gamma(i)\overline{\Gamma(i)} = \Gamma(i)\Gamma(-i) = \frac{\Gamma(i)\Gamma(1-i)}{-i} = \frac{\pi}{-i\sin(i\pi)} = \frac{2\pi}{e^{\pi} - e^{-\pi}}.$$

D.W. Masser



D.W. Masser, Auxiliary Polynomials in Number Theory, Cambridge Tracts in Mathematics **207** (2016), Cambridge University Press.

Standard relations among Gamma values

(Translation) :

$$\Gamma(a+1) = a\Gamma(a).$$

(Reflection) :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}.$$

(Multiplication) : for any positive number n,

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Conjecture of Rohrlich

Conjecture (D. Rohrlich)
Any multiplicative relation

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

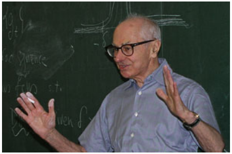
with b *and* m_a *in* $\mathbb{Z}$ *is in the ideal generated by the standard relations.*

David Rohrlich

S. Lang. *Relations de distributions et exemples classiques*
Séminaire Delange-Pisot-Poitou, 19e année : 1977/78, Théorie des nombres, Fasc. 2, Exp. N° 40, 6 p.
Collected Papers, vol. III, Springer (2000), 59–65.

Conjecture of Rohrlich–Lang

D. Rohrlich



S. Lang

1927 - 2005

Conjecture (D. Rohrlich–S. Lang, 1978) *Any algebraic dependence relation among* $(2\pi)^{-1/2}\Gamma(a)$ *with* $a \in \mathbb{Q}$ *is in the ideal generated by the standard relations* (universal odd distribution).

Consequence of the conjecture of Rohrlich–Lang
(F. Adiceam) : the three numbers $\Gamma(1/5)$, $\Gamma(2/5)$ and $\mathrm{e}^{\pi\sqrt{5}}$ are algebraically independent. *(Not yet know)*.

Euler Gamma function

Is the number

$$\Gamma(1/5) = 4.590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\ \dots$$

irrational ?

$$\Gamma(z) = \mathrm{e}^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} \mathrm{e}^{z/n} = \int_0^{\infty} \mathrm{e}^{-t} t^{z} \cdot \frac{dt}{t}$$

Here is the set of rational values $r \in (0,1)$ for which the answer is known (and, for these arguments, the Gamma value $\Gamma(r)$ is a transcendental number) :

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \pmod 1.$$

Gamma values

$$\frac{\Gamma(1/24)\Gamma(11/24)}{\Gamma(5/24)\Gamma(7/24)} = \sqrt{3}\sqrt{2+\sqrt{3}}.$$

Yves André — Groupes de Galois motiviques et périodes.
Séminaire N. Bourbaki, Samedi 7 novembre 2015, 68ème année, 2015-2016, n° 1104.
http://www.bourbaki.ens.fr/TEXTES/1104.pdf

Y. André

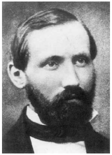
J. Ayoub : analogs for function fields of the periods conjectures of Grothendieck and Kontsevich–Zagier.
Une version relative de la conjecture des périodes de Kontsevich-Zagier.
Ann. of Math. (2) **181** (2015), no. 3, 905–992.

Riemann zeta function

L. Euler

1707 - 1783

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$



B. Riemann

1826 - 1866

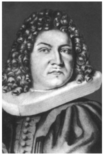
Euler : $s \in \mathbb{R}$.

Riemann : $s \in \mathbb{C}$.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Euler.html

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Riemann.html

Special values of the Riemann zeta function



Jacques Bernoulli
1655–1705

Leonhard Euler
1707 – 1783

$\pi^{-2k}\zeta(2k) \in \mathbb{Q}$ *for* $k \in \mathbb{Z}$, $k \geq 1$ (Bernoulli numbers).

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$,
$\zeta(8) = \pi^8/9450 \cdots$

Values of the Riemann zeta function at the positive integers

Even positive integers

$$\zeta(2n) = (-1)^{n-1} 2^{2n-1} \frac{B_{2n}}{(2n)!} \pi^{2n} \quad (n \geq 1).$$

Odd positive integers : $\zeta(2n+1)$, $n \geq 1$?

Question : *for* $n \geq 1$, *is the number*

$$\frac{\zeta(2n+1)}{\pi^{2n+1}}$$

rational ?

Diophantine question

Determine all algebraic relations among the numbers

$$\zeta(2),\ \zeta(3),\ \zeta(5),\ \zeta(7),\ \ldots,\ \zeta(2n+1),\ \ldots$$

Conjecture. *There is no algebraic relation : the numbers*

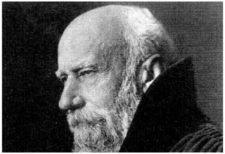
$$\zeta(2),\ \zeta(3),\ \zeta(5),\ \zeta(7),\ \ldots,\ \zeta(2n+1),\ \ldots$$

are algebraically independent.

As a consequence, one expects the numbers $\zeta(2n+1)$ and $\zeta(2n+1)/\pi^{2n+1}$ for $n \geq 1$ to be transcendental.

Values of ζ at the even positive integers

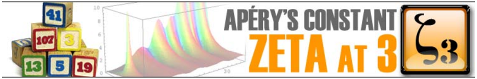
• F. Lindemann : π *is a transcendental number, hence* $\zeta(2k)$ *also for* $k \geq 1$.



F. Lindemann

1852 - 1939

Values of ζ at the odd positive integers



- Apéry (1978) : *The number*

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\,\ldots$$

is irrational.

- Open problem : *Is the number*

$$\zeta(5) = \sum_{n \geq 1} \frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

irrational?

Tanguy Rivoal

Let $\epsilon > 0$. *For any sufficiently large odd integer* a, *the dimension of the* $\mathbb{Q}$*–space spanned by the numbers* $1,\ \zeta(3),\ \zeta(5),\ \cdots,\zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

T. Rivoal

Rivoal (2000) + Ball, Zudilin, Fischler,. . .

Wadim Zudilin (2002)

- *At least one of the four numbers*
  $\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$
  is irrational.

- *There exists an odd number* j *in the interval* $[5, 69]$ *such that the three numbers*
  $1, \zeta(3), \zeta(j)$
  are $\mathbb{Q}$*–linearly independent.*

W. Zudilin

References

S. Fischler
Irrationalité de valeurs de zêta,
(d'après Apéry, Rivoal, . . .),
Sém. Nicolas Bourbaki, 2002-2003,
N° 910 (Novembre 2002).
http//www.math.u-psud.fr/~fischler/publi.html

C. Krattenthaler & T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, Mem. Amer. Math. Soc. **186** (2007), 93 p.
http://www-fourier.ujf-grenoble.fr/~rivoal/articles.html

S. Fischler, J. Sprang, W. Zudilin. *Many odd zeta values are irrational*, (2018).
https://arxiv.org/abs/1803.08905

S. Fischler

Hurwitz zeta function

T. Rivoal (2006) : consider the Hurwitz zeta function

$$\zeta(s,z) = \sum_{k\geq 1} \frac{1}{(k+z)^s}.$$

Expand $\zeta(2,z)$ as a series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}.$$

The coefficients of the expansion belong to $\mathbb{Q}+\mathbb{Q}\zeta(3)$. This produces a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.

In the same way : new proof of the irrationality of $\log 2$ by expanding

$$\sum_{k\geq 1} \frac{(-1)^k}{k+z}.$$

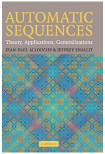
On $\sum z^{2^n}$ and $\sum z^{n^2}$

The name *Fredholm series* is often wrongly attributed to the power series

$$\sum_{n \geq 0} z^{2^n}$$

(see Allouche & Shallit, Notes on chapter 13). However Fredholm studied rather the theta series

$$\sum_{n \geq 0} z^{n^2}.$$



J-P. Allouche | J. Shallit

Automatic sequences, Theory, applications, generalizations, Cambridge University Press, Cambridge, 2003,

On $\sum z^{2^n}$

Let $\chi(z) = \sum_{n\geq 0} z^{2^n}$.

E. Catalan (1875), J. Sondow and W. Zudilin (2006) :

$$\gamma = \int_0^1 (\chi(x) - x) \frac{\mathrm{d}x}{x(1+x)}\cdot$$

E. Catalan

1814 - 1894

J. Sondow

W. Zudilin

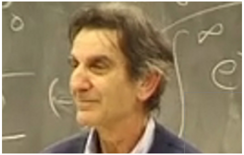
J. Sondow & W. Zudilin. Euler's constant, q-logarithms, and formulas of Ramanujan and Gosper. Ramanujan J. **12** (2006) 225–244.

On $\sum z^{2^n}$ and $\sum z^{n^2}$

Connection between the series $\sum z^{2^n}$, $\sum z^{n^2}$, theta functions, modular functions and paperfolding.

Michel Mendes-France & Ahmed Sebbar, Pliages de papier, fonctions thêta et méthode du cercle, Acta Math. **183** (1999), 101–139.

Ahmed Sebbar, Paperfolding and modular functions. in Exponential Analysis of Differential Equations and Related Topics (ed. Y. Takei). RIMS Kôkyûroku Bessatsu, B **52** (2014), 97–126.



M. Mendes–France
1936–2018

A. Sebbar

Mahler functions

A. J. Kempner (1916) proved the transcendence of the number

$$\chi(1/2) = \sum_{n\geq 0} 2^{-2^n}.$$

Kurt Mahler (1930, 1969) :
For $d \geq 2$, transcendence of the values at algebraic points of $\chi_d(z) = \sum_{n\geq 0} z^{d^n}$.

K. Mahler
1903 - 1988

`Tool:` The function χ_d satisfies the functional equation $\chi_d(z) = z + \chi_d(z^d)$ for $|z| < 1$.

Mahler functions : another example

The function

$$h(z) = \prod_{n \geq 0} (1 - z^{2^n})$$

satisfies the functional equation

$$h(z) = (1 - z^2) h(z^2).$$

The coefficients of its Taylor series at the origin

$$h(z) = \sum_{n \geq 0} (-1)^{t_n} z^n \qquad |z| < 1$$

are given by the Prouhet–Thue–Morse sequence $(t_n)_{n \geq 0} = (0110100110010110\dots)$:

$$t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n.$$

The Prouhet–Thue–Morse number

Let $(t_n)_{n\geq 0}$ be the Prouhet–Thue–Morse sequence and let $g \geq 2$.

- K. Mahler (1929) : *the so-called Prouhet–Thue–Morse number*

$$\sum_{n\geq 0} t_n g^{-n}$$

is transcendental.

Transcendence of the Prouhet–Thue–Morse number

Sketch of proof : for $|z| < 1$ the function

$$h(z) = \prod_{n\geq 0}(1 - z^{2^n}) \quad \text{satisfies} \quad h(z) = \sum_{n\geq 0}(-1)^{t_n} z^n$$

where $(t_n)_{n\geq 0}$ is the Prouhet–Thue–Morse number. For $a \in \{0, 1\}$, we have $(-1)^a = 1 - 2a$. Hence

$$h(z) = \sum_{n\geq 0}(1 - 2t_n)z^n = \frac{1}{1-z} - 2\sum_{n\geq 0} t_n z^n.$$

Using the functional equation $h(z) = (1 - z)h(z^2)$, Mahler shows that $h(\alpha)$ is transcendental for α algebraic for $0 < |\alpha| < 1$.

□

B. Adamczewski, Y. Bugeaud, F. Luca

A consequence of the Subspace Theorem (related to results by G. Christol, T. Kamae, M. Mendès-France, G. Rauzy (1980)) :

Corollary. *Let* $g \geq 2$ *be an integer,* p *a prime number and* $(u_k)_{k\geq 1}$ *a sequence of integers in the interval* $\{0, \ldots, p-1\}$. *The formal series*

$$\sum_{k\geq 1} u_k X^k \in \mathbb{F}_p((X))$$

and the real number

$$\sum_{k\geq 1} u_k g^{-k} \in \mathbb{R}$$

are simultaneously algebraic (over $\mathbb{F}_p(X)$ *et* $\mathbb{Q}$*, respectively) if and only if both are rational.*

The Prouhet-Thue-Morse series

Let $(t_n)_{n\geq 0}$ be the Prouhet-Thue-Morse sequence. The series

$$F(X) = \sum_{n\geq 0} t_n X^n$$

is algebraic over $F_2(X)$:

$$(1+X)^3F^2 + (1+X)^2F + X = 0.$$

One deduces another proof of Mahler's Theorem on the transcendence of

$$\sum_{n\geq 0} t_n g^{-n}.$$

Mahler theory : general framework

ω_{ij} $(1 \leq i, j \leq n)$ nonnegative integers,

$$\begin{array}{rcl} \Omega : \mathbb{C}^n & \rightarrow & \mathbb{C}^n \\ \underline{z} = (z_1, \ldots, z_n) & \mapsto & \left(\prod_{j=1}^{n} z_i{}^{\omega_{ij}} \right)_{1 \leq i \leq n} \end{array}$$

$f_1, \ldots, f_m$ power series with algebraic coefficients in n variables,

A a $m \times m$ matrix with algebraic entries,

$B(\underline{z})$ a $1 \times m$ matrix with entries rational fractions with algebraic coefficients.

Functional equations :

$$\begin{pmatrix} f_1(\underline{z}) \\ \vdots \\ f_m(\underline{z}) \end{pmatrix} = A \begin{pmatrix} f_1(\Omega \underline{z}) \\ \vdots \\ f_m(\Omega \underline{z}) \end{pmatrix} + B(\underline{z}).$$

Mahler theory

The proof of algebraic independence results for values of Mahler functions reduces to the proof of algebraic independence of the functions.

K. Mahler

1903 - 1988

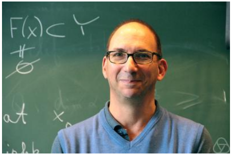
Kumiko Nishioka,
Mahler Functions and Transcendence,
Lecture Notes in Mathematics **1631** (1996), Springer Verlag.

Kumiko Nishioka

Rational and algebraic points on graphs

- E. Bombieri and J. Pila (1989) : on the number of integral points on arcs and ovals.
- J. Pila (2004) : on integer points on the dilation of a subanalytic surface.
- Upper bound for the number of points and the density of algebraic points of bounded degree and height on graphs of transcendental analytic functions.

E. Bombieri



J. Pila

Rational and algebraic points on graphs

- A. Surroca (2006) : transcendence method related to Schneider's work.
- D.W. Masser (2011) : *There exists a constant C such that for any $D \geq 3$ there are at most $C(\log D)(\log\log D)^2$ rational numbers s with $2 < s < 3$ and denominator at most D such that the Riemann zeta function $\zeta(s)$ is also rational with denominator at most D.*
- E. Besson (2014) : Euler Gamma function.
- G. Boxall, G. Jones, P. Villemot, T.P. Chalebgwa...

A. Surroca

D.W. Masser

o-minimal structures from model theory

- 2006, J. Pila and A. Wilkie : rational points of a definable set
- 2011, J. Pila : contribution to the André–Oort conjecture for $\mathbb{C}^n$ using o-minimality.

J. Pila and A. Wilkie

A. Beshenov & M. Bilu & Yu. Bilu & P. Rath.
Rational points on analytic varieties.
EMS Surv. Math. Sci. **2** (2015), 109–130.
http://dx.doi.org/10.4171/EMSS/10

Conclusion

The transcendence theory of values of Siegel E–functions and of functions satisfying Mahler equations is strong, but a lot remains to be done in the other situations. One open problem is to prove the Hermite–Lindemann Theorem on the transcendence of $\log \alpha$ for nonzero algebraic number α by using the logarithmic function (i.e. the theory of G–functions) instead of the exponential function (i.e. the theory of E–functions).

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April 12 - 23, 2021: Hanoi (Vietnam) (online)
CIMPA School on Functional Equations: Theory, Practice and Interaction.

Introduction to Transcendental Number Theory **7**

Transcendence of values of special functions

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