Yogyakarta, CIMPA School UGM, February 27, 2020

Linear recurrence sequences,

Michel Waldschmidt

Sorbonne University, Paris
Institut de Mathématiques de Jussieu

http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

Linear recurrence sequences are ubiquitous. They occur in biology, economics, computer science (analysis of algorithms), digital signal processing and number theory. We give a survey of this subject, together with connections with linear combinations of powers, with powers of matrices and with linear differential equations.

We first work over a field of any characteristic. Next we consider linear recurrence sequences over finite fields.

Applications of linear recurrence sequences

Combinatorics

Elimination

Symmetric functions

Hypergeometric series

Language

Communication, shift registers

Finite difference equations

Logic

Approximation

Pseudo–random sequences

Applications of linear recurrence sequences

- Biology (Integrodifference equations, spatial ecology).

- Computer science (analysis of algorithms).

- Digital signal processing (infinite impulse response (IIR) digital filters).

- Economics (time series analysis).

https://en.wikipedia.org/wiki/Recurrence_relation

Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer d together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n \geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension d of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences.

A basis of this space is obtained by taking for the initial d values $(u_0, u_1, \dots, u_{d-1})$ the elements of the canonical basis of $\mathbb{K}^d$.

Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer d together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n\geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension d of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences.

A basis of this space is obtained by taking for the initial d values $(u_0, u_1, \dots, u_{d-1})$ the elements of the canonical basis of $\mathbb{K}^d$.

Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer d together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n\geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension d of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences.

A basis of this space is obtained by taking for the initial d values $(u_0, u_1, \dots, u_{d-1})$ the elements of the canonical basis of $\mathbb{K}^d$.

Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer d together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n\geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension d of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences.

A basis of this space is obtained by taking for the initial d values $(u_0, u_1, \dots, u_{d-1})$ the elements of the canonical basis of $\mathbb{K}^d$.

Generating series, characteristic polynomial

The generating series is the formal series

$$\sum_{n\geq 0} u_n X^n.$$

Let $\gamma \in K^\times$; the sequence $(\gamma^n)_{n\geq 0}$ satisfies the linear recurrence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

if and only if $\gamma^d = a_1\gamma^{d-1} + \cdots + a_d$.
The characteristic (or companion) polynomial of the linear recurrence is

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Recall that 0 is not a root of this polynomial ($a_d \neq 0$).

Generating series, characteristic polynomial

The generating series is the formal series

$$\sum_{n\geq 0} u_n X^n.$$

Let $\gamma \in K^\times$; the sequence $(\gamma^n)_{n\geq 0}$ satisfies the linear recurrence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

if and only if $\gamma^d = a_1\gamma^{d-1} + \cdots + a_d$.

The characteristic (or companion) polynomial of the linear recurrence is

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Recall that 0 is not a root of this polynomial ($a_d \neq 0$).

Generating series, characteristic polynomial

The generating series is the formal series

$$\sum_{n\geq 0} u_n X^n.$$

Let $\gamma \in K^\times$; the sequence $(\gamma^n)_{n\geq 0}$ satisfies the linear recurrence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

if and only if $\gamma^d = a_1\gamma^{d-1} + \cdots + a_d$.
The characteristic (or companion) polynomial of the linear recurrence is

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Recall that 0 is not a root of this polynomial ($a_d \neq 0$).

Linear recurrence sequences : examples

- Constant sequence : $u_n = u_0$.

Linear recurrence sequence of order 1 : $u_{n+1} = u_n$.
Characteristic polynomial : $f(X) = X - 1$.
Generating series :

$$\sum_{n \geq 0} u_0 X^n = \frac{u_0}{1 - X}.$$

- Geometric progression : $u_n = u_0 \gamma^n$.

Linear recurrence sequence of order 1 : $u_n = \gamma u_{n-1}$.
Characteristic polynomial $f(X) = X - \gamma$.
Generating series :

$$\sum_{n \geq 0} u_0 \gamma^n X^n = \frac{u_0}{1 - \gamma X}.$$

Linear recurrence sequences : examples

- Constant sequence : $u_n = u_0$.

Linear recurrence sequence of order 1 : $u_{n+1} = u_n$.
Characteristic polynomial : $f(X) = X - 1$.
Generating series :

$$\sum_{n\geq 0} u_0 X^n = \frac{u_0}{1-X}.$$

- Geometric progression : $u_n = u_0\gamma^n$.

Linear recurrence sequence of order 1 : $u_n = \gamma u_{n-1}$.
Characteristic polynomial $f(X) = X - \gamma$.
Generating series :

$$\sum_{n\geq 0} u_0 \gamma^n X^n = \frac{u_0}{1-\gamma X}.$$

Linear recurrence sequences : examples

- $u_n = n$. This is a linear recurrence sequence of order 2 :

$$n + 2 = 2(n + 1) - n.$$

Characteristic polynomial

$$f(X) = X^2 - 2X + 1 = (X - 1)^2.$$

Generating series

$$\sum_{n \geq 0} nX^n = \frac{1}{1 - 2X + X^2}.$$

Power of matrices :

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix}.$$

Linear recurrence sequences : examples

- $u_n = p(n)$, where p is a polynomial of degree d. This is a linear recurrence sequence of order $d+1$.

Proof. The sequences

$$(p(n))_{n\geq 0}, \quad (p(n+1))_{n\geq 0}, \quad \cdots, \quad (p(n+k))_{n\geq 0}$$

are $\mathbb{K}$–linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d-1$ and linearly dependent for $k = d$.

A basis of the space of polynomials of degree d is given by the $d+1$ polynomials

$$p(X),\ p(X+1),\ \ldots,\ p(X+d).$$

`Question` : *which is the characteristic polynomial ?*

Linear recurrence sequences : examples

• $u_n = p(n)$, where p is a polynomial of degree d. This is a linear recurrence sequence of order $d+1$.

Proof. The sequences

$$(p(n))_{n\geq 0}, \quad (p(n+1))_{n\geq 0}, \quad \cdots, \quad (p(n+k))_{n\geq 0}$$

are $\mathbb{K}$–linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d-1$ and linearly dependent for $k = d$.

A basis of the space of polynomials of degree d is given by the $d+1$ polynomials

$$p(X),\ p(X+1),\ \ldots,\ p(X+d).$$

Question : *which is the characteristic polynomial ?*

Linear recurrence sequences : examples

• $u_n = p(n)$, where p is a polynomial of degree d. This is a linear recurrence sequence of order $d+1$.

Proof. The sequences

$$(p(n))_{n\geq 0}, \quad (p(n+1))_{n\geq 0}, \quad \cdots, \quad (p(n+k))_{n\geq 0}$$

are $\mathbb{K}$–linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d-1$ and linearly dependent for $k = d$.

A basis of the space of polynomials of degree d is given by the $d+1$ polynomials

$$p(X),\ p(X+1),\ \ldots,\ p(X+d).$$

`Question` : *which is the characteristic polynomial ?*

Linear sequences which are ultimately recurrent

The sequence

$$(1, 0, 0, \dots)$$

is not a linear recurrence sequence.

The condition

$$u_{n+1} = u_n$$

is satisfied only for $n \geq 1$.

The relation

$$u_{n+2} = u_{n+1} + 0u_n$$

with $d = 2$, $a_d = 0$ does not fulfil the requirement $a_d \neq 0$.

Linear sequences which are ultimately recurrent

The sequence

$$(1, 0, 0, \dots)$$

is not a linear recurrence sequence.

The condition

$$u_{n+1} = u_n$$

is satisfied only for $n \geq 1$.

The relation

$$u_{n+2} = u_{n+1} + 0u_n$$

with $d = 2$, $a_d = 0$ does not fulfil the requirement $a_d \neq 0$.

Linear sequences which are ultimately recurrent

The sequence

$$(1, 0, 0, \dots)$$

is not a linear recurrence sequence.

The condition

$$u_{n+1} = u_n$$

is satisfied only for $n \geq 1$.

The relation

$$u_{n+2} = u_{n+1} + 0u_n$$

with $d = 2$, $a_d = 0$ does not fulfil the requirement $a_d \neq 0$.

Order of a linear recurrence sequence

If $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is f, then, for any monic polynomial $g \in \mathbb{K}[X]$ with $g(0) \neq 0$, this sequence $\mathbf{u}$ also satisfies the linear recurrence, the characteristic polynomial of which is fg.
`Example` : for $g(X) = X - \gamma$ with $\gamma \neq 0$, from

$$(\star) \qquad u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n = 0$$

we deduce

$$\begin{aligned} &u_{n+d+1} - a_1 u_{n+d} - \cdots - a_d u_{n+1} \\ &\qquad - \gamma\big(u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n\big) = 0. \end{aligned}$$

The *order* of a linear recurrence sequence is the smallest d such that $(\star)$ holds for all $n \geq 0$.

Order of a linear recurrence sequence

If $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is f, then, for any monic polynomial $g \in \mathbb{K}[X]$ with $g(0) \neq 0$, this sequence $\mathbf{u}$ also satisfies the linear recurrence, the characteristic polynomial of which is fg.
`Example` : for $g(X) = X - \gamma$ with $\gamma \neq 0$, from

$$(\star) \qquad u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n = 0$$

we deduce

$$\begin{aligned} & u_{n+d+1} - a_1 u_{n+d} - \cdots - a_d u_{n+1} \\ & \qquad - \gamma(u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n) = 0. \end{aligned}$$

The *order* of a linear recurrence sequence is the smallest d such that $(\star)$ holds for all $n \geq 0$.

Generating series of a linear recurrence sequence

Let $\mathbf{u} = (u_n)_{n\geq 0}$ be a linear recurrence sequence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

with characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Denote by f^- the reciprocal polynomial of f :

$$f^-(X) = X^d f(X^{-1}) = 1 - a_1 X - \cdots - a_d X^d.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)},$$

where r is a polynomial of degree less than d determined by the initial values of $\mathbf{u}$.

Generating series of a linear recurrence sequence

Assume

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)}.$$

Proof. Comparing the coefficients of X^n for $n \geq d$ shows that

$$f^-(X) \sum_{n=0}^{\infty} u_n X^n$$

is a polynomial of degree less than d.

Generating series of a linear recurrence sequence

Assume

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)}\cdot$$

Proof. Comparing the coefficients of X^n for $n \geq d$ shows that

$$f^-(X) \sum_{n=0}^{\infty} u_n X^n$$

is a polynomial of degree less than d.

Taylor coefficients of rational functions

Conversely, the sequence of coefficients in the Taylor expansion of any rational fraction $a(X)/b(X)$ with $\deg a < \deg b$ and $b(0) \neq 0$ satisfies the recurrence relation with characteristic polynomial $f \in K[X]$ given by $f(X) = b^{-}(X)$.

Therefore a sequence $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the recurrence relation $(\star)$ with characteristic polynomial $f \in K[X]$ if and only if

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^{-}(X)},$$

where r is a polynomial of degree less than d determined by the initial values of $\mathbf{u}$.

Taylor coefficients of rational functions

Conversely, the sequence of coefficients in the Taylor expansion of any rational fraction $a(X)/b(X)$ with $\deg a < \deg b$ and $b(0) \neq 0$ satisfies the recurrence relation with characteristic polynomial $f \in K[X]$ given by $f(X) = b^{-}(X)$.

Therefore a sequence $\mathbf{u} = (u_n)_{n \geq 0}$ satisfies the recurrence relation ($\star$) with characteristic polynomial $f \in K[X]$ if and only if

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^{-}(X)},$$

where r is a polynomial of degree less than d determined by the initial values of $\mathbf{u}$.

Linear differential equations

Given a sequence $(u_n)_{n\geq 0}$ of numbers, its exponential generating power series is

$$\psi(z) = \sum_{n\geq 0} u_n \frac{z^n}{n!}.$$

For $k \geq 0$, the k-the derivative $\psi^{(k)}$ of ψ satisfies

$$\psi^{(k)}(z) = \sum_{n\geq 0} u_{n+k} \frac{z^n}{n!}.$$

Hence the sequence satisfies the linear recurrence relation

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

if and only if ψ is a solution of the homogeneous linear differential equation

$$y^{(d)} = a_1 y^{(d-1)} + \cdots + a_{d-1} y' + a_d y.$$

Linear differential equations

Given a sequence $(u_n)_{n\geq 0}$ of numbers, its exponential generating power series is

$$\psi(z) = \sum_{n\geq 0} u_n \frac{z^n}{n!}.$$

For $k \geq 0$, the k-the derivative $\psi^{(k)}$ of ψ satisfies

$$\psi^{(k)}(z) = \sum_{n\geq 0} u_{n+k} \frac{z^n}{n!}.$$

Hence the sequence satisfies the linear recurrence relation

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

if and only if ψ is a solution of the homogeneous linear differential equation

$$y^{(d)} = a_1 y^{(d-1)} + \cdots + a_{d-1} y' + a_d y.$$

Linear differential equations

Given a sequence $(u_n)_{n\geq 0}$ of numbers, its exponential generating power series is

$$\psi(z) = \sum_{n\geq 0} u_n \frac{z^n}{n!}.$$

For $k \geq 0$, the k-the derivative $\psi^{(k)}$ of ψ satisfies

$$\psi^{(k)}(z) = \sum_{n\geq 0} u_{n+k} \frac{z^n}{n!}.$$

Hence the sequence satisfies the linear recurrence relation

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

if and only if ψ is a solution of the homogeneous linear differential equation

$$y^{(d)} = a_1 y^{(d-1)} + \cdots + a_{d-1} y' + a_d y.$$

Matrix notation for a linear recurrence sequence

The linear recurrence sequence

$(\star)$ $$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

can be written

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}.$$

Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The determinant of $I_d X - A$ (the characteristic polynomial of A) is nothing but

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d,$$

the characteristic polynomial of the linear recurrence sequence. By induction

$$U_n = A^n U_0.$$

Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The determinant of $I_d X - A$ (the characteristic polynomial of A) is nothing but

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d,$$

the characteristic polynomial of the linear recurrence sequence.

By induction

$$U_n = A^n U_0.$$

Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The determinant of $I_d X - A$ (the characteristic polynomial of A) is nothing but

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d,$$

the characteristic polynomial of the linear recurrence sequence. By induction

$$U_n = A^n U_0.$$

Powers of matrices

Let $A = (a_{ij})_{1 \le i,j \le d} \in \mathrm{GL}_{d \times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in $\mathbb{K}$ and nonzero determinant. For $n \ge 0$, define

$$A^n = \left(a_{ij}^{(n)}\right)_{1 \le i,j \le d}.$$

Then each of the d^2 sequences $\left(a_{ij}^{(n)}\right)_{n \ge 0}$, $(1 \le i, j \le d)$ is a linear recurrence sequence. The roots of the characteristic polynomial of these linear recurrences are the eigenvalues of A.

In particular the sequence $\left(\mathrm{Tr}(A^n)\right)_{n \ge 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix A.

Powers of matrices

Let $A = (a_{ij})_{1\le i,j\le d} \in \mathrm{GL}_{d\times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in $\mathbb{K}$ and nonzero determinant. For $n \ge 0$, define

$$A^n = \left(a_{ij}^{(n)}\right)_{1\le i,j\le d}.$$

Then each of the d^2 sequences $\left(a_{ij}^{(n)}\right)_{n\ge 0}$, $(1 \le i, j \le d)$ is a linear recurrence sequence. The roots of the characteristic polynomial of these linear recurrences are the eigenvalues of A.

In particular the sequence $\left(\mathrm{Tr}(A^n)\right)_{n\ge 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix A.

Powers of matrices

Let $A = (a_{ij})_{1\le i,j\le d} \in \mathrm{GL}_{d\times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in $\mathbb{K}$ and nonzero determinant. For $n \geq 0$, define

$$A^n = \left(a_{ij}^{(n)}\right)_{1\le i,j\le d}.$$

Then each of the d^2 sequences $\left(a_{ij}^{(n)}\right)_{n\geq 0}$, $(1 \leq i, j \leq d)$ is a linear recurrence sequence. The roots of the characteristic polynomial of these linear recurrences are the eigenvalues of A.

In particular the sequence $\left(\mathrm{Tr}(A^n)\right)_{n\geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix A.

Conversely :

Given a linear recurrence sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$, there exist an integer $d \geq 1$ and a matrix $A \in \mathrm{GL}_d(\mathbb{K})$ such that, for each $n \geq 0$,

$$u_n = a_{11}^{(n)}.$$

The characteristic polynomial of A is the characteristic polynomial of the linear recurrence sequence.

Everest G., van der Poorten A., Shparlinski I., Ward T. – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104.

Conversely :

Given a linear recurrence sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$, there exist an integer $d \geq 1$ and a matrix $A \in \mathrm{GL}_d(\mathbb{K})$ such that, for each $n \geq 0$,

$$u_n = a_{11}^{(n)}.$$

The characteristic polynomial of A is the characteristic polynomial of the linear recurrence sequence.

Everest G., van der Poorten A., Shparlinski I., Ward T. – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104.

Linear recurrence sequences : simple roots

A basis of $E_{\underline{a}}$ over $\mathbb{K}$ is obtained by attributing to the initial values $u_0, \dots, u_{d-1}$ the values given by the canonical basis of $\mathbb{K}^d$.

Given γ in $\mathbb{K}^\times$, a necessary and sufficient condition for a sequence $(\gamma^n)_{n\geq 0}$ to satisfy $(\star)$ is that γ is a root of the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

If this polynomial has d distinct roots $\gamma_1, \dots, \gamma_d$ in $\mathbb{K}$,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \neq \gamma_j,$$

then a basis of $E_{\underline{a}}$ over $\mathbb{K}$ is given by the d sequences $({\gamma_i}^n)_{n\geq 0}$, $i = 1, \dots, d$.

Linear recurrence sequences : simple roots

A basis of $E_{\underline{a}}$ over $\mathbb{K}$ is obtained by attributing to the initial values $u_0, \dots, u_{d-1}$ the values given by the canonical basis of $\mathbb{K}^d$.

Given γ in $\mathbb{K}^\times$, a necessary and sufficient condition for a sequence $(\gamma^n)_{n\geq 0}$ to satisfy $(\star)$ is that γ is a root of the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

If this polynomial has d distinct roots $\gamma_1, \dots, \gamma_d$ in $\mathbb{K}$,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \neq \gamma_j,$$

then a basis of $E_{\underline{a}}$ over $\mathbb{K}$ is given by the d sequences $({\gamma_i}^n)_{n\geq 0}$, $i = 1, \dots, d$.

Linear recurrence sequences : simple roots

A basis of $E_{\underline{a}}$ over $\mathbb{K}$ is obtained by attributing to the initial values $u_0, \dots, u_{d-1}$ the values given by the canonical basis of $\mathbb{K}^d$.

Given γ in $\mathbb{K}^\times$, a necessary and sufficient condition for a sequence $(\gamma^n)_{n\geq 0}$ to satisfy $(\star)$ is that γ is a root of the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

If this polynomial has d distinct roots $\gamma_1, \dots, \gamma_d$ in $\mathbb{K}$,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \neq \gamma_j,$$

then a basis of $E_{\underline{a}}$ over $\mathbb{K}$ is given by the d sequences $({\gamma_i}^n)_{n\geq 0}$, $i = 1, \dots, d$.

Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root γ.

The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that γ is a root of multiplicity ≥ 2 of $f(X)$.

Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root γ.

The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that γ is a root of multiplicity ≥ 2 of $f(X)$.

Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root γ.

The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that γ is a root of multiplicity ≥ 2 of $f(X)$.

Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root γ.

The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that γ is a root of multiplicity ≥ 2 of $f(X)$.

Linear recurrence sequences : multiple roots

In general, when the characteristic polynomial splits as

$$X^d - a_1 X^{d-1} - \cdots - a_d = \prod_{i=1}^{\ell} (X - \gamma_i)^{t_i},$$

a basis of $E_{\underline{a}}$ is given by the d sequences

$$(n^k {\gamma_i}^n)_{n \geq 0}, \qquad 0 \leq k \leq t_i - 1, \quad 1 \leq i \leq \ell.$$

Polynomial combinations of powers

The sum and the product of any two linear recurrence sequences are linear recurrence sequences.

The set $\cup_{\underline{a}} E_{\underline{a}}$ of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials $p_1, \ldots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \ldots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\big(p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n\big)_{n\geq 0}$$

is a linear recurrence sequence.

Conversely, any linear recurrence sequence is of this form.

Polynomial combinations of powers

The sum and the product of any two linear recurrence sequences are linear recurrence sequences.

The set $\cup_{\underline{a}} E_{\underline{a}}$ of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials $p_1, \dots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \dots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\left(p_1(n)\gamma_1{}^n + \cdots + p_\ell(n)\gamma_\ell^n\right)_{n\geq 0}$$

is a linear recurrence sequence.

Conversely, any linear recurrence sequence is of this form.

Polynomial combinations of powers

The sum and the product of any two linear recurrence sequences are linear recurrence sequences.

The set $\cup_{\underline{a}} E_{\underline{a}}$ of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials $p_1, \dots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \dots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\bigl(p_1(n)\gamma_1{}^n + \cdots + p_\ell(n)\gamma_\ell^n\bigr)_{n\geq 0}$$

is a linear recurrence sequence.

Conversely, any linear recurrence sequence is of this form.

Polynomial combinations of powers

The sum and the product of any two linear recurrence sequences are linear recurrence sequences.

The set $\cup_{\underline{a}} E_{\underline{a}}$ of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials $p_1, \dots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \dots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\left(p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n\right)_{n\geq 0}$$

is a linear recurrence sequence.

Conversely, any linear recurrence sequence is of this form.

Consequence

• When p is a polynomial of degree $< d$, the characteristic polynomial of the sequence $u_n = p(n)$ divides $(X-1)^d$.

`Proof.`
Set

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = I_d + N$$

where I_d is the $d \times d$ identity matrix and N is nilpotent : $N^d = 0$.

Consequence

The characteristic polynomial of A is $(X-1)^d$. Hence for $1 \leq i, j \leq d$, the sequence u_n of the coefficient $a_{ij}^{(n)}$ of A^n satisfies the linear recurrence relation

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n,$$

that is

$$u_{n+d} = d u_{n+d-1} - \binom{d}{2} u_{n+d-2} + \cdots + (-1)^{d-2} d u_{n+1} + (-1)^{d-1} u_n.$$

The characteristic polynomial of this recurrence relation is $(X-1)^d$.

Characteristic polynomial of the recurrence sequence $p(n)$.

Since, for $1 \le i, j \le d$ and $n \ge 0$, we have

$$a_{ij}^{(n)} = \binom{n}{j-i}$$

(where we agree that $\binom{n}{k} = 0$ for $k < 0$ and for $k > n$, while $\binom{d}{0} = \binom{d}{d} = 1$), we deduce that each of the d polynomials

$$1, \quad \frac{X(X+1)\cdots(X+k-1)}{k!} \qquad k = 1, 2, \dots, d-1$$

namely

$$1, X, \frac{X(X+1)}{2}, \dots, \frac{X(X+1)\cdots(X+d-2)}{(d-1)!},$$

satisfies the recurrence $(\star)$. These d polynomials constitute a basis of the space of polynomials of degree $< d$.

Sum of polynomial combinations of powers

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two linear recurrence sequences of characteristic polynomials f_1 and f_2 respectively, then $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\frac{f_1 f_2}{\gcd(f_1, f_2)}.$$

Product of polynomial combinations of powers

If the characteristic polynomials of the two linear recurrence sequences $\mathbf{u}_1$ and $\mathbf{u}_2$ are respectively

$$f_1(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j} \quad \text{and} \quad f_2(T) = \prod_{k=1}^{\ell'} (T - \gamma'_k)^{t'_k},$$

then $\mathbf{u}_1\mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\prod_{j=1}^{\ell} \prod_{k=1}^{\ell'} (T - \gamma_j \gamma'_k)^{t_j + t'_k - 1}.$$

Linear recurrence sequences and Brahmagupta–Pell–Fermat Equation

Let d be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d} y_n = (x_1 + \sqrt{d} y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d} y_1)^n + (x_1 - \sqrt{d} y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for y_n, and also for $n \leq 0$.

Linear recurrence sequences and Brahmagupta–Pell–Fermat Equation

Let d be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d} y_n = (x_1 + \sqrt{d} y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d} y_1)^n + (x_1 - \sqrt{d} y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for y_n, and also for $n \leq 0$.

Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by d consecutive values.

Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by d consecutive values.

Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by d consecutive values.

Discrete version of linear differential equations

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ can be viewed as a linear map $\mathbb{N} \longrightarrow \mathbb{K}$. Define the discrete derivative $\mathcal{D}$ by

$$\mathcal{D}\mathbf{u} : \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{K} \\ n & \longmapsto & u_{n+1} - u_n. \end{array}$$

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ is a linear recurrence sequence if and only if there exists $Q \in \mathbb{K}[T]$ with $Q(1) \neq 1$ such that

$$Q(\mathcal{D})\mathbf{u} = 0.$$

Linear recurrence sequences are a discrete version of linear differential equations with constant coefficients.

The condition $Q(1) \neq 0$ reflects $a_d \neq 0$ – otherwise one gets *ultimately* recurrent sequences.

Conclusion

The same mathematical object occurs in a different guise :

- Linear recurrence sequences

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

- Linear combinations with polynomial coefficients of powers

$$p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n.$$

- Taylor coefficients of rational functions.

- Coefficients of power series which are solutions of homogeneous linear differential equations.

- Sequence of coefficients of powers of a matrix.

Reference

EVEREST, GRAHAM ; VAN DER POORTEN, ALF ; SHPARLINSKI, IGOR ; WARD, TOM – *Recurrence sequences*, Mathematical Surveys and Monographs (AMS, 2003), volume 104. `1290 references.`

Graham Everest

Alf van der Poorten

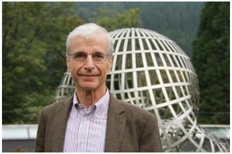
Igor Shparlinski

Tom Ward

Linear recurrence sequences over finite fields

`Reference:` Chapter 8 : *Linear recurring sequences* of

LIDL, RUDOLF ; NIEDERREITER, HARALD.
Finite fields. Paperback reprint of the hardback 2nd edition 1996. (English)
Encyclopedia of Mathematics and Its Applications 20.
Cambridge University Press (ISBN 978-0-521-06567-2/pbk).
xiv, 755 p. (2008).



Harald Niederreiter

Linear recurring sequences

Given $a, a_0, \ldots, a_{k-1}$ in a finite field $\mathbb{F}_q$, consider a k–*th order linear recurrence relation* : for $n = 0, 1, 2, \ldots$,

$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1 u_{n+1} + a_0 u_n + a$$

Homogeneous : $a = 0$.

Initial values : $u_0, u_1, \ldots, u_{k-1}$.

State vector : $\mathbf{u}_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})$.

Initial state vector : $\mathbf{u}_0 = (u_0, u_1, \ldots, u_{k-1})$.

Linear recurring sequences

Given $a, a_0, \ldots, a_{k-1}$ in a finite field $\mathbb{F}_q$, consider a k–*th order linear recurrence relation* : for $n = 0, 1, 2, \ldots$,

$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1u_{n+1} + a_0u_n + a$$

Homogeneous : $a = 0$.

Initial values : $u_0, u_1, \ldots, u_{k-1}$.

State vector : $\mathbf{u}_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})$.

Initial state vector : $\mathbf{u}_0 = (u_0, u_1, \ldots, u_{k-1})$.

Linear recurring sequences

Given $a, a_0, \ldots, a_{k-1}$ in a finite field $\mathbb{F}_q$, consider a k–*th order linear recurrence relation* : for $n = 0, 1, 2, \ldots$,

$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1 u_{n+1} + a_0 u_n + a$$

Homogeneous : $a = 0$.

Initial values : $u_0, u_1, \ldots, u_{k-1}$.

State vector : $\mathbf{u}_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})$.

Initial state vector : $\mathbf{u}_0 = (u_0, u_1, \ldots, u_{k-1})$.

Linear recurring sequences

Given $a, a_0, \ldots, a_{k-1}$ in a finite field $\mathbb{F}_q$, consider a k–*th order linear recurrence relation* : for $n = 0, 1, 2, \ldots$,

$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1 u_{n+1} + a_0 u_n + a$$

Homogeneous : $a = 0$.

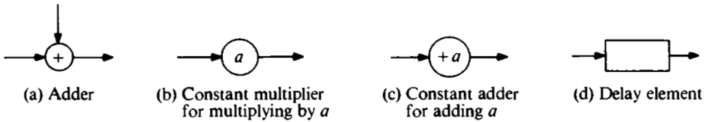
Initial values : $u_0, u_1, \ldots, u_{k-1}$.

State vector : $\mathbf{u}_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})$.

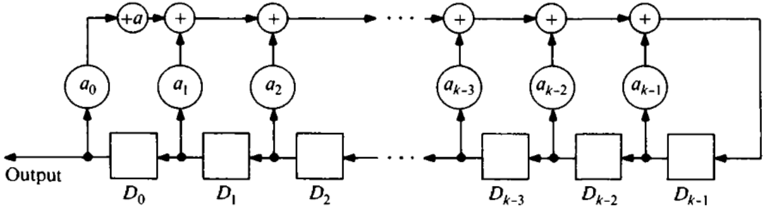
Initial state vector : $\mathbf{u}_0 = (u_0, u_1, \ldots, u_{k-1})$.

Feedback shift register

Electronic switching circuit : adder, constant multiplier, constant adder, delay element (*flip-flop*)



$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1u_{n+1} + a_0u_n + a$$



The least period of a linear recurrence sequence

Since $\mathbb{F}_q$ is finite, any linear recurrence sequence $(u_n)_{n\geq 0}$ in $\mathbb{F}_q$ is *ultimately periodic* : there exists $r > 0$ and $n_0 \geq 0$ such that $u_n = u_{n+r}$ for $n \geq n_0$. The least n_0 for which this relation holds is the *preperiod*.

Any period is a multiple of the least period.

A linear recurrence sequence $(u_n)_{n\geq 0}$ is periodic if there exists a period $r > 0$ such that $u_n = u_{n+r}$ for $n \geq 0$. In this case this relation holds for the least period ; the preperiod is 0. If $a_0 \neq 0$, then the sequence is periodic.

The least period r of a (homogeneous) linear recurrence sequence in $\mathbb{F}_q$ of order k satisfies $r \leq q^k - 1$.

The least period of a linear recurrence sequence

Since $\mathbb{F}_q$ is finite, any linear recurrence sequence $(u_n)_{n\geq 0}$ in $\mathbb{F}_q$ is *ultimately periodic* : there exists $r > 0$ and $n_0 \geq 0$ such that $u_n = u_{n+r}$ for $n \geq n_0$. The least n_0 for which this relation holds is the *preperiod*.

Any period is a multiple of the least period.

A linear recurrence sequence $(u_n)_{n\geq 0}$ is periodic if there exists a period $r > 0$ such that $u_n = u_{n+r}$ for $n \geq 0$. In this case this relation holds for the least period ; the preperiod is 0. If $a_0 \neq 0$, then the sequence is periodic.

The least period r of a (homogeneous) linear recurrence sequence in $\mathbb{F}_q$ of order k satisfies $r \leq q^k - 1$.

The least period of a linear recurrence sequence

Since $\mathbb{F}_q$ is finite, any linear recurrence sequence $(u_n)_{n\geq 0}$ in $\mathbb{F}_q$ is *ultimately periodic* : there exists $r > 0$ and $n_0 \geq 0$ such that $u_n = u_{n+r}$ for $n \geq n_0$. The least n_0 for which this relation holds is the *preperiod*.

Any period is a multiple of the least period.

A linear recurrence sequence $(u_n)_{n\geq 0}$ is periodic if there exists a period $r > 0$ such that $u_n = u_{n+r}$ for $n \geq 0$. In this case this relation holds for the least period ; the preperiod is 0. If $a_0 \neq 0$, then the sequence is periodic.

The least period r of a (homogeneous) linear recurrence sequence in $\mathbb{F}_q$ of order k satisfies $r \leq q^k - 1$.

The least period of a linear recurrence sequence

Since $\mathbb{F}_q$ is finite, any linear recurrence sequence $(u_n)_{n\geq 0}$ in $\mathbb{F}_q$ is *ultimately periodic* : there exists $r > 0$ and $n_0 \geq 0$ such that $u_n = u_{n+r}$ for $n \geq n_0$. The least n_0 for which this relation holds is the *preperiod*.

Any period is a multiple of the least period.

A linear recurrence sequence $(u_n)_{n\geq 0}$ is periodic if there exists a period $r > 0$ such that $u_n = u_{n+r}$ for $n \geq 0$. In this case this relation holds for the least period ; the preperiod is 0. If $a_0 \neq 0$, then the sequence is periodic.

The least period r of a (homogeneous) linear recurrence sequence in $\mathbb{F}_q$ of order k satisfies $r \leq q^k - 1$.

The companion matrix

The linear recurrence sequence

$$u_{n+k} = a_{k-1}u_{n+k-1} + \cdots + a_0 u_n \quad \text{for} \quad n \geq 0$$

can be written

$$\mathbf{u}_n = \mathbf{u}_0 A^n$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{k-1} \end{pmatrix}.$$

The least period

Assume $a_0 \neq 0$

The least period of the linear recurrence sequence divides the order of the matrix A in the general linear group $\mathrm{GL}_k(\mathbb{F}_q)$.

The *impulse response sequence* is the linear recurrence sequence with the initial state $(0, 0, \dots, 0, 1)$.

The least period of a linear recurrence sequence divides the least period of the corresponding impulse response sequence.

The least period

Assume $a_0 \neq 0$

The least period of the linear recurrence sequence divides the order of the matrix A in the general linear group $\mathrm{GL}_k(\mathbb{F}_q)$.

The *impulse response sequence* is the linear recurrence sequence with the initial state $(0, 0, \ldots, 0, 1)$.

The least period of a linear recurrence sequence divides the least period of the corresponding impulse response sequence.

The least period

Assume $a_0 \neq 0$

The least period of the linear recurrence sequence divides the order of the matrix A in the general linear group $\mathrm{GL}_k(\mathbb{F}_q)$.

The *impulse response sequence* is the linear recurrence sequence with the initial state $(0, 0, \ldots, 0, 1)$.

The least period of a linear recurrence sequence divides the least period of the corresponding impulse response sequence.

The least period

Assume $a_0 \neq 0$

The least period of the linear recurrence sequence divides the order of the matrix A in the general linear group $\mathrm{GL}_k(\mathbb{F}_q)$.

The *impulse response sequence* is the linear recurrence sequence with the initial state $(0, 0, \ldots, 0, 1)$.

The least period of a linear recurrence sequence divides the least period of the corresponding impulse response sequence.

Further examples of linear recurrence sequences

- Fibonacci
- Lucas
- Perrin
- Padovan
- Narayana

References
Linear recurrence sequences : an introduction.
http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/LinearRecurrenceSequencesIntroduction.pdf
Linear recurrence sequences, exponential polynomials and Diophantine approximation.
http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/LinRecSeqDiophAppxVI.pdf

Leonardo Pisano (Fibonacci)

Fibonacci sequence $(F_n)_{n\geq 0}$,

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233,\ldots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

http://oeis.org/A000045

Leonardo Pisano (Fibonacci)
(1170–1250)

Lucas sequence

http://oeis.org/000032

The Lucas sequence $(L_n)_{n\geq 0}$ satisfies the same recurrence relation as the Fibonacci sequence, namely

$$L_{n+2} = L_{n+1} + L_n \quad \text{for} \quad n \geq 0,$$

only the initial values are different :

$$L_0 = 2,\ L_1 = 1.$$

The sequence of Lucas numbers starts with

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322,\dots$$

A closed form involving the Golden ratio Φ is

$$L_n = \Phi^n + (-\Phi)^{-n},$$

from which it follows that for $n \geq 2$, L_n is the nearest integer to Φ^n.

Lucas sequence

http://oeis.org/000032

The Lucas sequence $(L_n)_{n\geq 0}$ satisfies the same recurrence relation as the Fibonacci sequence, namely

$$L_{n+2} = L_{n+1} + L_n \quad \text{for} \quad n \geq 0,$$

only the initial values are different :

$$L_0 = 2,\ L_1 = 1.$$

The sequence of Lucas numbers starts with

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322, \dots$$

A closed form involving the Golden ratio Φ is

$$L_n = \Phi^n + (-\Phi)^{-n},$$

from which it follows that for $n \geq 2$, L_n is the nearest integer to Φ^n.

Perrin sequence

http://oeis.org/A001608

The Perrin sequence (also called *skiponacci sequence*) is the linear recurrence sequence $(P_n)_{n\geq 0}$ defined by

$$P_{n+3} = P_{n+1} + P_n \quad \text{for} \quad n \geq 0,$$

with the initial conditions

$$P_0 = 3,\ P_1 = 0,\ P_2 = 2.$$

It starts with

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68, \dots$$

François Olivier Raoul Perrin (1841-1910) :
https://en.wikipedia.org/wiki/Perrin_number

Perrin sequence

http://oeis.org/A001608

The Perrin sequence (also called *skiponacci sequence*) is the linear recurrence sequence $(P_n)_{n\geq 0}$ defined by

$$P_{n+3} = P_{n+1} + P_n \quad \text{for} \quad n \geq 0,$$

with the initial conditions

$$P_0 = 3,\ P_1 = 0,\ P_2 = 2.$$

It starts with

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68, \dots$$

François Olivier Raoul Perrin (1841-1910) :
https://en.wikipedia.org/wiki/Perrin_number

Narayana sequence https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.

It starts with

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277, \ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

Narayana sequence https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.

It starts with

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277, \ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

Narayana sequence https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.
It starts with

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277, \ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

Padovan sequence

https://oeis.org/A000931

The Padovan sequence $(p_n)_{n\geq 0}$ satisfies the same recurrence

$$p_{n+3} = p_{n+1} + p_n$$

as the Perrin sequence but has different initial values :

$$p_0 = 1, \quad p_1 = p_2 = 0.$$

It starts with

$$1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16, \ldots$$

Richard Padovan

http://mathworld.wolfram.com/LinearRecurrenceEquation.html

Linear recurrence sequences,

Michel Waldschmidt

Sorbonne University, Paris
Institut de Mathématiques de Jussieu

http://www.imj-prg.fr/~michel.waldschmidt/