भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान बरहमपुर
Indian Institute of Science Education and Research Berhampur

Mathematics Online Seminar Series 2020

Linear recurrence sequences: part II

Michel Waldschmidt

Institut de Mathématiques de Jussieu — Sorbonne Université
http://www.math.jussieu.fr/~michel.waldschmidt/

Abstract

Linear recurrence sequences are ubiquitous. They occur in biology, economics, computer science (analysis of algorithms), digital signal processing. We give a survey of this subject, together with connections with linear combinations of powers, with powers of matrices and with linear differential equations.

The first part was devoted to examples.

Here we develop the general theory and we quote some connections with Diophantine questions.

Linear recurrence sequences : definition

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer d together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$ of zero characteristic.
Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set of linear recurrence sequences $\mathbf{u} = (u_n)_{n\geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension d of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences.
If c is an element of $\mathbb{K}$ and $\mathbf{u}$ and $\mathbf{v}$ satisfy $(\star)$, then so do

$$\mathbf{u}+\mathbf{v} = (u_0+v_0, u_1+v_1, u_2+v_2, \dots) \quad \text{and} \quad c\mathbf{u} = (cu_0, cu_1, cu_2, \dots).$$

Ultimately recurrent sequences

The sequence

$$(1, 0, 0, 0, \dots)$$

is not a recurrent sequence : it satisfies

$$u_{n+1} = u_n \quad \text{for} \quad n \geq 1$$

but not for $n = 0$. It also satisfies

$$u_{n+2} = u_{n+1} + 0 \cdot u_n \quad \text{for} \quad n \geq 0$$

with $d = 2$, $a_1 = 1$, $a_2 = 0$ but the condition $a_d \neq 0$ is not satisfied.
If we remove the condition $a_d \neq 0$, we get *ultimately recurrent sequences*.

The characteristic polynomial

Let γ be an element of $\mathbb{K}^\times$. The sequence $\mathbf{u} = (u_0, u_1, u_2, \dots)$ with

$$u_n = \gamma^n$$

is a solution of the linear recurrence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n$$

if and only if

$$\gamma^d = a_1 \gamma^{d-1} + \cdots + a_d$$

The *characteristic* (or *companion*) polynomial of the linear recurrence is

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Linear recurrence sequences : examples

- Constant (not zero) sequence : $u_n = u_0$.
Linear recurrence sequence of order 1 : $u_{n+1} = u_n$.
Characteristic polynomial : $f(X) = X - 1$.
Generating series : $\displaystyle\sum_{n\geq 0} X^n = \frac{u_0}{1-X}$.
Differential equation : $y' = y$.

- Geometric progression : $u_n = u_0\gamma^n$.
Linear recurrence sequence of order 1 : $u_n = \gamma u_{n-1}$.
Characteristic polynomial : $f(X) = X - \gamma$.
Generating series : $\displaystyle\sum_{n\geq 0} u_0\gamma^n X^n = \frac{u_0}{1-\gamma X}$.
Differential equation : $y' = \gamma y$.

Linear recurrence sequences : examples

• The sequence $u_n = n$ is a linear recurrence sequence of order 2 :

$$n + 2 = 2(n + 1) - n.$$

Characteristic polynomial

$$f(X) = X^2 - 2X + 1 = (X - 1)^2.$$

Generating series $\displaystyle\sum_{n\geq 0} nX^n = \frac{1}{1 - 2X + X^2}$.

Differential equation $y'' - 2y' + y = 0$.

Power of matrices :

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix}.$$

Linear recurrence sequences : examples

- The sequence $u_n = p(n)$, where p is a polynomial of degree d, is a linear recurrence sequence of order $d+1$.

Proof. The sequences

$$(p(n))_{n\geq 0}, \quad (p(n+1))_{n\geq 0}, \quad \cdots, \quad (p(n+k))_{n\geq 0}$$

are $\mathbb{K}$–linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d-1$ and linearly dependent for $k = d$.

A basis of the space of polynomials of degree d is given by the $d+1$ polynomials

$$p(X),\ p(X+1),\ \ldots,\ p(X+d). \qquad \square$$

`Exercise` : *what is the characteristic polynomial of the sequence* $u_n = p(n)$ *?*

Generating series of a linear recurrence sequence

A sequence $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the linear recurrence relation

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

if and only if its generating series can be written

$$\sum_{n=0}^{\infty} u_n X^n = \frac{B(X)}{A(X)},$$

where $A(X)$ is the *reciprocal polynomial* or *reflected polynomial* of the characteristic polynomial

$$A(X) = X^d f(X^{-1}) = 1 - a_1 X - \cdots - a_d X^d,$$

while $B(X)$ is a polynomial of degree less than d.

Generating series : sketch of proof

Multiply

$$1 - a_1 X - \cdots - a_d X^d \quad \text{and} \quad \sum_{n=0}^{\infty} u_n X^n$$

$$(1 - a_1 X - a_2 X^2 - \cdots - a_d X^d)(u_0 + u_1 X + u_2 X^2 + \cdots)$$
$$= u_0 + (u_1 - a_1 u_0) X + (u_2 - a_1 u_1 - a_2 u_0) X^2 + \cdots$$

The coefficient of X^j is

$$\begin{cases} u_j - a_1 u_{j-1} - \cdots - a_j u_0 & \text{for } 0 \le j \le d-1, \\ u_j - a_1 u_{j-1} - \cdots - a_d u_{j-d} = 0 & \text{for } j \ge d. \end{cases}$$

(telescoping series). Hence the product is

$$B(X) = \sum_{j=0}^{d-1} (u_j - a_1 u_{j-1} - \cdots - a_j u_0) X^j.$$

Exponential generating series and homogeneous linear differential equations

A sequence $(u_n)_{n\geq 0}$ satisfies the linear recurrence sequence

$(\star)$ $$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

if and only if its exponential power series

$$y(x) = \sum_{n\geq 0} u_n \frac{x^n}{n!}$$

satisfies the homogeneous linear differential equations

$$y^{(d)} - a_1 y^{(d-1)} - a_2 y^{(d-2)} - \cdots - a_{d-1} y' - a_d y = 0.$$

Matrix notation for a linear recurrence sequence

The linear recurrence sequence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

can be written

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}.$$

Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The characteristic polynomial of A is the characteristic polynomial of the sequence :

$$\det(I_d X - A) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

By induction

$$U_n = A^n U_0.$$

Powers of matrices

Let $A = (a_{ij})_{1 \le i,j \le d} \in \mathrm{GL}_{d \times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in $\mathbb{K}$ and nonzero determinant. For $n \ge 0$, let

$$A^n = \left(a_{ij}^{(n)}\right)_{1 \le i,j \le d}.$$

Then each of the d^2 sequences $\left(a_{ij}^{(n)}\right)_{n \ge 0}$, $(1 \le i, j \le d)$ is a linear recurrence sequence.

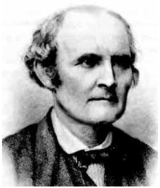
In particular the sequence $\left(\mathrm{Tr}(A^n)\right)_{n \ge 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix A.

Theorem of Cayley – Hamilton

$$A^d = a_1 A^{d-1} + \cdots + a_{d-1} A + a_d I_d$$

where I_d is the identity $d \times d$ matrix.

Arthur Cayley
(1821 – 1895)



Sir William Rowan Hamilton
(1805 – 1865)

Hence, for $n \geq 0$,

$$A^{n+d} = a_1 A^{n+d-1} + \cdots + a_{d-1} A^{n+1} + a_d A^n.$$

It follows that each entry $a_{ij}(n)$, $1 \leq i, j \leq d$, satisfies a linear recurrence relation, the same for all i, j.

Conversely :

Given a linear recurrence sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$, there exist an integer $d \geq 1$ and a matrix $A \in \mathrm{GL}_d(\mathbb{K})$ such that, for each $n \geq 0$,

$$u_n = a_{11}^{(n)}.$$

The characteristic polynomial of A is the characteristic polynomial of the linear recurrence sequence.

Everest G., van der Poorten A., Shparlinski I., Ward T. – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104.

Linear recurrence sequences : simple roots

A basis of $E_{\underline{a}}$ over $\mathbb{K}$ is obtained by attributing to the initial values $u_0, \ldots, u_{d-1}$ the values given by the canonical basis of $\mathbb{K}^d$.

Given γ in $\mathbb{K}^\times$, a necessary and sufficient condition for a sequence $(\gamma^n)_{n\geq 0}$ to satisfy ($\star$) is that γ is a root of the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

If this polynomial has d distinct roots $\gamma_1, \ldots, \gamma_d$ in $\mathbb{K}$,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \neq \gamma_j,$$

then a basis of $E_{\underline{a}}$ over $\mathbb{K}$ is given by the d sequences $({\gamma_i}^n)_{n\geq 0}$, $i = 1, \ldots, d$.

Dominant root

$$u_n = \lambda_1 {\gamma_1}^n + \cdots + \lambda_d {\gamma_d}^n$$

If $\gamma_1 > \max_{2\le i\le d} |\gamma_i|$ and $\lambda_1 \ne 0$, then $u_n \sim \lambda_1 {\gamma_1}^n$ as $n \to \infty$.

Exercise (Pierre Arnoux) :
$u_n = u_{n-1} + u_{n-2}$,
$u_0 = 1$, $u_1 = (1-\sqrt{5})/2$.
Use a calculator to estimate u_{100}.

Pierre Arnoux

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/Ariane5VI.pdf

Excel file Column A : n Column B : u_n

$$u_0 = 1,\ u_1 = (1-\sqrt{5})/2, \quad u_n = u_{n-1} + u_{n-2}$$

	A	B
1	0	1
2	1	`=(1-RACINE(5))/2`

	A	B
1	0	1
2	1	-0.618034

	A	B
1	0	1
2	1	-0.618034
3	=1+A2	=B1+B2

	A	B
1	0	1
2	1	-0.618034
3	2	0.38196601

Copy $A3$ $B3$ down

Excel file : u_1 to u_{39}

1	-0,61803399
2	0,381966011
3	-0,23606798
4	0,145898034
5	-0,09016994
6	0,05572809
7	-0,03444185
8	0,021286236
9	-0,01315562
10	0,008130619
11	-0,005025
12	0,00310562
13	-0,00191938
14	0,001186241
15	-0,00073314
16	0,000453104
17	-0,00028003
18	0,00017307
19	-0,00010696
20	6,6107E-05
21	-4,0856E-05
22	2,52506E-05
23	-1,5606E-05
24	9,64487E-06
25	-5,9609E-06
26	3,68401E-06
27	-2,2769E-06
28	1,40715E-06
29	-8,6971E-07
30	5,37445E-07
31	-3,3226E-07
32	2,05185E-07
33	-1,2708E-07
34	7,8109E-08
35	-4,8967E-08
36	2,91423E-08
37	-1,9824E-08
38	9,31784E-09
39	-1,0507E-08

Excel file : u_1 to u_{39}

1	-0,61803399
2	0,381966011
3	-0,23606798
4	0,145898034
5	-0,09016994
6	0,05572809
7	-0,03444185
8	0,021286236
9	-0,01315562
10	0,008130619
11	-0,005025
12	0,00310562
13	-0,00191938
14	0,001186241
15	-0,00073314
16	0,000453104
17	-0,00028003
18	0,00017307
19	-0,00010696
20	6,6107E-05
21	-4,0856E-05
22	2,52506E-05
23	-1,5606E-05
24	9,64487E-06
25	-5,9609E-06
26	3,68401E-06
27	-2,2769E-06
28	1,40715E-06
29	-8,6971E-07
30	5,37445E-07
31	-3,3226E-07
32	2,05185E-07
33	-1,2708E-07
34	7,8109E-08
35	-4,8967E-08
36	2,91423E-08
37	-1,9824E-08
38	9,31784E-09
39	-1,0507E-08

Excel answer : $u_{100} = -19\,241.901\,833\,167\ldots$

38	9,31784E-09
39	-1,05066E-08
40	-1,18878E-09
41	-1,16954E-08
42	-1,28842E-08
43	-2,45796E-08
44	-3,74637E-08
45	-6,20433E-08
46	-9,9507E-08
47	-1,6155E-07
48	-2,61057E-07
49	-4,22608E-07
50	-6,83665E-07
51	-1,10627E-06
52	-1,78994E-06

85	-14,10695857
86	-22,82553845
87	-36,93249702
88	-59,75803546
89	-96,69053248
90	-156,4485679
91	-253,1391004
92	-409,5876684
93	-662,7267688
94	-1072,314437
95	-1735,041206
96	-2807,355643
97	-4542,396849
98	-7349,752492
99	-11892,14934
100	-19241,90183

u_{39} : -10^{-8} vs $-7 \cdot 10^{-9}$

Numerical values :

$$\tilde{\Phi} = -0.618\,033\,988\,749\,895\ldots,$$

$$\log|\tilde{\Phi}| = -0.481\,211\,825\,059\,603\,4\ldots$$

$$u_{39} = \tilde{\Phi}^{39} = -\mathrm{e}^{-18.767\,261\,177\,324{,}453\ldots} = -7.071\,019\ldots 10^{-9}.$$

PARI GP : https://pari.math.u-bordeaux.fr/ PARI GP

Comparing the excel values with the exact values

	excel value	exact value
30	5,37445E-07	5,3749E-07
31	-3,32261E-07	-3,32187E-07
32	2,05185E-07	2,05303E-07
33	-1,27076E-07	-1,26884E-07
34	7,8109E-08	7,84188E-08
35	-4,89667E-08	-4,84655E-08
36	2,91423E-08	2,99533E-08
37	-1,98244E-08	-1,85122E-08
38	9,31784E-09	1,14411E-08
39	-1,05066E-08	-7,07102E-09
40	-1,18878E-09	4,37013E-09
41	-1,16954E-08	-2,70089E-09

Exact value of u_n

Observations : The signs of u_n alternate, the absolute value is decreasing.

Set $\tilde{\Phi} = (1 - \sqrt{5})/2$. Notice that $\tilde{\Phi}$ is a root of $X^2 - X - 1$, the other root is $\Phi = (1 + \sqrt{5})/2$, the golden ratio.

From $\tilde{\Phi}^n = \tilde{\Phi}^{n-1} + \tilde{\Phi}^{n-2}$ with $u_0 = 1$, $u_1 = \tilde{\Phi}$, we deduce by induction $u_n = \tilde{\Phi}^n$.

Exact value of u_{100}

The answer to initial question is

$$u_{100} = \tilde{\Phi}^{100}$$

$\tilde{\Phi} = -0.618\,033\,988\,749\,895\ldots$, $\log|\tilde{\Phi}| = -0.481\,211\,825\,059\,603\,4\ldots$

$\tilde{\Phi}^{100} = \mathrm{e}^{-48.121\,182\,505\,960\,34\ldots} = 1.262\,513\,338\,064\ldots 10^{-21}$.

For $u_0 = 1$ and $u_1 = \tilde{\Phi} + \varepsilon$ we have

$$u_n = \frac{\varepsilon}{\Phi - \tilde{\Phi}} \Phi^n + \left(1 - \frac{\varepsilon}{\Phi - \tilde{\Phi}}\right) \tilde{\Phi}^n$$

with $\tilde{\Phi}^n \to 0$. Hence $u_n \to 0$ if and only if $\epsilon = 0$

The linear recurrence sequence $u_n = u_{n-1} + u_{n-2}$

From the two solutions Φ^n and $\tilde{\Phi}^n$ one deduces that any solution is of the form $u_n = a\Phi^n + b\tilde{\Phi}^n$.

Since $|\Phi| > 1$, the term Φ^n tends to ∞.

Since $|\tilde{\Phi}| < 1$, the term $b\tilde{\Phi}^n$ tends to 0.

If $a = 0$, then $u_n = b\tilde{\Phi}^n$ tends to 0 and takes alternatively positive and negative values.

If $a \neq 0$, then $|u_n|$ tends to infinity like $a\Phi^n$.

If two consecutive terms are of the same sign, then $a \neq 0$, all the next ones have the same sign and $|u_n|$ tends to infinity.

Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root γ.
The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

$$u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2} \quad \Longleftrightarrow \quad u_n = (\lambda + \mu n)\gamma^n.$$

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that γ is a root of multiplicity ≥ 2 of $f(X)$.

Linear recurrence sequences : multiple roots

In general, when the characteristic polynomial splits as

$$X^d - a_1 X^{d-1} - \cdots - a_d = \prod_{i=1}^{\ell} (X - \gamma_i)^{t_i},$$

a basis of $E_{\underline{a}}$ is given by the d sequences

$$({n^k \gamma_i}^n)_{n \geq 0}, \qquad 0 \leq k \leq t_i - 1, \quad 1 \leq i \leq \ell.$$

Polynomial combinations of powers

Given polynomials $p_1, \dots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \dots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\left(p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n\right)_{n\geq 0}$$

is a linear recurrence sequence, the minimal polynomial of which is of the form

$$X^d - a_1X^{d-1} - \cdots - a_d = \prod_{i=1}^{\ell}(X-\gamma_i)^{t_i},$$

Fact : any linear recurrence sequence is of this form.
Consequence : the sum and the product of any two linear recurrence sequences are linear recurrence sequences.
The set of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Summary

The same mathematical object occurs in a different guise :

- Linear recurrence sequences

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

- Linear combinations with polynomial coefficients of powers

$$p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n.$$

- Taylor coefficients of rational functions.

- Coefficients of power series which are solutions of homogeneous linear differential equations.

- Sequence of coefficients of powers of a matrix.

Sum of polynomial combinations of powers

For $\mathbf{u} = (u_n)_{n\geq 0}$ and $\mathbf{v} = (v_n)_{n\geq 0}$,

$$\mathbf{u} + \mathbf{v} = (u_n + v_n)_{n\geq 0}.$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two linear recurrence sequences of characteristic polynomials f_1 and f_2 respectively, then $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\frac{f_1 f_2}{\gcd(f_1, f_2)}.$$

Product of polynomial combinations of powers

For $\mathbf{u} = (u_n)_{n\geq 0}$ and $\mathbf{v} = (v_n)_{n\geq 0}$,

$$\mathbf{uv} = (u_n v_n)_{n\geq 0}.$$

If the characteristic polynomials of the two linear recurrence sequences $\mathbf{u}_1$ and $\mathbf{u}_2$ are respectively

$$f_1(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j} \quad \text{and} \quad f_2(T) = \prod_{k=1}^{\ell'} (T - \gamma'_k)^{t'_k},$$

then $\mathbf{u}_1\mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\prod_{j=1}^{\ell} \prod_{k=1}^{\ell'} (T - \gamma_j \gamma'_k)^{t_j + t'_k - 1}.$$

Linear recurrence sequences and Brahmagupta–Pell–Fermat Equation

Let d be a positive integer, not a square. The solutions $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d}y_n = (x_1 + \sqrt{d}y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d}y_1)^n + (x_1 - \sqrt{d}y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for y_n, and also for $n \geq 0$.

Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by d consecutive values.

Example of a doubly infinite sequence

The sequence of Lucas numbers beginning at $(2,1)$ is given by $u_n = \Phi^n + \tilde{\Phi}^n$, $n \geq 0$ http://oeis.org/A000032

$$2, 1, 3, 4, 7, 11, \ldots$$

The *Fibonacci-type sequence based on subtraction* of http://oeis.org/A061084

$$u_{-n} = \Phi^{-n} + \tilde{\Phi}^{-n} = (-1)^n(\Phi^n + \tilde{\Phi}^n), \quad n \geq 0$$

starts with

$$2, -1, 3, -4, 7, -11, \ldots$$

This gives the doubly infinite linear recurrence sequence

$$\ldots, -11, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, \ldots$$

Discrete version of linear differential equations

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ can be viewed as a linear map $\mathbb{N} \longrightarrow \mathbb{K}$.
Define the discrete derivative $\mathcal{D}$ by

$$\begin{array}{rccc} \mathcal{D}\mathbf{u}: & \mathbb{N} & \longrightarrow & \mathbb{K} \\ & n & \longmapsto & u_{n+1} - u_n. \end{array}$$

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ is a linear recurrence sequence if and only if there exists $Q \in \mathbb{K}[T]$ with $Q(1) \neq 1$ such that

$$Q(\mathcal{D})\mathbf{u} = 0.$$

Linear recurrence sequences are a discrete version of linear differential equations with constant coefficients.

The condition $Q(1) \neq 0$ reflects $a_d \neq 0$ – otherwise one gets *ultimately* recurrent sequences.

Skolem – Mahler – Lech Theorem

Theorem (Skolem 1934 – Mahler 1935 – Lech 1953). *Given a linear recurrence sequence, the set of indices* $n \geq 0$ *such that* $u_n = 0$ *is a finite union of arithmetic progressions.*

Thoralf Albert Skolem (1887 – 1963)

Kurt Mahler (1903 – 1988)

Christer Lech

An *arithmetic progression* is a set of positive integers of the form $\{n_0, n_0 + k, n_0 + 2k, \ldots\}$. Here, we allow $k = 0$.

A dynamical system

Let V be a finite dimensional vector space over a field of zero characteristic, H an hyperplane of V, $f : V \to V$ an endomorphism (linear map) and x an element in V.

Theorem. *If there exist infinitely many* $n \geq 1$ *such that* $f^n(x) \in H$, *then there is an (infinite) arithmetic progression of* n *for which it is so.*

A. J. Parameswaran

S.G. Dani

A dynamical system

Let V be a finite dimensional vector space over a field of zero characteristic, W a subspace of V, $f : V \to V$ an endomorphism (linear map) and x an element in V.

Corollary of the Skolem – Mahler – Lech Theorem. *The set of $n \geq 0$ such that $f^n(x) \in W$ is a finite union of arithmetic progressions.*

By induction, it suffices to consider the case where $W = H$ is an hyperplane of V.

Proof of the corollary

Choose a basis of V. The endomorphism f is given by a square $d \times d$ matrix A, where d is the dimension of V. Consider the characteristic polynomial of A, say

$$X^d - a_1 X^{d-1} - \cdots - a_{d-1} X - a_d.$$

By the Theorem of Cayley – Hamilton,

$$A^d = a_1 A^{d-1} + \cdots + a_{d-1} A + a_d I_d$$

where I_d is the identity $d \times d$ matrix.

Hyperplane membership

Let $b_1x_1 + \cdots + b_dx_d = 0$ be an equation of the hyperplane H in the selected basis of V. Let ${}^t\underline{b}$ denote the $1 \times d$ matrix $(b_1, \ldots, b_d)$ (transpose of a column matrix $\underline{b}$). Using the notation $\underline{v}$ for the $d \times 1$ (column) matrix given by the coordinates of an element v in V, the condition $v \in H$ can be written ${}^t\underline{b}\,\underline{v} = 0$.

Let x be an element in V and $\underline{x}$ the $d \times 1$ (column) matrix given by its coordinates. The condition $f^n(x) \in H$ can now be written

$$ {}^t\underline{b}A^n\underline{x} = 0. $$

The entry u_n of the 1×1 matrix ${}^t\underline{b}A^n\underline{x}$ satisfies a linear recurrence relation, hence, the Skolem – Mahler – Lech Theorem applies.

Remark on the theorem of Skolem–Mahler–Lech

T.A. Skolem treated the case $K = \mathbb{Q}$ of in 1934.

K. Mahler the case $\mathbb{K} = \overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, in 1935.

The general case was settled by C. Lech in 1953.

Finite characteristic

C. Lech pointed out in 1953 that such a result may not hold if the characteristic of $\mathbb{K}$ is positive : he gave as an example the sequence $u_n = (1+x)^n - x^n - 1$, a third-order linear recurrence over the field of rational functions in one variable over the field $\mathbb{F}_p$ with p elements, where $u_n = 0$ for $n \in \{1, p, p^2, p^3, \dots\}$. A substitute is provided by a result of Harm Derksen (2007), who proved that the zero set in characteristic p is a p–automatic sequence. Further results by Boris Adamczewski and Jason Bell.

Harm Derksen

Boris Adamczewski

Jason Bell

Polynomial-linear recurrence relation

A generalization of the Theorem of Skolem–Mahler–Lech has been achieved by Jason P. Bell, Stanley Burris and Karen Yeats who prove that the same conclusion holds if the sequence $(u_n)_{n\geq 0}$ satisfies a polynomial-linear recurrence relation

$$u_n = \sum_{i=1}^{d} P_i(n) u_{n-i}$$

where d is a positive integer and $P_1, \dots, P_d$ are polynomials with coefficient in the field $\mathbb{K}$ of zero characteristic, provided that $P_d(x)$ is a nonzero constant.

Algebraic maps, algebraic groups

There are also analogues of the Theorem of Skolem–Mahler–Lech for algebraic maps on varieties (Jason Bell).

A version of the Skolem–Mahler–Lech Theorem for any algebraic group is due to Umberto Zannier.

Jason Bell

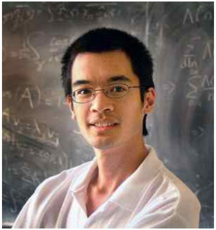
Umberto Zannier

Open problem

One main open problem related with Theorem of Skolem–Mahler–Lech is that it is not effective : explicit upper bounds for the number of arithmetic progressions, depending only on the order d of the linear recurrence sequence, are known (W.M. Schmidt, U. Zannier), but no upper bound for the arithmetic progressions themselves is known. A related open problem raised by T.A. Skolem and C. Pisot is :

> *Given an integer linear recurrence sequence, is the truth of the statement "$x_n \neq 0$ for all n" decidable in finite time ?*

Open problem



Terry Tao

T. Tao, *Effective Skolem Mahler Lech theorem*. In "Structure and Randomness : pages from year one of a mathematical blog", American Mathematical Society (2008), 298 pages.

https://mathshistory.st-andrews.ac.uk/Biographies/Tao/

http://terrytao.wordpress.com/2007/05/25/open-question-effective-skolem-mahler-lech-theorem/

Zeros of linear recurrence sequences

Jean Berstel et Maurice Mignotte. – Deux propriétés décidables des suites récurrentes linéaires Bulletin de la S.M.F., tome 104 (1976), p. 175-184.
http://www.numdam.org/item?id=BSMF_1976__104__175_0
Given a linear recurrence sequence with integer coefficients ; are there finitely or infinitely many zeroes ?

Philippe Robba. – Zéros de suites récurrentes linéaires. Groupe de travail d'analyse ultramétrique (1977-1978) Volume : 5, page 1-5.

L. Cerlienco, M. Mignotte, F. Piras. Suites récurrentes linéaires. Propriétés algébriques et arithmétiques. L'Enseignement Mathématique **33** (1987).

Zeros of linear recurrence sequences

Maurice Mignotte Propriétés arithmétiques des suites récurrentes linéaires. Besançon, 1989
http://pmb.univ-fcomte.fr/1989/Mignotte.pdf

E. Bavencoffe and J-P. Bézivin *Une famille remarquable de suites recurrentes lineaires*. – Monatshefte für Mathematik, (1995) **120** 3, 189–203

Karim Samake. – Suites récurrentes linéaires, problème d'effectivité. Inst. de Recherche Math. Avancée, 1996 - 62 pages

Reference

Everest, Graham ; van der Poorten, Alf ; Shparlinski, Igor ; Ward, Tom – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104.

`1290 references.`

Berstel's sequence

http://oeis.org/A007420

$$0,\ 0,\ 1,\ 2,\ 0,\ -4,\ 0,\ 16,\ 16,\ -32,\ -64,\ 64,\ 256,\ 0,\ -768,\ \dots$$

Jean Berstel

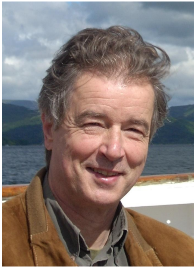
$b_0 = b_1 = 0$, $b_2 = 1$,
$b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n$
for $n \geq 0$.

Linear recurrence sequence of order 3 with exactly 6 zeros :
$n = 0,\ 1,\ 4,\ 6,\ 13,\ 52.$

http://www-igm.univ-mlv.fr/~berstel/

Ternary linear recurrences

Berstel's sequence is a linear recurrence sequence of order 3 with 6 zeroes.



Frits Beukers

Frits Beukers (1991) : up to trivial transformation, any other linear recurrence of order 3 with finitely many zeroes has at most 5 zeros.

Edgard Bavencoffe and Jean-Paul Bézivin

Let $n \geq 2$. The sequence with initial values

$$u_0 = 1,\ u_1 = \cdots = u_{n-1} = 0$$

satisfying the recurrence relation of order n with characteristic polynomial

$$\frac{X^{n+1} - (-2)^{n-1}X + (-2)^n}{X+2}$$

has at least

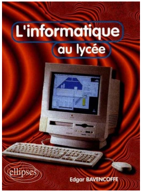
$$\frac{n(n+1)}{2} - 1$$

zeroes.

Edgard Bavencoffe and Jean-Paul Bézivin

For $n = 3$ one obtains Berstel's sequence which happens to have an extra zero.

$$\frac{X^4 + 4X - 8}{X + 2} = X^3 - 2X^2 + 4X - 4.$$



Edgard Bavencoffe

Jean-Paul Bézivin

Berstel's sequence

$$0,\ 0,\ 1,\ 2,\ 0,\ -4,\ 0,\ 16,\ 16,\ -32,\ -64,\ 64,\ 256,\ 0,\ -768,\ \ldots$$

$b_0 = b_1 = 0$, $b_2 = 1$, $b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n$ for $n \geq 0$.

Maurice Mignotte

The equation $b_m = \pm b_n$ has exactly 21 solutions (m, n) with $m \neq n$.

The equation $b_n = \pm 2^r 3^s$ has exactly 44 solutions (n, r, s).

Hankel determinants

To test an arbitrary sequence $\mathbf{u} = (u_n)_{n\geq 0}$ of elements of a field $\mathbb{K}$ for the property of being a linear recurrence sequence, consider the Hankel determinants

$$\Delta_{N,d}(\mathbf{u}) = \det\left(u_{d+i+j}\right)_{0\leq i,j\leq N}.$$

Hermann Hankel

1839–1873

The sum $f(z) = \displaystyle\sum_{n=0}^{\infty} u_n z^n$ represents a rational function if and only if for some d, $\Delta_{N,d}(\mathbf{u}) = 0$ for all sufficiently large N.

https://mathshistory.st-andrews.ac.uk/Biographies/Hankel/

Hankel determinants

Alan Haynes, Wadim Zudilin. – Hankel determinants of zeta values
(Submitted on 7 Oct 2015)

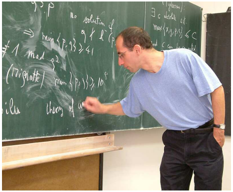
`Abstract:` *We study the asymptotics of Hankel determinants constructed using the values* $\zeta(an+b)$ *of the Riemann zeta function at positive integers in an arithmetic progression. Our principal result is a Diophantine application of the asymptotics.*

Alan Haynes

Wadim Zudilin

Perfect powers in the Fibonacci sequence

Yann Bugeaud, Maurice Mignotte, Samir Siksek (2004) : The only perfect powers (squares, cubes, etc.) in the Fibonacci sequence are 1, 8 and 144.



Yann Bugeaud

Maurice Mignotte

Samir Siksek

Powers in recurrence sequences

Mike Bennett

M. A. Bennett, Powers in recurrence sequences : Pell equations, Trans. Amer. Math. Soc. **357** (2005), 1675-1691.

http://www.math.ubc.ca/~bennett/paper31.pdf

Intersection of two linear recurrence sequences

Let $\mathbf{u} = (u_n)_{n \geq 0}$ and $\mathbf{v} = (v_m)_{m \geq 0}$ be two linear recurrence sequences. The intersection is given by the set of (n, m) such that $u_n = v_m$.

Examples :

- *Empty intersection :*

  $\mathbf{u} = (2, 4, 8, 16, \ldots)$, $u_n = 2^{n+1}$ $(n \geq 0)$

  $\mathbf{v} = (3, 9, 27, 81, \ldots)$, $v_m = 3^{m+1}$ $(m \geq 0)$
- *Non empty finite intersection :*

  $\mathbf{u} = (1, 2, 4, 8, 16, \ldots)$, $u_n = 2^n$ $(n \geq 0)$

  $\mathbf{v} = (1, 3, 9, 27, 81, \ldots)$, $v_m = 3^m$ $(m \geq 0)$
- *Infinite intersection :*

  $\mathbf{u} = (1, 2, 4, 8, 16, \ldots)$, $u_n = 2^n$ $(n \geq 0)$

  $\mathbf{v} = (1, 4, 16, 64, 256, \ldots)$, $v_m = 4^m$ $(m \geq 0)$

A non trivial example : $u_n = P(n), \quad v_m = b^m.$

Let $P \in \mathbb{Z}[X]$ and $b \in \mathbb{Z}$. Consider the set (n, m) of pairs of nonnegative integers satisfying the exponential Diophantine equation

$$P(n) = b^m.$$

$P \neq 0$, $b = 0$: finite set.

$P(X) = (X - a)^d$, $b \neq 0$: infinite set : $(n - a)^d = b^m$.

P has at least two distinct roots in $\mathbb{C}$, $b \neq 0$: finite set.

$u_n = v_m$

$$u_n = a_1(n){\alpha_1}^n + \cdots + a_d(n){\alpha_d}^n, \quad v_m = b_1(m){\beta_1}^m + \cdots + b_\ell(m)\alpha_\ell^m.$$

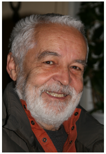
The equation $u_n = v_m$ is

$$a_1(n){\alpha_1}^n + \cdots + a_d(n){\alpha_d}^n = b_1(m){\beta_1}^m + \cdots + b_\ell(m)\alpha_\ell^m.$$

S–unit equation, Schmidt's Subspace Theorem.

Non effective.

Intersection of linear recurrence sequences



M. Mignotte

M. Laurent

Maurice Mignotte. *Intersection des images de certaines suites récurrentes linéaires* (French).
Theoret. Comput. Sci. **7** (1978), no. 1, 117–122.
https://doi.org/10.1016/0304-3975(78)90043-9

Michel Laurent. *Équations exponentielles–polynômes et suites récurrentes linéaires* (French).
Astérisque **147-148** (1987), 121–139, 343–344. Journées arithmétiques de Besançon (1985).
(II) *Journal of Number Theory* **31** 1(1989), 24–53.
https://doi.org/10.1016/0022-314X(89)90050-4

Intersection of linear recurrence sequences

H.P. Schlickewei

W.M. Schmidt

M. Bennett

Á. Pintér

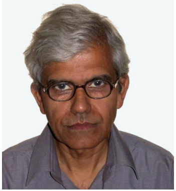
Hans Peter Schlickewei and Wolfgang M. Schmidt.
The intersection of recurrence sequences. Acta Arith. **72** 1 (1995), 1–44.
http://matwbn.icm.edu.pl/ksiazki/aa/aa72/aa7211.pdf

Michael A. Bennett and Ákos Pintér.
Intersection of linear recurrences. Proc. Amer. Math. Soc. **143**, 6 (2015), 2347–2353.
https://www.math.ubc.ca/~bennett/BePi-PAMS-2015.pdf

Exponential Diophantine equations

A. Schinzel

R. Tjideman

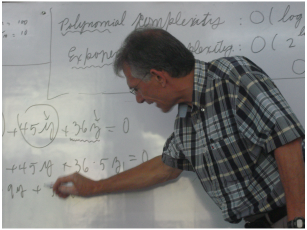


T.N. Shorey

Andrzej Schinzel, & Robert Tijdeman. *On the equation* $y^m = P(x)$. Acta Arith. **31** (1976), no. 2, 199–204.

Tarlok N. Shorey & Robert Tijdeman. *Exponential Diophantine equations*. Cambridge Tracts in Mathematics, **87**. Cambridge University Press, Cambridge, 1986.

Joint work with Claude Levesque



Claude Levesque

Linear recurrence sequences and twisted binary forms. Proceedings of the International Conference on Pure and Applied Mathematics ICPAM–GOROKA 2014, South Pacific Journal of Pure and Applied Mathematics, vol. **2**, No 3 (2015), 65–83. `arXiv:1802.05154 [math.NT]`.

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/ProcConfPNG2014.pdf

Families of binary forms

Consider a binary form $F_0(X, Y) \in \mathbb{C}[X, Y]$ which satisfies $F_0(1, 0) = 1$. We write it as

$$F_0(X, Y) = X^d + a_1 X^{d-1} Y + \cdots + a_d Y^d \; = \prod_{i=1}^{d} (X - \alpha_i Y).$$

Let $\epsilon_1, \ldots, \epsilon_d$ be d nonzero complex numbers not necessarily distinct. Twisting F_0 by the powers ${\epsilon_1}^n, \ldots, {\epsilon_d}^n$ ($n \in \mathbb{Z}$) boils down to considering the family of binary forms

$$F_n(X, Y) = \prod_{i=1}^{d} (X - \alpha_i {\epsilon_i}^n Y),$$

which we write as

$$X^d - U_1(n) X^{d-1} Y + \cdots + (-1)^d U_d(n) Y^d.$$

Therefore

$$U_h(0) = (-1)^h a_h \qquad (1 \leq h \leq d).$$

Families of Diophantine equations

With Claude Levesque, we considered some families of diophantine equations

$$F_n(x, y) = m$$

obtained in the same way from a given irreducible form $F(X, Y)$ with coefficients in $\mathbb{Z}$, when $\epsilon_1, \dots, \epsilon_d$ are algebraic units and when the algebraic numbers $\alpha_1\epsilon_1, \dots, \alpha_d\epsilon_d$ are Galois conjugates with $d \geq 3$.

Theorem. *Let $\mathbb{K}$ be a number field of degree $d \geq 3$, S a finite set of places of $\mathbb{K}$ containing the places at infinity. Denote by $\mathcal{O}_S$ the ring of S–integers of $\mathbb{K}$ and by ${\mathcal{O}_S}^\times$ the group of S–units of $\mathbb{K}$. Assume $\alpha_1, \dots, \alpha_d, \epsilon_1, \dots, \epsilon_d$ belong to $\mathbb{K}^\times$ Then there are only finitely many (x, y, n) in $\mathcal{O}_S \times \mathcal{O}_S \times \mathbb{Z}$ satisfying*

$$F_n(x, y) \in {\mathcal{O}_S}^\times, \quad xy \neq 0 \quad \textit{and} \quad \mathrm{Card}\{\alpha_1{\epsilon_1}^n, \dots, \alpha_d{\epsilon_d}^n\} \geq 3.$$

Families of Diophantine equations

Each of the sequences $\big(U_h(n)\big)_{n\in\mathbb{Z}}$ coming from the coefficients of the relation

$$F_n(X,Y) = X^d - U_1(n)X^{d-1}Y + \cdots + (-1)^d U_d(n)Y^d$$

is a linear recurrence sequence.
For example, for $n \in \mathbb{Z}$,

$$U_1(n) = \sum_{i=1}^{d} \alpha_i {\epsilon_i}^n, \quad U_d(n) = \prod_{i=1}^{d} \alpha_i {\epsilon_i}^n.$$

For $1 \le h \le d$, the sequence $\big(U_h(n)\big)_{n\in\mathbb{Z}}$ is a linear combination of the sequences

$$\big((\epsilon_{i_1}\cdots\epsilon_{i_h})^n\big)_{n\in\mathbb{Z}}, \quad (1 \le i_1 < \cdots < i_h \le d).$$

Some units of Bernstein and Hasse

Let t and s be two positive integers, D an integer ≥ 1, and $c \in \{-1, +1\}$. Let $\omega > 1$ satisfy

$$\omega^{st} = D^{st} + c,$$

where it is assumed that $\mathbb{Q}(\omega)$ is of degree st.
Consider

$$\alpha = D - \omega, \quad \epsilon = D^t - \omega^t.$$

L. Bernstein and H. Hasse noticed that α and ϵ are units of degree st and s respectively, and showed that these units can be obtained from the Jacobi–Perron algorithm. H.-J. Stender proved that for $s = t = 2$, $\{\alpha, \epsilon\}$ is a fundamental system of units of the quartic field $\mathbb{Q}(\omega)$.

Helmut Hasse

$D > 0$, $s \geq 1$, $t \geq 1$,
$c \in \{-1, +1\}$, $\omega > 0$,

$$\omega^{st} = D^{st} + c,$$

$$\alpha = D - \omega,$$

$$\epsilon = D^t - \omega^t.$$

Helmut Hasse
1898-1979

$$(\alpha - D)^{st} = (-1)^{st}(D^{st} + c).$$

Diophantine equations associated with some units of Bernstein and Hasse

The irreducible polynomial of α is $F_0(X,1)$, with

$$F_0(X,Y) = (X - DY)^{st} - (-1)^{st}(D^{st} + c)Y^{st}.$$

For $n \in \mathbb{Z}$, the binary form $F_n(X,Y)$, obtained by twisting $F_0(X,Y)$ with the powers ϵ^n of ϵ, is the homogeneous version of the irreducible polynomial $F_n(X,1)$ of $\alpha\epsilon^n$. So F_n depends of the parameters n, D, s, t and c.

Theorem (LW). *Suppose $st \geq 3$. There exists an effectively computable constant κ, depending only on D, s and t, with the following property. Let m, a, x, y be rational integers satisfying $m \geq 2$, $xy \neq 0$, $[\mathbb{Q}(\alpha\epsilon^a) : \mathbb{Q}] = st$ and*

$$|F_n(x,y)| \leq m.$$

Then

$$\max\{\log|x|, \log|y|, |n|\} \leq \kappa \log m.$$

Bases of the space of linear recurrence sequences

Given $a_1, \ldots, a_d$ with $a_d \neq 0$, consider the vector space of linear recurrence sequences satisfying, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Assuming the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d$$

of the recurrence splits completely in $\mathbb{K}$,

$$f(X) = \prod_{j=1}^{\ell} (X - \gamma_j)^{t_i}$$

we have two bases. The first one given by the initial conditions $(u_0, \ldots, u_{d-1})$, and the second one is given by the sequences

$$(n^i {\gamma_j}^n)_{n \geq 0}, \quad 0 \leq i \leq t_j - 1, \; 1 \leq j \leq \ell.$$

Change of basis

The matrix of change of bases is

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_\ell \end{pmatrix}$$

where

$$M_j = \begin{pmatrix} 1 & \gamma_j & {\gamma_j}^2 & \dots & {\gamma_j}^{t_j-1} & {\gamma_j}^{t_j} & \dots & {\gamma_j}^{d-1} \\ 0 & 1 & \binom{2}{1}\gamma_j & \dots & \binom{t_j-1}{1}{\gamma_j}^{t_j-2} & \binom{t_j}{1}{\gamma_j}^{t_j-1} & \dots & \binom{d-1}{1}{\gamma_j}^{d-2} \\ 0 & 0 & 1 & \dots & \binom{t_j-1}{2}{\gamma_j}^{t_j-3} & \binom{t_j}{2}{\gamma_j}^{t_j-2} & \dots & \binom{d-1}{2}{\gamma_j}^{d-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \binom{t_j}{t_j-1}\gamma_j & \dots & \binom{d-1}{t_j-1}{\gamma_j}^{d-t_j} \end{pmatrix}.$$

Exponential polynomials

The sequence of derivatives of an exponential polynomial evaluated at one point satisfies a linear recurrence relation.
Let $p_1(z), \ldots, p_\ell(z)$ be nonzero polynomials of $\mathbb{C}[z]$ of degrees smaller than $t_1, \ldots, t_\ell$ respectively. Let $\gamma_1, \ldots, \gamma_\ell$ be distinct complex numbers. Suppose that the function

$$F(z) = p_1(z)e^{\gamma_1 z} + \cdots + p_\ell(z)e^{\gamma_\ell z}$$

is not identically 0. Then its vanishing order at a point z_0 is smaller than or equal to $t_1 + \cdots + t_\ell - 1$.
In other terms, when the complex numbers γ_j are distinct, the determinant

$$\left| \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^a \left(z^i e^{\gamma_j z}\right)_{z=0} \right|_{\substack{0 \le i \le t_j - 1,\ 1 \le j \le \ell \\ 0 \le a \le d-1}}$$

is different from 0. This is no surprise that we come across the determinant of the matrix M.

The matrix M

The determinant of M is

$$\det M = \prod_{1 \le i < j \le \ell} (\gamma_j - \gamma_i)^{t_i t_j}.$$

For $1 \le j \le \ell$, $0 \le i \le t_j - 1, \;\; 0 \le k \le d-1$, the $(s_j + i, k)$ entry of the matrix M is

$$\frac{1}{i!} \left(\frac{\mathrm{d}}{\mathrm{d}T} \right)^i T^k \Bigg|_{T=\gamma_j} = \binom{k}{i} {\gamma_j}^{k-i}.$$

The matrix M is associated with the linear system of d equations in d unknowns which amounts to finding a polynomial $f \in K[z]$ of degree $< d$ for which the d numbers

$$\frac{\mathrm{d}^i f}{\mathrm{d}z^i}(\gamma_j), \qquad (1 \le j \le \ell, \; 0 \le i \le t_j - 1)$$

take prescribed values.

Interpolation

Let γ_j $(1 \leq j \leq \ell)$ be distinct elements in $\mathbb{K}$, t_j $(1 \leq j \leq \ell)$ be positive integers, η_{ij} $(1 \leq j \leq \ell,\ 0 \leq i \leq t_j - 1)$ be elements in $\mathbb{K}$. Set $d = t_1 + \cdots + t_\ell$. There exists a unique polynomial $f \in \mathbb{K}[z]$ of degree $< d$ satisfying

$$\frac{\mathrm{d}^i f}{\mathrm{d}z^i}(\gamma_j) = \eta_{ij}, \qquad (1 \leq j \leq \ell,\ 0 \leq i \leq t_j - 1).$$

Truncated Taylor expansion

Let $g \in \mathbb{K}(z)$, let $z_0 \in \mathbb{K}$ and let $t \geq 1$. Assume z_0 is not a pole of g. We set

$$T_{g,z_0,t}(z) = \sum_{i=0}^{t-1} \frac{\mathrm{d}^i g}{\mathrm{d}z^i}(z_0) \frac{(z - z_0)^i}{i!}.$$

In other words, $T_{g,z_0,t}$ is the unique polynomial in $\mathbb{K}[z]$ of degree $< t$ such that there exists $r(z) \in \mathbb{K}(z)$ having no pole at z_0 with

$$g(z) = T_{g,z_0,t}(z) + (z - z_0)^t r(z).$$

Notice that if g is a polynomial of degree $< t$, then $g = T_{g,z_0,t}$ for any $z_0 \in \mathbb{K}$.

Explicit solution to the interpolation problem

For $j = 1, \ldots, \ell$, define

$$h_j(z) = \prod_{\substack{1 \le k \le \ell \\ k \ne j}} \left(\frac{z - \gamma_k}{\gamma_j - \gamma_k} \right)^{t_k} \quad \text{and} \quad p_j(z) = \sum_{i=0}^{t_j - 1} \eta_{ij} \frac{(z - \gamma_j)^i}{i!}.$$

Then the solution f of the interpolation problem

$$\frac{\mathrm{d}^i f}{\mathrm{d} z^i}(\gamma_j) = \eta_{ij}, \qquad (1 \le j \le \ell,\ 0 \le i \le t_j - 1).$$

is given by

$$f = \sum_{j=1}^{\ell} h_j T_{\frac{p_j}{h_j}, \gamma_j, t_j}.$$

Further references

H.P. Schlickewei, W.M. Schmidt and MW.
Zeros of linear recurrence sequences.
Manuscripta Mathematica, **98** N°2 (1999), 225–241.
http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/ManuscriptaMath98-1999.pdf

C. Levesque and MW.
Linear recurrence sequences and twisted binary forms.
Proceedings of the International Conference on Pure and Applied Mathematics ICPAM–GOROKA 2014,
South Pacific Journal of Pure and Applied Mathematics, vol. **2**, No 3 (2015), 65–83.
arXiv:1802.05154 [math.NT].

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान बरहमपुर

Indian Institute of Science Education and Research Berhampur

Mathematics Online Seminar Series 2020

Linear recurrence sequences: part II

Michel Waldschmidt

Institut de Mathématiques de Jussieu — Sorbonne Université

http://www.math.jussieu.fr/~michel.waldschmidt/